Foam Rheology Near the Jamming Transition

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Chapter 5

Normal Stress

Up till now, we have only dealt with the scaling of the shear stress, σ_{xy}. We will now turn our attention to the normal components of the stress: σ_{xx} and σ_{yy}. Our 3E model can be naturally extended to these two components, although we will need to include a new empirical relation for the stress as a function of the strain.

5.1 Scaling Model

Before we introduce the extension to our 3E model and test its predictions, we note that the two normal stresses, σ_{xx} and σ_{yy} are close (not identical) within the regime in which we consider the data. This can be seen in figure 5.1. While there is a bump in σ_{xx} compared to σ_{yy}, its magnitude is small and we have not investigated it further. We will first discuss whether and how the three ingredients of the 3E model for shear stresses, Eqs. 3.1, 3.2, 3.3, 3.4 and 3.5, need to be changed for the normal stress. The first ingredient, energy conservation, Eq. 3.1, remains:

$$L_x L_y \sigma_{xy} \dot{\gamma} = \frac{NZb}{2} \left\langle \Delta v^2 \right\rangle . \tag{5.1}$$

Energy is conserved no matter what component of the stress we consider. Note that this suggests that the shear stress enters the description of the normal stresses.

The second set of ingredients for our 3E model, the two expressions for the yield strain, similar to Eq. 3.2, and the dynamic strain, similar to Eq. 3.3, will also be unchanged, as we can still assign an effective strain based on the compression and the relaxation time scale. We will, however, allow for the balance between these contributions to be different from the shear case. Therefore we have:

$$\gamma_{\text{eff}} = B_{\text{eff}}^{xx} \Delta\phi + \frac{\dot{\gamma} d}{\Delta v}, \tag{5.2}$$

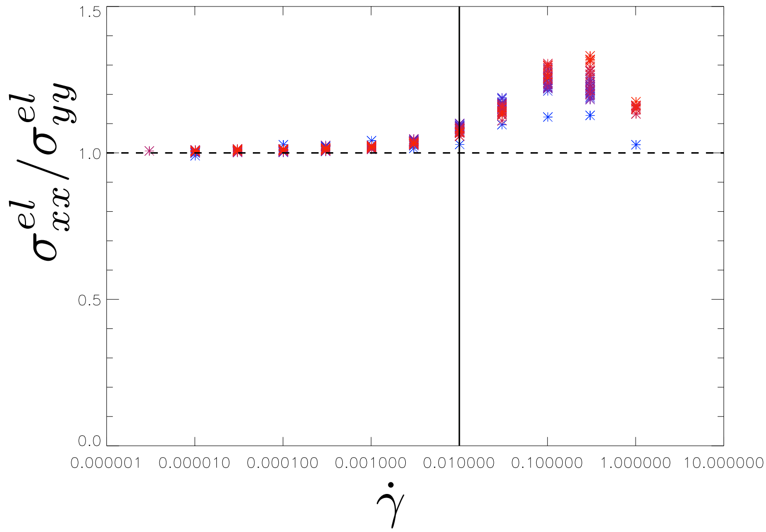

Figure 5.1: The ratio of the xx and yy components of the elastic stress. In the regime where we consider the data, $\dot{\gamma} < 10^{-2}$, left of the black line, the two are equal. Red means high density, blue means low density, see the legend in table 3.3

with B^{xx}_{eff} a new undetermined coefficient. We will test this assumption together with the elasticity relation that we introduce next, just as we did for the shear stress above.

The third ingredient describes the relationship between the stress and the effective strain. For shear stresses, we proposed a linear, $\sigma_{xy} \sim G\gamma_{\text{eff}}$ (Eq. 3.4), and a quadratic, $\sigma_{xy} \sim \gamma^2_{\text{eff}}$ (Eq. 3.5), regime. However, since the result of Wyart *et al.* for the quadratic part is not necessarily valid for the normal stress, we do not have a natural prediction for the elasticity relation. The simplest relation is a linear one:

$$\sigma_{xx} = A_{xx} k \gamma_{\text{eff}} \tag{5.3}$$

The best way to find out, however, is to determine it empirically.

