Light with a twist : ray aspects in singular wave and quantum optics

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An exact quantum theory of rotating light

7.1 Introduction

During the past decades, both the propagation and the diffraction of light through optical set-ups with rotating optical elements [104, 105, 48, 69], as well as the physical properties of rotating beams of light [19, 106, 20] have attracted a steady amount of attention. So far, both theoretical and experimental work has focused mainly on classical aspects of rotating light. Only recently, van Enk and Nienhuis have proposed a first quantum theory of rotating photons [107]. They construct rotating field operators as coherent superpositions of the field operators corresponding to the rotational Doppler-shifted [48] angular-momentum components of the field. In leading order of the paraxial approximation, the spin and orbital degrees of freedom of the radiation field decouple [45] and fields with a rotating polarization and a stationary spatial pattern can be constructed as superpositions of rotational Doppler-shifted circular-polarization states. Similarly, fields with a rotating mode pattern and a stationary polarization can be built up from the rotational Doppler-shifted angular-momentum components of the spatial field distribution. It is, of course, also possible to construct fields with both a rotating polarization and a rotating spatial pattern. Since, in the paraxial approximation, the polarization and spatial degrees of freedom are decoupled, the rotation frequencies may even have different values. The rotation of the polarization and spatial patterns of the fields that are thus constructed are uniform only in the paraxial limit. Moreover, the approach requires that the differences in diffraction of the Doppler-shifted angular-momentum components of the field are negligible, i.e that the the rotation frequency is small compared to the optical frequency.

In this chapter, we introduce the first exact quantum theory of rotating light. We show that Maxwell's equations in free space have complete sets of solutions that rotate uniformly as a function of time, i.e., that are monochromatic in a rotating frame. Our approach does not necessarily involve paraxial approximations and both the spatial structure and the polarization of the rotating modes of free space rotate at a uniform velocity about the rotation axis. Once such rotating solutions have been obtained, quantization is relatively straightforward. We follow the standard procedure of canonical quantization and show that quantization in the co-rotating frame is consistent with quantization in the stationary frame. We show how this approach can be applied to obtain a quantum-mechanical description of the dynamics of the set of modes that obey rotating boundary conditions. We derive the paraxial counterpart of the exact theory and discuss quantization of the rotating cavity modes that we have studied in chapters 3 and 5 as an example.

The material in this chapter is organized as follows. In the next section, we summarize the equations of motion of the radiation field, show how they may be derived from the standard Lagrangian for the free electromagnetic field and discuss canonical quantization in the Coulomb gauge [5]. In section 7.3 we study the dynamics of light in a rotating frame and derive complete sets of monochromatic solutions of the wave equation in such a frame. The corresponding field operators in a stationary frame are introduced and discussed in section 7.4, where we also discuss quantization in the rotating frame. In the final section we summarize our results and draw our conclusions.

7.2 Preliminaries

7.2.1 Equations of motion of the free radiation field

It is well-known from textbook electrodynamics that the electric and magnetic fields are fully characterized by a scalar potential $\Phi(\mathbf{r}, t)$ and a vector potential $\mathbf{A}(\mathbf{r}, t)$. In terms of these potentials the fields are given by [4]:

$$\mathbf{B}(\mathbf{r}, t) = \nabla \times \mathbf{A}(\mathbf{r}, t) \qquad \text{and} \qquad \mathbf{E}(\mathbf{r}, t) = -\nabla\Phi(\mathbf{r}, t) - \frac{\partial \mathbf{A}}{\partial t} , \tag{7.1}$$

where c is the speed light. These definitions ensure that the homogeneous Maxwell equations are obeyed [4]. Although the fields are fully specified by the potentials $\Phi(\mathbf{r}, t)$ and $\mathbf{A}(\mathbf{r}, t)$, the reverse is not true; there is some arbitrariness (gauge freedom) in the choice of the potentials. The dynamics of the free radiation field is most conveniently described in the Coulomb gauge, which is defined by the requirement that [4]

$$\nabla \cdot \mathbf{A}(\mathbf{r}, t) = 0 . \tag{7.2}$$

In the absence of electric charges and currents, it follows from the inhomogeneous Maxwell equations that the scalar potential Φ vanishes while the vector potential obeys the wave equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{A}(\mathbf{r},t) = 0\ , \tag{7.3}$$

which, together with the requirement that the field is divergence free (7.2), fully describes the dynamics of the free radiation field in the Coulomb gauge.

