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Appendix A

G-Problem Suite Description

In recent years a large number of optimization methods including constrained solvers are developed to tackle real-world optimization problems efficiently. Suitable benchmark suites are necessary for evaluating new algorithms, comparing their performances with each other and easing the algorithm development procedure.

G-problem suite is a challenging set of 24 constrained optimization problems used as a benchmark for an optimization competition in the special session of constrained real-parameter optimization at CEC 2006 conference. A subset of these problems, G01 – G11, were initially suggested by Michalewicz and Schoenauer in 1996 [114] as a handy reference test set for future methods. The test problems were mainly taken from Floudas and Pardalos 1990 [60] and Michalewicz et al. 1996 [115]. Later, Runarsson and Yao [150] extended the list to 13 problems by adding G12 [100] and G13 [81]. The remaining 11 problems were added later to the list in [107].

A constrained optimization problem can be defined by the minimization of an objective function $f(.)$ subject to inequality constraint function(s) $g_j(.)$ and equality constraint function(s) $h_k(.)$:

$$\begin{aligned} \text{Minimize} \quad & f(\vec{x}), && \vec{x} \in [\vec{l}, \vec{u}] \subset \mathbb{R}^d \\ \text{subject to} \quad & g_j(\vec{x}) \leq 0, && j = 1, 2, \ldots, m \\ & h_k(\vec{x}) = 0, && k = 1, 2, \ldots, r, \end{aligned} \tag{A.1}$$

where $\vec{l}$ is the lower bound of the search space $\mathbb{S} \subseteq \mathbb{R}^d$ and the $\vec{u}$ is the upper bound. $\vec{x} = [x_1, x_2, \cdots, x_d]$ is a vector with the length of the parameter space size d. The x_i refers to the i-th element of the vector $\vec{x}$. The goal is to find $\vec{x}^*$ which minimizes the fitness function $f(.)$ in the feasible space $\mathbb{F} \subseteq \mathbb{R}^{d'} \subseteq \mathbb{S} \subseteq \mathbb{R}^d$, where $d' \leq d$. Maximization problems can be transformed to minimization without loss of generality.

2D-Radviz for G-problem Charactresitics

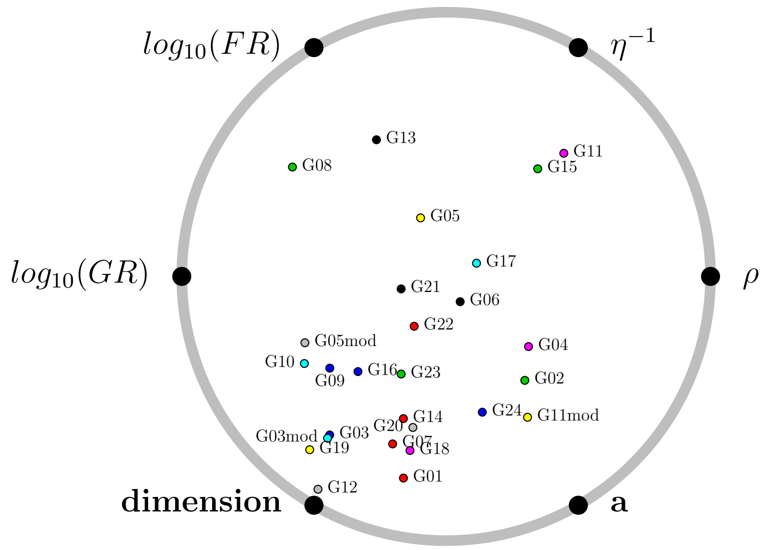

Figure A.1: Normalized radial visualization of G-problem's properties.

Diversity in characteristics of G-problem suite makes this test set challenging, see Tab. A.1. Due to different type and level of difficulty each G-problem has, finding an optimizer which can solve the whole set efficiently remains a challenge. High dimensionality, multimodality and being highly constrained are several challenges that we should deal with, addressing these problems. Small or zero feasibility ratio $\rho = \frac{|\mathbb{F}|}{|\mathbb{S}|}$ is also another characteristics that makes many of G-problems hard to solve. In Tab. A.1, the feasibility ratio ρ is determined experimentally by evaluating 10^6 random points in the search space. Furthermore, problems with low feasibility subspace ratio $\eta = \frac{d'}{d}$ appear to be burdensome.

In this appendix we describe all 24 G-problems plus four modified problems G03mod, G05mod, G11mod and G15mod, for which the equality constraints are transformed to inequality constraints[1]. These problems are often addressed in literature. The 2-dimensional problems are followed with visualization. The active constraints are highlighted in blue. For each problem the best known solution is reported and the regarding challenges are mentioned.

[1]The implementation of these problems can be found at github link

Table A.1: Characteristics of the G-functions: d: dimension, ρ: feasibility rate (%), η: feasibility subspace ratio, FR: range of the fitness values, GR: ratio of largest to smallest constraint range, LI/NI: number of linear/nonlinear inequalities, LE/NE: number of linear/nonlinear equalities, a: number of constraints active at the optimum.

Fct.	d	ρ	η	FR	GR	LI / NI	LE / NE	a
G01	13	0.0003%	1	298.14	1.969	9 / 0	0 / 0	6
G02	20	99.997%	1	0.57	2.632	1 / 1	0 / 0	1
G03	20	0.0000%	0.95	$9.27 \cdot 10^{10}$	1	0 / 0	0 / 1	1
G03mod	20	2.46e-6%	1	$9.27 \cdot 10^{10}$	1	0 / 1	0 / 0	1
G04	5	26.9217%	1	9832.45	2.161	0 / 6	0 / 0	2
G05	4	0.0000%	0.25	8863.69	1788.74	2 / 0	0 / 3	3
G05mod	4	0.0919%	1	8863.69	1788.74	2 / 3	0 / 0	3
G06	2	0.0072%	1	1246828.23	1.010	0 / 2	0 / 0	2
G07	10	0.0000%	1	5928.19	12.671	3 / 5	0 / 0	6
G08	2	0.8751%	1	1821.61	2.393	0 / 2	0 / 0	0
G09	7	0.5207%	1	10013016.18	25.05	0 / 4	0 / 0	2
G10	8	0.0008%	1	27610.89	3842702	3 / 3	0 / 0	6
G11	2	0.0000%	0.5	4.99	1	0 / 0	0 / 1	1
G11mod	2	66.7240%	1	4.99	1	0 / 1	0 / 0	1
G12	3	0.0482%	1	0.72813	1	0 / 1	0 / 0	0
G13	5	0.0000%	0.4	$1.91 \cdot 10^{75}$	2.94	0 / 0	0 / 3	3
G14	10	0.0000%	0.7	1813.3	1.343	0 / 0	3 / 0	3
G15	3	0.0000%	0.3	586.0	1.034	0 / 0	1 / 1	2
G15mod	3	0.0337%	0.3	586.0	1.034	1 / 1	0 / 0	2
G16	5	0.0000%	1	811263.1	75.73	4 / 34	0 / 0	4
G17	6	0.0000%	0.3	42278.85	4.09	0 / 0	0 / 4	4
G18	9	0.0000%	1	5584.5	4.9	0 / 13	0 / 0	6
G19	15	0.33592%	1	55659.1	1.95	9 / 5	0 / 0	0
G20	24	0.0000%	0.42	28.99	858.19	0 / 6	2 / 12	16
G21	7	0.0000%	0.28	1000	23516.64	0 / 1	0 / 5	6
G22	22	0.0000%	0.14	2000	$3.1 \cdot 10^{9}$	0 / 1	8 / 11	19
G23	9	0.0000%	0.55	13044.3	82.56	0 / 2	3 / 1	6
G24	2	0.44250%	1	6.97	1.82	0 / 2	0 / 0	2

G01

This problem has a 13-dimensional parameter space and is restricted to 9 constraints 6 of which are active.

$$
\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = 5\sum_{i=1}^{4} x_i - 5\sum_{i=1}^{4} x_i^2 - \sum_{i=5}^{13} x_i, \\
\text{subject to} \quad & g_1(\vec{x}) = 2x_1 + 2x_2 + x_{10} + x_{11} - 10 \leq 0, \\
& g_2(\vec{x}) = 2x_1 + 2x_3 + x_{10} + x_{12} - 10 \leq 0, \\
& g_3(\vec{x}) = 2x_2 + 2x_3 + x_{11} + x_{12} - 10 \leq 0, \\
& g_4(\vec{x}) = -8x_1 + x_{10} \leq 0, \\
& g_5(\vec{x}) = -8x_2 + x_{11} \leq 0, \\
& g_6(\vec{x}) = -8x_3 + x_{12} \leq 0, \\
& g_7(\vec{x}) = -2x_4 - x_5 + x_{10} \leq 0, \\
& g_8(\vec{x}) = -2x_6 - x_7 + x_{11} \leq 0, \\
& g_9(\vec{x}) = -2x_8 - x_9 + x_{12} \leq 0.
\end{aligned}
$$

The lower bound is at $\vec{l}_{01} = \vec{0}$ and the upper bound is at $\vec{u}_{01} = [1, 1, 1, 1, 1, 1, 1, 1, 1, 100, 100, 100, 1]$. The global optimal solution is at $\vec{x}^*_{01} = [1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 1]$ and $f(\vec{x}^*_{01}) = -15$.
Challenges: High-dimensionality, highly constrained.