5.1.1 Testing the Elasticity Relation

We test the elasticity relation in exactly the same way as we did for the normal stress: we plot σ_{xx} vs. γ_{eff} in figure 5.2 **a**. Since our expression for σ_{xx}, Eq. 5.3 does not depend on the density, no further rescaling should be necessary to attain collapse. It is immediately clear, however, that the data does not collapse. This means at the very least that the expression we used for the effective strain, $\gamma_{\text{eff}} = B_{\text{eff}}\Delta\phi + \dot{\gamma}d/\Delta v$ with $B_{\text{eff}} = 2.2$, is not correct. If it

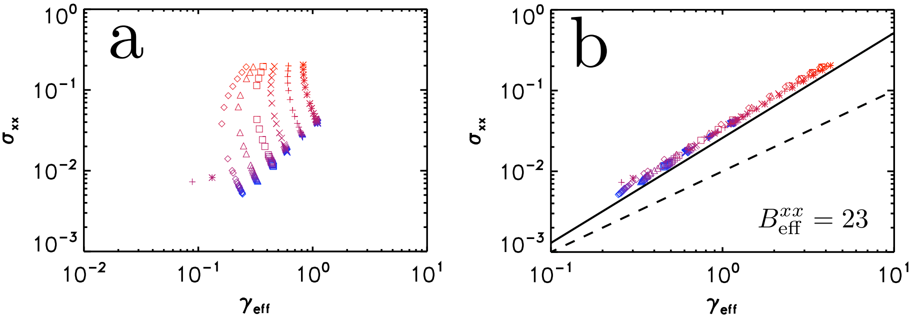

Figure 5.2: Plot of the normal stress vs. the effective strain. **a**: using exactly the same expression for γ_{eff} as we do in the shear stress case. **b**: changing the value of B_{eff} to achieve collapse. The dashed line is a power law with exponent 1, the black line with exponent 1.3. Colors and symbols correspond to the legend of table 3.3.

were correct we would have seen collapse. The shape of the collapsed curve would then have told us the functional form of the dependence of σ_{xx} on γ_{eff}.

The most simple way in which our proposed formulation of the effective strain can be incorrect is if the value of B_{eff} is not the same in the shear and normal cases. In other words: the normal stress is still determined by an effective strain given by two contributions, a yield strain and a dynamic strain, but the contributions balance differently. If this is the case, we should be able to get better collapse by adjusting B_{eff} to a new value that we call B^{xx}_{eff} to distinguish it from the previous result, which we will cal B^{xy}_{eff}. The panel **b** of figure 5.2 shows that we are indeed able to get collapse for $B^{xx}_{\text{eff}} = 23 \pm 2$, which is an order of magnitude larger than B^{xy}_{eff}, for which we found a value of 2.2. We note that it is not surprising that the static, compression based strain is more important in determining the compressive component of the stress.

Now we can also see that the expression for the elasticity that we formulated in Eq. 5.3 is not correct. The dashed line in figure 5.2 **b** is a power law of exponent 1, the expected linear behavior. Clearly, this does not match the slope of the data. The data can still be described by a simple power law, but it will have an exponent different from 1. In fact, as shown by the black line in the figure 5.3 **b**, a power law with exponent 1.3 works well. This means that the correct formulation of the elasticity relation is:

$$\sigma_{xx} = A_{xx} k \gamma_{\text{eff}}^{1.3} \tag{5.4}$$

We stress that this is an empirical result.

5.2 Regimes

With the three ingredients formulated and checked, Eqs. 5.1, 5.2 and 5.4, we can now derive the expressions for the normal stress as function of the density and strain rate. Since there is now only one ingredient, the effective strain, that has different behavior in different regimes, we expect to have only two distinct scaling regimes for the normal stress, while we had four regimes for the shear stress. We will call these regimes the Normal Yield regime, for 'large' $\Delta\phi$ and 'small' $\dot{\gamma}$ so that Eq. 5.2 is dominated by the yield contribution $B_{\text{eff}}^{xx}\Delta\phi$, and the Normal Dynamic regime, for 'small' $\Delta\phi$ and 'large' $\dot{\gamma}$ so that Eq. 5.2 is dominated by the dynamic contribution $\dot{\gamma}d/\Delta v$. Our power balance expression complicates this though, as we will illustrate below.

