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Self-adjusting surrogate-assisted optimization techniques for expensive constrained black box problems

Bagheri, S.

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Bibliography

- [1] Abbot, I.H.A., von Doenhoff, A.E.: Theory of wing sections, including a summary of airfoil data. Dover Publications, New York (1959)
- [2] Abraham, F., Behr, M., Heinkenschloss, M.: Shape optimization in steady blood flow: a numerical study of non-newtonian effects. *Computer methods in biomechanics and biomedical engineering* 8(2), 127–137 (2005)
- [3] Arato, K., Takashima, T.: A study on reduction of heat loss by optimizing combustion chamber shape. *SAE International Journal of Engines* 8(2), 596–608 (2015)
- [4] Arnold, D.V.: An active-set evolution strategy for optimization with known constraints. In: *Parallel Problem Solving from Nature*. pp. 192–202. Springer (2016)
- [5] Arnold, D.V.: Reconsidering constraint release for active-set evolution strategies. In: *Proceedings of the Genetic and Evolutionary Computation Conference*. pp. 665–672. GECCO '17, ACM, New York, NY, USA (2017), <http://doi.acm.org/10.1145/3071178.3071294>
- [6] Arnold, D.V., Hansen, N.: A (1+1)-CMA-ES for constrained optimisation. In: *Proceedings of the 14th conference on Genetic and evolutionary computation (GECCO)*. pp. 297–304. ACM (2012)
- [7] Arnold, D., Hansen, N.: A (1+1)-CMA-ES for Constrained Optimisation. In: Soule, T., Moore, J.H. (eds.) *Proceedings of the 14th International Conference on Genetic and Evolutionary Computation*. pp. 297–304. ACM (2012)
- [8] Audet, C., Booker, A.J., Dennis, Jr, Frank, P.D., Moore, D.W.: A Surrogate-Model-Based Method For Constrained Optimization. In: *AIAA/ISSMO*. pp. 2000–4891 (2000)

-
- [9] Auger, A.: Benchmarking the (1+1) evolution strategy with one-fifth success rule on the bbob-2009 function testbed. In: ACM-GECCO Genetic and Evolutionary Computation Conference (2009)
- [10] Auger, A., Teytaud, O.: Continuous lunches are free plus the design of optimal optimization algorithms. *Algorithmica* 57(1), 121–146 (2010)
- [11] Bäck, T.: *Evolutionary Algorithms in Theory and Practice: Evolution Strategies, Evolutionary Programming, Genetic Algorithms*. Oxford University Press, Oxford, UK (1996)
- [12] Bäck, T., Foussette, C., Krause, P.: *Contemporary evolution strategies*. Springer (2013)
- [13] Bäck, T., Hoffmeister, F., Schwefel, H.: A survey of evolution strategies. In: Belew, R.K., Booker, L.B. (eds.) *Proceedings of the 4th International Conference on Genetic Algorithms*, San Diego, CA, USA. pp. 2–9. Morgan Kaufmann (1991)
- [14] Bagheri, S., Konen, W., Allmendinger, R., Branke, J., Deb, K., Fieldsend, J., Quagliarella, D., Sindhya, K.: Constraint handling in efficient global optimization. In: *Proc. Genetic and Evolutionary Computation Conference GECCO'17*. pp. 673–680. ACM, New York (2017)
- [15] Bagheri, S., Konen, W., Bäck, T.: Online selection of surrogate models for constrained black-box optimization. In: Jin, Y. (ed.) *IEEE SSCI'2016*, Athens. p. 1 (2016)
- [16] Bagheri, S., Konen, W., Bäck, T.: Equality constraint handling for surrogate-assisted constrained optimization. In: Tan, K.C. (ed.) *WCCI'2016*, Vancouver. p. 1. IEEE (2016), <http://www.gm.fh-koeln.de/~konen/Publikationen/Bagh16-WCCI.pdf>
- [17] Bagheri, S., Konen, W., Bäck, T.: Comparing Kriging and radial basis function surrogates. In: Hoffmann, F., Hüllermeier, E. (eds.) *Proc. 27. Workshop Computational Intelligence*. pp. 243–259. Universitätsverlag Karlsruhe (November 2017)
- [18] Bagheri, S., Konen, W., Bäck, T.: How to solve the dilemma of margin-based equality handling methods. In: Hoffmann, F., Hüllermeier, E., Mikut, R. (eds.) *Proc. 28. Workshop Computational Intelligence*. pp. 257–270. KIT Scientific Publishing, Karlsruhe (November 2018)

BIBLIOGRAPHY

- [19] Bagheri, S., Konen, W., Emmerich, M., Bäck, T.: Self-adjusting parameter control for surrogate-assisted constrained optimization under limited budgets. *Applied Soft Computing* 61, 377 – 393 (2017)
- [20] Bagheri, S., Konen, W., Foussette, C., Krause, P., Bäck, T., Koch, P.: SACO-BRA: Self-adjusting constrained black-box optimization with RBF. In: Hoffmann, F., Hüllermeier, E. (eds.) *Proc. 25. Workshop Computational Intelligence*. pp. 87–96. Universitätsverlag Karlsruhe (2015)
- [21] Bajer, L., Pitra, Z., Holeňa, M.: Benchmarking Gaussian processes and random forests surrogate models on the BBOB noiseless testbed. In: *Proc. Genetic and Evolutionary Computation Conf. GECCO’15*. pp. 1143–1150. ACM, New York (2015)
- [22] Basudhar, A., Dribusch, C., Lacaze, S., Missoum, S.: Constrained Efficient Global Optimization with Support Vector Machines. *Structural and Multidisciplinary Optimization* 46(2), 201–221 (2012)
- [23] Bates, D.M., Watts, D.G.: *Nonlinear regression analysis and its applications*. Wiley series in probability and mathematical statistics, Wiley, New York [u.a.] (1988)
- [24] Beale, E.M.L.: On an iterative method for finding a local minimum of a function of more than one variable. No. 25, *Statistical Techniques Research Group, Section of Mathematical Statistics ...* (1958)
- [25] Beyer, H.G., Finck, S.: On the design of constraint covariance matrix self-adaptation evolution strategies including a cardinality constraint. *Trans. Evol. Comp* 16(4), 578–596 (Aug 2012)
- [26] Bhattacharjee, K.S., Ray, T.: A novel constraint handling strategy for expensive optimization problems. In: *11th World Congress on Structural and Multidisciplinary Optimization*. Sydney, Australia (June 2015)
- [27] Bhattacharjee, K.S., Singh, H.K., Ray, T.: Multi-objective optimization with multiple spatially distributed surrogates. *Journal of Mechanical Design* 138(9), 091401 (2016)
- [28] Bohn, B., Garcke, J., Iza-Teran, R., Paprotny, A., Peherstorfer, B., Schepmeier, U., Thole, C.A.: Analysis of car crash simulation data with nonlinear machine learning methods. *Procedia Computer Science* 18, 621–630 (2013)

-
- [29] Booker, A.J., Dennis, J.E., Frank, P.D., Serafini, D.B., Torczon, V., Trosset, M.W.: A rigorous framework for optimization of expensive functions by surrogates. *Structural Optimization* 17(1), 1–13 (1999)
- [30] Box, G.E.: Evolutionary operation: A method for increasing industrial productivity. *Applied statistics* pp. 81–101 (1957)
- [31] Brest, J., Greiner, S., Boskovic, B., Mernik, M., Zumer, V.: Self-adapting control parameters in differential evolution: A comparative study on numerical benchmark problems. *Trans. Evol. Comp* 10(6), 646–657 (Dec 2006)
- [32] Brockhoff, D., Tran, T.D., Hansen, N.: Benchmarking numerical multiobjective optimizers revisited. In: *Proceedings of the 17th conference on Genetic and Evolutionary Computation (GECCO)*. pp. 639–646. ACM, Madrid, Spain (2015)
- [33] Carr, J.C., Beatson, R.K., et al.: Reconstruction and representation of 3D objects with radial basis functions. In: *Proc. of the 28th conference on Computer Graphics and Interactive Techniques*. pp. 67–76. ACM (2001)
- [34] Carter, R., Gablonsky, J., Patrick, A., Kelley, C.T., Eslinger, O.: Algorithms for noisy problems in gas transmission pipeline optimization. *Optimization and engineering* 2(2), 139–157 (2001)
- [35] Cecilia, J.M., García, J.M., Nisbet, A., Amos, M., Ujaldón, M.: Enhancing data parallelism for ant colony optimization on gpus. *Journal of Parallel and Distributed Computing* 73(1), 42 – 51 (2013), <http://www.sciencedirect.com/science/article/pii/S0743731512000032>, metaheuristics on GPUs
- [36] Chen, Y., Hoffman, M.W., Colmenarejo, S.G., Denil, M., Lillicrap, T.P., Botvinick, M., de Freitas, N.: Learning to learn without gradient descent by gradient descent. In: *Proceedings of the 34th International Conference on Machine Learning-Volume 70*. pp. 748–756. JMLR. org (2017)
- [37] Chen, Y., Hoffman, M.W., Colmenarejo, S.G., Denil, M., Lillicrap, T.P., de Freitas, N.: Learning to learn for global optimization of black box functions (2018)
- [38] Chootinan, P., Chen, A.: Constraint handling in genetic algorithms using a gradient-based repair method. *Computers & Operations Research* 33(8), 2263–2281 (2006)

BIBLIOGRAPHY

- [39] Coello Coello, C.A.: Use of a self-adaptive penalty approach for engineering optimization problems. *Computers in Industry* 41(2), 113–127 (2000)
- [40] Conn, A.R., Le Digabel, S.: Use of quadratic models with mesh-adaptive direct search for constrained black box optimization. *Optimization Methods and Software* 28(1), 139–158 (2013)
- [41] Corne, D., Knowles, J.: Some multiobjective optimizers are better than others. In: *Proceedings of the IEEE Congress on Evolutionary Computation*. vol. 4, pp. 2506–2512 (2003)
- [42] Couckuyt, I., Turck, F.D., Dhaene, T., Gorissen, D.: Automatic surrogate model type selection during the optimization of expensive black-box problems. In: *Proceedings of the 2011 Winter Simulation Conference (WSC)*. pp. 4269–4279 (Dec 2011)
- [43] Curtis, P.C., et al.: n -parameter families and best approximation. *Pacific Journal of Mathematics* 9(4), 1013–1027 (1959)
- [44] Deb, K.: An efficient constraint handling method for genetic algorithms. *Computer Methods in Applied Mechanics and Engineering* 186(2–4), 311–338 (2000)
- [45] Deb, K.: An efficient constraint handling method for genetic algorithms. *Computer Methods in Applied Mechanics and Engineering* 186(2), 311 – 338 (2000)
- [46] Digabel, S.L., Wild, S.M.: A taxonomy of constraints in simulation-based optimization. *arXiv preprint arXiv:1505.07881* (2015)
- [47] Drela, M.: XFOIL: An analysis and design system for low reynolds number airfoils. In: *Conference on Low Reynolds Number Airfoil Aerodynamics*. University of Notre Dame (Jun 1989), http://web.mit.edu/drela/Public/papers/xfoil_sv.pdf
- [48] Drela, M., Giles, M.B.: Viscous-inviscid analysis of transonic and low reynolds number airfoils. *AIAA Journal* 25(10), 1347–1355 (Oct 1987)
- [49] Droste, S., Jansen, T., Wegener, I.: Perhaps not a free lunch but at least a free appetizer. In: *Proceedings of the 1st Annual Conference on Genetic and Evolutionary Computation-Volume 1*. pp. 833–839. Morgan Kaufmann Publishers Inc. (1999)

-
- [50] Duchon, J.: Splines minimizing rotation-invariant semi-norms in sobolev spaces. In: *Constructive theory of functions of several variables*, pp. 85–100. Springer (1977)
- [51] Durantin, C., Marzat, J., Balesdent, M.: Analysis of multi-objective kriging-based methods for constrained global optimization. *Computational Optimization and Applications* 63(3), 903–926 (2016)
- [52] Eiben, A.E., van Hemert, J.I.: SAW-ing EAs: Adapting the fitness function for solving constrained problems. In: Corne, D., Dorigo, M., Glover, F. (eds.) *New Ideas in Optimization*, pp. 389–402. McGraw-Hill (1999)
- [53] Eiben, A.E., Hinterding, R., Michalewicz, Z.: Parameter control in evolutionary algorithms. *Evolutionary Computation, IEEE Transactions on* 3(2), 124–141 (1999)
- [54] Emmerich, M., Giannakoglou, K.C., Naujoks, B.: Single- and multiobjective evolutionary optimization assisted by Gaussian random field metamodels. *Evolutionary Computation, IEEE Transactions on* 10(4), 421–439 (2006)
- [55] Erlich, I., Venayagamoorthy, G.K., Worawat, N.: A mean-variance optimization algorithm. In: *IEEE Congress on Evolutionary Computation*. pp. 1–6 (July 2010)
- [56] Farmani, R., Wright, J.: Self-adaptive fitness formulation for constrained optimization. *Evolutionary Computation, IEEE Transactions on* 7(5), 445–455 (2003)
- [57] Field, R.V.: A decision-theoretic method for surrogate model selection. *Journal of Sound and Vibration* 311(3–5), 1371 – 1390 (2008)
- [58] Finck, S., Hansen, N., Ros, R., Auger, A.: Real-parameter black-box optimization benchmarking 2009: Presentation of the noiseless functions. Tech. Rep. 2009/20, Research Center PPE (2009)
- [59] Fister Jr, I., Yang, X.S., Fister, I., Brest, J., Fister, D.: A brief review of nature-inspired algorithms for optimization. arXiv preprint arXiv:1307.4186 (2013)
- [60] Floudas, C.A., Pardalos, P.M.: *A Collection of Test Problems for Constrained Global Optimization Algorithms*. Springer-Verlag, New York, USA (1990)