G02

This problem is scalable in dimension. G02 problem is commonly investigated with $d = 20$ in different related research works.

$$
\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = -\left| \frac{\sum_{i=1}^{n} \cos^4(x_i) - \prod_{i=1}^{n} \cos^2(x_i)}{\sqrt{\sum_{i=1}^{n} i x_i^2}} \right|, \\
\text{subject to} \quad & g_1(\vec{x}) = 0.75 - \prod_{i=1}^{n} x_i \leq 0, \\
& g_2(\vec{x}) = \sum_{i=1}^{n} x_i - 7.5n \leq 0,
\end{aligned}
$$

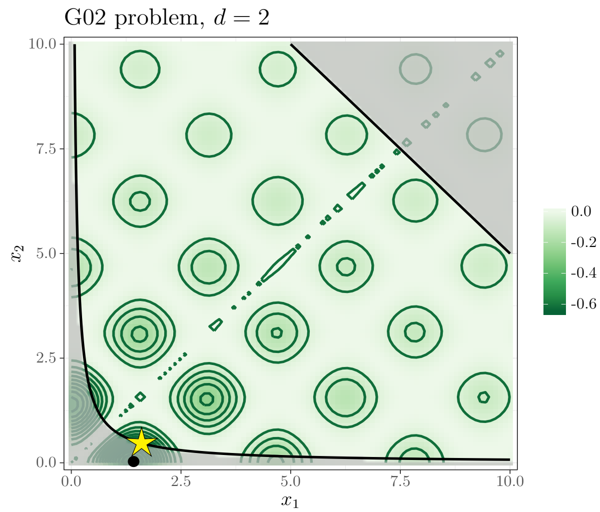

Figure A.2: G02 problem description. A 2d optimization problem with two inequality constraints. The shaded (green) contours depict the fitness function f (darker = smaller). The black curves show the borders of the inequality constraints. The infeasible area is shaded gray. The black point shows the global optimum of the fitness function which is different from the optimum of the constrained problem shown as the gold star.

where n is the size of the parameter space d. As the problem is scalable size of the parameter space can be any arbitrary integer larger than $1n = d > 2$. The lower bound is at $\vec{l}_{02} = \vec{0}$ and the upper bound is at $\vec{u}_{02} = \vec{10}$. The optimal solution for $d = 20$ is at $\vec{x}^*_{02} = [3.16246061, 3.12833142, 3.09479213, 3.06145059, 3.02792916, 2.99382607, 2.95866872, 2.92184227, 0.49482511, 0.48835711, 0.48231643, 0.47664475, 0.47129551, 0.46623099, 0.46142005, 0.45683665, 0.45245877, 0.44826762, 0.44424701, 0.44038286]$. Fig. A.2 shows G02 problem in the 2-dimensional space. As shown in Fig. A.2 only one of constraint function is active and the problem has a pretty large feasible region. The multimodality of the fitness function, makes this problem very challenging for surrogate-assisted optimizers. The complexity of this problem grows as the dimension grows.
Challenges: High-dimensionality, multimodality.

G03

This problem is scalable in dimension and has only one equality constraint. G03 is commonly investigated with $d = 20$ in different related research works.

$$\text{Minimize} \quad f(\vec{x}) = -(\sqrt{n})^n \prod_{i=1}^{n} x_i,$$

$$\text{subject to} \quad h_1(\vec{x}) = \sum_{i=1}^{n} x_i^2 - 1 = 0,$$

where n is the size of parameter space d. As the problem is scalable $n = d \geq 2$. The lower bound is $\vec{l}_{03} = \vec{0}$ and the upper bound is $\vec{u}_{03} = \vec{1}$. The optimal solution can easily be calculated analytically for any arbitrary dimension n. $\vec{x}^*_{03} = \frac{1}{\sqrt{n}}\vec{I} = \overrightarrow{\frac{1}{\sqrt{n}}}$ which means the optimal value is -1 for any n.

$$f(\vec{x}^*_{03}) = f(\overrightarrow{\frac{1}{\sqrt{n}}}) = -(\sqrt{n})^n \cdot (\frac{1}{\sqrt{n}})^n = -1$$

The solution suggested in CEC2006 [107] is not fully feasible and has a value better than the optimal value. Fig. A.3 shows G03 problem in the 2-dimensional space.
Challenges: High-dimensionality, small feasible space ($\rho = 0$ due to existence of an equality constraint).

G03mod

This problem is a modified version of G03 which transforms the equality constraint to an inequality constraint by assuming one side of the constraint being feasible.

$$\text{Minimize} \quad f(\vec{x}) = -(\sqrt{n})^n \prod_{i=1}^{n} x_i,$$

$$\text{subject to} \quad g_1(\vec{x}) = \sum_{i=1}^{n} x_i^2 - 1 \leq 0$$

The optimum is calculated exactly same as the G03 problem. Several papers address G03mod instead of G03 due to difficulties that many optimizers have in handling equality constraints. Fig. A.4 shows G03mod problem in the 2-dimensional space.
Challenges: High-dimensionality.

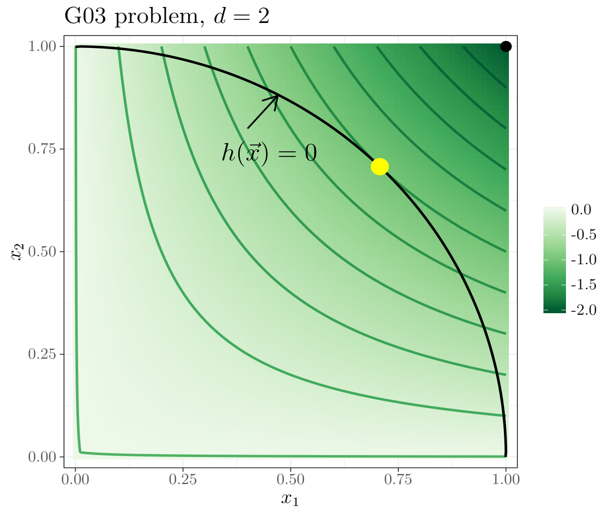

Figure A.3: G03 problem description. A $2d$ optimization problem with only one equality constraint. The shaded (green) contours depict the fitness function f (darker = smaller). The black curve shows the equality constraint. Feasible solutions are restricted to this line. The black point shows the global optimum of the fitness function which is different from the optimum of the constrained problem shown as the gold star.

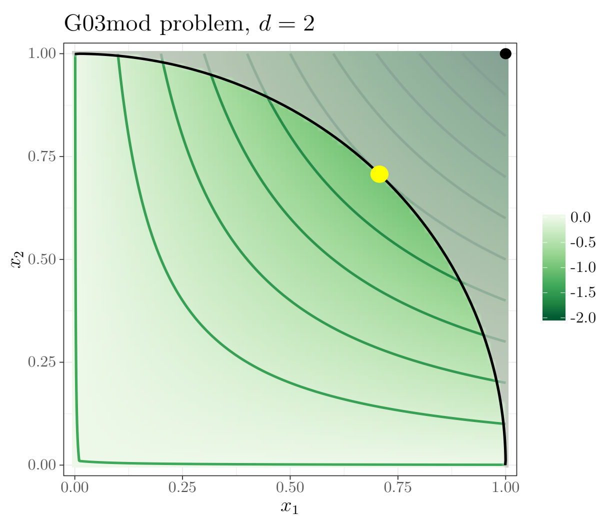

Figure A.4: G03mod problem description. A $2d$ optimization problem with one inequality constraint. The shaded (green) contours depict the fitness function f (darker = smaller). The black curve shows the borders of the inequality constraint. The infeasible area is shaded gray. The black point shows the global optimum of the fitness function which is different from the optimum of the constrained problem shown as the gold star.

G04

G04 is a 5-dimensional COP subject to 6 constraints two of which are active.

$$
\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = 5.3578547x_3^2 + 0.8356891x_1x_5 + 37.293239x_1 - 40792.141, \\
\text{subject to} \quad & g_1(\vec{x}) = 85.334407 + 0.0056858x_2x_5 + 0.0006262x_1x_4 - 0.0022053x_3x_5 - 92 \leq 0, \\
& g_2(\vec{x}) = -85.334407 - 0.0056858x_2x_5 - 0.0006262x_1x_4 + 0.0022053x_3x_5 \leq 0, \\
& g_3(\vec{x}) = 80.51249 + 0.0071317x_2x_5 + 0.0029955x_1x_2 + 0.0021813x_3^2 - 110 \leq 0, \\
& g_4(\vec{x}) = -80.51249 - 0.0071317x_2x_5 - 0.0029955x_1x_2 - 0.0021813x_3^2 + 90 \leq 0, \\
& g_5(\vec{x}) = 9.300961 + 0.0047026x_3x_5 + 0.0012547x_1x_3 + 0.0019085x_3x_4 - 25 \leq 0, \\
& g_6(\vec{x}) = -9.300961 - 0.0047026x_3x_5 - 0.0012547x_1x_3 - 0.0019085x_3x_4 + 20 \leq 0.
\end{aligned}
$$

The lower bound is at $\vec{l}_{04} = [78, 33, 27, 27, 27]$ and the upper bound is at $\vec{u}_{04} = [102, 45, 45, 45, 45]$. The optimal solution is at $\vec{x}^*_{04} = [78, 33, 29.99525602, 45, 36.77581290]$ and $f(\vec{x}_{04}) = -30665.53867178332$.
Challenges: Highly constrained.