In general, the dynamics of the free electromagnetic field may be described by the Lagrangian [5]

$$L = \int d_3\mathbf{r}\ \mathcal{L}(\mathbf{A},\dot{\mathbf{A}}) = \frac{\epsilon_0}{2}\int d_3\mathbf{r}\left\{|\dot{\mathbf{A}}|^2 - c^2|(\nabla\times\mathbf{A})|^2\right\} = \frac{\epsilon_0}{2}\left\{\langle\dot{\mathbf{A}}|\dot{\mathbf{A}}\rangle - c^2\langle\nabla\times\mathbf{A}|\nabla\times\mathbf{A}\rangle\right\}\ , \tag{7.4}$$

where $\mathcal{L}$ is the Lagrangian density in real space and we have adopted the Dirac notation of quantum mechanics to denote the state of the classical radiation field. In case of the free radiation field it is natural to assume that the field $\mathbf{A}$ and its derivatives vanish at infinity while, for the radiation field enclosed by an ideal cavity with a perfectly conducting boundary, the Maxwell boundary conditions [4] require that $\mathbf{A}$ at the boundary is locally normal to it. In both cases, and under the assumption that the field is locally transverse so that it obeys equation (7.2), partial integration of the second term in equation (7.4) yields $\int d_3\mathbf{r}\ |\nabla\times\mathbf{A}|^2 = -\int d_3\mathbf{r}\ \mathbf{A}\cdot\left(\nabla^2\mathbf{A}\right)$. Using this, one may show that the Euler-Lagrange equation that derives from the Lagrangian (7.4) reproduces the wave equation (7.3). The canonical momentum density corresponding to the field $\mathbf{A}$ is given by

$$\mathbf{\Pi_A} = \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{A}}} = \epsilon_0\dot{\mathbf{A}}\ . \tag{7.5}$$

The Hamiltonian may be obtained as

$$H = \int d_3\mathbf{r}\left\{\mathbf{\Pi_A}\cdot\dot{\mathbf{A}} - \mathcal{L}\right\} = \frac{1}{2\epsilon_0}\left\{\langle\mathbf{\Pi_A}|\mathbf{\Pi_A}\rangle + \epsilon_0^2c^2\langle\nabla\times\mathbf{A}|\nabla\times\mathbf{A}\rangle\right\} = \frac{1}{2\epsilon_0}\left\{\langle\mathbf{\Pi_A}|\mathbf{\Pi_A}\rangle - \epsilon_0^2c^2\langle\mathbf{A}|\nabla^2\mathbf{A}\rangle\right\}\ . \tag{7.6}$$

The second equality only holds in the Coulomb gauge as defined by equation (7.2). In this gauge, the corresponding Hamilton equations reproduce the wave equation (7.3).

7.2.2 Modes and quantization

Optical modes are usually defined as monochromatic solutions of the wave equation (7.3). Although the vector potential is real, it is convenient to allow for complex mode functions $\mathbf{F}_\lambda(\mathbf{r})$ so that the vector potential corresponding to a mode $\mathbf{F}_\lambda$ is given by $\mathbf{A}(\mathbf{r},t) = \mathrm{Re}\{\mathbf{F}_\lambda(\mathbf{r})\exp(-i\omega_\lambda t)\}$. The subscript λ denotes a set of mode indices, which characterizes the spatial and polarization structure of the mode function $\mathbf{F}_\lambda$. For a given set of modes, the amplitudes $\langle\mathbf{F}_\lambda|\mathbf{A}\rangle$ and their derivatives $\langle\mathbf{F}_\lambda|\dot{\mathbf{A}}\rangle$ obey harmonic equations of motion and it follows that the radiation field can be quantized as a set of harmonic oscillators. In case of the free field it is customary to quantize the field in a basis of plane waves. It is well-known, however, that quantization can be performed in a manifestly basis-independent manner, which also applies to the case of a set of cavity modes with finite spatial extent [108]. For later reference, we briefly summarize the quantization of the radiation field in an orthonormal but otherwise arbitrary set of modes $\{\mathbf{F}_\lambda\}$. Since the complex vector potential corresponding to a mode $\mathbf{F}_\lambda$ is given by $\mathbf{F}_\lambda(\mathbf{r})\exp(-i\omega_\lambda t)$, it follows from the wave equation (7.3) that the mode functions obey the Helmholtz equation

$$\left(\nabla^2 + k_\lambda^2\right)\mathbf{F}_\lambda(\mathbf{r}) = 0\,, \tag{7.7}$$