BIBLIOGRAPHY

- [61] Fornberg, B., Flyer, N.: Accuracy of radial basis function interpolation and derivative approximations on 1-d infinite grids. *Advances in Computational Mathematics* 23(1), 5–20 (2005)
- [62] Fornberg, B., Flyer, N.: Solving pdes with radial basis functions. *Acta Numerica* 24, 215–258 (2015)
- [63] Forrester, A.I., Keane, A.J.: Recent advances in surrogate-based optimization. *Progress in Aerospace Sciences* 45(1), 50–79 (2009)
- [64] Founti, M., Tomboulides, A.: Report of the ercoftac greek pilot centre, 2010–2015 (2015)
- [65] Franke, R.: Scattered data interpolation: tests of some methods. *Mathematics of computation* 38(157), 181–200 (1982)
- [66] Friese, M., Zaefferer, M., Bartz-Beielstein, T., Flasch, O., Koch, P., Konen, W., Naujoks, B.: Ensemble based optimization and tuning algorithms. *Schriftenreihe des Instituts für Angewandte Informatik, Automatisierungstechnik am Karlsruher Institut für Technologie* p. 119 (2011)
- [67] Giannakoglou, K.: Design of optimal aerodynamic shapes using stochastic optimization methods and computational intelligence. *Progress in Aerospace Sciences* 38(1), 43 – 76 (2002)
- [68] Ginsbourger, D., Le Riche, R., Carraro, L.: A multi-points criterion for deterministic parallel global optimization based on gaussian processes (2008)
- [69] Ginsbourger, D., Le Riche, R., Carraro, L.: Kriging Is Well-Suited to Parallelize Optimization, pp. 131–162. Springer Berlin Heidelberg, Berlin, Heidelberg (2010)
- [70] Goldberg, D.E.: *Genetic Algorithms in Search, Optimization and Machine Learning*. Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA, 1st edn. (1989)
- [71] González-Barbosa, J.E., Guarín-Arenas, F., García-Chinchilla, C.A., de Jesús Cotes-León, E., Díaz-Prada, C.A., Rodríguez-Walteros, C.: Optimization of electrical submersible pump artificial lift system for extraheavy oils through and analysis of bottom dilution scheme. *CT& F Ciencia, Tecnología y Futuro* 4, 63–73 (06 2010)

-
- [72] Gorissen, D., Dhaene, T., Turck, F.D.: Evolutionary model type selection for global surrogate modeling. *J. Mach. Learn. Res.* 10, 2039–2078 (Dec 2009)
- [73] Gramacy, R.B., Lee, H.K.H.: Optimization under unknown constraints. In: *Bayesian Statistics*, vol. 9, pp. 229–247. Oxford University Press (2011)
- [74] Haar, A.: Die minkowskische geometrie und die annäherung an stetige funktionen. *Mathematische Annalen* 78(1), 294–311 (Dec 1917), <https://doi.org/10.1007/BF01457106>
- [75] Hansen, N.: The cma evolution strategy: a comparing review. In: *Towards a new evolutionary computation*, pp. 75–102. Springer (2006)
- [76] Hansen, N., Ostermeier, A.: Adapting arbitrary normal mutation distributions in evolution strategies: The covariance matrix adaptation. In: *Proc. of 1996 IEEE International Conference on Evolutionary Computation*, Nayoya University, Japan. pp. 312–317 (1996)
- [77] Hansen, N., Ros, R., Mauny, N., Schoenauer, M., Auger, A.: Impacts of Invariance in Search: When CMA-ES and PSO Face Ill-Conditioned and Non-Separable Problems. *Applied Soft Computing* 11, 5755–5769 (2011), <https://hal.inria.fr/inria-00583669>
- [78] Hardy, R.: Theory and applications of the multiquadric-biharmonic method 20 years of discovery 1968–1988. *Computers & Mathematics with Applications* 19(8), 163 – 208 (1990)
- [79] Hardy, R.L.: Multiquadric equations of topography and other irregular surfaces. *Journal of Geophysical Research* 76(8), 1905–1915 (1971)
- [80] Hicks, R., Henne, P.A.: Wing design by numerical optimization. *Journal of Aircraft* 15(7), 407–412 (1978)
- [81] Hock, W., Schittkowski, K.: Test examples for nonlinear programming codes. *Journal of Optimization Theory and Applications* 30(1), 127–129 (1980)
- [82] Holmström, K., Quttineh, N.H., Edvall, M.: An adaptive radial basis algorithm (arbf) for expensive black-box mixed-integer constrained global optimization. *Optimization and Engineering* 9(4), 311–339 (2008)
- [83] Hooke, R., Jeeves, T.: Direct search solution of numerical and statistical problems. *Journal of the ACM (JACM)* 8(2), 212–229 (1961)

BIBLIOGRAPHY

- [84] Hussein, R., Deb, K.: A generative kriging surrogate model for constrained and unconstrained multi-objective optimization. In: GECCO '16. pp. 573–580. ACM, New York, NY, USA (2016)
- [85] Igel, C., Suttorp, T., Hansen, N.: A computational efficient covariance matrix update and a (1+1)-cma for evolution strategies. In: Proceedings of the 8th annual conference on Genetic and evolutionary computation. pp. 453–460. ACM (2006)
- [86] Igel, C., Toussaint, M.: A no-free-lunch theorem for non-uniform distributions of target functions. *Journal of Mathematical Modelling and Algorithms* 3(4), 313–322 (2005)
- [87] Iuliano, E., Quagliarella, D.: Evolutionary optimization of benchmark aerodynamic cases using physics-based surrogate models. In: AIAA SciTech, pp. –. American Institute of Aeronautics and Astronautics (Jan 2015)
- [88] Jiao, L., Li, L., Shang, R., Liu, F., Stolkin, R.: A novel selection evolutionary strategy for constrained optimization. *Information Sciences* 239, 122 – 141 (2013)
- [89] Jiao, R., Zeng, S., Li, C., Jiang, Y., Wang, J.: Expected improvement of constraint violation for expensive constrained optimization. In: GECCO (2018)
- [90] Jones, D.R., Schonlau, M., Welch, W.J.: Efficient global optimization of expensive black-box functions. *J. of Global Optimization* 13(4), 455–492 (Dec 1998)
- [91] Jones, D.R., Perttunen, C.D., Stuckman, B.E.: Lipschitzian optimization without the lipschitz constant. *Journal of optimization Theory and Applications* 79(1), 157–181 (1993)
- [92] Jones, D.: Large-scale multi-disciplinary mass optimization in the auto industry. *Modeling and Optimization: Theory and Applications Conference (MOPTA)* (2008)
- [93] Kannan, B., Kramer, S.N.: An augmented lagrange multiplier based method for mixed integer discrete continuous optimization and its applications to mechanical design. *Journal of mechanical design* 116(2), 405–411 (1994)
- [94] Kennedy, J.: Particle swarm optimization. *Encyclopedia of machine learning* pp. 760–766 (2010)

-
- [95] Kessy, A., Lewin, A., Strimmer, K.: Optimal whitening and decorrelation. *The American Statistician* (2017), accepted
- [96] Kirkpatrick, S., Gelatt, C.D., Vecchi, M.P.: Optimization by simulated annealing. *science* 220(4598), 671–680 (1983)
- [97] Koch, P., Bagheri, S., Konen, W., Foussette, C., Krause, P., Bäck, T.: Constrained optimization with a limited number of function evaluations. In: Hoffmann, F., Hüllermeier, E. (eds.) *Proc. 24. Workshop Computational Intelligence*. pp. 119–134. Universitätsverlag Karlsruhe (2014)
- [98] Koch, P., Bagheri, S., Konen, W., Foussette, C., Krause, P., Bäck, T.: A new repair method for constrained optimization. In: *Proceedings of the 2015 on Genetic and Evolutionary Computation Conference (GECCO)*. pp. 273–280. ACM (2015)
- [99] Koch, P., Wagner, T., Emmerich, M.T.M., Bäck, T., Konen, W.: Efficient multi-criteria optimization on noisy machine learning problems. *Applied Soft Computing* 29, 357–370 (2015), <http://www.gm.fh-koeln.de/~konen/Publikationen/Koch2015a-ASOC.pdf>
- [100] Koziel, S., Michalewicz, Z.: Evolutionary algorithms, homomorphous mappings, and constrained parameter optimization. *Evolutionary computation* 7(1), 19–44 (1999)
- [101] Kramer, O., Schwefel, H.P.: On Three New Approaches To Handle Constraints Within Evolution Strategies. *Natural Computing* 5(4), 363–385 (2006)
- [102] Kramer, O.: *Self-Adaptive Heuristics for Evolutionary Computation, Studies in Computational Intelligence*, vol. 147. Springer Berlin Heidelberg (2008)
- [103] Krige, D.G.: A statistical approach to some basic mine valuation problems on the witwatersrand. *Journal of the Southern African Institute of Mining and Metallurgy* 52(6), 119–139 (1951)
- [104] Kvasov, D.E., Sergeyev, Y.D.: Deterministic approaches for solving practical black-box global optimization problems. *Advances in Engineering Software* 80, 58–66 (2015)
- [105] Lagarias, J.C., Reeds, J.A., Wright, M.H., Wright, P.E.: Convergence properties of the nelder–mead simplex method in low dimensions. *SIAM Journal on optimization* 9(1), 112–147 (1998)

BIBLIOGRAPHY

- [106] Larrañaga, P., Lozano, J.A.: Estimation of distribution algorithms: A new tool for evolutionary computation, vol. 2. Springer Science & Business Media (2001)
- [107] Liang, J., Runarsson, T.P., Mezura-Montes, E., Clerc, M., Suganthan, P., Coello, C.C., Deb, K.: Problem definitions and evaluation criteria for the CEC 2006 special session on constrained real-parameter optimization. *Journal of Applied Mechanics* 41, 8 (2006)
- [108] Loshchilov, I., Schoenauer, M., Sebag, M.: Self-adaptive surrogate-assisted covariance matrix adaptation evolution strategy. *CoRR abs/1204.2356* (2012)
- [109] Mairhuber, J.C.: On haar's theorem concerning chebychev approximation problems having unique solutions. *Proceedings of the American Mathematical Society* 7(4), 609–615 (1956)
- [110] Matheron, G.: Krigeage d'un panneau rectangulaire par sa périphérie. *Note géostatistique* 28 (1960)
- [111] Mehmani, A., Chowdhury, S., Messac, A.: A novel approach to simultaneous selection of surrogate models, constitutive kernels, and hyper-parameter values. *10th AIAA Multidisciplinary Design Optimization Conference* (2014)
- [112] Micchelli, C.A.: Interpolation of scattered data: Distance matrices and conditionally positive definite functions. *Constructive Approximation* 2(1), 11–22 (Dec 1986)
- [113] Michalewicz, Z., Nazhiyath, G.: Genocop III: a co-evolutionary algorithm for numerical optimization problems with nonlinear constraints. In: *IEEE International Conference on Evolutionary Computation*. vol. 2, pp. 647–651 vol.2. IEEE., Piscataway, NJ (1995)
- [114] Michalewicz, Z., Schoenauer, M.: Evolutionary Algorithms for Constrained Parameter Optimization Problems. *Evolutionary Computation* 4(1), 1–32 (1996)
- [115] Michalewicz, Z., Nazhiyath, G., Michalewicz, M.: A note on usefulness of geometrical crossover for numerical optimization problems. In: *Evolutionary Programming* (1996)
- [116] Moćkus, J.: On bayesian methods for seeking the extremum and their application. In: *IFIP Congress*. pp. 195–200 (1977)

-
- [117] Moćkus, J.: Bayesian approach to global optimization: theory and applications, vol. 37. Springer Science & Business Media (2012)
- [118] Moré, J.J., Wild, S.M.: Benchmarking derivative-free optimization algorithms. *SIAM J. Optimization* 20(1), 172–191 (2009)
- [119] Moritz, H.: Advanced physical geodesy. Sammlung Wichmann: Neue Folge, Buchreihe, Wichmann (1980)
- [120] Müller, J., Shoemaker, C.A.: Influence of ensemble surrogate models and sampling strategy on the solution quality of algorithms for computationally expensive black-box global optimization problems. *Journal of Global Optimization* 60(2), 123–144 (2014)
- [121] Murphy, K.P.: *Machine Learning: A Probabilistic Perspective*. The MIT Press (2012), pp. 118–121
- [122] Nelder, J.A., Mead, R.: A simplex method for function minimization. *The Computer Journal* 7(4), 308–313 (1965)
- [123] Papoutsis-Kiachagias, E., Andrejašić, M., Porziani, S., Groth, C., Erzen, D., Biancolini, M., Costa, E., Giannakoglou, K.: Combining an rbf-based morpher with continuous adjoint for low-speed aeronautical optimization applications. ECCOMAS, Crete, Greece (2016)
- [124] Parr, J., Holden, C.M., Forrester, A.I., Keane, A.J.: Review of efficient surrogate infill sampling criteria with constraint handling. In: 2nd International Conference on Engineering Optimization. pp. 1–10 (2010)
- [125] Parr, J.M., Keane, A.J., Forrester, A.I., Holden, C.M.: Infill sampling criteria for surrogate-based optimization with constraint handling. *Engineering Optimization* 44(10), 1147–1166 (2012)
- [126] Phelps, R., Krasnicki, M., Rutenbar, R.A., Carley, L.R., Hellums, J.R.: Anaconda: simulation-based synthesis of analog circuits via stochastic pattern search. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems* 19(6), 703–717 (2000)
- [127] Picheny, V.: A stepwise uncertainty reduction approach to constrained global optimization. In: Proc 7th International Conference on Artificial Intelligence and Statistics, Reykjavik, Iceland. pp. 787–795 (2014)

BIBLIOGRAPHY

- [128] Poloczek, J., Kramer, O.: Local SVM Constraint Surrogate Models for Self-adaptive Evolution Strategies. In: Timm, I.J., Thimm, M. (eds.) KI 2013: Advances in Artificial Intelligence. Lecture Notes in Computer Science, vol. 8077, pp. 164–175. Springer Berlin Heidelberg (2013)
- [129] Pošík, P., Klemš, V.: Jade, an adaptive differential evolution algorithm, benchmarked on the bbob noiseless testbed. In: Proceedings of the 14th Annual Conference Companion on Genetic and Evolutionary Computation. pp. 197–204. GECCO '12, ACM, New York, NY, USA (2012), <http://doi.acm.org/10.1145/2330784.2330814>
- [130] Powell, M.J.: The bobyqa algorithm for bound constrained optimization without derivatives. Cambridge NA Report NA2009/06, University of Cambridge, Cambridge pp. 26–46 (2009)
- [131] Powell, M.: A direct search optimization method that models the objective and constraint functions by linear interpolation. In: Gomez, S., Hennart, J.P. (eds.) Optimization And Numerical Analysis, pp. 51–67. Kluwer Academic, Dordrecht (1994)
- [132] Press, W.H.: Numerical recipes 3rd edition: The art of scientific computing. Cambridge university press (2007)
- [133] Price, K., Storn, R., Lampinen, J.: Differential Evolution: A Practical Approach to Global Optimization. Natural Computing Series, Springer (2005)
- [134] Qin, A.K., Suganthan, P.N.: Self-adaptive differential evolution algorithm for numerical optimization. In: IEEE Congress on Evolutionary Computation (CEC), 2005. vol. 2, pp. 1785–1791. IEEE (2005)
- [135] Quagliarella, D., Petrone, G., Iaccarino, G.: Optimization under uncertainty using the generalized inverse distribution function. In: Fitzgibbon, W. (ed.) Modeling, Simulation and Optimization for Science and Technology, Computational Methods in Applied Sciences, vol. 34, pp. 171–190. Springer, NL (June 2014)
- [136] R Core Team: R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria (2013), <http://www.R-project.org/>