In the Normal Dynamic regime we have

$$\sigma_{xx} \sim \gamma_{\text{eff}}^{1.3} \tag{5.5}$$

$$\gamma_{\text{eff}} \sim \frac{\dot{\gamma}}{\Delta v} \tag{5.6}$$

$$\sigma_{xy}\dot{\gamma} \sim \Delta v^2, \tag{5.7}$$

where we have used scaling expressions for simplicity. Substituting Eq. 5.7 into Eq. 5.6 yields

$$\gamma_{\text{eff}} \sim \frac{\dot{\gamma}}{\sqrt{\sigma_{xy}\dot{\gamma}}}. \tag{5.8}$$

Substituting this into Eq. 5.5 then yields

$$\sigma_{xx} \sim \left(\sqrt{\frac{\dot{\gamma}}{\sigma_{xy}}}\right)^{1.3} \tag{5.9}$$

Due to the dependence of the supplied power on the shear stress, σ_{xy}, the shear stress enters the expression for the normal stress. We need to substitute the expressions we found for the shear stress to express the normal stress as a function of $\dot{\gamma}$ and $\Delta\phi$. The shear stress, however, also has regimes. In principle, this could split the Normal Dynamic regime into three regimes, one for each regime of the shear stress. Fortunately, the crossover between the Yield and Transition regimes is the same as the crossover between the Normal Yield and the Normal Dynamic regimes: in both cases you are in the Yield regime if $\Delta\phi > \dot{\gamma}/\Delta v$ [1]. We will therefore refine our description to include three regimes: the Normal Yield regime, in which the effective strain is determined by the yield strain and the shear stress does not enter, the Normal Transition regime, in which the effective strain is in the dynamic regime and the shear stress in the Transition regime, and the Critical regime, in which the effective strain is in the dynamic regime and the shear stress is in the Critical regime.

[1] In fact, we have seen that $B_{\text{eff}}^{xx} > B_{\text{eff}}^{xy}$, meaning that the crossover from the Normal Yield to the Normal Dynamic regime takes place for much higher density than the crossover from Yield to Transition regime

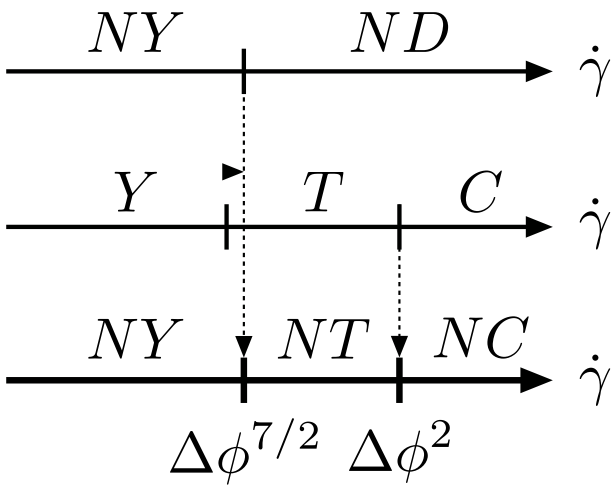

Figure 5.3: Top: the regimes in the normal stress without taking into account the shear stress dependence that enters through power balance. There are two regimes in this case the Normal Yield (NY) regime and the Normal Dynamic (ND) regime. Middle: the regimes of the shear stress. Bottom: the final regimes of the normal stress. Vertical dashed arrows indicate the inheritance of a crossover. NT denotes the Normal Transition regime and NC denotes the Normal Critical regime.

	Normal Critical			Normal Transition			Normal Yield		
effective strain	γ_{eff}	$\sim$	$\dot{\gamma}/\Delta v$	γ_{eff}	$\sim$	$\dot{\gamma}/\Delta v$	γ_{eff}	$\sim$	$\Delta\phi$
shear stress	σ	$\sim$	$\dot{\gamma}^{1/2}$	σ	$\sim$	$\Delta\phi^{1/3}\dot{\gamma}^{1/3}$			
rheology	σ	$\sim$	$\dot{\gamma}^{0.33}$	σ	$\sim$	$\Delta\phi^{-0.22}\dot{\gamma}^{0.43}$	σ	$\sim$	$\Delta\phi^{1.3}$
range	$\Delta\phi^2$	$<$	$\dot{\gamma}$	$\Delta\phi^{7/2}$	$<$	$\dot{\gamma}$	$\Delta\phi^{7/2}$	$>$	$\dot{\gamma}$
				$\Delta\phi^2$	$>$	$\dot{\gamma}$			

Table 5.1: The three rheological regimes with their definitions, results and ranges of validity.