where $k_\lambda^2 = \omega_\lambda^2/c^2$ so that also $\mathbf{F}_\lambda(\mathbf{r})\exp(i\omega_\lambda t)$ is a solution of the wave equation (7.3). However, since also $\mathbf{F}^*_\lambda(\mathbf{r})$ obeys the Helmholtz equation (7.7), and since $\mathrm{Re}\{\mathbf{F}_\lambda(\mathbf{r})e^{i\omega_\lambda t}\} = \mathrm{Re}\{\mathbf{F}^*_\lambda(\mathbf{r})e^{-i\omega_\lambda t}\}$, it follows that without loss of generality we can assume that $\omega_\lambda > 0$. It is convenient to define λ^* such that $\mathbf{F}^*_\lambda(\mathbf{r}) = \mathbf{F}_{\lambda^*}(\mathbf{r})$. Notice that this convention implies that, in general, λ^* is not the complex conjugate of λ. In the specific case of real mode functions $\mathbf{F}_\lambda = \mathbf{F}^*_\lambda$ it implies that $\lambda^* = \lambda$. The mode functions $\{\mathbf{F}_\lambda\}$ are eigenfunctions of the Hermitian operator ∇^2 and form, therefore, a complete basis in real space. This implies that any solution of the wave equation (7.3) can be expanded as

$$\mathbf{A}(\mathbf{r},t) = \sum_\lambda \langle\mathbf{F}_\lambda|\mathbf{A}(t)\rangle\mathbf{F}_\lambda(\mathbf{r})\,. \tag{7.8}$$

In order to quantize the field, we introduce the normal variables, which are defined as

$$a_\lambda(t) = \left(\frac{\epsilon_0}{2\hbar\omega_\lambda}\right)^{1/2}\left(i\langle\mathbf{F}_\lambda|\dot{\mathbf{A}}(t)\rangle + \omega_\lambda\langle\mathbf{F}_\lambda|\mathbf{A}(t)\rangle\right) \tag{7.9}$$

and

$$(a_{\lambda^*})^*(t) = \left(\frac{\epsilon_0}{2\hbar\omega_\lambda}\right)^{1/2}\left(-i\langle\mathbf{F}_\lambda|\dot{\mathbf{A}}(t)\rangle + \omega_\lambda\langle\mathbf{F}_\lambda|\mathbf{A}(t)\rangle\right)\,, \tag{7.10}$$

where we used that the physical field $\mathbf{A}$ and its time derivative $\dot{\mathbf{A}}$ are real. Notice that in case of real mode functions $\mathbf{F}_\lambda = \mathbf{F}_{\lambda^*}$ it follows that $a_\lambda = a_{\lambda^*}$. Inverting the definitions (7.9) and (7.10) yields

$$\mathbf{A}(\mathbf{r},t) = \sum_\lambda\left(\frac{\hbar}{2\epsilon_0\omega_\lambda}\right)^{1/2}\left(a_\lambda(t)\mathbf{F}_\lambda(\mathbf{r}) + a^*_\lambda(t)\mathbf{F}^*_\lambda(\mathbf{r})\right) \tag{7.11}$$

and

$$\dot{\mathbf{A}}(\mathbf{r},t) = -i\sum_\lambda\left(\frac{\hbar\omega_\lambda}{2\epsilon_0}\right)^{1/2}\left(a_\lambda(t)\mathbf{F}_\lambda(\mathbf{r}) - a^*_\lambda(t)\mathbf{F}^*_\lambda(\mathbf{r})\right)\,. \tag{7.12}$$

The corresponding expressions for the electric and magnetic fields can be obtained by applying equation (7.1). From equation (7.5), it follows that the canonical momentum density can be expressed as

$$\mathbf{\Pi}(\mathbf{r},t) = -i\sum_\lambda\left(\frac{\hbar\omega_\lambda\epsilon_0}{2}\right)^{1/2}\left(a_\lambda(t)\mathbf{F}_\lambda(\mathbf{r}) - a^*_\lambda(t)\mathbf{F}^*_\lambda(\mathbf{r})\right)\,. \tag{7.13}$$

Since the mode functions $\mathbf{F}_\lambda$ do not depend on time, equations (7.11) and (7.12) imply that

$$\dot{a}_\lambda(t) = -i\omega_\lambda a_\lambda(t) \; , \tag{7.14}$$

which also follows from the fact that the field (7.12) obeys the wave equation (7.3). Substitution in the Hamiltonian (7.6) gives

$$H = \sum_\lambda \frac{\hbar\omega_\lambda}{2} \left(a_\lambda^* a_\lambda + a_\lambda a_\lambda^*\right) \; , \tag{7.15}$$

which, in view of equation (7.14), does not depend on time and takes the form of the Hamiltonian of a harmonic oscillator for each mode $\mathbf{F}_\lambda$.