-
- [137] R Core Team: R: A Language and Environment for Statistical Computing. R Foundation for Statistical Computing, Vienna, Austria (2014), <http://www.R-project.org>
- [138] Rasmussen, C.E., Williams, C.K.I.: Gaussian Processes for Machine Learning. the MIT Press (2006), pp. 114-117
- [139] Rees, T., Dollar, H.S., Wathen, A.J.: Optimal solvers for pde-constrained optimization. *SIAM Journal on Scientific Computing* 32(1), 271–298 (2010)
- [140] Regis, R.G.: Stochastic radial basis function algorithms for large-scale optimization involving expensive black-box objective and constraint functions. *Computers & OR* 38(5), 837–853 (2011)
- [141] Regis, R.G.: Constrained optimization by radial basis function interpolation for high-dimensional expensive black-box problems with infeasible initial points. *Engineering Optimization* 46(2), 218–243 (2014)
- [142] Regis, R.G.: Trust regions in surrogate-assisted evolutionary programming for constrained expensive black-box optimization. In: Datta, R., Deb, K. (eds.) *Evolutionary Constrained Optimization*, pp. 51–94. Springer (2015)
- [143] Regis, R.G., Shoemaker, C.A.: Constrained global optimization of expensive black box functions using radial basis functions. *J. of Global Optimization* 31(1), 153–171 (Jan 2005), <http://dx.doi.org/10.1007/s10898-004-0570-0>
- [144] Regis, R.G., Shoemaker, C.A.: Parallel radial basis function methods for the global optimization of expensive functions. *European Journal of Operational Research* 182(2), 514–535 (2007)
- [145] Regis, R.G., Shoemaker, C.A.: A quasi-multistart framework for global optimization of expensive functions using response surface models. *Journal of Global Optimization* 56(4), 1719–1753 (2013)
- [146] Rice, J.R.: The algorithm selection problem. In: *Advances in computers*, vol. 15, pp. 65–118. Elsevier (1976)
- [147] Rios, L.M., Sahinidis, N.V.: Derivative-free optimization: a review of algorithms and comparison of software implementations. *Journal of Global Optimization* 56(3), 1247–1293 (2013)

BIBLIOGRAPHY

- [148] Rocha, H.: On the selection of the most adequate radial basis function. *Applied Mathematical Modelling* 33(3), 1573 – 1583 (2009)
- [149] Roustant, O., Ginsbourger, D., Deville, Y.: DiceKriging, DiceOptim: Two R packages for the analysis of computer experiments by Kriging-based metamodeling and optimization. *Journal of Statistical Software* 21, 1–55 (2012)
- [150] Runarsson, T.P., Yao, X.: Stochastic ranking for constrained evolutionary optimization. *IEEE Transactions on Evolutionary Computation* 4(3), 284–294 (2000)
- [151] Runarsson, T.P., Yao, X.: Search biases in constrained evolutionary optimization. *IEEE Transactions on Systems, Man, and Cybernetics, Part C: Applications and Reviews* 35(2), 233–243 (2005)
- [152] Sasena, M.J., Papalambros, P., Goovaerts, P.: Exploration of metamodeling sampling criteria for constrained global optimization. *Engineering optimization* 34(3), 263–278 (2002)
- [153] Sasena, M.J., Papalambros, P.Y., Goovaerts, P.: The use of surrogate modeling algorithms to exploit disparities in function computation time within simulation-based optimization. In: *4th World Congress of Structural and Multidisciplinary Optimization*. pp. 5–11 (2001)
- [154] Sasena, M.J.: Flexibility and efficiency enhancements for constrained global design optimization with kriging approximations (2002)
- [155] Sawyerr, B.A., Adewumi, A.O., Ali, M.M.: Benchmarking rcgau on the noiseless bbob testbed. *The Scientific World Journal* 2015 (2015)
- [156] Sawyerr, B.A., Adewumi, A.O., Ali, M.M.: Benchmarking projection-based real coded genetic algorithm on bbob-2013 noiseless function testbed. In: *Proceedings of the 15th Annual Conference Companion on Genetic and Evolutionary Computation*. pp. 1193–1200. *GECCO '13 Companion*, ACM, New York, NY, USA (2013), <http://doi.acm.org/10.1145/2464576.2482698>
- [157] Saxena, N., Tripathi, A., Mishra, K.K., Misra, A.K.: Dynamic-pso: An improved particle swarm optimizer. In: *2015 IEEE Congress on Evolutionary Computation (CEC)*. pp. 212–219 (May 2015)
- [158] Schagen, I.: Interpolation in two dimensions—a new technique. *IMA Journal of Applied Mathematics* 23(1), 53–59 (1979)

-
- [159] Schonlau, M., Welch, W.J., Jones, D.R.: Global versus local search in constrained optimization of computer models. In: Flournoy, N., Rosenberger, W.F., Wong, W.K. (eds.) *New developments and applications in experimental design*, Lecture Notes–Monograph Series, vol. 34, pp. 11–25. Institute of Mathematical Statistics, Hayward, CA (1998)
- [160] Schwefel, H.P.P.: *Evolution and Optimum Seeking: The Sixth Generation*. John Wiley & Sons, Inc., New York, USA (1993)
- [161] Shir, O.M., Roslund, J., Whitley, D., Rabitz, H.: Evolutionary Hessian learning: Forced optimal covariance adaptive learning (FOCAL). CoRR (arXiv) abs/1112.4454 (2011)
- [162] Shir, O.M., Roslund, J., Whitley, D., Rabitz, H.: Efficient retrieval of landscape Hessian: Forced optimal covariance adaptive learning. *Physical Review E* 89(6), 063306 (2014)
- [163] Shubert, B.O.: A sequential method seeking the global maximum of a function. *SIAM Journal on Numerical Analysis* 9(3), 379–388 (1972)
- [164] Singh, H., Ray, T., Smith, W.: Surrogate assisted simulated annealing (sasa) for constrained multi-objective optimization. In: *Evolutionary Computation (CEC), 2010 IEEE Congress on*. pp. 1–8 (July 2010)
- [165] Spethmann, P., Thomke, S.H., Herstatt, C.: The impact of crash simulation on productivity and problem-solving in automotive r&d. Tech. rep., Working Paper (2006)
- [166] Spettel, P., Beyer, H.G., Hellwig, M.: A Covariance Matrix Self-Adaptation Evolution Strategy for Linear Constrained Optimization. ArXiv e-prints (Jun 2018)
- [167] Storn, R., Price, K.: Differential evolution—a simple and efficient heuristic for global optimization over continuous spaces. *Journal of global optimization* 11(4), 341–359 (1997)
- [168] Sutton, A.M., Lunacek, M., Whitley, L.D.: Differential evolution and non-separability: using selective pressure to focus search. In: *Proceedings of the 9th annual conference on Genetic and evolutionary computation*. pp. 1428–1435. ACM (2007)

BIBLIOGRAPHY

- [169] Swann, W.: Direct search methods. Numerical methods for unconstrained optimization pp. 13–28 (1972)
- [170] Tabios III, G.Q., Salas, J.D.: A comparative analysis of techniques for spatial interpolation of precipitation 1. JAWRA Journal of the American Water Resources Association 21(3), 365–380 (1985)
- [171] Takahama, T., Sakai, S.: Constrained optimization by the ϵ constrained differential evolution with gradient-based mutation and feasible elites. In: 2006 IEEE International Conference on Evolutionary Computation. pp. 1–8 (July 2006)
- [172] Tenne, Y., Armfield, S.W.: A memetic algorithm assisted by an adaptive topology RBF network and variable local models for expensive optimization problems. In: Kosinski, W. (ed.) Advances in Evolutionary Algorithms, p. 468. INTECH Open Access Publisher (2008), <http://cdn.intechweb.org/pdfs/5230.pdf>
- [173] Tessema, B., Yen, G.G.: An adaptive penalty formulation for constrained evolutionary optimization. Systems, Man and Cybernetics, Part A: Systems and Humans, IEEE Transactions on 39(3), 565–578 (2009)
- [174] Torczon, V.: On the convergence of pattern search algorithms. SIAM Journal on optimization 7(1), 1–25 (1997)
- [175] Tsopelas, I.I.: Integration of the gpu-enabled cfd solver puma into the workflow of a turbomachinery industry. testing and validation.
- [176] Tutum, C.C., Deb, K., Baran, I.: Constrained efficient global optimization for pultrusion process. Materials and Manufacturing Processes 30(4), 538–551 (2015), <http://dx.doi.org/10.1080/10426914.2014.994752>
- [177] Villanueva, D., Le Riche, R., Picard, G., Haftka, R.: Surrogate-based agents for constrained optimization. In: In 14th AIAA Non-Deterministic Approaches Conference. p. 1935 (2012)
- [178] Wang, G.G., Dong, Z., Aitchison, P.: Adaptive response surface method – a global optimization scheme for computation-intensive design problems. Journal of Optimization and Engineering 33, 707–734 (2001)

-
- [179] Wei, W., Wang, J., Tao, M.: Constrained differential evolution with multiobjective sorting mutation operators for constrained optimization. *Applied Soft Computing* 33, 207 – 222 (2015)
- [180] de Winter, R., van Stein, B., Dijkman, M., Bäck, T.: Designing ships using constrained multi-objective efficient global optimization. In: Nicosia, G., Pardalos, P., Giuffrida, G., Umeton, R., Sciacca, V. (eds.) *Machine Learning, Optimization, and Data Science*. pp. 191–203. Springer International Publishing, Cham (2019)
- [181] Wolpert, D.H., Macready, W.G.: No free lunch theorems for optimization. *IEEE Transactions on Evolutionary Computation* 1(1), 67–82 (1997)
- [182] Wright, G.B.: *Radial basis function interpolation: numerical and analytical developments* (2003)
- [183] Xie, X.F., Zhang, W.J., Bi, D.C.: Handling equality constraints by adaptive relaxing rule for swarm algorithms. *Congress on Evolutionary Computation (CEC)* pp. 2012–2016 (2004)
- [184] Yang, X.S.: *Nature-inspired optimization algorithms*. Elsevier (2014)
- [185] Yang, X.S., Deb, S.: Cuckoo search via lévy flights. In: *2009 World Congress on Nature & Biologically Inspired Computing (NaBIC)*. pp. 210–214. IEEE (2009)
- [186] Zahara, E., Kao, Y.T.: Hybrid Nelder–Mead simplex search and particle swarm optimization for constrained engineering design problems. *Expert Systems with Applications* 36(2), 3880–3886 (2009)
- [187] Zhang, H., Rangaiah, G.: An efficient constraint handling method with integrated differential evolution for numerical and engineering optimization. *Computers & Chemical Engineering* 37, 74–88 (2012)
- [188] Zhang, T., Choi, K., Rahman, S., Cho, K., Baker, P., Shakil, M., Heitkamp, D.: A hybrid surrogate and pattern search optimization method and application to microelectronics. *Structural and Multidisciplinary Optimization* 32(4), 327–345 (2006)

Appendix A

G-Problem Suite Description

In recent years a large number of optimization methods including constrained solvers are developed to tackle real-world optimization problems efficiently. Suitable benchmark suites are necessary for evaluating new algorithms, comparing their performances with each other and easing the algorithm development procedure.

G-problem suite is a challenging set of 24 constrained optimization problems used as a benchmark for an optimization competition in the special session of constrained real-parameter optimization at CEC 2006 conference. A subset of these problems, G01 – G11, were initially suggested by Michalewicz and Schoenauer in 1996 [114] as a handy reference test set for future methods. The test problems were mainly taken from Floudas and Pardalos 1990 [60] and Michalewicz et al. 1996 [115]. Later, Runarsson and Yao [150] extended the list to 13 problems by adding G12 [100] and G13 [81]. The remaining 11 problems were added later to the list in [107].

A constrained optimization problem can be defined by the minimization of an objective function $f(\cdot)$ subject to inequality constraint function(s) $g_j(\cdot)$ and equality constraint function(s) $h_k(\cdot)$:

$$\begin{aligned} \text{Minimize} \quad & f(\vec{x}), & \vec{x} \in [\vec{l}, \vec{u}] \subset \mathbb{R}^d & \quad (\text{A.1}) \\ \text{subject to} \quad & g_j(\vec{x}) \leq 0, & j = 1, 2, \dots, m \\ & h_k(\vec{x}) = 0, & k = 1, 2, \dots, r, \end{aligned}$$

where \vec{l} is the lower bound of the search space $\mathbb{S} \subseteq \mathbb{R}^d$ and the \vec{u} is the upper bound. $\vec{x} = [x_1, x_2, \dots, x_d]$ is a vector with the length of the parameter space size d . The x_i refers to the i -th element of the vector \vec{x} . The goal is to find \vec{x}^* which minimizes the fitness function $f(\cdot)$ in the feasible space $\mathbb{F} \subseteq \mathbb{R}^{d'} \subseteq \mathbb{S} \subseteq \mathbb{R}^d$, where $d' \leq d$. Maximization problems can be transformed to minimization without loss of generality.

2D-Radviz for G-problem Characteristics

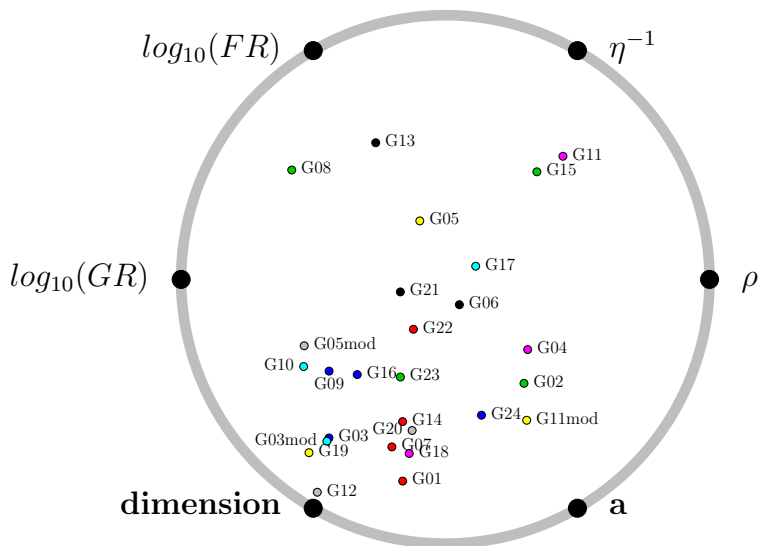


Figure A.1: Normalized radial visualization of G-problem’s properties.