These regimes are shown in figure 5.3. At the top we have Normal Yield and Normal Dynamic regimes that follow from the analysis of the Normal stress without taking into account the shear stress dependence of Δv that enters through power balance in the Normal Dynamic regime. In the middle we show the Yield, Transition and Critical regime from the shear stress. Note that the crossover from the Normal Yield to the Normal Dynamic regime and the crossover from the Yield and the Transition regime have the same scaling ($\dot{\gamma} \sim \Delta\phi^2$) as they are based on the same crossover from yield dominated γ_{eff} to dynamic dominated γ_{eff}. However, since $B_{\text{eff}}^{xx} > B_{\text{eff}}$, the numerical value for the crossover between the Normal Yield and Normal Dynamic regime is higher. On the bottom of figure 5.3 we show the final regimes for the normal stress. The final Normal Yield regime is not influenced by the shear stress and therefore the crossover to the next regime is not changed. The crossover between the Normal Transition and the Normal Critical regime is directly inherited from the shear stress. Now that we have defined the three regimes we can derive the rheological expressions in the same way we derived the results fro the normal stress. The results are summarised in table 5.1 and given in a little more detail below.

Normal Yield In the Normal Yield regime we have:

$$\begin{cases} \sigma_{xy}\dot{\gamma} \sim \Delta v^2 \\ \gamma_{\text{eff}} \sim \Delta\phi \\ \sigma_{xx} \sim \gamma_{\text{eff}}^{1.3} \\ \sigma_{xy} \sim \Delta\phi^{3/2} \end{cases} \Rightarrow \begin{cases} \Delta v \sim \dot{\gamma}^{1/2}\Delta\phi^{3/4} \\ \gamma_{\text{eff}} \sim \Delta\phi \\ \sigma_{xx} \sim \Delta\phi^{1.3} \end{cases} \tag{5.10}$$

Note that our scaling for the normal stress in the limit of zero strain rate, $\sigma_{xx} \sim \Delta\phi^{1.3}$ is not the same as our prediction for the shear stress in the limit of zero strain rate, $\sigma_{xy} \sim \Delta\phi^{3/2}$. This is in contrast to the expectations and findings of many [23, 38]. We note that we do not have strong empirical data that corroborates that either $\sigma_{xy} \sim \Delta\phi^{3/2}$ or $\sigma_{xx} \sim \Delta\phi^{1.3}$.

Normal Transition In the normal Transition regime we have:

$$\begin{cases} \sigma_{xy}\dot{\gamma} \sim \Delta v^2 \\ \gamma_{\text{eff}} \sim \frac{\dot{\gamma}}{\langle\Delta v\rangle} \\ \sigma_{xx} \sim \gamma_{\text{eff}}^{1.3} \\ \sigma_{xy} \sim \Delta\phi^{1/3}\dot{\gamma}^{1/3} \end{cases} \Rightarrow \begin{cases} \Delta v \sim \Delta\phi^{1/6}\dot{\gamma}^{2/3} \\ \gamma_{\text{eff}} \sim \Delta\phi^{-1/6}\dot{\gamma}^{1/3} \\ \sigma_{xx} \sim \Delta\phi^{-1.3/6}\dot{\gamma}^{1.3/3} \end{cases} \tag{5.11}$$

Normal Critical In the Normal Critical regime we have:

Regime Combination	Rescaled Axes
Critical and Transition	$\sigma/\Delta\phi^{0.65}$ vs. $\dot{\gamma}/\Delta\phi^{2}$
Yield and Transition	$\sigma/\Delta\phi^{1.3}$ vs. $\dot{\gamma}/\Delta\phi^{7/2}$
Yield and Critical	$\sigma/\Delta\phi^{1.3}$ vs. $\dot{\gamma}/\Delta\phi^{3.9}$

Table 5.2: Prescriptions of what to plot for collapse of the normal stress in the indicated regimes.