Canonical quantization of the field involves replacing the field and the canonical momentum density by hermitian vector operators $\hat{\mathbf{A}}$ and $\hat{\mathbf{\Pi}}$ whose components obey canonical commutation relations. In the Coulomb gauge these take the following form [5]

$$\left[\hat{A}_i(\mathbf{r}, t), \hat{\Pi}_j(\mathbf{r}', t)\right] = i\hbar\delta_\perp(\mathbf{r} - \mathbf{r}')\delta_{ij} \; , \tag{7.16}$$

where the indices i and j run over the vector components, $\delta_\perp(\mathbf{r} - \mathbf{r}')$ denotes the transverse delta function [5] and δ_{ij} denotes the Kronecker delta. As opposed to, for instance, ∇^2, which acts as an operator in the Hilbert space of physical states of the classical radiation field $\mathbf{A}(\mathbf{r}, t)$, the field and momentum operators $\hat{\mathbf{A}}(\mathbf{r}, t)$ and $\hat{\mathbf{\Pi}}(\mathbf{r}, t)$ are operators in the Hilbert space of quantum states of the radiation field. By replacing the classical field and momentum in the definitions of the normal variables (7.9) and (7.10) by the corresponding operators, one finds the operators $\hat{a}_\lambda$ and $\hat{a}_\lambda^\dagger$ that correspond to these variables. They obey boson commutation rules

$$\left[\hat{a}_\lambda, \hat{a}_{\lambda'}^\dagger\right] = \delta_{\lambda\lambda'} \; . \tag{7.17}$$

The operators $\hat{a}_\lambda^\dagger$ and $\hat{a}_\lambda$ respectively create and annihilate a photon in the mode $\mathbf{F}_\lambda$. The vacuum state, which is the quantum state of the field in which none of the modes $\mathbf{F}_\lambda$ contains photons, is defined by

$$\hat{a}_\lambda|\text{vac}) = 0 \qquad \forall\lambda \; , \tag{7.18}$$

where a bra vector $|...)$ with a round bracket denotes a vector in the Hilbert space of quantum states of the radiation field. Other states can be generated by acting with (functions of) the creation operators $\hat{a}_\lambda^\dagger$ on the vacuum. The quantum dynamics of the radiation field is governed by the Heisenberg equation of motion for the field operators, or, equivalently, the Schrödinger equation for the quantum states. The Hamilton operator takes the form of equation (7.15), the normal variables being replaced by the creation and annihilation operators. Similarly, the field and momentum operators take the form of equation (7.12) and (7.13), the creation and annihilation operators replacing the normal variables. The canonical commutation relations (7.16) are ensured by the boson commutation rules (7.17). The definition of the vacuum state (7.18), the field and momentum operators (7.16) and the Hamiltonian (7.15) provide a complete description of the quantum dynamics of the radiation field.

So far, we have assumed that $\{\mathbf{F}_\lambda\}$ constitutes a discrete set of modes. In case of a continuous set, the mode functions are normalized to δ functions and the summations over λ are replaced by integrals over the continuous variables that characterize the modes. In the particular case of normalized plane waves $\exp(i\mathbf{k}\cdot\mathbf{r})/(2\pi)^{3/2}$ the summations are replaced by $\sum_\lambda \to (2\pi)^{-3/2}\int d_3\mathbf{k}$.

7.3 Wave optics in a rotating frame

7.3.1 Equations of motion

In chapter 3, we have shown that the modes of an optical cavity that is put into uniform rotation about its optical axis can be defined as solutions of the time-dependent wave equation that rotate along with the mirrors. These solutions are monochromatic in the co-rotating frame. The corresponding complex fields, whose real parts correspond to the physical fields, are separable in space and time and, therefore, stationary in the co-rotating frame. We shall generalize the rotating-mode concept to the case of a freely propagating non-paraxial field and obtain complete sets of rotating modes of the free radiation field as monochromatic solutions in a rotating frame. First, we derive the equations of motion for light in a rotating frame.

Analogous to the discussion in chapter 5, we express the time-dependent vector potential in the stationary frame in terms of the vector potential in a rotating frame. The latter is denoted $\mathbf{C}(\mathbf{r},t)$. Since rotation of both the vector components and their spatial structure of a vector field in $\mathbb{R}^3$ is a real transformation of the field, it follows that $\mathbf{C}(\mathbf{r},t)$ can be defined real. It is related to the vector potential in the stationary frame by the identity

$$\mathbf{A}(\mathbf{r},t) = \langle\mathbf{r}|e^{-i\Omega t\hat{J}_z}|\mathbf{C}\rangle \,, \tag{7.19}$$

