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## Dirac and Majorana edge states in graphene and topological superconductors

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# Dirac and Majorana edge states in graphene and topological superconductors

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*To my parents.*



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