$$\begin{cases} \sigma_{xy}\dot{\gamma} \sim \Delta v^2 \\ \gamma_{\text{eff}} \sim \frac{\dot{\gamma}}{\langle \Delta v \rangle} \\ \sigma_{xx} \sim \gamma_{\text{eff}}^{1.3} \\ \sigma_{xy} \sim \dot{\gamma}^{1/2} \end{cases} \Rightarrow \begin{cases} \Delta v \sim \dot{\gamma}^{3/4} \\ \gamma_{\text{eff}} \sim \dot{\gamma}^{1/4} \\ \sigma_{xx} \sim \dot{\gamma}^{1.3/4} \end{cases} \tag{5.12}$$

5.2.1 Crossovers

As we have discussed above and illustrated in figure 5.3, the crossover between the Normal Transition and the Normal Critical regimes is the same as the crossover from the Transition to the Critical regime by construction and therefore scales as $\dot{\gamma} \sim \Delta\phi^{7/2}$. The crossover from the Normal Yield to the normal Transition regime scales the same as the crossover from the Yield to the Transition regime, $\dot{\gamma} \sim \Delta\phi^2$, as both take place where the yield contribution and the dynamic contribution to the effective strain are equal. Since, however, the yield contribution is bigger for the normal stress, the numerical value of the crossover will be different.

5.3 Plotting and Results

5.3.1 Collapse Plots

Just as in the case of the shear stress, we can now determine the exponents Δ and Γ for which a plot of $\sigma_{xx}/\Delta\phi^{\Delta}$ vs. $\dot{\gamma}/\Delta\phi^{\Gamma}$ results in collapse in the various regimes. The analysis is completely analogous to that of section 3.3.1 and no new interesting physics is found; in fact, due to the inherited crossovers, some of the results are identical to the shear case. The results are presented in table 5.2.

Normal Yield and Normal Transition Regimes As mentioned above, the strain rate needs to be rescaled to make the crossover between the Normal Yield and Normal Transition regimes, which is the same as for the shear tress,

Regime	Full Expression
Normal Critical	$\sigma_{xx} = A_{xx} \left(d^2NZ/2L_xL_yA_1A_2\right)^{1.3/4} \dot{\gamma}^{1.3/4}$
Normal Transition	$\sigma_{xx} = A_{xx} \left(NZd^2/2L_xL_yA_1\right)^{1.3/3} \Delta\phi^{-1.3/6}\dot{\gamma}^{1.3/3}$
Normal Yield	$\sigma_{xx} = A_{xx}B_{\text{eff}}^{xx\ 1.3}\Delta\phi^{1.3}$
crossover Transition Yield	$\dot{\gamma} = 2A_1B_{\text{eff}}^{xx\ 3}L_xL_y/d^2NZ\ \Delta\phi^{7/2}$
crossover Critical Transition	$\dot{\gamma} = 2L_xL_yA_1/NZA_2^3d^2\ \Delta\phi^2$

Table 5.3: Full expressions for the three regimes and their crossovers in dimensionless form.

collapse: $\tilde{\dot{\gamma}} \sim \dot{\gamma}/\Delta\phi^{7/2}$. Since the stress in the Normal Yield regime depends only on the density, this prescribes the rescaling of the stress: $\sigma_{xx} \sim \Delta\phi^{1.3}$.

Normal Transition and Normal Critical Regimes Between the Normal Transition and the Normal Critical regime there is again a crossover that determines the strain rate rescaling: $\tilde{\dot{\gamma}} \sim \dot{\gamma}/\Delta\phi^2$. Substituting this into the expression for the stress in the Normal Critical regime yields:

$$\sigma_{xx} \sim \dot{\gamma}^{1.3/4} = \tilde{\dot{\gamma}}^{1.3/4}\Delta\phi^{1.3/2} \Rightarrow \tilde{\sigma}_{xx} \sim \sigma_{xx}/\Delta\phi^{1.3/2}. \tag{5.13}$$

Normal Yield and Normal Critical Regimes Again, the dependence of the Normal Yield stress on the density prescribes the rescaling of the stress: $\sigma_{xx} \sim \Delta\phi^{1.3}$. This can be substituted into the expression for the stress in the Critical regime to deduce the rescaling of the strain rate:

$$\sigma_{xx} \sim \dot{\gamma}^{1.3/4} \Rightarrow \tilde{\sigma}_{xx} \sim \dot{\gamma}^{1.3/4}/\Delta\phi^{1.3} \Rightarrow \tilde{\dot{\gamma}} \sim \dot{\gamma}/\Delta\phi^4 \tag{5.14}$$

5.3.2 Prefactors

The final detail that is necessary to complete our Q3E model for the normal stress is the value of any new prefactors that were introduced. Since we already have determined that $B_{\text{eff}}^{xx} = 23 \pm 2$, the only remaining new parameter is A_{xx} in Eq. 5.4. The approach for determining this fit factor is the same as for the shear case: comparing the data to our full expression for the stress in the Critical regime. This and all the other full expressions are given in table 5.3. The next step is to plot the data for collapse, so that we can fit a power law.