where $|\mathbf{r}\rangle$ is an eigenket of the position operator so that $\mathbf{C}(\mathbf{r},t) = \langle\mathbf{r}|\mathbf{C}\rangle$ is the real-space representation of the vector potential in the rotating frame. The operator $\exp(-i\hat{J}_z\Omega t)$ describes a time-dependent rotation of both the spatial structure and the polarization of a vector field, where Ω is the rotation frequency and $\hat{J}_z$ is the corresponding generator. By considering infinitesimal rotations $\exp(-i\alpha\hat{J}_z)\mathbf{A} = \mathbf{A} - i\alpha\hat{J}_z\mathbf{A} + O(\alpha^2)$ of both the vector components of a field $\mathbf{A}$ and their spatial structure, we find that $\hat{J}_z$ may be expressed as

$$\langle\mathbf{r}|\hat{J}_z|\mathbf{A}\rangle = \langle\mathbf{r}|\hat{L}_z + \hat{S}_z|\mathbf{A}\rangle \doteq -i\frac{\partial}{\partial\phi}\mathbf{A}(\mathbf{r},t) + i\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\mathbf{A}(\mathbf{r},t) \,. \tag{7.20}$$

The spin part $\hat{S}_z$ acts upon the vector nature of the field and generates rotations of the vector components. The orbital part, on the other hand, solely acts upon the spatial structure of each of the vector components and generates rotations of their spatial patterns. Notice that the form of the real-space representation of $\hat{J}_z$, which figures in the second right-hand side in equation (7.20), confirms that the rotation of a vector field in $\mathbb{R}^3$ is a real transformation so that the vector potential can be assumed real in both frames.

Substitution of the rotating field $|\mathbf{A}\rangle = e^{-i\Omega t\hat{J}_z}|\mathbf{C}\rangle$ and its time derivative $|\dot{\mathbf{A}}\rangle = e^{-i\Omega t\hat{J}_z}\left(|\dot{\mathbf{C}}\rangle - i\Omega\hat{J}_z|\mathbf{C}\rangle\right)$ in the Lagrangian (7.4) yields the Lagrangian in the rotating frame

$$L_{\mathrm{rot}} = \frac{\epsilon_0}{2}\left\{\langle\dot{\mathbf{C}}|\dot{\mathbf{C}}\rangle + i\Omega\langle\mathbf{C}|\hat{J}_z|\dot{\mathbf{C}}\rangle - i\Omega\langle\dot{\mathbf{C}}|\hat{J}_z|\mathbf{C}\rangle + \Omega^2\langle\mathbf{C}|\hat{J}_z^2|\mathbf{C}\rangle - c^2\langle\nabla\times\mathbf{C}|\nabla\times\mathbf{C}\rangle\right\} , \quad (7.21)$$

where we have used that $\nabla\times(\hat{J}_z\mathbf{C}) = \hat{J}_z(\nabla\times\mathbf{C})$ so that $\langle\nabla\times\mathbf{A}|\nabla\times\mathbf{A}\rangle = \langle\nabla\times\mathbf{C}|\nabla\times\mathbf{C}\rangle$. Using the real-space representation of $\hat{J}_z$, which figures in equation (7.20), one may show that $\nabla\cdot(\hat{J}_z\mathbf{C}) = \hat{L}_z(\nabla\cdot\mathbf{C})$. It follows that the transversality condition (7.2) is not affected by a transformation to a rotating frame so that

$$\nabla\cdot\mathbf{C}(\mathbf{r},t) = 0 . \quad (7.22)$$

By using that, for a transverse field, $\langle\nabla\times\mathbf{C}|\nabla\times\mathbf{C}\rangle = -\langle\mathbf{C}|\nabla^2\mathbf{C}\rangle$, the Euler-Lagrange equation for $\mathbf{C}(\mathbf{r},t)$ yields the wave equation in the rotating frame

$$\left(\nabla^2 + \frac{\Omega^2\hat{J}_z^2}{c^2} + \frac{2i\Omega\hat{J}_z}{c^2}\frac{\partial}{\partial t} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{C}(\mathbf{r},t) = 0 . \quad (7.23)$$

This equation can also be obtained directly from substitution of the rotating field (7.19) in the wave equation in the stationary frame (7.3). Notice that, $i\hat{J}_z$ is real so that the wave equation (7.23) in the rotating frame is real.

The canonical-momentum density in the rotating frame is given by

$$\mathbf{\Pi}_{\mathbf{C}}(\mathbf{r},t) = \frac{\partial\mathcal{L}}{\partial\dot{\mathbf{C}}} = \epsilon_0\left(\dot{\mathbf{C}}(\mathbf{r},t) - i\Omega\langle\mathbf{r}|\hat{J}_z|\mathbf{C}\rangle\right) = \epsilon_0\left(\dot{\mathbf{C}}(\mathbf{r},t) + i\Omega\langle\mathbf{C}|\hat{J}_z|\mathbf{r}\rangle\right) , \quad (7.24)$$

which is also real. By using the expression (7.5) of the momentum in a stationary frame and $|\dot{\mathbf{A}}\rangle = e^{-i\Omega t\hat{J}_z}\left(|\dot{\mathbf{C}}\rangle - i\Omega\hat{J}_z|\mathbf{C}\rangle\right)$, we find that

$$\mathbf{\Pi}_{\mathbf{A}}(\mathbf{r},t) = \langle\mathbf{r}|e^{-i\Omega t\hat{J}_z}|\mathbf{\Pi}_{\mathbf{C}}\rangle . \quad (7.25)$$