Diversity in characteristics of G-problem suite makes this test set challenging, see Tab. A.1. Due to different type and level of difficulty each G-problem has, finding an optimizer which can solve the whole set efficiently remains a challenge. High dimensionality, multimodality and being highly constrained are several challenges that we should deal with, addressing these problems. Small or zero feasibility ratio $\rho = \frac{|F|}{|S|}$ is also another characteristics that makes many of G-problems hard to solve. In Tab. A.1, the feasibility ratio ρ is determined experimentally by evaluating 10^6 random points in the search space. Furthermore, problems with low feasibility subspace ratio $\eta = \frac{d'}{a}$ appear to be burdensome.

In this appendix we describe all 24 G-problems plus four modified problems G03mod, G05mod, G11mod and G15mod, for which the equality constraints are transformed to inequality constraints¹. These problems are often addressed in literature. The 2-dimensional problems are followed with visualization. The active constraints are highlighted in blue. For each problem the best known solution is reported and the regarding challenges are mentioned.

¹The implementation of these problems can be found at github link

Table A.1: Characteristics of the G-functions: d : dimension, ρ : feasibility rate (%), η : feasibility subspace ratio, FR : range of the fitness values, GR : ratio of largest to smallest constraint range, LI/NL: number of linear/nonlinear inequalities, LE/NE: number of linear/nonlinear equalities, a : number of constraints active at the optimum.

Fct.	d	ρ	η	FR	GR	LI / NI	LE / NE	a
G01	13	0.0003%	1	298.14	1.969	9 / 0	0 / 0	6
G02	20	99.997%	1	0.57	2.632	1 / 1	0 / 0	1
G03	20	0.0000%	0.95	$9.27 \cdot 10^{10}$	1	0 / 0	0 / 1	1
G03mod	20	2.46e-6%	1	$9.27 \cdot 10^{10}$	1	0 / 1	0 / 0	1
G04	5	26.9217%	1	9832.45	2.161	0 / 6	0 / 0	2
G05	4	0.0000%	0.25	8863.69	1788.74	2 / 0	0 / 3	3
G05mod	4	0.0919%	1	8863.69	1788.74	2 / 3	0 / 0	3
G06	2	0.0072%	1	1246828.23	1.010	0 / 2	0 / 0	2
G07	10	0.0000%	1	5928.19	12.671	3 / 5	0 / 0	6
G08	2	0.8751%	1	1821.61	2.393	0 / 2	0 / 0	0
G09	7	0.5207%	1	10013016.18	25.05	0 / 4	0 / 0	2
G10	8	0.0008%	1	27610.89	3842702	3 / 3	0 / 0	6
G11	2	0.0000%	0.5	4.99	1	0 / 0	0 / 1	1
G11mod	2	66.7240%	1	4.99	1	0 / 1	0 / 0	1
G12	3	0.0482%	1	0.72813	1	0 / 1	0 / 0	0
G13	5	0.0000%	0.4	$1.91 \cdot 10^{75}$	2.94	0 / 0	0 / 3	3
G14	10	0.0000%	0.7	1813.3	1.343	0 / 0	3 / 0	3
G15	3	0.0000%	0.3	586.0	1.034	0 / 0	1 / 1	2
G15mod	3	0.0337%	0.3	586.0	1.034	1 / 1	0 / 0	2
G16	5	0.0000%	1	811263.1	75.73	4 / 34	0 / 0	4
G17	6	0.0000%	0.3	42278.85	4.09	0 / 0	0 / 4	4
G18	9	0.0000%	1	5584.5	4.9	0 / 13	0 / 0	6
G19	15	0.33592%	1	55659.1	1.95	9 / 5	0 / 0	0
G20	24	0.0000%	0.42	28.99	858.19	0 / 6	2 / 12	16
G21	7	0.0000%	0.28	1000	23516.64	0 / 1	0 / 5	6
G22	22	0.0000%	0.14	2000	$3.1 \cdot 10^9$	0 / 1	8 / 11	19
G23	9	0.0000%	0.55	13044.3	82.56	0 / 2	3 / 1	6
G24	2	0.44250%	1	6.97	1.82	0 / 2	0 / 0	2

G01

This problem has a 13-dimensional parameter space and is restricted to 9 constraints 6 of which are active.

$$\begin{aligned} \text{Minimize } f(\vec{x}) &= 5 \sum_{i=1}^4 x_i - 5 \sum_{i=1}^4 x_i^2 - \sum_{i=5}^{13} x_i, \\ \text{subject to } g_1(\vec{x}) &= 2x_1 + 2x_2 + x_{10} + x_{11} - 10 \leq 0, \\ g_2(\vec{x}) &= 2x_1 + 2x_3 + x_{10} + x_{12} - 10 \leq 0, \\ g_3(\vec{x}) &= 2x_2 + 2x_3 + x_{11} + x_{12} - 10 \leq 0, \\ g_4(\vec{x}) &= -8x_1 + x_{10} \leq 0, \\ g_5(\vec{x}) &= -8x_2 + x_{11} \leq 0, \\ g_6(\vec{x}) &= -8x_3 + x_{12} \leq 0, \\ g_7(\vec{x}) &= -2x_4 - x_5 + x_{10} \leq 0, \\ g_8(\vec{x}) &= -2x_6 - x_7 + x_{11} \leq 0, \\ g_9(\vec{x}) &= -2x_8 - x_9 + x_{12} \leq 0. \end{aligned}$$

The lower bound is at $\vec{l}_{01} = \vec{0}$ and the upper bound is at $\vec{u}_{01} = [1, 1, 1, 1, 1, 1, 1, 1, 1, 100, 100, 100, 1]$. The global optimal solution is at $\vec{x}_{01}^* = [1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 1]$ and $f(\vec{x}_{01}^*) = -15$.

Challenges: High-dimensionality, highly constrained.

G02

This problem is scalable in dimension. G02 problem is commonly investigated with $d = 20$ in different related research works.

$$\begin{aligned} \text{Minimize } f(\vec{x}) &= - \left| \frac{\sum_{i=1}^n \cos^4(x_i) - \prod_{i=1}^n \cos^2(x_i)}{\sqrt{\sum_{i=1}^n ix_i^2}} \right|, \\ \text{subject to } g_1(\vec{x}) &= 0.75 - \prod_{i=1}^n x_i \leq 0, \\ g_2(\vec{x}) &= \sum_{i=1}^n x_i - 7.5n \leq 0, \end{aligned}$$

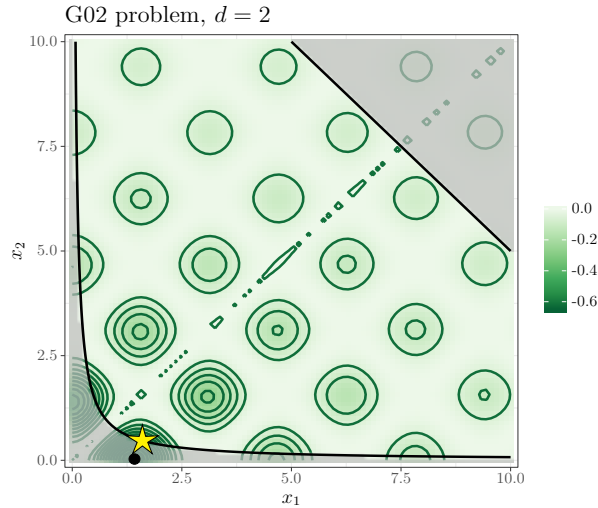


Figure A.2: G02 problem description. A $2d$ optimization problem with two inequality constraints. The shaded (green) contours depict the fitness function f (darker = smaller). The black curves show the borders of the inequality constraints. The infeasible area is shaded gray. The black point shows the global optimum of the fitness function which is different from the optimum of the constrained problem shown as the gold star.

where n is the size of the parameter space d . As the problem is scalable size of the parameter space can be any arbitrary integer larger than $1n = d > 2$. The lower bound is at $\vec{l}_{02} = \vec{0}$ and the upper bound is at $\vec{u}_{02} = \vec{10}$. The optimal solution for $d = 20$ is at $\vec{x}_{02}^* = [3.16246061, 3.12833142, 3.09479213, 3.06145059, 3.02792916, 2.99382607, 2.95866872, 2.92184227, 0.49482511, 0.48835711, 0.48231643, 0.47664475, 0.47129551, 0.46623099, 0.46142005, 0.45683665, 0.45245877, 0.44826762, 0.44424701, 0.44038286]$. Fig. A.2 shows G02 problem in the 2-dimensional space. As shown in Fig. A.2 only one of constraint function is active and the problem has a pretty large feasible region. The multimodality of the fitness function, makes this problem very challenging for surrogate-assisted optimizers. The complexity of this problem grows as the dimension grows.

Challenges: High-dimensionality, multimodality.

G03

This problem is scalable in dimension and has only one equality constraint. G03 is commonly investigated with $d = 20$ in different related research works.

$$\begin{aligned} \text{Minimize} \quad & f(\vec{x}) = -(\sqrt{n})^n \prod_{i=1}^n x_i, \\ \text{subject to} \quad & h_1(\vec{x}) = \sum_{i=1}^n x_i^2 - 1 = 0, \end{aligned}$$

where n is the size of parameter space d . As the problem is scalable $n = d \geq 2$. The lower bound is $\vec{l}_{03} = \vec{0}$ and the upper bound is $\vec{u}_{03} = \vec{1}$. The optimal solution can easily be calculated analytically for any arbitrary dimension n . $\vec{x}_{03}^* = \frac{1}{\sqrt{n}} \vec{1} = \frac{\vec{1}}{\sqrt{n}}$ which means the optimal value is -1 for any n .

$$f(\vec{x}_{03}^*) = f\left(\frac{\vec{1}}{\sqrt{n}}\right) = -(\sqrt{n})^n \cdot \left(\frac{1}{\sqrt{n}}\right)^n = -1$$

The solution suggested in CEC2006 [107] is not fully feasible and has a value better than the optimal value. Fig. A.3 shows G03 problem in the 2-dimensional space.

Challenges: High-dimensionality, small feasible space ($\rho = 0$ due to existence of an equality constraint).

G03mod

This problem is a modified version of G03 which transforms the equality constraint to an inequality constraint by assuming one side of the constraint being feasible.

$$\begin{aligned} \text{Minimize} \quad & f(\vec{x}) = -(\sqrt{n})^n \prod_{i=1}^n x_i, \\ \text{subject to} \quad & g_1(\vec{x}) = \sum_{i=1}^n x_i^2 - 1 \leq 0 \end{aligned}$$

The optimum is calculated exactly same as the G03 problem. Several papers address G03mod instead of G03 due to difficulties that many optimizers have in handling equality constraints. Fig. A.4 shows G03mod problem in the 2-dimensional space.

Challenges: High-dimensionality.

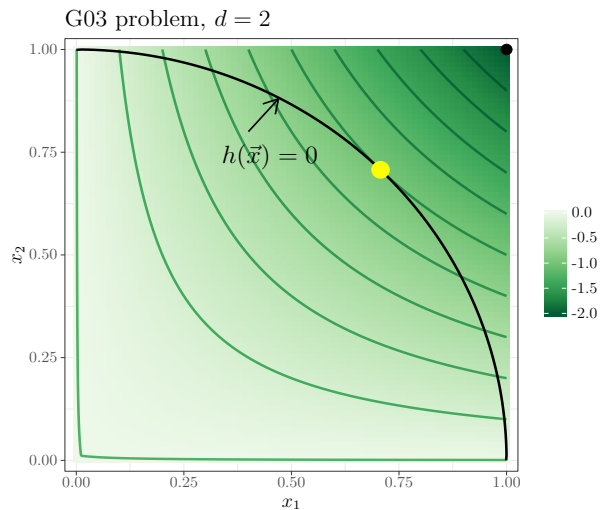


Figure A.3: G03 problem description. A $2d$ optimization problem with only one equality constraint. The shaded (green) contours depict the fitness function f (darker = smaller). The black curve shows the equality constraint. Feasible solutions are restricted to this line. The black point shows the global optimum of the fitness function which is different from the optimum of the constrained problem shown as the gold star.

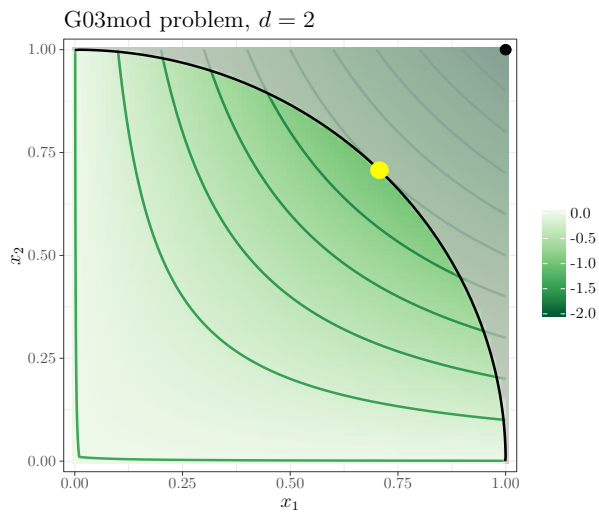


Figure A.4: G03mod problem description. A $2d$ optimization problem with one inequality constraint. The shaded (green) contours depict the fitness function f (darker = smaller). The black curve shows the borders of the inequality constraint. The infeasible area is shaded gray. The black point shows the global optimum of the fitness function which is different from the optimum of the constrained problem shown as the gold star.