G05

G05 is a 4-dimensional COP subject to 5 constraints including 3 equality constraints.

$$
\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = 3x_1 + 0.000001x_1^3 + 2x_2 + (0.000002/3)x_2^3, \\
\text{subject to} \quad & g_1(\vec{x}) = -x_4 + x_3 - 0.55 \leq 0, \\
& g_2(\vec{x}) = -x_3 + x_4 - 0.55 \leq 0, \\
& h_1(\vec{x}) = 1000\sin(-x_3 - 0.25) + 1000\sin(-x_4 - 0.25) + 894.8 - x_1 = 0, \\
& h_2(\vec{x}) = 1000\sin(x_3 - 0.25) + 1000\sin(x_3 - x_4 - 0.25) + 894.8 - x_2 = 0, \\
& h_3(\vec{x}) = 1000\sin(x_4 - 0.25) + 1000\sin(x_4 - x_3 - 0.25) + 1294.8 = 0.
\end{aligned}
$$

The lower bound is at $\vec{l}_{05} = [0, 0, -0.55, 0.55]$ and the upper bound is at $\vec{u}_{05} = [1200, 1200, 0.55, 0.55]$. The optimal solution is at $\vec{x}^*_{05} = [679.94531749, 1026.06713513, 0.11887637, -0.39623355]$ and $f(\vec{x}_{05}) = 5126.498109$. The solution suggested in CEC2006 [107] is not feasible and all equality constraints have a violation of size 10^{-4}, that's why the result reported in CEC2006 is better than the real optimal value.
Challenges: Highly constrained, zero feasible ratio.

G05mod

G05mod is a 4-dimensional COP subject to 5 inequality constraints.

$$\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = 3x_1 + 0.000001x_1^3 + 2x_2 + (0.000002/3)x_2^3, \\
\text{subject to} \quad & g_1(\vec{x}) = -x_4 + x_3 - 0.55 \leq 0, \\
& g_2(\vec{x}) = -x_3 + x_4 - 0.55 \leq 0, \\
& g_3(\vec{x}) = 1000\sin(-x_3 - 0.25) + 1000\sin(-x_4 - 0.25) + 894.8 - x_1 \leq 0, \\
& g_4(\vec{x}) = 1000\sin(x_3 - 0.25) + 1000\sin(x_3 - x_4 - 0.25) + 894.8 - x_2 \leq 0, \\
& g_5(\vec{x}) = 1000\sin(x_4 - 0.25) + 1000\sin(x_4 - x_3 - 0.25) + 1294.8 \leq 0.
\end{aligned}$$

The lower bound is at $\vec{l}_{05} = [0, 0, -0.55, 0.55]$ and the upper bound is at $\vec{u}_{05} = [1200, 1200, 0.55, 0.55]$. The optimal solution is at $\vec{x}^*_{05} = [679.94531749, 1026.06713513, 0.11887637, -0.39623355]$ and $f(\vec{x}_{05}) = 5126.498109$. The solution suggested in CEC2006 [107] is not feasible and all equality constraints have a violation of size 10^{-4}, that's why the result reported in CEC2006 is better than the real optimal value.
Challenges: Highly constrained, zero feasible ratio.

G06

A 2-dimensional COP with two active inequality constraints.

$$\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = (x_1 - 10)^3 + (x_2 - 20)^3, \\
\text{subject to} \quad & g_1(\vec{x}) = -(x_1 - 5)^2 - (x_2 - 5)^2 + 100 \leq 0, \\
& g_2(\vec{x}) = (x_1 - 6)^2 + (x_2 - 5)^2 - 82.81 \leq 0.
\end{aligned}$$

The lower bound is at $\vec{l}_{06} = [13, 0]$ and the upper bound is at $\vec{u}_{06} = [100, 100]$. The optimal solution is at $\vec{x}^*_{06} = [14.095, 0.8429608]$, where $f(\vec{x}_{06}) = -6961.813875580$. Fig. A.5 shows the G06 problem with three different zoomed in level. As shown in Fig. A.5 it is difficult to spot the feasible region in the original large space. As we zoom in about 10 times into the interesting region, the feasible area appears as a moon-shaped. G06 is a challenging COP due to its small feasible region, a very steep fitness function and two active constraints.
Challenges: Small feasible ratio ρ.

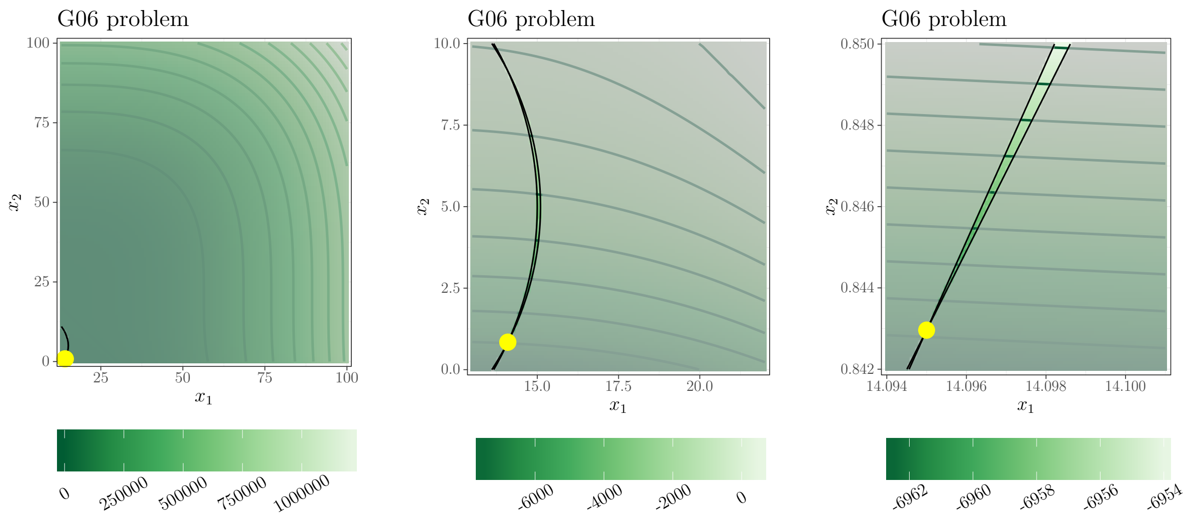

Figure A.5: G06 problem description. A $2d$ optimization problem with two inequality constraints. The shaded (green) contours depict the fitness function f (darker = smaller). The black curves show the borders of the inequality constraints. The infeasible area is shaded gray. The optimum of the constrained problem is shown as the gold star. The plots from left to right show the G06 problem with different zoom-in levels. Left: the original search space. Most of the search space seems to be infeasible and the interesting region is hardly detectable. Middle: $\approx 10\times$ zoomed in the interesting region. In the middle plot a tiny moon-shaped feasible region is observable. Right: $\approx 1000\times$ zoomed in.

G07

A 10-dimensional problem subjected to 8 inequality constraints 6 of which are active.

$$
\begin{aligned}
\text{Minimize}\quad & f(\vec{x}) = x_1^2 + x_2^2 + x_1x_2 - 14x_1 - 16x_2 + (x_3-10)^2 + 4(x_4-5)^2 + (x_5-3)^2 \\
& \quad + 2(x_6-1)^2 + 5x_7^2 + 7(x_8-11)^2 + 2(x_9-10)^2 + (x_{10}-7)^2 + 45, \\
\text{subject to}\quad & g_1(\vec{x}) = -105 + 4x_1 + 5x_2 - 3x_7 + 9x_8 \leq 0, \\
& g_2(\vec{x}) = 10x_1 - 8x_2 - 17x_7 + 2x_8 \leq 0, \\
& g_3(\vec{x}) = -8x_1 + 2x_2 + 5x_9 - 2x_{10} - 12 \leq 0, \\
& g_4(\vec{x}) = 3(x_1-2)^2 + 4(x_2-3)^2 + 2x_3^2 - 7x_4 - 120 \leq 0, \\
& g_5(\vec{x}) = 5x_1^2 + 8x_2 + (x_3-6)^2 - 2x_4 - 40 \leq 0, \\
& g_6(\vec{x}) = x_1^2 + 2(x_2-2)^2 - 2x_1x_2 + 14x_5 - 6x_6 \leq 0, \\
& g_7(\vec{x}) = 0.5(x_1-8)^2 + 2(x_2-4)^2 + 3x_5^2 - x_6 - 30 \leq 0, \\
& g_8(\vec{x}) = -3x_1 + 6x_2 + 12(x_9-8)^2 - 7x_{10} \leq 0.
\end{aligned}
$$

The lower bound is at $\vec{l}_{07} = -\overrightarrow{10}$ and the upper bound is at $\vec{u}_{07} = \overrightarrow{10}$. The optimal solution is at $\vec{x}^*_{07} = [2.17199783, 2.36367936, 8.77392512, 5.09598421, 0.99065597, 1.43057843, 1.32164704, 9.82872811, 8.28009420, 8.37592351]$. $f(\vec{x}^*_{07}) = 24.3062090689$
Challenges: High-dimensionality, highly constrained.

G08

A 2-dimensional problem subjected to 2 inequality constraints none of which are active at the optimum.

$$
\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = -\frac{\sin^3(2\pi x_1)\sin(2\pi x_2)}{x_1^3(x_1 + x_2)} \\
\text{subject to} \quad & g_1(\vec{x}) = x_1^2 - x_2 + 1 \leq 0, \\
& g_2(\vec{x}) = 1 - x_1 + (x_2 - 4)^2 \leq 0.
\end{aligned}
$$

The lower bound is at $\vec{l}_{08} = \vec{0}$ and the upper bound is at $\vec{u}_{08} = \overrightarrow{10}$. The optimal solution is at $\vec{x}^*_{08} = [1.2279713, 4.2453732]$and $f(\vec{x}^*_{08}) = -0.095825041418$. Fig. A.6 shows the G08 problem in two zoomed in levels. As shown in Fig. A.6 the fitness function of G08 is highly multimodal, therefore this COP is challenging to solve with surrogate-assisted optimizers.
Challenges: Multimodality.