The Hamiltonian in the rotating frame can be expressed as

$$H_{\mathrm{rot}} = \mathbf{\Pi}_{\mathbf{C}}\cdot\dot{\mathbf{C}} - L = \frac{1}{2\epsilon_0}\langle\mathbf{\Pi}_{\mathbf{C}}|\mathbf{\Pi}_{\mathbf{C}}\rangle + i\Omega\langle\mathbf{\Pi}_{\mathbf{C}}|\hat{J}_z|\mathbf{C}\rangle - i\Omega\langle\mathbf{C}|\hat{J}_z|\mathbf{\Pi}_{\mathbf{C}}\rangle + \frac{\epsilon_0 c^2}{2}\langle\nabla\times\mathbf{C}|\nabla\times\mathbf{C}\rangle . \quad (7.26)$$

By using that the field is transverse (7.22), one may show that the Hamilton equations that derive from this Hamiltonian are equivalent to the wave equation in the rotating frame (7.23).

7.3.2 Rotating modes in free space

Analogous to the discussion in chapter 3, rotating modes of the free radiation field are defined as solutions of the wave equation (7.3) that are monochromatic in a rotating frame. In complex notation, such solutions can be expressed $\mathbf{C}(\mathbf{r},t) = \mathrm{Re}\{\mathbf{C}(\mathbf{r})\exp(-i\omega t)\}$, where $\mathbf{C}(\mathbf{r})$ is the complex spatial vector potential in the rotating frame. Substitution in the wave equation in the rotating frame (7.23) gives

$$\left(\nabla^2 + \frac{\Omega^2 \hat{J}_z^2}{c^2} + \frac{2\omega\Omega\hat{J}_z}{c^2} + \frac{\omega^2}{c^2}\right)\mathbf{C}(\mathbf{r}) = 0 \ . \tag{7.27}$$

This equation plays the role analogous to that of the Helmholtz equation (7.7) in the rotating frame. Notice that due to the presence of $\hat{J}_z$, which is a purely imaginary operator, equation (7.27) is not real so that $\mathbf{C}(\mathbf{r})$ is, in general, a complex vector field.

Since $[\nabla^2, \hat{J}_z] = 0$, it follows that ∇^2 and $\hat{J}_z$ must have simultaneous eigenfunctions for which the wave equation (7.23) reduces to an algebraic equation, which can be solved to obtain a dispersion relation. We shall derive the simultaneous eigenfunctions of ∇^2 and $\hat{J}_z$, which allow us to obtain exact expressions of rotating modes in free space. For reasons of convenience, we first discuss the analogous case of a rotating scalar field $A(\mathbf{r}, t)$. Later on, we shall construct rotating complex vector fields $\mathbf{A}(\mathbf{r}, t)$ from these scalar ones. Analogous to equation (7.19), the negative frequency part of a rotating scalar field is defined as

$$A(\mathbf{r}, t) = e^{-i\hat{L}_z\Omega t} C(\mathbf{r}) e^{-i\omega t} \ , \tag{7.28}$$

where $C(\mathbf{r})$ is spatial field in the rotating frame. In case of a scalar field, equation (7.27) reduces to

$$\left(\nabla^2 + \frac{\Omega^2 \hat{L}_z^2}{c^2} + \frac{2\omega\Omega\hat{L}_z}{c^2} + \frac{\omega^2}{c^2}\right) C(\mathbf{r}) = 0 \ . \tag{7.29}$$

Since also $[\nabla^2, \hat{L}_z] = 0$ and since $\hat{L}_z$ and ∇^2 are both hermitian, $\hat{L}_z$ and ∇^2 must have an orthonormal set of simultaneous eigenfunctions for which the wave equation (7.29) becomes an algebraic equation. Since $[\hat{L}_z, -i\partial/\partial z] = 0$ and since the eigenfunctions of $\hat{L}_z$ are proportional to $\exp(il\phi)$ while the eigenfunctions of $-i\partial/\partial z$ are proportional to $\exp(iqz)$, it is natural to introduce cylindrical coordinates (R, ϕ, z) and look for solutions of the following type