Figure 5.4 plots the normal stress vs. the strain rate rescaled for collapse in the Yield and Critical regime. Firstly, we note that collapse looks good over the entire range, we will return to this below. For now, we focus on the fact that the black line, given by $0.03\left(\dot{\gamma}/\Delta\phi^{3.9}\right)^{1.3/4}$ describes the data in the Critical asymptote very well. This allows us to derive the value of A_{xx},

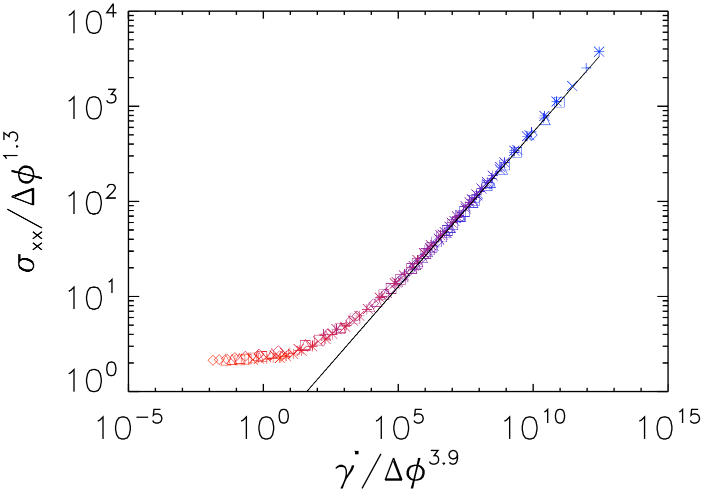

Figure 5.4: Plot of the stress vs. the strain rate rescaled for collapse in the Yield and Transition regime. The black line is given by $0.03\left(\dot{\gamma}/\Delta\phi^{3.9}\right)^{1.3/4}$. Colors and symbols correspond to density and strain rate as in table 3.3.

because

$$A_{xx}\left(\frac{d^2NZ}{2L_xL_yA_1A_2}\right)^{1.3/4} = 0.3 \pm 0.05, \tag{5.15}$$

in which A_{xx} is the only unknown. Substituting the values for d, N, Z, L_x, L_y, A_1 and A_2 that we set, approximated or fitted before, we find that $A_{xx} = 0.004 \pm 0.001$. This completes the normal extension to the Q3E model and we can focus on the width of the regimes to see where we would expect collapse when rescaling the data.

5.3.3 Regimes and Collapse

Figure 5.5 shows the same data as figure 5.4 but now coloured according to the regimes in which the various data points are located: Yield data points in black, Transition data points in red and Critical data points in blue. We can clearly see that the Transition regime is much smaller for the normal stress than for the shear stress, see figure 4.7, which is completely expected as B_{eff}^{xx} is about an order of magnitude larger than B_{eff}^{xy}. As we discussed above in section 4.4.1, data that is 'purely' in the Transition regime is already rare for the shear stress, but it will be virtually non-existent for the normal stress. This is why plotting for collapse in the Yield and Critical regime, as we have done in figure 5.5 is the appropriate way to plot the data and we expect collapse nearly everywhere.