G09

A 7-dimensional problem subjected to 4 inequality constraints 2 of which are active.

$$
\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = (x_1 - 10)^2 + 5(x_2 - 12)^2 + x_3^4 + 3(x_4 - 11)^2 + 10x_5^6 + 7x_6^2 \\
& + x_7^4 - 4x_6x_7 - 10x_6 - 8x_7 \\
\text{subject to} \quad & g_1(\vec{x}) = -127 + 2x_1^2 + 3x_2^4 + x_3 + 4x_4^2 + 5x_5 \leq 0, \\
& g_2(\vec{x}) = -282 + 7x_1 + 3x_2 + 10x_3^2 + x_4 - x_5 \leq 0, \\
& g_3(\vec{x}) = -196 + 23x_1 + x_2^2 + 6x_6^2 - 8x_7 \leq 0, \\
& g_4(\vec{x}) = 4x_1^2 + x_2^2 - 3x_1x_2 + 2x_3^2 + 5x_6 - 11x_7 \leq 0.
\end{aligned}
$$

The lower bound is at $\vec{l}_{09} = -\overrightarrow{10}$ and the upper bound is at $\vec{u}_{09} = \overrightarrow{10}$. The optimal solution is at $\vec{x}^*_{09} = [2.33049949323300210, 1.95137240, -0.47754042, 4.36572613,$

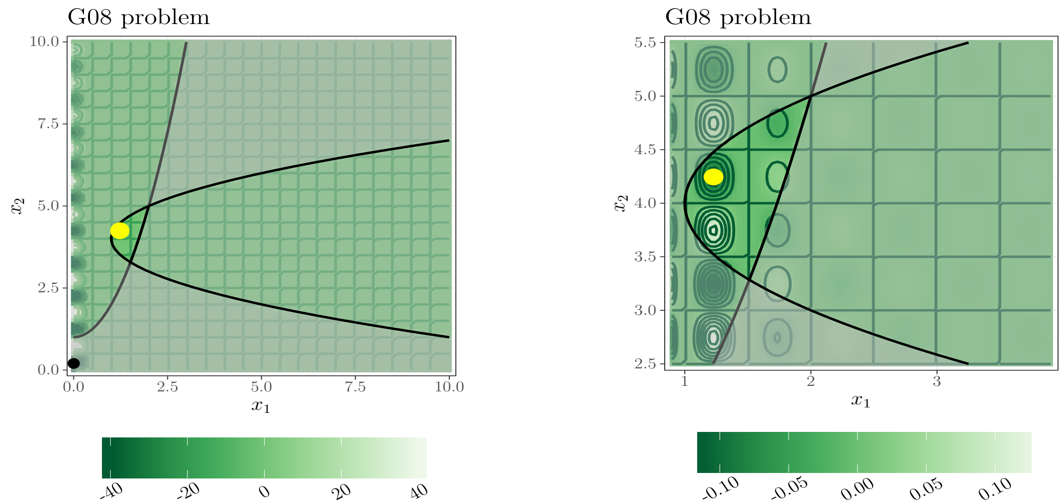

Figure A.6: G08 problem description. A $2d$ optimization problem with two inequality constraints. The shaded (green) contours depict the fitness function f (darker = smaller). The black curves show the borders of the inequality constraints. The infeasible area is shaded gray. The optimum of the constrained problem is shown as the gold star. The plots show the G08 problem with different zoom-in levels. Left: the original search space. Right: $\approx 2\times$ zoomed in. G08's fitness function has a large range. The local minima and maxima of G08's fitness function (out of the feasible area) have large values in the order of 1000 and -1000. For the visualization purposes we restricted the fitness range to $[-40; 40]$.

$-0.62448707, 1.03813092, 1.59422663]$ and $f(\vec{x}_{09}^{*}) = 680.63005737440$.
Challenges: Small feasibility ratio ρ.

G10

An 8-dimensional COP subjected to 6 constraints 3 of which are active.

$$\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = x_1 + x_2 + x_3 \\
\text{subject to} \quad & g_1(\vec{x}) = -1 + 0.0025(x_4 + x_6) \leq 0, \\
& g_2(\vec{x}) = -1 + 0.0025(x_5 + x_7 - x_4) \leq 0, \\
& g_3(\vec{x}) = -1 + 0.01(x_8 - x_5) \leq 0, \\
& g_4(\vec{x}) = -x_1 x_6 + 833.33252 x_4 + 100 x_1 - 83333.333 \leq 0, \\
& g_5(\vec{x}) = -x_2 x_7 + 1250 x_5 + x_2 x_4 - 1250 x_4 \leq 0, \\
& g_6(\vec{x}) = -x_3 x_8 + 1250000 + x_3 x_5 - 2500 x_5 \leq 0.
\end{aligned}$$

The lower bound is at $\vec{l}_{10} = -[100, 1000, 1000, 10, 10, 10, 10, 10]$ and the upper bound is at $\vec{u}_{10} = [10000, 10000, 10000, 1000, 1000, 1000, 1000, 1000]$. The optimal solution is at $\vec{x}^*_{10} = [579.29340270, 1359.97691009, 5109.97770901, 182.01659025, 295.60089166, 217.98340974, 286.41569858, 395.60089165]$ and $f(\vec{x}^*_{10}) = 7049.2480218071796$
Challenges: Small feasibility ratio ρ, highly constrained.

G11

A 2-dimensional COP subject to an equality constraint.

$$\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = x_1^2 + (x_2 - 1)^2, \\
\text{subject to} \quad & h_1(\vec{x}) = x_2 - x_1^2 = 0.
\end{aligned}$$

The lower bound is at $\vec{l}_{11} = -\vec{1}$ and the upper bound is at $\vec{u}_{11} = \vec{1}$. The optimal solution is at $\vec{x}^*_{11} = [-0.7071068, 0.5]$ or $\vec{x}^*_{11} = [0.7071068, 0.5]$ and $f(\vec{x}^*_{11}) = 0.75$
Challenges: Zero feasibility ratio $\rho = 0$.

G11mod

This problem is the modified version of G11 which transforms the equality constraint to an inequality constraint by assuming one side of the constraint being feasible.

$$\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = x_1^2 + (x_2 - 1)^2, \\
\text{subject to} \quad & g_1(\vec{x}) = x_2 - x_1^2 \leq 0.
\end{aligned}$$

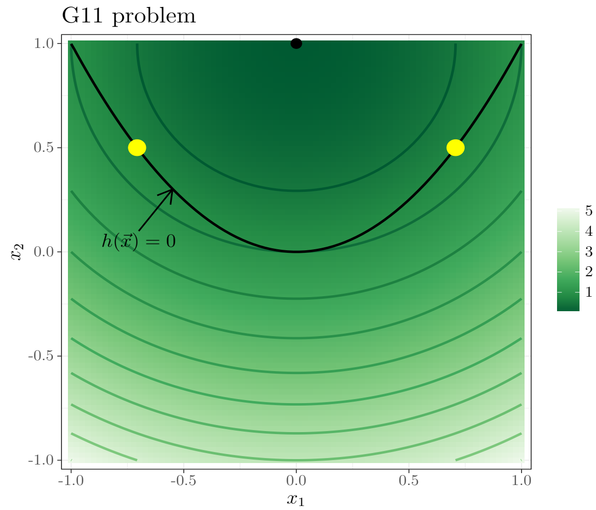

Figure A.7: G11 problem description. A $2d$ optimization problem with only one equality constraint. The shaded (green) contours depict the fitness function f (darker = smaller). The black curve shows the equality constraint. Feasible solutions are restricted to this line. The black point shows the global optimum of the fitness function which is different from the optimum of the constrained problem shown as the gold star.

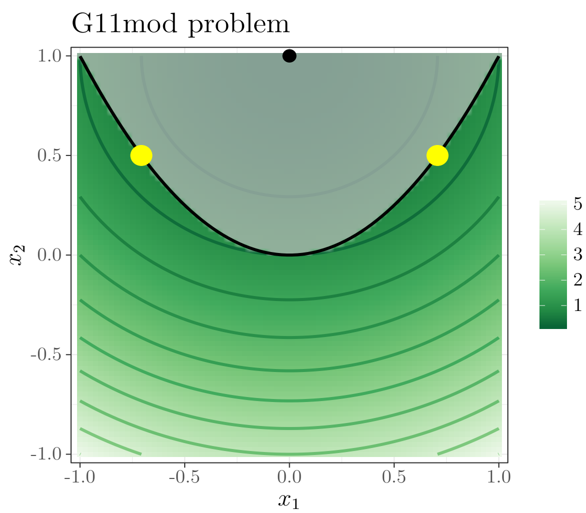

Figure A.8: G11 problem description. A $2d$ optimization problem with only one inequality constraint. The shaded (green) contours depict the fitness function f (darker = smaller). The black curve shows the inequality constraint. The infeasible area is shaded gray. The black point shows the global optimum of the fitness function which is different from the optimum of the constrained problem shown as the gold star.

The lower and upper bounds and the optimum are exactly same as the G11 problem. Several papers address G11mod instead of G11 due to difficulties that many optimizers have in handling equality constraints. Fig. A.8 shows G11mod problem.