$$C(R, \phi, z) = G(R) e^{il\phi} e^{iqz} \ , \tag{7.30}$$

with $l \in \mathbb{Z}$ and $q \in \mathbb{R}$. Substitution in the scalar wave equation (7.29) yields after rearranging the terms

$$\left(R^2 \frac{\partial^2}{\partial R^2} + R\frac{\partial}{\partial R} + (\kappa R)^2 - l^2\right) G(R) = 0 \ , \tag{7.31}$$

where $\kappa \in \mathbb{R}^+$ is defined by the dispersion relation

$$(\omega + l\Omega)^2 = c^2(\kappa^2 + q^2) \ . \tag{7.32}$$

The solutions of equation (7.31) are Bessel functions of the first kind $G_{\kappa l}(R) = J_l(\kappa R)$ so that a set of scalar Bessel mode functions can be introduced as

$$G_\lambda(\mathbf{r}) = \left(\frac{1}{2\pi}\right) J_l(\kappa R) e^{il\phi} e^{iqz} \ , \tag{7.33}$$

where $\lambda = (\kappa, l, q)$ denotes the set of spatial mode indices and the factor $1/(2\pi)$ is introduced for reasons of normalization. The corresponding frequencies can be obtained from (7.32)

$$\omega_{\lambda\pm} = \pm c\sqrt{\kappa^2 + q^2} - l\Omega \ . \tag{7.34}$$

For every solution G_λ with $\omega_{\lambda\pm}$, the mode function $G^*_\lambda = G_{\lambda^*}$ with $\lambda^* = (\kappa, -l, -q)$ obeys the scalar wave equation (7.29) with the frequencies $\omega_{\lambda^*\pm} = \pm c\sqrt{\kappa^2 + q^2} + l\Omega$. Since the mode functions are in general complex, the real scalar field in the rotating frame corresponding to the mode G_λ with $\omega_{\lambda+}$ is given by

$$C(\mathbf{r}, t) = \frac{e^{i\Omega t\hat{J}_z}\left(G_\lambda e^{-ic\sqrt{\kappa^2+q^2}t} + G_{\lambda^*}e^{ic\sqrt{\kappa^2+q^2}t}\right)}{2} \ , \tag{7.35}$$

where we have used that $\exp(-i\Omega t\hat{J}_z)$ is real. Since $\mathrm{Re}\left(G_\lambda e^{-ic\sqrt{\kappa^2+q^2}t}\right) = \mathrm{Re}\left(G_{\lambda^*}e^{ic\sqrt{\kappa^2+q^2}t}\right)$, it follows that without loss of generality we can choose

$$\omega_\lambda = c\sqrt{\kappa^2 + q^2} - l\Omega \qquad \text{so that} \qquad \omega_{\lambda^*} = c\sqrt{\kappa^2 + q^2} + l\Omega \ . \tag{7.36}$$

By using the orthonormality property of Bessel functions of the first kind [47], one may show that the mode functions $\{G_\lambda\}$ are normalized to δ functions

$$\langle G_\lambda | G_{\lambda'}\rangle = \int_0^\infty RdR \int_0^{2\pi} d\phi \int_{-\infty}^{\infty} dz\, G^*_\lambda(R, \phi, z)G_{\lambda'}(R, \phi, z) = \frac{1}{\kappa}\delta(\kappa - \kappa')\delta(q - q')\delta_{ll'} \ , \tag{7.37}$$

where $\delta(\kappa - \kappa')$ and $\delta(q - q')$ denote Dirac delta functions while $\delta_{ll'}$ denotes the Kronecker delta. By the Fourier-Bessel theorem and the Fourier theorem [47], the set of mode functions $\{G_\lambda\}$ constitutes a complete basis in real space.

It is clear that the vector field $(\mathbf{e}_z \times \nabla)A$ is locally transverse so that it obeys the transversality condition (7.2). It is easy to show that this is an exact solution of the wave equation (7.3) if (and only if) $A(\mathbf{r}, t)$ obeys the scalar equivalent of the wave equation. Since the corresponding electric field has a vanishing z component, it is customary to call this a transverse electric (TE) mode [4, 6]. The transversality condition (7.2) allows for two linearly independent polarization states. The other, for which the magnetic field is transverse (TM), can be constructed as $-(ic/\omega)\nabla \times (\mathbf{e}_z \times \nabla)A$, where ω is the frequency in a stationary frame. In general, the TE and TM mode functions corresponding to a set of scalar modes A are globally orthonormal. The vectorial mode functions corresponding the TE and TM Bessel modes can be expressed as

$$\mathbf{G}^{\mathrm{TE}}_\lambda(\mathbf{r}) = (\mathbf{e}_z \times \nabla)G_\lambda(\mathbf{r}) \tag{7.38}$$