G04

G04 is a 5-dimensional COP subject to 6 constraints two of which are active.

$$\begin{aligned} \text{Minimize } & f(\vec{x}) = 5.3578547x_3^2 + 0.8356891x_1x_5 + 37.293239x_1 - 40792.141, \\ \text{subject to } & g_1(\vec{x}) = 85.334407 + 0.0056858x_2x_5 + 0.0006262x_1x_4 - 0.0022053x_3x_5 - 92 \leq 0, \\ & g_2(\vec{x}) = -85.334407 - 0.0056858x_2x_5 - 0.0006262x_1x_4 + 0.0022053x_3x_5 \leq 0, \\ & g_3(\vec{x}) = 80.51249 + 0.0071317x_2x_5 + 0.0029955x_1x_2 + 0.0021813x_3^2 - 110 \leq 0, \\ & g_4(\vec{x}) = -80.51249 - 0.0071317x_2x_5 - 0.0029955x_1x_2 - 0.0021813x_3^2 + 90 \leq 0, \\ & g_5(\vec{x}) = 9.300961 + 0.0047026x_3x_5 + 0.0012547x_1x_3 + 0.0019085x_3x_4 - 25 \leq 0, \\ & g_6(\vec{x}) = -9.300961 - 0.0047026x_3x_5 - 0.0012547x_1x_3 - 0.0019085x_3x_4 + 20 \leq 0. \end{aligned}$$

The lower bound is at $\vec{l}_{04} = [78, 33, 27, 27, 27]$ and the upper bound is at $\vec{u}_{04} = [102, 45, 45, 45, 45]$. The optimal solution is at $\vec{x}_{04}^* = [78, 33, 29.99525602, 45, 36.77581290]$ and $f(\vec{x}_{04}^*) = -30665.53867178332$.

Challenges: Highly constrained.

G05

G05 is a 4-dimensional COP subject to 5 constraints including 3 equality constraints.

$$\begin{aligned} \text{Minimize } & f(\vec{x}) = 3x_1 + 0.000001x_1^3 + 2x_2 + (0.000002/3)x_2^3, \\ \text{subject to } & g_1(\vec{x}) = -x_4 + x_3 - 0.55 \leq 0, \\ & g_2(\vec{x}) = -x_3 + x_4 - 0.55 \leq 0, \\ & h_1(\vec{x}) = 1000 \sin(-x_3 - 0.25) + 1000 \sin(-x_4 - 0.25) + 894.8 - x_1 = 0, \\ & h_2(\vec{x}) = 1000 \sin(x_3 - 0.25) + 1000 \sin(x_3 - x_4 - 0.25) + 894.8 - x_2 = 0, \\ & h_3(\vec{x}) = 1000 \sin(x_4 - 0.25) + 1000 \sin(x_4 - x_3 - 0.25) + 1294.8 = 0. \end{aligned}$$

The lower bound is at $\vec{l}_{05} = [0, 0, -0.55, 0.55]$ and the upper bound is at $\vec{u}_{05} = [1200, 1200, 0.55, 0.55]$. The optimal solution is at $\vec{x}_{05}^* = [679.94531749, 1026.06713513, 0.11887637, -0.39623355]$ and $f(\vec{x}_{05}^*) = 5126.498109$. The solution suggested in CEC2006 [107] is not feasible and all equality constraints have a violation of size 10^{-4} , that's why the result reported in CEC2006 is better than the real optimal value.

Challenges: Highly constrained, zero feasible ratio.

G05mod

G05mod is a 4-dimensional COP subject to 5 inequality constraints.

$$\begin{aligned} \text{Minimize } & f(\vec{x}) = 3x_1 + 0.000001x_1^3 + 2x_2 + (0.000002/3)x_2^3, \\ \text{subject to } & g_1(\vec{x}) = -x_4 + x_3 - 0.55 \leq 0, \\ & g_2(\vec{x}) = -x_3 + x_4 - 0.55 \leq 0, \\ & g_3(\vec{x}) = 1000 \sin(-x_3 - 0.25) + 1000 \sin(-x_4 - 0.25) + 894.8 - x_1 \leq 0, \\ & g_4(\vec{x}) = 1000 \sin(x_3 - 0.25) + 1000 \sin(x_3 - x_4 - 0.25) + 894.8 - x_2 \leq 0, \\ & g_5(\vec{x}) = 1000 \sin(x_4 - 0.25) + 1000 \sin(x_4 - x_3 - 0.25) + 1294.8 \leq 0. \end{aligned}$$

The lower bound is at $\vec{l}_{05} = [0, 0, -0.55, 0.55]$ and the upper bound is at $\vec{u}_{05} = [1200, 1200, 0.55, 0.55]$. The optimal solution is at $\vec{x}_{05}^* = [679.94531749, 1026.06713513, 0.11887637, -0.39623355]$ and $f(\vec{x}_{05}^*) = 5126.498109$. The solution suggested in CEC2006 [107] is not feasible and all equality constraints have a violation of size 10^{-4} , that's why the result reported in CEC2006 is better than the real optimal value.

Challenges: Highly constrained, zero feasible ratio.

G06

A 2-dimensional COP with two active inequality constraints.

$$\begin{aligned} \text{Minimize } & f(\vec{x}) = (x_1 - 10)^3 + (x_2 - 20)^3, \\ \text{subject to } & g_1(\vec{x}) = -(x_1 - 5)^2 - (x_2 - 5)^2 + 100 \leq 0, \\ & g_2(\vec{x}) = (x_1 - 6)^2 + (x_2 - 5)^2 - 82.81 \leq 0. \end{aligned}$$

The lower bound is at $\vec{l}_{06} = [13, 0]$ and the upper bound is at $\vec{u}_{06} = [100, 100]$. The optimal solution is at $\vec{x}_{06}^* = [14.095, 0.8429608]$, where $f(\vec{x}_{06}^*) = -6961.813875580$. Fig. A.5 shows the G06 problem with three different zoomed in level. As shown in Fig. A.5 it is difficult to spot the feasible region in the original large space. As we zoom in about 10 times into the interesting region, the feasible area appears as a moon-shaped. G06 is a challenging COP due to its small feasible region, a very steep fitness function and two active constraints.

Challenges: Small feasible ratio ρ .

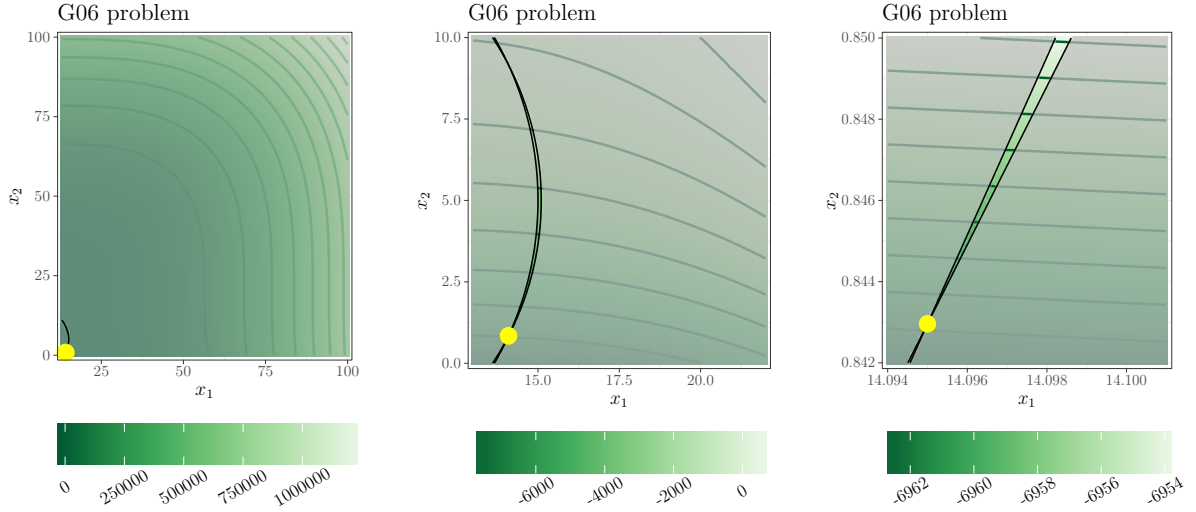


Figure A.5: G06 problem description. A $2d$ optimization problem with two inequality constraints. The shaded (green) contours depict the fitness function f (darker = smaller). The black curves show the borders of the inequality constraints. The infeasible area is shaded gray. The optimum of the constrained problem is shown as the gold star. The plots from left to right show the G06 problem with different zoom-in levels. Left: the original search space. Most of the search space seems to be infeasible and the interesting region is hardly detectable. Middle: $\approx 10\times$ zoomed in the interesting region. In the middle plot a tiny moon-shaped feasible region is observable. Right: $\approx 1000\times$ zoomed in.

G07

A 10-dimensional problem subjected to 8 inequality constraints 6 of which are active.

$$\begin{aligned}
 \text{Minimize } & f(\vec{x}) = x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3 - 10)^2 + 4(x_4 - 5)^2 + (x_5 - 3)^2 \\
 & + 2(x_6 - 1)^2 + 5x_7^2 + 7(x_8 - 11)^2 + 2(x_9 - 10)^2 + (x_{10} - 7)^2 + 45, \\
 \text{subject to } & g_1(\vec{x}) = -105 + 4x_1 + 5x_2 - 3x_7 + 9x_8 \leq 0, \\
 & g_2(\vec{x}) = 10x_1 - 8x_2 - 17x_7 + 2x_8 \leq 0, \\
 & g_3(\vec{x}) = -8x_1 + 2x_2 + 5x_9 - 2x_{10} - 12 \leq 0, \\
 & g_4(\vec{x}) = 3(x_1 - 2)^2 + 4(x_2 - 3)^2 + 2x_3^2 - 7x_4 - 120 \leq 0, \\
 & g_5(\vec{x}) = 5x_1^2 + 8x_2 + (x_3 - 6)^2 - 2x_4 - 40 \leq 0, \\
 & g_6(\vec{x}) = x_1^2 + 2(x_2 - 2)^2 - 2x_1x_2 + 14x_5 - 6x_6 \leq 0, \\
 & g_7(\vec{x}) = 0.5(x_1 - 8)^2 + 2(x_2 - 4)^2 + 3x_5^2 - x_6 - 30 \leq 0, \\
 & g_8(\vec{x}) = -3x_1 + 6x_2 + 12(x_9 - 8)^2 - 7x_{10} \leq 0.
 \end{aligned}$$

The lower bound is at $\vec{l}_{07} = -\vec{10}$ and the upper bound is at $\vec{u}_{07} = \vec{10}$. The optimal solution is at $\vec{x}_{07}^* = [2.17199783, 2.36367936, 8.77392512, 5.09598421, 0.99065597, 1.43057843, 1.32164704, 9.82872811, 8.28009420, 8.37592351]$. $f(\vec{x}_{07}^*) = 24.3062090689$

Challenges: High-dimensionality, highly constrained.

G08

A 2-dimensional problem subjected to 2 inequality constraints none of which are active at the optimum.

$$\begin{aligned} \text{Minimize } f(\vec{x}) &= -\frac{\sin^3(2\pi x_1) \sin(2\pi x_2)}{x_1^3(x_1 + x_2)} \\ \text{subject to } g_1(\vec{x}) &= x_1^2 - x_2 + 1 \leq 0, \\ g_2(\vec{x}) &= 1 - x_1 + (x_2 - 4)^2 \leq 0. \end{aligned}$$

The lower bound is at $\vec{l}_{08} = \vec{0}$ and the upper bound is at $\vec{u}_{08} = \vec{10}$. The optimal solution is at $\vec{x}_{08}^* = [1.2279713, 4.2453732]$ and $f(\vec{x}_{08}^*) = -0.095825041418$. Fig. A.6 shows the G08 problem in two zoomed in levels. As shown in Fig. A.6 the fitness function of G08 is highly multimodal, therefore this COP is challenging to solve with surrogate-assisted optimizers.

Challenges: Multimodality.

G09

A 7-dimensional problem subjected to 4 inequality constraints 2 of which are active.

$$\begin{aligned} \text{Minimize } f(\vec{x}) &= (x_1 - 10)^2 + 5(x_2 - 12)^2 + x_3^4 + 3(x_4 - 11)^2 + 10x_5^6 + 7x_6^2 \\ &\quad + x_7^4 - 4x_6x_7 - 10x_6 - 8x_7 \\ \text{subject to } g_1(\vec{x}) &= -127 + 2x_1^2 + 3x_2^4 + x_3 + 4x_4^2 + 5x_5 \leq 0, \\ g_2(\vec{x}) &= -282 + 7x_1 + 3x_2 + 10x_3^2 + x_4 - x_5 \leq 0, \\ g_3(\vec{x}) &= -196 + 23x_1 + x_2^2 + 6x_6^2 - 8x_7 \leq 0, \\ g_4(\vec{x}) &= 4x_1^2 + x_2^2 - 3x_1x_2 + 2x_3^2 + 5x_6 - 11x_7 \leq 0. \end{aligned}$$

The lower bound is at $\vec{l}_{09} = -\vec{10}$ and the upper bound is at $\vec{u}_{09} = \vec{10}$. The optimal solution is at $\vec{x}_{09}^* = [2.33049949323300210, 1.95137240, -0.47754042, 4.36572613,$

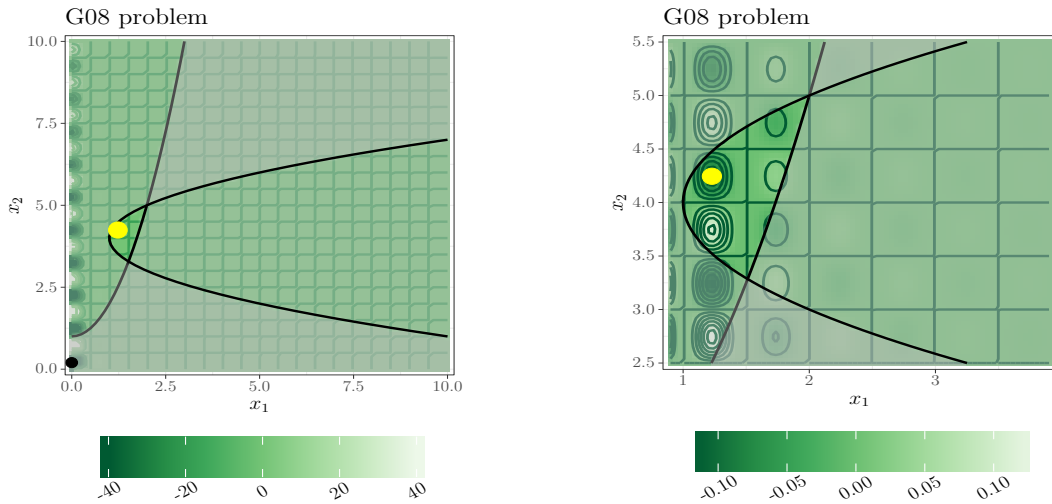


Figure A.6: G08 problem description. A $2d$ optimization problem with two inequality constraints. The shaded (green) contours depict the fitness function f (darker = smaller). The black curves show the borders of the inequality constraints. The infeasible area is shaded gray. The optimum of the constrained problem is shown as the gold star. The plots show the G08 problem with different zoom-in levels. Left: the original search space. Right: $\approx 2\times$ zoomed in. G08's fitness function has a large range. The local minima and maxima of G08's fitness function (out of the feasible area) have large values in the order of 1000 and -1000. For the visualization purposes we restricted the fitness range to $[-40; 40]$.