G12

A 3-dimensional COP subject to 1 constraint. This problem has a disjoint feasible region.

$$\begin{aligned} \text{Minimize} \quad & f(\vec{x}) = -0.01(100 - (x_1 - 5)^2 - (x_2 - 5)^2 - (x_3 - 5)^2) \\ \text{subject to} \quad & g_1(\vec{x}) = (x_1 - p)^2 - (x_2 - q)^2 - (x_3 - r)^2 - 0.0625 \leq 0 \end{aligned}$$

The lower bound is $\vec{l}_{12} = \vec{0}$ and the upper bound is $\vec{u}_{12} = \vec{10}$. $p, q, r = 1, 2 \cdots 9$. These are 729 disjoint spheres and a solution is feasible if it is within one of the 729 spheres. Therefore we take the min over $g_1(.)$. The optimal solution is at $\vec{x}^*_{12} = [5, 5, 5]$ and $f(\vec{x}^*_{12}) = -1$

Challenges: Disjoint feasible region

G13

A 5-dimensional COP subject to 3 equality constraints.

$$\begin{aligned} \text{Minimize} \quad & f(\vec{x}) = e^{\prod_{i=1}^{d} x_i}, \\ \text{subject to} \quad & h_1(\vec{x}) = \sum_{i=1}^{d} x_i^2 - 10 = 0, \\ & h_2(\vec{x}) = x_2 x_3 - 5 x_4 x_5 = 0, \\ & h_3(\vec{x}) = x_1^3 + x_2^3 + 1 = 0. \end{aligned}$$

The lower bound is at $\vec{l}_{13} = [-2.3, -2.3, -3.2, -3.2, -3.2]$ and the upper bound is at $\vec{u}_{13} = -\vec{l}_{13}$. One of the optimal solution is at $\vec{x}^*_{13} = [-1.71714359, 1.59570973, 1.82724569, -0.76364228, -0.76364390]$ and $f(\vec{x}^*_{13}) = 0.05394984069520585$. The solution is invariant against a sign flip in both x_4 and x_5, a sign flip in both x_3 and x_4, a sign flip in both x_3 and x_5 or exchanging x_4 and x_5.

Challenges: Multimodality, zero feasibility ratio $\rho = 0$.

G14

A 10-dimensional COP subject to 3 equality constraints.

$$\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = \sum_{i=1}^{10} x_i \left(c_i + \ln \frac{x_i}{\sum_{j=1}^{1} 0 x_j} \right), \\
\text{subject to} \quad & h_1(\vec{x}) = x_1 + 2x_2 + 2x_3 + x_6 + x_{10} - 2 = 0, \\
& h_2(\vec{x}) = x_4 + 2x_5 + x_6 + x_7 - 1 = 0, \\
& h_3(\vec{x}) = x_3 + x_7 + x_8 + 2x_9 + x_{10} - 1 = 0,
\end{aligned}$$

The lower bound is at $\vec{l}_{14} = \vec{0}$, the upper bound is at $\vec{u}_{14} = \vec{10}$ and $\vec{c} = [-6.089, -17.164, -34.054, -5.914, -24.721, -14.986, -24.1, -10.708, -26.662, -22.179]$. The optimal solution is at $\vec{x}^*_{14} = [0.04066841, 0.14772124, 0.78320573, 0.00141434, 0.48529364, 0.00069318, 0.02740520, 0.01795097, 0.03732682, 0.09688446]$ and $f(\vec{x}^*_{14}) = -47.764888459491459$.
Challenges: High dimensionality, zero feasibility ratio $\rho = 0$.

G15

A 3-dimensional COP subject to 2 equality constraints.

$$\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = 1000 - x_1^2 - 2x_2^2 - x_3^2 - x_1 x_2 - x_1 x_3, \\
\text{subject to} \quad & h_1(\vec{x}) = \sum_{i=1}^{3} x_i^2 - 25 = 0, \\
& h_2(\vec{x}) = 8x_1 + 14x_2 + 7x_3 - 56 = 0.
\end{aligned}$$

The lower bound is at $\vec{l}_{15} = \vec{0}$ and the upper bound is at $\vec{u}_{15} = \vec{10}$. The optimal solution is at $\vec{x}^*_{15} = [3.51212813, 0.21698751, 3.55217855$. $f(\vec{x}^*_{15}) = 961.71502228996087$
Challenges: Zero feasibility ratio $\rho = 0$.

G15mod

A 3-dimensional COP subject to 2 inequality constraints.

$$\begin{aligned} \text{Minimize} \quad & f(\vec{x}) = 1000 - x_1^2 - 2x_2^2 - x_3^2 - x_1x_2 - x_1x_3, \\ \text{subject to} \quad & g_1(\vec{x}) = \sum_{i=1}^{3} x_i^2 - 25 \leq 0, \\ & g_2(\vec{x}) = 8x_1 + 14x_2 + 7x_3 - 56 \leq 0. \end{aligned}$$

The lower bound is at $\vec{l}_{15} = \vec{0}$ and the upper bound is at $\vec{u}_{15} = \vec{10}$. The optimal solution is at $\vec{x}^*_{15} = [3.51212813, 0.21698751, 3.55217855$. $f(\vec{x}^*_{15}) = 961.71502228996087$

G16

A 5-dimensional problem subject to 38 constraints 4 of which are active.

$$\begin{aligned} \text{Minimize} \quad & f(\vec{x}) = 0.000117y_{14} + 0.1365 + 0.00002358y_{13} + 0.000001502y_{16} + 0.0321y_{12} \\ & +0.004324y_5 + 0.0001\tfrac{c_{15}}{c_{16}} + 37.48\tfrac{y_2}{c_{12}} - 0.0000005843y_{17}, \\ \text{subject to} \quad & g_1(\vec{x}) = \tfrac{0.28}{0.72}y_5 - y_4 \leq 0, \qquad g_2(\vec{x}) = x_3 - 1.5x_2 \leq 0, \\ & g_3(\vec{x}) = 3496\tfrac{y_2}{c_{12}} - 21 \leq 0, \qquad g_4(\vec{x}) = 110.6 + y_1 - \tfrac{62212}{c_{17}} \leq 0, \\ & g_5(\vec{x}) = 213.1 - y_1 \leq 0, \qquad g_6(\vec{x}) = y_1 - 405.23 \leq 0, \\ & g_7(\vec{x}) = 17.505 - y_2 \leq 0, \qquad g_8(\vec{x}) = y_2 - 1053.6667 \leq 0, \\ & g_9(\vec{x}) = 11.275 - y_3 \leq 0, \qquad g_{10}(\vec{x}) = y_3 - 35.03 \leq 0, \\ & g_{11}(\vec{x}) = 214.228 - y_4 \leq 0, \qquad g_{12}(\vec{x}) = y_4 - 665.585 \leq 0, \\ & g_{13}(\vec{x}) = 7.458 - y_5 \leq 0, \qquad g_{14}(\vec{x}) = y_5 - 584.463 \leq 0, \\ & g_{15}(\vec{x}) = 0.961 - y_6 \leq 0, \qquad g_{16}(\vec{x}) = y_6 - 265.916 \leq 0, \\ & g_{17}(\vec{x}) = 1.612 - y_7 \leq 0, \qquad g_{18}(\vec{x}) = y_7 - 7.046 \leq 0, \\ & g_{19}(\vec{x}) = 0.146 - y_8 \leq 0, \qquad g_{20}(\vec{x}) = y_8 - 0.222 \leq 0, \\ & g_{21}(\vec{x}) = 107.99 - y_9 \leq 0, \qquad g_{22}(\vec{x}) = y_9 - 273.366 \leq 0, \\ & g_{23}(\vec{x}) = 922.693 - y_{10} \leq 0, \qquad g_{24}(\vec{x}) = y_{10} - 1286.105 \leq 0, \\ & g_{25}(\vec{x}) = 926.832 - y_{11} \leq 0, \qquad g_{26}(\vec{x}) = y_{11} - 1444.046 \leq 0, \\ & g_{27}(\vec{x}) = 18.766 - y_{12} \leq 0, \qquad g_{28}(\vec{x}) = y_{12} - 537.141 \leq 0, \\ & g_{29}(\vec{x}) = 1072.163 - y_{13} \leq 0, \qquad g_{30}(\vec{x}) = y_{13} - 3247.039 \leq 0, \\ & g_{31}(\vec{x}) = 8961.448 - y_{14} \leq 0, \qquad g_{32}(\vec{x}) = y_{14} - 26844.086 \leq 0, \\ & g_{33}(\vec{x}) = 0.063 - y_{15} \leq 0, \qquad g_{34}(\vec{x}) = y_{15} - 0.386 \leq 0, \\ & g_{35}(\vec{x}) = 71084.33 - y_{16} \leq 0, \qquad g_{36}(\vec{x}) = -140000 - y_{16} \leq 0, \\ & g_{37}(\vec{x}) = 2802713 - y_{17} \leq 0, \qquad g_{38}(\vec{x}) = y_{17} - 12146108 \leq 0, \end{aligned}$$