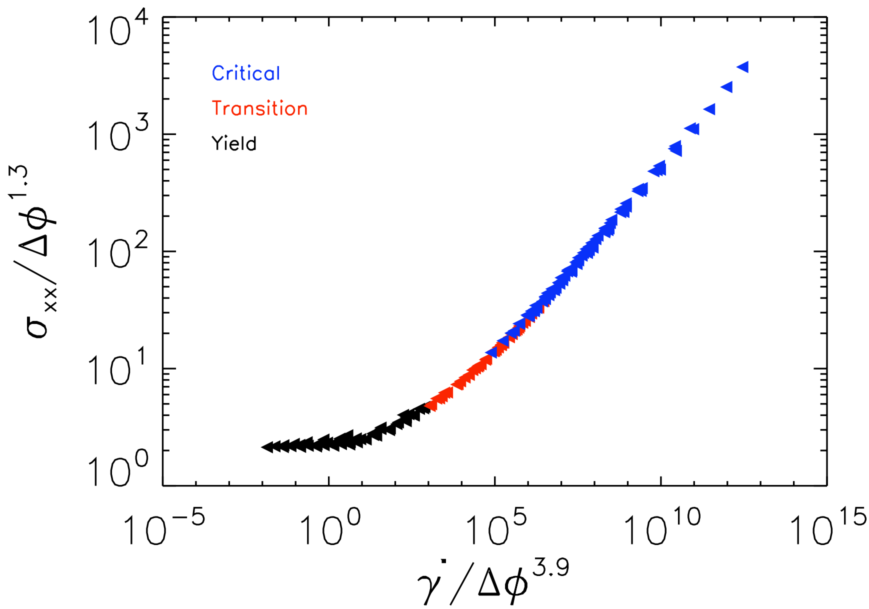

Figure 5.5: Collapse plot of the stress vs. the strain rate. Data points are coloured according to the regimes they are in: blue for Critical regime, red for Transition regime and black for Yield regime.

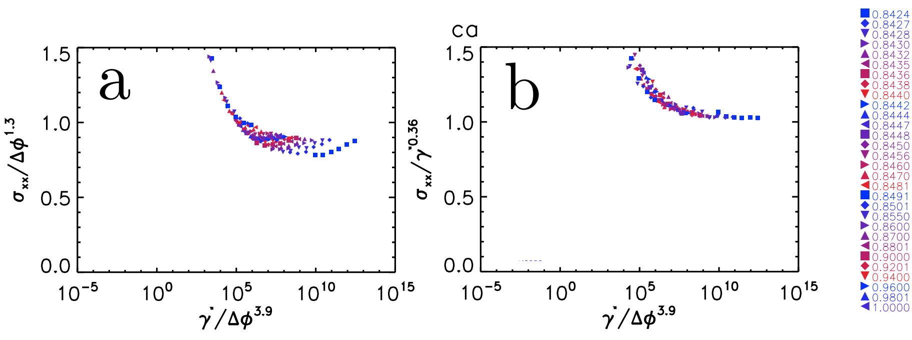

Figure 5.6: Plots of the stress vs. the strain rate with a Critical power law divided out. **a** with the model prediction of $\dot{\gamma}^{1.3/4} \approx \dot{\gamma}^{0.34}$ divided out. **b** with the power law that best yields a horizontal residue: $\dot{\gamma}^{0.36}$ divided out. The legend indicates which density corresponds to each symbol-color-combination.

However, just as was the case for the shear stress, this is not a reliable way to determine whether our predicted exponent in the Critical regime, β, is exactly correct. The real β cannot be too different from $1.3/4 \approx 0.34$, otherwise we would not have had good collapse. However, it can be slightly different. And in fact, in figure 5.6 **b** we see that dividing out our model prediction of $\dot{\gamma}^{1.3/4}$ does not result in perfectly flat residual data. Instead, as we show in figure 5.6 **b**, we best achieve flat behavior when dividing out $\dot{\gamma}^{0.36}$. Just as in the case of the shear stress, this exponent is slightly different from the model prediction but it is close enough to be considered consistent with our model.

It should be noted that this results is different from the most recent result by Olsson & Teitel [23] in two ways. First, the numerical value of the exponent in the Critical regime is different, 0.36 for us as opposed to 0.28 for Olsson & Teitel. And second, Olsson and Teitel find that the exponents for the shear stress and normal stress in the critical regime are the same, while we find that they are different: 0.47 vs. 0.36.

5.4 Conclusion

We have successfully extended our Q3E model to the normal component of the stress tensor. While there were some surprises, a different balance between the components of the effective strain and especially an unexpected elasticity relation, we have been able to account for these in our Q3E model. With these inclusions the Q3E model describes the simulation data well.The best numerical estimate for the value of the critical exponent β is within 5% our prediction.