$[-0.62448707, 1.03813092, 1.59422663]$ and $f(\vec{x}_{09}^*) = 680.63005737440$.

Challenges: Small feasibility ratio ρ .

G10

An 8-dimensional COP subjected to 6 constraints 3 of which are active.

$$\begin{aligned} \text{Minimize} \quad & f(\vec{x}) = x_1 + x_2 + x_3 \\ \text{subject to} \quad & g_1(\vec{x}) = -1 + 0.0025(x_4 + x_6) \leq 0, \\ & g_2(\vec{x}) = -1 + 0.0025(x_5 + x_7 - x_4) \leq 0, \\ & g_3(\vec{x}) = -1 + 0.01(x_8 - x_5) \leq 0, \\ & g_4(\vec{x}) = -x_1x_6 + 833.33252x_4 + 100x_1 - 83333.333 \leq 0, \\ & g_5(\vec{x}) = -x_2x_7 + 1250x_5 + x_2x_4 - 1250x_4 \leq 0, \\ & g_6(\vec{x}) = -x_3x_8 + 1250000 + x_3x_5 - 2500x_5 \leq 0. \end{aligned}$$

The lower bound is at $\vec{l}_{10} = -[100, 1000, 1000, 10, 10, 10, 10, 10]$ and the upper bound is at $\vec{u}_{10} = [10000, 10000, 10000, 1000, 1000, 1000, 1000, 1000]$. The optimal solution is at $\vec{x}_{10}^* = [579.29340270, 1359.97691009, 5109.97770901, 182.01659025, 295.60089166, 217.98340974, 286.41569858, 395.60089165]$ and $f(\vec{x}_{10}^*) = 7049.2480218071796$

Challenges: Small feasibility ratio ρ , highly constrained.

G11

A 2-dimensional COP subject to an equality constraint.

$$\begin{aligned} \text{Minimize} \quad & f(\vec{x}) = x_1^2 + (x_2 - 1)^2, \\ \text{subject to} \quad & h_1(\vec{x}) = x_2 - x_1^2 = 0. \end{aligned}$$

The lower bound is at $\vec{l}_{11} = -\vec{1}$ and the upper bound is at $\vec{u}_{11} = \vec{1}$. The optimal solution is at $\vec{x}_{11}^* = [-0.7071068, 0.5]$ or $\vec{x}_{11}^* = [0.7071068, 0.5]$ and $f(\vec{x}_{11}^*) = 0.75$

Challenges: Zero feasibility ratio $\rho = 0$.

G11mod

This problem is the modified version of G11 which transforms the equality constraint to an inequality constraint by assuming one side of the constraint being feasible.

$$\begin{aligned} \text{Minimize} \quad & f(\vec{x}) = x_1^2 + (x_2 - 1)^2, \\ \text{subject to} \quad & g_1(\vec{x}) = x_2 - x_1^2 \leq 0. \end{aligned}$$

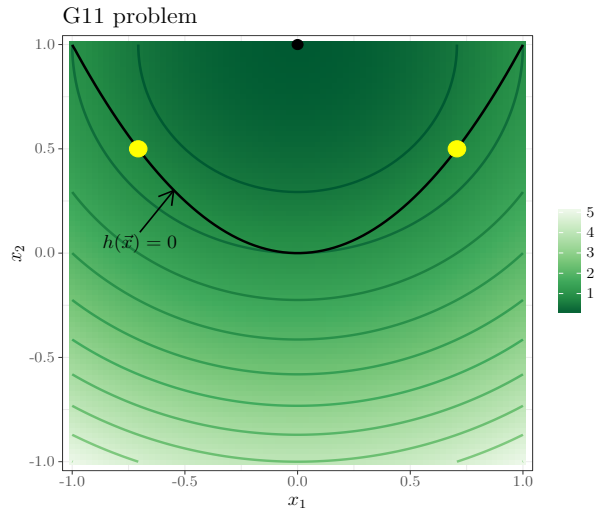


Figure A.7: G11 problem description. A $2d$ optimization problem with only one equality constraint. The shaded (green) contours depict the fitness function f (darker = smaller). The black curve shows the equality constraint. Feasible solutions are restricted to this line. The black point shows the global optimum of the fitness function which is different from the optimum of the constrained problem shown as the gold star.

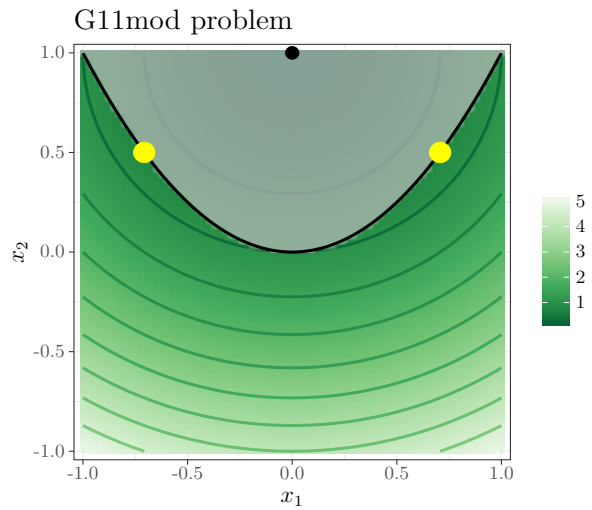


Figure A.8: G11 problem description. A $2d$ optimization problem with only one inequality constraint. The shaded (green) contours depict the fitness function f (darker = smaller). The black curve shows the inequality constraint. The infeasible area is shaded gray. The black point shows the global optimum of the fitness function which is different from the optimum of the constrained problem shown as the gold star.

The lower and upper bounds and the optimum are exactly same as the G11 problem. Several papers address G11mod instead of G11 due to difficulties that many optimizers have in handling equality constraints. Fig. A.8 shows G11mod problem.

G12

A 3-dimensional COP subject to 1 constraint. This problem has a disjoint feasible region.

$$\begin{aligned} \text{Minimize } & f(\vec{x}) = -0.01(100 - (x_1 - 5)^2 - (x_2 - 5)^2 - (x_3 - 5)^2) \\ \text{subject to } & g_1(\vec{x}) = (x_1 - p)^2 - (x_2 - q)^2 - (x_3 - r)^2 - 0.0625 \leq 0 \end{aligned}$$

The lower bound is $\vec{l}_{12} = \vec{0}$ and the upper bound is $\vec{u}_{12} = \vec{10}$. $p, q, r = 1, 2 \dots 9$. These are 729 disjoint spheres and a solution is feasible if it is within one of the 729 spheres. Therefore we take the min over $g_1(\cdot)$. The optimal solution is at $\vec{x}_{12}^* = [5, 5, 5]$ and $f(\vec{x}_{12}^*) = -1$

Challenges: Disjoint feasible region

G13

A 5-dimensional COP subject to 3 equality constraints.

$$\begin{aligned} \text{Minimize } & f(\vec{x}) = e^{\prod_{i=1}^d x_i}, \\ \text{subject to } & h_1(\vec{x}) = \sum_{i=1}^d x_i^2 - 10 = 0, \\ & h_2(\vec{x}) = x_2 x_3 - 5 x_4 x_5 = 0, \\ & h_3(\vec{x}) = x_1^3 + x_2^3 + 1 = 0. \end{aligned}$$

The lower bound is at $\vec{l}_{13} = [-2.3, -2.3, -3.2, -3.2, -3.2]$ and the upper bound is at $\vec{u}_{13} = -\vec{l}_{13}$. One of the optimal solution is at $\vec{x}_{13}^* = [-1.71714359, 1.59570973, 1.82724569, -0.76364228, -0.76364390]$ and $f(\vec{x}_{13}^*) = 0.05394984069520585$. The solution is invariant against a sign flip in both x_4 and x_5 , a sign flip in both x_3 and x_4 , a sign flip in both x_3 and x_5 or exchanging x_4 and x_5 .

Challenges: Multimodality, zero feasibility ratio $\rho = 0$.

G14

A 10-dimensional COP subject to 3 equality constraints.

$$\begin{aligned} \text{Minimize } f(\vec{x}) &= \sum_{i=1}^{10} x_i \left(c_i + \ln \frac{x_i}{\sum_{j=1}^{10} 0x_j} \right), \\ \text{subject to } h_1(\vec{x}) &= x_1 + 2x_2 + 2x_3 + x_6 + x_{10} - 2 = 0, \\ h_2(\vec{x}) &= x_4 + 2x_5 + x_6 + x_7 - 1 = 0, \\ h_3(\vec{x}) &= x_3 + x_7 + x_8 + 2x_9 + x_{10} - 1 = 0, \end{aligned}$$

The lower bound is at $\vec{l}_{14} = \vec{0}$, the upper bound is at $\vec{u}_{14} = \vec{10}$ and $\vec{c} = [-6.089, -17.164, -34.054, -5.914, -24.721, -14.986, -24.1, -10.708, -26.662, -22.179]$. The optimal solution is at $\vec{x}_{14}^* = [0.04066841, 0.14772124, 0.78320573, 0.00141434, 0.48529364, 0.00069318, 0.02740520, 0.01795097, 0.03732682, 0.09688446]$ and $f(\vec{x}_{14}^*) = -47.764888459491459$.

Challenges: High dimensionality, zero feasibility ratio $\rho = 0$.

G15

A 3-dimensional COP subject to 2 equality constraints.

$$\begin{aligned} \text{Minimize } f(\vec{x}) &= 1000 - x_1^2 - 2x_2^2 - x_3^2 - x_1x_2 - x_1x_3, \\ \text{subject to } h_1(\vec{x}) &= \sum_{i=1}^3 x_i^2 - 25 = 0, \\ h_2(\vec{x}) &= 8x_1 + 14x_2 + 7x_3 - 56 = 0. \end{aligned}$$

The lower bound is at $\vec{l}_{15} = \vec{0}$ and the upper bound is at $\vec{u}_{15} = \vec{10}$. The optimal solution is at $\vec{x}_{15}^* = [3.51212813, 0.21698751, 3.55217855]$. $f(\vec{x}_{15}^*) = 961.71502228996087$

Challenges: Zero feasibility ratio $\rho = 0$.

G15mod

A 3-dimensional COP subject to 2 inequality constraints.

$$\begin{aligned} \text{Minimize } & f(\vec{x}) = 1000 - x_1^2 - 2x_2^2 - x_3^2 - x_1x_2 - x_1x_3, \\ \text{subject to } & g_1(\vec{x}) = \sum_{i=1}^3 x_i^2 - 25 \leq 0, \\ & g_2(\vec{x}) = 8x_1 + 14x_2 + 7x_3 - 56 \leq 0. \end{aligned}$$

The lower bound is at $\vec{l}_{15} = \vec{0}$ and the upper bound is at $\vec{u}_{15} = \vec{10}$. The optimal solution is at $\vec{x}_{15}^* = [3.51212813, 0.21698751, 3.55217855]$. $f(\vec{x}_{15}^*) = 961.71502228996087$

G16

A 5-dimensional problem subject to 38 constraints 4 of which are active.

$$\begin{aligned} \text{Minimize } & f(\vec{x}) = 0.000117y_{14} + 0.1365 + 0.00002358y_{13} + 0.000001502y_{16} + 0.0321y_{12} \\ & + 0.004324y_5 + 0.0001\frac{c_{15}}{c_{16}} + 37.48\frac{y_2}{c_{12}} - 0.0000005843y_{17}, \\ \text{subject to } & g_1(\vec{x}) = \frac{0.28}{0.72}y_5 - y_4 \leq 0, & g_2(\vec{x}) = x_3 - 1.5x_2 \leq 0, \\ & g_3(\vec{x}) = 3496\frac{y_2}{c_{12}} - 21 \leq 0, & g_4(\vec{x}) = 110.6 + y_1 - \frac{62212}{c_{17}} \leq 0, \\ & g_5(\vec{x}) = 213.1 - y_1 \leq 0, & g_6(\vec{x}) = y_1 - 405.23 \leq 0, \\ & g_7(\vec{x}) = 17.505 - y_2 \leq 0, & g_8(\vec{x}) = y_2 - 1053.6667 \leq 0, \\ & g_9(\vec{x}) = 11.275 - y_3 \leq 0, & g_{10}(\vec{x}) = y_3 - 35.03 \leq 0, \\ & g_{11}(\vec{x}) = 214.228 - y_4 \leq 0, & g_{12}(\vec{x}) = y_4 - 665.585 \leq 0, \\ & g_{13}(\vec{x}) = 7.458 - y_5 \leq 0, & g_{14}(\vec{x}) = y_5 - 584.463 \leq 0, \\ & g_{15}(\vec{x}) = 0.961 - y_6 \leq 0, & g_{16}(\vec{x}) = y_6 - 265.916 \leq 0, \\ & g_{17}(\vec{x}) = 1.612 - y_7 \leq 0, & g_{18}(\vec{x}) = y_7 - 7.046 \leq 0, \\ & g_{19}(\vec{x}) = 0.146 - y_8 \leq 0, & g_{20}(\vec{x}) = y_8 - 0.222 \leq 0, \\ & g_{21}(\vec{x}) = 107.99 - y_9 \leq 0, & g_{22}(\vec{x}) = y_9 - 273.366 \leq 0, \\ & g_{23}(\vec{x}) = 922.693 - y_{10} \leq 0, & g_{24}(\vec{x}) = y_{10} - 1286.105 \leq 0, \\ & g_{25}(\vec{x}) = 926.832 - y_{11} \leq 0, & g_{26}(\vec{x}) = y_{11} - 1444.046 \leq 0, \\ & g_{27}(\vec{x}) = 18.766 - y_{12} \leq 0, & g_{28}(\vec{x}) = y_{12} - 537.141 \leq 0, \\ & g_{29}(\vec{x}) = 1072.163 - y_{13} \leq 0, & g_{30}(\vec{x}) = y_{13} - 3247.039 \leq 0, \\ & g_{31}(\vec{x}) = 8961.448 - y_{14} \leq 0, & g_{32}(\vec{x}) = y_{14} - 26844.086 \leq 0, \\ & g_{33}(\vec{x}) = 0.063 - y_{15} \leq 0, & g_{34}(\vec{x}) = y_{15} - 0.386 \leq 0, \\ & g_{35}(\vec{x}) = 71084.33 - y_{16} \leq 0, & g_{36}(\vec{x}) = -140000 - y_{16} \leq 0, \\ & g_{37}(\vec{x}) = 2802713 - y_{17} \leq 0, & g_{38}(\vec{x}) = y_{17} - 12146108 \leq 0, \end{aligned}$$