where,

$$
\begin{aligned}
y_1 &= x_2 + x_3 + 41.6, & c_1 &= 0.024x_4 - 4.62,\\
y_2 &= \tfrac{12.5}{c_1} + 12, & c_2 &= 0.0003535x_1^2 + 0.5311x_1 + 0.08705y_2x_1,\\
y_3 &= \tfrac{c_2}{c_3}, & c_3 &= 0.052x_1 + 78 + 0.002377y_2x_1,\\
y_4 &= 19y_3, & c_5 &= 100x_2,\\
c_4 &= 0.04782(x_1 - x_3) + \tfrac{0.1956(x_1-y_3)^2}{x_2} + 0.6376y_4 + 1.594y_3, \\
y_5 &= c_6c_7, & c_6 &= x_1 - y_3 - y_4\\
y_6 &= x_1 - y_5 - y_4 - y_3, & c_7 &= 0.950 - \tfrac{c_4}{c_5},\\
y_7 &= \tfrac{c_8}{y_1}, & c_8 &= (y_5 + y_4)0.995,\\
y_8 &= \tfrac{c_8}{3798}, & c_9 &= y_7 - \tfrac{0.0663y_7}{y_8} - 0.3153,\\
y_9 &= \tfrac{96.82}{c_9} + 0.321y_1 & c_{10} &= \tfrac{12.3}{752.3},\\
y_{10} &= 1.29y_5 + 1.258y_4 + 2.29y_3 + 1.71y_6, & c_{11} &= (1.75y_2)(0.995x_1),\\
y_{11} &= 1.71x_1 - 0.452y_4 + 0.580y_3, & c_{12} &= 0.995y_{10} + 1998,\\
y_{12} &= c_{10}x_1 + \tfrac{c_{11}}{c_{12}}, & c_{14} &= 2324y_{10} - 28740000y_2,\\
y_{13} &= c_{12} - 1.75y_2, & c_{15} &= \tfrac{y_{13}}{y_{15}} - \tfrac{y_{13}}{0.52},\\
c_{13} &= 0.995y_{10} + 60.8x_2 + 48x_4 - 0.1121y_{14} - 5095, \\
y_{14} &= 3623 + 64.4x_2 + 58.4x_3 + \tfrac{146312}{y_9+x_5}, & c_{16} &= 1.104 - 0.72y_{15},\\
y_{15} &= \tfrac{y_{13}}{c_{13}}, & c_{17} &= y_9 + y_5,\\
y_{16} &= 148000 - 331000y_{15} + 40y_{13} - 61y_{15}y_{13}, \\
y_{17} &= y_9 + x_5.
\end{aligned}
$$

The lower bound is at $\vec{l}_{16} = [704.4148, 68.6, 0.0, 193, 25$ and the upper bound is at $\vec{u}_{16} = [906.3855, 288.88, 134.75, 287.0966, 84.1988]$. The optimal solution is at $\vec{x}^*_{16} = [705.17454, 68.6, 102.9, 282.32493, 37.58412]$ and $f(\vec{x}^*_{16}) = -1.905155$
Challenges: Highly constrained

G17

A 6-dimensional problem subject to 4 equality constraints.

$$\text{Minimize} \quad f(\vec{x}) = f_1(x_1) + f_2(x_2)$$

$$f_1(x_1) = \begin{cases} 30x_1, & \text{if } 0 \leq x_1 < 300 \\ 31x_1, & \text{if } 300 \leq x_1 \leq 400 \end{cases}$$

$$f_2(x_2) = \begin{cases} 28x_2, & \text{if } 0 \leq x_2 < 100 \\ 29x_2, & \text{if } 100 \leq x_2 \leq 200 \\ 30x_2, & \text{if } 200 \leq x_2 \leq 1000 \end{cases}$$

$$\begin{aligned} \text{subject to} \quad & h_1(\vec{x}) = -x_1 + 300 - \frac{x_3 x_4}{131.078}\cos(1.48477 - x_6) + \frac{0.90798 x_3^2}{131.078}\cos(1.47588) = 0, \\ & h_2(\vec{x}) = -x_2 - \frac{x_3 x_4}{131.078}\cos(1.48477 + x_6) + \frac{0.90798 x_4^2}{131.078}\cos(1.47588) = 0, \\ & h_3(\vec{x}) = -x_5 - \frac{x_3 x_4}{131.078}\sin(1.48477 + x_6) + \frac{0.90798 x_4^2}{131.078}\sin(1.47588) = 0, \\ & h_4(\vec{x}) = 200 - \frac{x_3 x_4}{131.078}\sin(1.48477 - x_6) + \frac{0.90798 x_3^2}{131.078}\sin(1.47588) = 0. \end{aligned}$$

The lower bound is at $\vec{l}_{17} = [0, 0, 340, 340, -1000, 0]$ and the upper bound is at$\vec{u}_{17} = [400, 1000, 420, 420, 1000, 0.5236]$. The optimal solution is at $\vec{x}^*_{17} = [201.78446721, 99.99999999, 383.07103485420, -10.90765845, 0.07314823]$ and $f(\vec{x}^*_{17}) = 8853.534$.
Challenges: Zero feasibility ratio $\rho = 0$.

G18

A 9-dimenstional problem subject to 13 constraints 6 of which are active.

$$
\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = -0.5(x_1x_4 - x_2x_3 + x_3x_9 - x_5x_9 + x_5x_8 - x_6x_7), \\
\text{subject to} \quad & g_1(\vec{x}) = x_3^2 + x_4^2 - 1 \le 0, \\
& g_2(\vec{x}) = x_9^2 - 1 \le 0, \\
& g_3(\vec{x}) = x_5^2 + x_6^2 - 1 \le 0, \\
& g_4(\vec{x}) = x_1^2 + (x_2 - x_9)^2 - 1 \le 0, \\
& g_5(\vec{x}) = (x_1 - x_5)^2 + (x_2 - x_6)^2 - 1 \le 0, \\
& g_6(\vec{x}) = (x_1 - x_7)^2 + (x_2 - x_8)^2 - 1 \le 0, \\
& g_7(\vec{x}) = (x_3 - x_5)^2 + (x_4 - x_6)^2 - 1 \le 0, \\
& g_8(\vec{x}) = (x_3 - x_7)^2 + (x_4 - x_8)^2 - 1 \le 0, \\
& g_9(\vec{x}) = x_7^2 + (x_8 - x_9)^2 - 1 \le 0, \\
& g_{10}(\vec{x}) = x_2x_3 - x_1x_4 \le 0, \\
& g_{11}(\vec{x}) = -x_3x_9 \le 0, \\
& g_{12}(\vec{x}) = x_5x_9 \le 0, \\
& g_{13}(\vec{x}) = x_6x_7 - x_5x_8 \le 0.
\end{aligned}
$$

The lower bound is at $\vec{l}_{18} = [-10, -10, -10, -10, -10, -10, -10, -10, 0]$ and the upper bound is at $\vec{u}_{18} = [10, 10, 10, 10, 10, 10, 10, 10, 20]$. The optimal solution is at $\vec{x}^*_{18} = [-0.98900055, 0.14791184, -0.62428976, -0.78118417, -0.98761593, 0.15047783, -0.62259598, -0.78254342, 0.0]$ and $f(\vec{x}^*_{18}) = -0.86573533494888033$.
Challenges: Highly constrained.

G19

A 15-dimensional problem subject to 5 constraints.

$$
\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = \sum_{j=1}^{5}\sum_{i=1}^{5} c_{ij}x_{(10+i)}x_{(10+j)} + 2\sum_{j=1}^{5} d_j x^3_{(10+j)} - \sum_{i=1}^{10} b_i x_i \\
\text{subject to} \quad & g_j(\vec{x}) = -2\sum_{i=1}^{5} c_{ij}x_{(10+i)} - 3d_j x^2_{(10+j)} - e_j + \sum_{i=1}^{10} a_{ij}x_i \le 0 \quad j = 1, \cdots, 5
\end{aligned}
$$

,where $\overrightarrow{b} = [-40, -2, -0.25, -4, -4, -1, -40, -60, 5, 1], \overrightarrow{d} = [4, 8, 10, 6, 2]$ and $\overrightarrow{e} = [-15, -27, -36, -18, -12]$. The lower bound is at $\vec{l}_{19} = \overrightarrow{0}$ and the upper bound is at $\vec{u}_{19} = \overrightarrow{10}$.

$$\mathbf{a} = \begin{bmatrix} -16 & 2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0.4 & 2 \\ -3.5 & 0 & 2 & 0 & 0 \\ 0 & -2 & 0 & -4 & -1 \\ 0 & -9 & -2 & 1 & -2.8 \\ 2 & 0 & -4 & 0 & 0 \\ -1 & -1 & -1 & -1 & -1 \\ -1 & -2 & -3 & -2 & -1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

$$\mathbf{c} = \begin{bmatrix} 30 & -20 & -10 & 32 & -10 \\ -20 & 39 & -6 & -31 & 32 \\ -10 & -6 & 10 & -6 & -10 \\ 32 & -31 & -6 & 39 & -20 \\ -10 & 32 & -10 & -20 & 30 \end{bmatrix}$$

The optimal solution is at $\vec{x}^*_{19} = [0, 6.08597252436373e-033, 3.94600628013917, -2.35103745208393e-032, 3.28318162727873, 10, 5.74431051614192e-033, -1.15517863716213e-032, -2.6336322104807e-032, -3.50389001765656e-033, 0.370762125835098, 0.278454209512692, 0.523838440499861, 0.388621589976956, 0.29815843730292]$ and $f(\vec{x}^*_{19}) = 32.655592950349401$.

Challenges: High-dimensionality.