where,

$$\begin{aligned}
y_1 &= x_2 + x_3 + 41.6, & c_1 &= 0.024x_4 - 4.62, \\
y_2 &= \frac{12.5}{c_1} + 12, & c_2 &= 0.0003535x_1^2 + 0.5311x_1 + 0.08705y_2x_1, \\
y_3 &= \frac{c_2}{c_3}, & c_3 &= 0.052x_1 + 78 + 0.002377y_2x_1, \\
y_4 &= 19y_3, & c_5 &= 100x_2, \\
c_4 &= 0.04782(x_1 - x_3) + \frac{0.1956(x_1 - y_3)^2}{x_2} + 0.6376y_4 + 1.594y_3, \\
y_5 &= c_6c_7, & c_6 &= x_1 - y_3 - y_4 \\
y_6 &= x_1 - y_5 - y_4 - y_3, & c_7 &= 0.950 - \frac{c_4}{c_5}, \\
y_7 &= \frac{c_8}{y_1}, & c_8 &= (y_5 + y_4)0.995, \\
y_8 &= \frac{c_8}{3798}, & c_9 &= y_7 - \frac{0.0663y_7}{y_8} - 0.3153, \\
y_9 &= \frac{96.82}{c_9} + 0.321y_1 & c_{10} &= \frac{12.3}{752.3}, \\
y_{10} &= 1.29y_5 + 1.258y_4 + 2.29y_3 + 1.71y_6, & c_{11} &= (1.75y_2)(0.995x_1), \\
y_{11} &= 1.71x_1 - 0.452y_4 + 0.580y_3, & c_{12} &= 0.995y_{10} + 1998, \\
y_{12} &= c_{10}x_1 + \frac{c_{11}}{c_{12}}, & c_{14} &= 2324y_{10} - 28740000y_2, \\
y_{13} &= c_{12} - 1.75y_2, & c_{15} &= \frac{y_{13}}{y_{15}} - \frac{y_{13}}{0.52}, \\
c_{13} &= 0.995y_{10} + 60.8x_2 + 48x_4 - 0.1121y_{14} - 5095, \\
y_{14} &= 3623 + 64.4x_2 + 58.4x_3 + \frac{146312}{y_9 + x_5}, & c_{16} &= 1.104 - 0.72y_{15}, \\
y_{15} &= \frac{y_{13}}{c_{13}}, & c_{17} &= y_9 + y_5, \\
y_{16} &= 148000 - 331000y_{15} + 40y_{13} - 61y_{15}y_{13}, \\
y_{17} &= y_9 + x_5.
\end{aligned}$$

The lower bound is at $\vec{l}_{16} = [704.4148, 68.6, 0.0, 193, 25]$ and the upper bound is at $\vec{u}_{16} = [906.3855, 288.88, 134.75, 287.0966, 84.1988]$. The optimal solution is at $\vec{x}_{16}^* = [705.17454, 68.6, 102.9, 282.32493, 37.58412]$ and $f(\vec{x}_{16}^*) = -1.905155$

Challenges: Highly constrained

G17

A 6-dimensional problem subject to 4 equality constraints.

Minimize $f(\vec{x}) = f_1(x_1) + f_2(x_2)$

$$f_1(x_1) = \begin{cases} 30x_1, & \text{if } 0 \leq x_1 < 300 \\ 31x_1, & \text{if } 300 \leq x_1 \leq 400 \end{cases}$$
$$f_2(x_2) = \begin{cases} 28x_2, & \text{if } 0 \leq x_2 < 100 \\ 29x_2, & \text{if } 100 \leq x_2 \leq 200 \\ 30x_2, & \text{if } 200 \leq x_2 \leq 1000 \end{cases}$$

subject to $h_1(\vec{x}) = -x_1 + 300 - \frac{x_3x_4}{131.078} \cos(1.48477 - x_6) + \frac{0.90798x_3^2}{131.078} \cos(1.47588) = 0,$

$$h_2(\vec{x}) = -x_2 - \frac{x_3x_4}{131.078} \cos(1.48477 + x_6) + \frac{0.90798x_4^2}{131.078} \cos(1.47588) = 0,$$
$$h_3(\vec{x}) = -x_5 - \frac{x_3x_4}{131.078} \sin(1.48477 + x_6) + \frac{0.90798x_4^2}{131.078} \sin(1.47588) = 0,$$
$$h_4(\vec{x}) = 200 - \frac{x_3x_4}{131.078} \sin(1.48477 - x_6) + \frac{0.90798x_3^2}{131.078} \sin(1.47588) = 0.$$

The lower bound is at $\vec{l}_{17} = [0, 0, 340, 340, -1000, 0]$ and the upper bound is at $\vec{u}_{17} = [400, 1000, 420, 420, 1000, 0.5236]$. The optimal solution is at $\vec{x}_{17}^* = [201.78446721, 99.99999999, 383.07103485420, -10.90765845, 0.07314823]$ and $f(\vec{x}_{17}^*) = 8853.534$.

Challenges: Zero feasibility ratio $\rho = 0$.

G18

A 9-dimensional problem subject to 13 constraints 6 of which are active.

Minimize $f(\vec{x}) = -0.5(x_1x_4 - x_2x_3 + x_3x_9 - x_5x_9 + x_5x_8 - x_6x_7)$,
subject to $g_1(\vec{x}) = x_3^2 + x_4^2 - 1 \leq 0$,
 $g_2(\vec{x}) = x_9^2 - 1 \leq 0$,
 $g_3(\vec{x}) = x_5^2 + x_6^2 - 1 \leq 0$,
 $g_4(\vec{x}) = x_1^2 + (x_2 - x_9)^2 - 1 \leq 0$,
 $g_5(\vec{x}) = (x_1 - x_5)^2 + (x_2 - x_6)^2 - 1 \leq 0$,
 $g_6(\vec{x}) = (x_1 - x_7)^2 + (x_2 - x_8)^2 - 1 \leq 0$,
 $g_7(\vec{x}) = (x_3 - x_5)^2 + (x_4 - x_6)^2 - 1 \leq 0$,
 $g_8(\vec{x}) = (x_3 - x_7)^2 + (x_4 - x_8)^2 - 1 \leq 0$,
 $g_9(\vec{x}) = x_7^2 + (x_8 - x_9)^2 - 1 \leq 0$,
 $g_{10}(\vec{x}) = x_2x_3 - x_1x_4 \leq 0$,
 $g_{11}(\vec{x}) = -x_3x_9 \leq 0$,
 $g_{12}(\vec{x}) = x_5x_9 \leq 0$,
 $g_{13}(\vec{x}) = x_6x_7 - x_5x_8 \leq 0$.

The lower bound is at $\vec{l}_{18} = [-10, -10, -10, -10, -10, -10, -10, -10, 0]$ and the upper bound is at $\vec{u}_{18} = [10, 10, 10, 10, 10, 10, 10, 10, 20]$. The optimal solution is at $\vec{x}_{18}^* = [-0.98900055, 0.14791184, -0.62428976, -0.78118417, -0.98761593, 0.15047783, -0.62259598, -0.78254342, 0.0]$ and $f(\vec{x}_{18}^*) = -0.86573533494888033$.

Challenges: Highly constrained.

G19

A 15-dimensional problem subject to 5 constraints.

Minimize $f(\vec{x}) = \sum_{j=1}^5 \sum_{i=1}^5 c_{ij}x_{(10+i)}x_{(10+j)} + 2 \sum_{j=1}^5 d_jx_{(10+j)}^3 - \sum_{i=1}^{10} b_ix_i$
subject to $g_j(\vec{x}) = -2 \sum_{i=1}^5 c_{ij}x_{(10+i)} - 3d_jx_{(10+j)}^2 - e_j + \sum_{i=1}^{10} a_{ij}x_i \leq 0 \quad j = 1, \dots, 5$

, where $\vec{b} = [-40, -2, -0.25, -4, -4, -1, -40, -60, 5, 1]$, $\vec{d} = [4, 8, 10, 6, 2]$ and $\vec{e} = [-15, -27, -36, -18, -12]$. The lower bound is at $\vec{l}_{19} = \vec{0}$ and the upper bound is at $\vec{u}_{19} = \vec{10}$.

$$\mathbf{a} = \begin{bmatrix} -16 & 2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0.4 & 2 \\ -3.5 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & -4 & -1 \\ 0 & -9 & -2 & 1 & -2.8 \\ 2 & 0 & -4 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 \\ -1 & -2 & -3 & -2 & -1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} 30 & -20 & -10 & 32 & -10 \\ -20 & 39 & -6 & -31 & 32 \\ -10 & -6 & 10 & -6 & -10 \\ 32 & -31 & -6 & 39 & -20 \\ -10 & 32 & -10 & -20 & 30 \end{bmatrix}$$

The optimal solution is at $\vec{x}_{19}^* = [0, 6.08597252436373e - 033, 3.94600628013917, -2.35103745208393e - 032, 3.28318162727873, 10, 5.74431051614192e - 033, -1.15517863716213e - 032, -2.6336322104807e - 032, -3.50389001765656e - 033, 0.370762125835098, 0.278454209512692, 0.523838440499861, 0.388621589976956, 0.29815843730292]$ and $f(\vec{x}_{19}^*) = 32.655592950349401$.

Challenges: High-dimensionality.

G20

A 24-dimensional problem subject to 20 constraints 16 of which are active.

$$\begin{aligned}
 \text{Minimize } & f(\vec{x}) = \sum_{i=1}^{24} a_i x_i, \\
 \text{subject to } & g_j(\vec{x}) = \frac{(x_j + x_{(j+12)})}{\sum_{i=1}^{24} x_i + e_j} \leq 0, \quad j = 1, 2, 3, \\
 & g_j(\vec{x}) = \frac{(x_{(j+3)} + x_{(j+15)})}{\sum_{i=1}^{24} x_i + e_j} \leq 0, \quad j = 4, 5, 6, \\
 & h_k(\vec{x}) = \frac{x_{(k+12)}}{b_{(k+12)} \sum_{k=13}^{24} \frac{x_k}{b_k}} - \frac{c_k x_k}{40 b_k \sum_{k=1}^{12} \frac{x_k}{b_k}} = 0, \quad k = 1, \dots, 12, \\
 & h_{13}(\vec{x}) = \sum_{i=1}^{24} x_i - 1 = 0, \\
 & h_{14}(\vec{x}) = \sum_{i=1}^{12} \frac{x_i}{d_i} + \alpha \sum_{i=13}^{24} \frac{x_i}{b_i} - 1.671 = 0.
 \end{aligned}$$

The lower bound is at $\vec{l}_{20} = \vec{0}$ and the upper bound is at $\vec{u}_{20} = \vec{10}$.

$$\alpha = (0.7302)(530)\left(\frac{14.7}{40}\right).$$

$$\vec{a} = [0.0693, 0.0577, 0.05, 0.2, 0.26, 0.55, 0.06, 0.1, 0.12, 0.18, 0.1, 0.09, 0.0693, 0.0577, 0.05, 0.2, 0.26, 0.55, 0.06, 0.1, 0.12, 0.18, 0.1, 0.09]$$

$$\vec{b} = [44.094, 58.12, 58.12, 137.4, 120.9, 170.9, 62.501, 84.94, 133.425, 82.507, 46.07, 60.097, 44.094, 58.12, 58.12, 137.4, 120.9, 170.9, 62.501, 84.94, 133.425, 82.507, 46.07, 60.097]$$

$$\vec{c} = [123.7, 31.7, 45.7, 14.7, 84.7, 27.7, 49.7, 7.1, 2.1, 17.7, 0.85, 0.64]$$

$$\vec{d} = [31.244, 36.12, 34.784, 92.7, 82.7, 91.6, 56.708, 82.7, 80.8, 64.517, 49.4, 49.1]$$

$$\vec{e} = [0.1, 0.3, 0.4, 0.3, 0.6, 0.3]$$

The optimal solution is at $\vec{x}_{20}^* = [9.53E - 7, 0, 4.21e - 3, 1.039e - 4, 0, 0, 2.072e - 1, 5.979e - 1, 1.298e - 1, 3.35e - 2, 1.711e - 2, 8.827e - 3, 4.657e - 10, 0, 0, 0, 0, 0, 2.868e - 4, 1.193e - 3, 8.332e - 5, 1.239e - 4, 2.07e - 5, 1.829e - 5]$.

Challenges: High-dimensionality, highly constrained, zero feasibility ratio $\rho = 0$.

G21

A 7-dimensional COP subject to 6 active equality and inequality constraints.

$$\begin{aligned} \text{Minimize } & f(\vec{x}) = x_1, \\ \text{subject to } & g_1(\vec{x}) = -x_1 + 35x_2^{0.6} + 35x_3^{0.6} \leq 0, \\ & h_1(\vec{x}) = -300x_3 + 7500x_5 - 7500x_6 - 25x_4x_5 + 25x_4x_6 + x_3x_4 = 0, \\ & h_2(\vec{x}) = 100x_2 + 155.365x_4 + 2500x_7 - x_2x_4 - 25x_4x_7 - 15536.5 = 0, \\ & h_3(\vec{x}) = -x_5 + \ln(-x_4 + 900) = 0, \\ & h_4(\vec{x}) = -x_6 + \ln(x_4 + 300) = 0, \\ & h_5(\vec{x}) = -x_7 + \ln(-2x_4 + 700) = 0. \end{aligned}$$

The lower bound is at $\vec{l}_{21} = [0.0, 0.0, 0.0, 100, 6.3, 5.9, 4.5]$ and the upper bound is at $\vec{u}_{21} = [1000, 40, 40, 300, 6.7, 6.4, 6.25]$. The optimal solution is at $\vec{x}_{21}^* = [193.724510070034967, 5.56944131553368433e-27, 17.3191887294084914, 100.047897801386839, 6.68445185362377892, 5.99168428444264833, 6.21451648886070451]$ and $f(\vec{x}_{21}^*) = 193.72451007003497$.