G20

A 24-dimensional problem subject to 20 constraints 16 of which are active.

$$\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = \sum_{i=1}^{24} a_i x_i, \\
\text{subject to} \quad & g_j(\vec{x}) = \frac{(x_j + x_{(j+12)})}{\sum_{i=1}^{24} x_i + e_j} \leq 0, \quad j = 1, 2, 3, \\
& g_j(\vec{x}) = \frac{(x_{(j+3)} + x_{(j+15)})}{\sum_{i=1}^{24} x_i + e_j} \leq 0, \quad j = 4, 5, 6, \\
& h_k(\vec{x}) = \frac{x_{(k+12)}}{b_{(k+12)} \sum_{k=13}^{24} \frac{x_k}{b_k}} - \frac{c_k x_k}{40 b_k \sum_{k=1}^{12} \frac{x_k}{b_k}} = 0, \quad k = 1, \cdots, 12, \\
& h_{13}(\vec{x}) = \sum_{i=1}^{24} x_i - 1 = 0, \\
& h_{14}(\vec{x}) = \sum_{i=1}^{12} \frac{x_i}{d_i} + \alpha \sum_{i=13}^{24} \frac{x_i}{b_i} - 1.671 = 0.
\end{aligned}$$

The lower bound is at $\vec{l}_{20} = \vec{0}$ and the upper bound is at $\vec{u}_{20} = \vec{10}$.

$\alpha = (0.7302)(530)(\frac{14.7}{40})$.
$\vec{a} = [0.0693, 0.0577, 0.05, 0.2, 0.26, 0.55, 0.06, 0.1, 0.12, 0.18, 0.1, 0.09, 0.0693, 0.0577, 0.05, 0.2, 0.26, 0.55, 0.06, 0.1, 0.12, 0.18, 0.1, 0.09]$
$\vec{b} = [44.094, 58.12, 58.12, 137.4, 120.9, 170.9, 62.501, 84.94, 133.425, 82.507, 46.07, 60.097, 44.094, 58.12, 58.12, 137.4, 120.9, 170.9, 62.501, 84.94, 133.425, 82.507, 46.07, 60.097]$
$\vec{c} = [123.7, 31.7, 45.7, 14.7, 84.7, 27.7, 49.7, 7.1, 2.1, 17.7, 0.85, 0.64]$
$\vec{d} = [31.244, 36.12, 34.784, 92.7, 82.7, 91.6, 56.708, 82.7, 80.8, 64.517, 49.4, 49.1]$
$\vec{e} = [0.1, 0.3, 0.4, 0.3, 0.6, 0.3]$

The optimal solution is at $\vec{x}^*_{20} = [9.53E-7, 0, 4.21e-3, 1.039e-4, 0, 0, 2.072e-1, 5.979e-1, 1.298e-1, 3.35e-2, 1.711e-2, 8.827e-3, 4.657e-10, 0, 0, 0, 0, 0, 2.868e-4, 1.193e-3, 8.332e-5, 1.239e-4, 2.07e-5, 1.829e-5]$.

Challenges: High-dimensionality, highly constrained, zero feasibility ratio $\rho = 0$.

G21

A 7-dimensional COP subject to 6 active equality and inequality constraints.

$$\begin{aligned}\text{Minimize} \quad & f(\vec{x}) = x_1, \\ \text{subject to} \quad & g_1(\vec{x}) = -x_1 + 35x_2^{0.6} + 35x_3^{0.6} \leq 0, \\ & h_1(\vec{x}) = -300x_3 + 7500x_5 - 7500x_6 - 25x_4x_5 + 25x_4x_6 + x_3x_4 = 0, \\ & h_2(\vec{x}) = 100x_2 + 155.365x_4 + 2500x_7 - x_2x_4 - 25x_4x_7 - 15536.5 = 0, \\ & h_3(\vec{x}) = -x_5 + \ln(-x_4 + 900) = 0, \\ & h_4(\vec{x}) = -x_6 + \ln(x_4 + 300) = 0, \\ & h_5(\vec{x}) = -x_7 + \ln(-2x_4 + 700) = 0.\end{aligned}$$

The lower bound is at $\vec{l}_{21} = [0.0, 0.0, 0.0, 100, 6.3, 5.9, 4.5]$ and the upper bound is at $\vec{u}_{21} = [1000, 40, 40, 300, 6.7, 6.4, 6.25]$. The optimal solution is at $\vec{x}_{21}^* = [193.724510070034967, 5.56944131553368433e{-}27, 17.3191887294084914, 100.047897801386839, 6.68445185362377892, 5.99168428444264833, 6.21451648886070451]$ and $f(\vec{x}_{21}^*) = 193.72451007003497$.
Challenges: Highly constrained, zero feasibility ratio $\rho = 0$.

G22

A 22-dimensional COP subject to 20 constraints 19 of which are active equality constraints.

$$
\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = x_1, \\
\text{subject to} \quad & g_1(\vec{x}) = -x_1 + x_2^{0.6} + x_3^{0.6} + x_4^{0.6} \leq 0, \\
& h_1(\vec{x}) = x_5 - 10^5 x_8 + 10^7 = 0, \\
& h_2(\vec{x}) = x_6 + 10^5 x_8 - 10^5 x_9 = 0, \\
& h_3(\vec{x}) = x_7 + 10^5 x_9 - 5 \cdot 10^7 = 0, \\
& h_4(\vec{x}) = x_5 + 10^5 x_{10} - 3.3 \cdot 10^7 = 0, \\
& h_5(\vec{x}) = x_6 + 10^5 x_{11} - 4.4 \cdot 10^7 = 0, \\
& h_6(\vec{x}) = x_7 + 10^5 x_{12} - 6.6 \cdot 10^7 = 0, \\
& h_7(\vec{x}) = x_5 - 120 x_2 x_{13} = 0, \\
& h_8(\vec{x}) = x_6 - 80 x_3 x_{14} = 0, \\
& h_9(\vec{x}) = x_7 - 40 x_4 x_{15} = 0, \\
& h_{10}(\vec{x}) = x_8 - x_{11} + x_{16} = 0, \\
& h_{11}(\vec{x}) = x_9 - x_{12} + x_{17} = 0, \\
& h_{12}(\vec{x}) = -x_{18} + \ln(x_{10} - 100) = 0, \\
& h_{13}(\vec{x}) = -x_{19} + \ln(-x_8 + 300) = 0, \\
& h_{14}(\vec{x}) = -x_{20} + \ln(x_{16}) = 0, \\
& h_{15}(\vec{x}) = -x_{21} + \ln(-x_9 + 400) = 0, \\
& h_{16}(\vec{x}) = -x_{22} + \ln(x_{17}) = 0, \\
& h_{17}(\vec{x}) = -x_8 - x_{10} + x_{13} x_{18} - x_{13} x_{19} + 400 = 0, \\
& h_{18}(\vec{x}) = x_8 - x_9 - x_{11} + x_{14} x_{20} - x_{14} x_{21} + 400 = 0, \\
& h_{19}(\vec{x}) = x_9 - x_{12} - 4.60517 x_{15} + x_{15} x_{22} + 100 = 0,
\end{aligned}
$$

The lower bound is at $\vec{l}_{22} = [0, 0, 0, 0, 0, 0, 0, 100.100, 100.01, 100, 100, 0, 0, 0, 0, 0.01, -4.7, -4.7, -4.7, -4.7, -4.7]$ and the upper bound is at $\vec{u}_{22} = [2e+04, 1e+06, 1e+06, 1e+06, 4e+07, 4e+07, 4e+07, 3e+02, 4e+02, 3e+02, 4e+02, 6e+02, 5e+02, 5e+02, 5e+02, 3e+02, 4e+02, 6.25, 6.250, 6.25, 6.25, 6.25]$. The best solution found is at $\vec{x}^*_{22} = [2.416091e+02, 1.354324e+02, 9.159755e+02, 4.850804e+03, 3.000000e+06, 1.223014e+07, 2.476986e+07, 1.300000e+02, 2.523014e+02, 3.000000e+02, 3.176986e+02, 4.123014e+02, 1.845939e+02, 1.669005e+02, 1.276585e+02, 1.876986e+02, 1.600000e+02, 5.298317e+00, 5.135798e+00, 5.234837e+00, 4.995174e+00, 5.075174e+00]$

and $f(\vec{x}^*_{22}) = 241.609$ The optimal value that we find is about larger than what is reported in CEC2006 [107] but fully feasible.
Challenges: High-dimensionality, zero feasibility ratio $\rho = 0$, highly constrained, small feasible subspace ratio $\eta = \frac{3}{22} \approx 0.14$.

G23

A 9-dimensional problem subject to 6 active constraints.

$$\begin{aligned} \text{Minimize} \quad & f(\vec{x}) = -9x_5 - 15x_8 + 6x_1 + 16x_2 + 10(x_6 + x_7), \\ \text{subject to} \quad & g_1(\vec{x}) = x_9x_3 + 0.02x_6 - 0.025x_5 \leq 0, \\ & g_2(\vec{x}) = x_9x_4 + 0.02x_7 - 0.015x_8 \leq 0, \\ & h_1(\vec{x}) = x_1 + x_2 - x_3 - x_4 = 0, \\ & h_2(\vec{x}) = 0.03x_1 + 0.01x_2 - x_9(x_3 + x_4) = 0, \\ & h_3(\vec{x}) = x_3 + x_6 - x_5 = 0, \\ & h_4(\vec{x}) = x_4 + x_7 - x_8 = 0. \end{aligned}$$

The lower bound is at $\vec{l}_{23} = [0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.0, 0.01]$ and the upper bound is at $\vec{u}_{23} = [300, 300, 100, 200, 100, 300, 100, 200, 0.03]$.

The optimal solution is at $\vec{x}^*_{23} = [0, 100, 0, 100, 0, 0, 100, 200, 0.01]$. $f(\vec{x}^*_{23}) = -400.0$

Challenges: Highly constrained, zero feasibility ratio $\rho = 0$.

G24

A 2-dimensional COP subject to 2 active inequality constraints.

$$\begin{aligned} \text{Minimize} \quad & f(\vec{x}) = -x_1 - x_2, \\ \text{subject to} \quad & g_1(\vec{x}) = -2x_1^4 + 8x_1^3 - 8x_1^2 + x_2 - 2 \leq 0, \\ & g_2(\vec{x}) = -4x_1^4 + 32x_1^3 - 88x_1^2 + 96x_1 + x_2 - 36 \leq 0. \end{aligned}$$

The lower bound is at $\vec{l}_{24} = \vec{0}$ and the upper bound is at $\vec{u}_{24} = [3, 4]$. The optimal solution is at $\vec{x}^*_{24} = [2.329520197477607, 3.17849307411768]$ and $f(\vec{x}^*_{24}) = -5.508$. Fig. A.9 shows G24 problem.

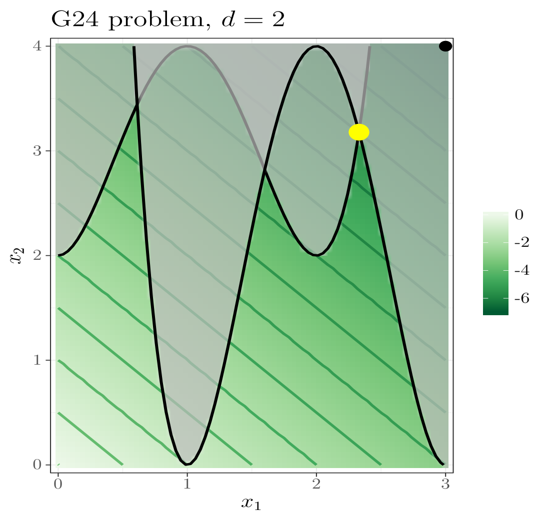

Figure A.9: G24 problem description. A $2d$ optimization problem with two inequality constraints. The shaded (green) contours depict the fitness function f (darker = smaller). The black curves show the borders of the inequality constraints. The infeasible area is shaded gray. The optimum of the constrained problem is shown as the gold star.

Appendix B

Transforming G22

$$
\begin{aligned}
\text{Minimize} \quad & f(\vec{x}) = x_1, \\
\text{subject to} \quad & g_1(\vec{x}) = -x_1 + x_2^{0.6} + x_3^{0.6} + x_4^{0.6} \leq 0, \\
& h_1(\vec{x}) = x_5 - 10^5 x_8 + 10^7 = 0, \\
& h_2(\vec{x}) = x_6 + 10^5 x_8 - 10^5 x_9 = 0, \\
& h_3(\vec{x}) = x_7 + 10^5 x_9 - 5 \cdot 10^7 = 0, \\
& h_4(\vec{x}) = x_5 + 10^5 x_{10} - 3.3 \cdot 10^7 = 0, \\
& h_5(\vec{x}) = x_6 + 10^5 x_{11} - 4.4 \cdot 10^7 = 0, \\
& h_6(\vec{x}) = x_7 + 10^5 x_{12} - 6.6 \cdot 10^7 = 0, \\
& h_7(\vec{x}) = x_5 - 120 x_2 x_{13} = 0, \\
& h_8(\vec{x}) = x_6 - 80 x_3 x_{14} = 0, \\
& h_9(\vec{x}) = x_7 - 40 x_4 x_{15} = 0, \\
& h_{10}(\vec{x}) = x_8 - x_{11} + x_{16} = 0, \\
& h_{11}(\vec{x}) = x_9 - x_{12} + x_{17} = 0, \\
& h_{12}(\vec{x}) = -x_{18} + \ln(x_{10} - 100) = 0, \\
& h_{13}(\vec{x}) = -x_{19} + \ln(-x_8 + 300) = 0, \\
& h_{14}(\vec{x}) = -x_{20} + \ln(x_{16}) = 0, \\
& h_{15}(\vec{x}) = -x_{21} + \ln(-x_9 + 400) = 0, \\
& h_{16}(\vec{x}) = -x_{22} + \ln(x_{17}) = 0, \\
& h_{17}(\vec{x}) = -x_8 - x_{10} + x_{13} x_{18} - x_{13} x_{19} + 400 = 0, \\
& h_{18}(\vec{x}) = x_8 - x_9 - x_{11} + x_{14} x_{20} - x_{14} x_{21} + 400 = 0, \\
& h_{19}(\vec{x}) = x_9 - x_{12} - 4.60517 x_{15} + x_{15} x_{22} + 100 = 0,
\end{aligned}
$$

By solving the first three equality constraints we get rid of three variables x_5, x_6 and x_7, so that we write them as a function of x_8 and x_9 as follows.

$$\begin{aligned} h_1(\vec{x}) \rightarrow &\quad x_5 = 10^5(x_8 - 10) \\ h_2(\vec{x}) \rightarrow &\quad x_6 = 10^5(x_9 - x_8) \\ h_3(\vec{x}) \rightarrow &\quad x_7 = 10^5(500 - x_9) \end{aligned}$$

Now that we have x_5, x_6 and x_7 we can substitute them in h_4, h_5 and h_6 in order to write x_{10}, x_{11} and x_{12} dependent on x_8 and x_9 as follows.

$$\begin{aligned} h_4(\vec{x}) \rightarrow &\quad x_{10} = \frac{3.3 \cdot 10^7 - x_5}{10^5} = 430 - x_8 \\ h_5(\vec{x}) \rightarrow &\quad x_{11} = \frac{4.4 \cdot 10^7 - x_6}{10^5} = 440 - x_9 + x_8 \\ h_6(\vec{x}) \rightarrow &\quad x_{12} = \frac{6.6 \cdot 10^7 - x_7}{10^5} = 160 + x_9 \end{aligned}$$

We can find 5 more variables (x_{16}, x_{17}, x_{18}, x_{19}, x_{20}) based on x_8 and x_9 by simply substitution of x_{10}, x_{11} and x_{12} in the following equality constraints. It turns out that one parameter $x_{17} = 160$ is equal to a constant.

$$\begin{aligned} h_{10}(\vec{x}) \rightarrow &\quad x_{16} = x_{11} - x_8 = 440 - x_9 \\ h_{11}(\vec{x}) \rightarrow &\quad x_{17} = x_{12} - x_9 = 160 \\ h_{12}(\vec{x}) \rightarrow &\quad x_{18} = \ln(x_{10} - 100) = \ln(330 - x_8) \\ h_{13}(\vec{x}) \rightarrow &\quad x_{19} = \ln(300 - x_8) \\ h_{15}(\vec{x}) \rightarrow &\quad x_{21} = \ln(400 - x_9) \end{aligned}$$

Now that we have x_{16} and x_{17} with the help of h_{14} and h_{15} equality constraints we can find x_{20} and x_{21}, where x_{21} has a constant value.

$$\begin{aligned} h_{14}(\vec{x}) \rightarrow &\quad x_{20} = \ln(x_{16}) = \ln(440 - x_9) \\ h_{16}(\vec{x}) \rightarrow &\quad x_{22} = \ln(x_{17}) = \ln(160) \end{aligned}$$

Reformulating the h_{17}, h_{18} and h_{19} equality constraints will give us x_{13}, x_{14}, x_{15}.

$$\begin{aligned}
h_{17}(\vec{x}) \rightarrow \quad x_{13} &= \frac{x_8 + x_{10} - 400}{x_{18} - x_{19}} = 30/\ln\left(\frac{330 - x_8}{300 - x_8}\right) \\
h_{18}(\vec{x}) \rightarrow \quad x_{14} &= \frac{x_9 - x_8 + x_{11} - 400}{x_{20} - x_{21}} = 40/\ln\left(\frac{160}{400 - x_9}\right) \\
h_{19}(\vec{x}) \rightarrow \quad x_{15} &= \frac{x_{12} - x_9 - 100}{x_{22} - 4.60517} = 60/\ln\left(\frac{160}{100}\right)
\end{aligned}$$

The last step is to reformulate h_7, h_8 and h_9 equality constraints in order to find x_2, x_3 and x_4.

$$\begin{aligned}
h_7(\vec{x}) \rightarrow \quad x_2 &= \frac{x_5}{120 \cdot x_{13}} = \frac{10^3}{36} \cdot (x_8 - 100) \cdot \ln\left(\frac{330 - x_8}{300 - x_8}\right) \\
h_8(\vec{x}) \rightarrow \quad x_3 &= \frac{x_6}{80 \cdot x_{14}} = \frac{10^3}{32} \cdot (x_9 - x_8) \cdot \ln\left(\frac{160}{400 - x_9}\right) \\
h_9(\vec{x}) \rightarrow \quad x_4 &= \frac{x_7}{40 \cdot x_{15}} = \frac{10^3}{24} \cdot (500 - x_9) \cdot \ln\left(\frac{160}{100}\right)
\end{aligned}$$

As we have seen, it is possible to describe 19 dimensions of the G22 problem only based on two parameters x_8 and x_9. This means that in presence of the analytical information for the equality constraints we can transform this 22-dimensional problem with one inequality and 19 equality constraint to a 3-dimensional problem (x_1, x_8 and x_9) with a single inequality constraints.