Challenges: Highly constrained, zero feasibility ratio $\rho = 0$.

G22

A 22-dimensional COP subject to 20 constraints 19 of which are active equality constraints.

Minimize $f(\vec{x}) = x_1$,
subject to $g_1(\vec{x}) = -x_1 + x_2^{0.6} + x_3^{0.6} + x_4^{0.6} \leq 0$,
 $h_1(\vec{x}) = x_5 - 10^5 x_8 + 10^7 = 0$,
 $h_2(\vec{x}) = x_6 + 10^5 x_8 - 10^5 x_9 = 0$,
 $h_3(\vec{x}) = x_7 + 10^5 x_9 - 5 \cdot 10^7 = 0$,
 $h_4(\vec{x}) = x_5 + 10^5 x_{10} - 3.3 \cdot 10^7 = 0$,
 $h_5(\vec{x}) = x_6 + 10^5 x_{11} - 4.4 \cdot 10^7 = 0$,
 $h_6(\vec{x}) = x_7 + 10^5 x_{12} - 6.6 \cdot 10^7 = 0$,
 $h_7(\vec{x}) = x_5 - 120x_2x_{13} = 0$,
 $h_8(\vec{x}) = x_6 - 80x_3x_{14} = 0$,
 $h_9(\vec{x}) = x_7 - 40x_4x_{15} = 0$,
 $h_{10}(\vec{x}) = x_8 - x_{11} + x_{16} = 0$,
 $h_{11}(\vec{x}) = x_9 - x_{12} + x_{17} = 0$,
 $h_{12}(\vec{x}) = -x_{18} + \ln(x_{10} - 100) = 0$,
 $h_{13}(\vec{x}) = -x_{19} + \ln(-x_8 + 300) = 0$,
 $h_{14}(\vec{x}) = -x_{20} + \ln(x_{16}) = 0$,
 $h_{15}(\vec{x}) = -x_{21} + \ln(-x_9 + 400) = 0$,
 $h_{16}(\vec{x}) = -x_{22} + \ln(x_{17}) = 0$,
 $h_{17}(\vec{x}) = -x_8 - x_{10} + x_{13}x_{18} - x_{13}x_{19} + 400 = 0$,
 $h_{18}(\vec{x}) = x_8 - x_9 - x_{11} + x_{14}x_{20} - x_{14}x_{21} + 400 = 0$,
 $h_{19}(\vec{x}) = x_9 - x_{12} - 4.60517x_{15} + x_{15}x_{22} + 100 = 0$,

The lower bound is at $\vec{l}_{22} = [0, 0, 0, 0, 0, 0, 0, 100.100, 100.01, 100, 100, 0, 0, 0, 0, 0.01, -4.7, -4.7, -4.7, -4.7, -4.7]$ and the upper bound is at $\vec{u}_{22} = [2e + 04, 1e + 06, 1e + 06, 1e + 06, 4e + 07, 4e + 07, 4e + 07, 3e + 02, 4e + 02, 3e + 02, 4e + 02, 6e + 02, 5e + 02, 5e + 02, 5e + 02, 3e + 02, 4e + 02, 6.25, 6.250, 6.25, 6.25, 6.25]$. The best solution found is at $\vec{x}_{22}^* = [2.416091e + 02, 1.354324e + 02, 9.159755e + 02, 4.850804e + 03, 3.000000e + 06, 1.223014e + 07, 2.476986e + 07, 1.300000e + 02, 2.523014e + 02, 3.000000e + 02, 3.176986e + 02, 4.123014e + 02, 1.845939e + 02, 1.669005e + 02, 1.276585e + 02, 1.876986e + 02, 1.600000e + 02, 5.298317e + 00, 5.135798e + 00, 5.234837e + 00, 4.995174e + 00, 5.075174e + 00]$

and $f(\vec{x}_{22}^*) = 241.609$ The optimal value that we find is about larger than what is reported in CEC2006 [107] but fully feasible.

Challenges: High-dimensionality, zero feasibility ratio $\rho = 0$, highly constrained, small feasible subspace ratio $\eta = \frac{3}{22} \approx 0.14$.

G23

A 9-dimensional problem subject to 6 active constraints.

$$\begin{aligned}
 &\text{Minimize} && f(\vec{x}) = -9x_5 - 15x_8 + 6x_1 + 16x_2 + 10(x_6 + x_7), \\
 &\text{subject to} && g_1(\vec{x}) = x_9x_3 + 0.02x_6 - 0.025x_5 \leq 0, \\
 &&& g_2(\vec{x}) = x_9x_4 + 0.02x_7 - 0.015x_8 \leq 0, \\
 &&& h_1(\vec{x}) = x_1 + x_2 - x_3 - x_4 = 0, \\
 &&& h_2(\vec{x}) = 0.03x_1 + 0.01x_2 - x_9(x_3 + x_4) = 0, \\
 &&& h_3(\vec{x}) = x_3 + x_6 - x_5 = 0, \\
 &&& h_4(\vec{x}) = x_4 + x_7 - x_8 = 0.
 \end{aligned}$$

The lower bound is at $\vec{l}_{23} = [0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.01]$ and the upper bound is at $\vec{u}_{23} = [300, 300, 100, 200, 100, 300, 100, 200, 0.03]$.

The optimal solution is at $\vec{x}_{23}^* = [0, 100, 0, 100, 0, 0, 100, 200, 0.01]$. $f(\vec{x}_{23}^*) = -400.0$

Challenges: Highly constrained, zero feasibility ratio $\rho = 0$.

G24

A 2-dimensional COP subject to 2 active inequality constraints.

$$\begin{aligned}
 &\text{Minimize} && f(\vec{x}) = -x_1 - x_2, \\
 &\text{subject to} && g_1(\vec{x}) = -2x_1^4 + 8x_1^3 - 8x_1^2 + x_2 - 2 \leq 0, \\
 &&& g_2(\vec{x}) = -4x_1^4 + 32x_1^3 - 88x_1^2 + 96x_1 + x_2 - 36 \leq 0.
 \end{aligned}$$

The lower bound is at $\vec{l}_{24} = \vec{0}$ and the upper bound is at $\vec{u}_{24} = [3, 4]$. The optimal solution is at $\vec{x}_{24}^* = [2.329520197477607, 3.17849307411768]$ and $f(\vec{x}_{24}^*) = -5.508$. Fig. A.9 shows G24 problem.

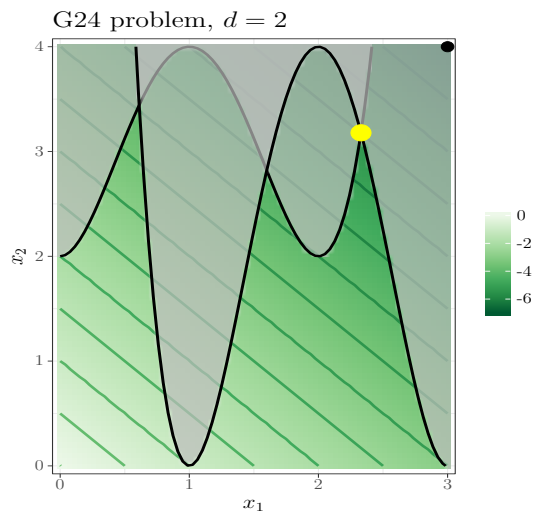


Figure A.9: G24 problem description. A $2d$ optimization problem with two inequality constraints. The shaded (green) contours depict the fitness function f (darker = smaller). The black curves show the borders of the inequality constraints. The infeasible area is shaded gray. The optimum of the constrained problem is shown as the gold star.

Appendix B

Transforming G22

$$\begin{aligned} & \text{Minimize} && f(\vec{x}) = x_1, \\ & \text{subject to} && g_1(\vec{x}) = -x_1 + x_2^{0.6} + x_3^{0.6} + x_4^{0.6} \leq 0, \\ & && h_1(\vec{x}) = x_5 - 10^5 x_8 + 10^7 = 0, \\ & && h_2(\vec{x}) = x_6 + 10^5 x_8 - 10^5 x_9 = 0, \\ & && h_3(\vec{x}) = x_7 + 10^5 x_9 - 5 \cdot 10^7 = 0, \\ & && h_4(\vec{x}) = x_5 + 10^5 x_{10} - 3.3 \cdot 10^7 = 0, \\ & && h_5(\vec{x}) = x_6 + 10^5 x_{11} - 4.4 \cdot 10^7 = 0, \\ & && h_6(\vec{x}) = x_7 + 10^5 x_{12} - 6.6 \cdot 10^7 = 0, \\ & && h_7(\vec{x}) = x_5 - 120 x_2 x_{13} = 0, \\ & && h_8(\vec{x}) = x_6 - 80 x_3 x_{14} = 0, \\ & && h_9(\vec{x}) = x_7 - 40 x_4 x_{15} = 0, \\ & && h_{10}(\vec{x}) = x_8 - x_{11} + x_{16} = 0, \\ & && h_{11}(\vec{x}) = x_9 - x_{12} + x_{17} = 0, \\ & && h_{12}(\vec{x}) = -x_{18} + \ln(x_{10} - 100) = 0, \\ & && h_{13}(\vec{x}) = -x_{19} + \ln(-x_8 + 300) = 0, \\ & && h_{14}(\vec{x}) = -x_{20} + \ln(x_{16}) = 0, \\ & && h_{15}(\vec{x}) = -x_{21} + \ln(-x_9 + 400) = 0, \\ & && h_{16}(\vec{x}) = -x_{22} + \ln(x_{17}) = 0, \\ & && h_{17}(\vec{x}) = -x_8 - x_{10} + x_{13} x_{18} - x_{13} x_{19} + 400 = 0, \\ & && h_{18}(\vec{x}) = x_8 - x_9 - x_{11} + x_{14} x_{20} - x_{14} x_{21} + 400 = 0, \\ & && h_{19}(\vec{x}) = x_9 - x_{12} - 4.60517 x_{15} + x_{15} x_{22} + 100 = 0, \end{aligned}$$

By solving the first three equality constraints we get rid of three variables x_5 , x_6 and x_7 , so that we write them as a function of x_8 and x_9 as follows.

$$\begin{aligned} h_1(\vec{x}) &\rightarrow x_5 = 10^5(x_8 - 10) \\ h_2(\vec{x}) &\rightarrow x_6 = 10^5(x_9 - x_8) \\ h_3(\vec{x}) &\rightarrow x_7 = 10^5(500 - x_9) \end{aligned}$$

Now that we have x_5 , x_6 and x_7 we can substitute them in h_4 , h_5 and h_6 in order to write x_{10} , x_{11} and x_{12} dependent on x_8 and x_9 as follows.

$$\begin{aligned} h_4(\vec{x}) &\rightarrow x_{10} = \frac{3.3 \cdot 10^7 - x_5}{10^5} = 430 - x_8 \\ h_5(\vec{x}) &\rightarrow x_{11} = \frac{4.4 \cdot 10^7 - x_6}{10^5} = 440 - x_9 + x_8 \\ h_6(\vec{x}) &\rightarrow x_{12} = \frac{6.6 \cdot 10^7 - x_7}{10^5} = 160 + x_9 \end{aligned}$$

We can find 5 more variables (x_{16} , x_{17} , x_{18} , x_{19} , x_{20}) based on x_8 and x_9 by simply substitution of x_{10} , x_{11} and x_{12} in the following equality constraints. It turns out that one parameter $x_{17} = 160$ is equal to a constant.

$$\begin{aligned} h_{10}(\vec{x}) &\rightarrow x_{16} = x_{11} - x_8 = 440 - x_9 \\ h_{11}(\vec{x}) &\rightarrow x_{17} = x_{12} - x_9 = 160 \\ h_{12}(\vec{x}) &\rightarrow x_{18} = \ln(x_{10} - 100) = \ln(330 - x_8) \\ h_{13}(\vec{x}) &\rightarrow x_{19} = \ln(300 - x_8) \\ h_{15}(\vec{x}) &\rightarrow x_{21} = \ln(400 - x_9) \end{aligned}$$

Now that we have x_{16} and x_{17} with the help of h_{14} and h_{15} equality constraints we can find x_{20} and x_{21} , where x_{21} has a constant value.

$$\begin{aligned} h_{14}(\vec{x}) &\rightarrow x_{20} = \ln(x_{16}) = \ln(440 - x_9) \\ h_{16}(\vec{x}) &\rightarrow x_{22} = \ln(x_{17}) = \ln(160) \end{aligned}$$

Reformulating the h_{17} , h_{18} and h_{19} equality constraints will give us x_{13} , x_{14} , x_{15} .

$$\begin{aligned}
h_{17}(\vec{x}) \rightarrow x_{13} &= \frac{x_8 + x_{10} - 400}{x_{18} - x_{19}} = 30 / \ln \left(\frac{330 - x_8}{300 - x_8} \right) \\
h_{18}(\vec{x}) \rightarrow x_{14} &= \frac{x_9 - x_8 + x_{11} - 400}{x_{20} - x_{21}} = 40 / \ln \left(\frac{160}{400 - x_9} \right) \\
h_{19}(\vec{x}) \rightarrow x_{15} &= \frac{x_{12} - x_9 - 100}{x_{22} - 4.60517} = 60 / \ln \left(\frac{160}{100} \right)
\end{aligned}$$

The last step is to reformulate h_7 , h_8 and h_9 equality constraints in order to find x_2 , x_3 and x_4 .

$$\begin{aligned}
h_7(\vec{x}) \rightarrow x_2 &= \frac{x_5}{120 \cdot x_{13}} = \frac{10^3}{36} \cdot (x_8 - 100) \cdot \ln \left(\frac{330 - x_8}{300 - x_8} \right) \\
h_8(\vec{x}) \rightarrow x_3 &= \frac{x_6}{80 \cdot x_{14}} = \frac{10^3}{32} \cdot (x_9 - x_8) \cdot \ln \left(\frac{160}{400 - x_9} \right) \\
h_9(\vec{x}) \rightarrow x_4 &= \frac{x_7}{40 \cdot x_{15}} = \frac{10^3}{24} \cdot (500 - x_9) \cdot \ln \left(\frac{160}{100} \right)
\end{aligned}$$

As we have seen, it is possible to describe 19 dimensions of the G22 problem only based on two parameters x_8 and x_9 . This means that in presence of the analytical information for the equality constraints we can transform this 22-dimensional problem with one inequality and 19 equality constraint to a 3-dimensional problem (x_1 , x_8 and x_9) with a single inequality constraints.