and

$$\mathbf{G}^{\mathrm{TM}}_\lambda(\mathbf{r}) = \frac{-i}{\sqrt{\kappa^2 + q^2}}\nabla \times (\mathbf{e}_z \times \nabla)G_\lambda(\mathbf{r}) \ , \tag{7.39}$$

where $\sqrt{\kappa^2 + q^2}$ arises as the length of the wave vector of the Bessel modes in a stationary frame. The frequencies in the rotating frame depend on Ω and are given by (7.36). It is convenient to define the subscript mode index of the vector fields such that it characterizes both the spatial and the polarization degrees of freedom associated with the modes. In order to do so, we introduce vectorial mode functions $\mathbf{G}_\mu$ with $\mu = (\lambda, \sigma)$, where σ runs over the TE and TM polarizations. One may prove that both the TE and TM mode are exact eigenstates of $\hat{J}_z$ with eigenvalues l, but not of $\hat{L}_z$ and $\hat{S}_z$ separately. Moreover, both are eigenfunctions of ∇^2 with eigenvalues $-(\kappa^2 + q^2)$, of $-i\partial/\partial z$ with eigenvalues q and of the transverse laplacian $\nabla^2_\rho = \nabla^2 - \partial^2/\partial z^2$ with eigenvalues $-\kappa^2$. Analogous to equation (7.37), the vectorial mode functions $\mathbf{G}_\mu$ obey the closure relation

$$\langle \mathbf{G}_\mu | \mathbf{G}_{\mu'} \rangle = \int_0^\infty R dR \int_0^{2\pi} d\phi \int_{-\infty}^\infty dz\, \mathbf{G}^*_\mu(R, \phi, z) \cdot \mathbf{G}_{\mu'}(R, \phi, z) = \frac{1}{\kappa} \delta(\kappa - \kappa')\delta(q - q')\delta_{ll'}\delta_{\sigma\sigma'} \,, \tag{7.40}$$

where $\mu = (\kappa, l, q, \sigma)$. It follows that the set $\{\mathbf{G}_\mu\}$ of vectorial Bessel mode functions constitutes a complete basis of transverse vector fields in $\mathbb{C}^3$ so that the general solution of the wave equation in the rotating frame (7.23) can be expanded as

$$\mathbf{C}(\mathbf{r}, t) = \sum_\sigma \int_0^\infty \kappa d\kappa \int_{-\infty}^\infty dq \sum_l \langle \mathbf{G}_\mu | \mathbf{C}(t) \rangle \mathbf{G}_\mu(\mathbf{r}) \,. \tag{7.41}$$

Since the vectorial Bessel modes are eigenfunctions of $\hat{J}_z$ and, therefore, of the rotation operator $\exp(-i\alpha\hat{J}_z)$, it follows that the corresponding fields are monochromatic both in the rotating and in the stationary frame. As such, the Bessel modes $\{\mathbf{G}_\mu\}$ accommodate the transformations from a stationary to a rotating frame and vice versa.

7.3.3 Basis transformations

In this section we discuss how an arbitrary set of rotating modes, in particular the set of mode functions that obey rotating boundary conditions, can be expanded in the vectorial Bessel modes. We consider an orthonormal set of mode functions $\{\mathbf{V}_\nu\}$ that correspond to transverse and monochromatic fields in the rotating frame, i.e., vector fields in $\mathbb{C}^3$ that obey equations (7.22) and (7.23). Again, the subscript mode index ν characterizes both the spatial and polarization degrees of freedom. The frequency of the mode $\mathbf{V}_\nu$ is denoted ω_ν. Analogous to equation (7.41), the modes can be expanded as

$$\mathbf{V}_\nu(\mathbf{r}) = \sum_\sigma \int_0^\infty \kappa d\kappa \sum_l \int_{-\infty}^\infty dq\, \langle \mathbf{G}_\mu | \mathbf{V}_\nu \rangle \mathbf{G}_\mu(\mathbf{r}) \,, \tag{7.42}$$

where $\mu = (\kappa, l, q, \sigma)$ and the coefficients of the expansion are given by

$$\langle \mathbf{G}_\mu | \mathbf{V}_\nu \rangle = \int_0^\infty R dR \int_0^{2\pi} d\phi \int_0^\infty dq\, \mathbf{G}^*_\mu(R, \phi, z) \cdot \mathbf{V}_\nu(R, \phi, z) \,. \tag{7.43}$$

By using that both $\mathbf{V}_\mu$ and $\mathbf{G}_\nu$ correspond to monochromatic solutions of the wave equation in the rotating frame (7.23), one may show by partial integration that the matrix elements (7.43) differ from 0 only if $\omega_\mu^2 = \omega_\nu^2$. This can be exploited by eliminating one of the spatial mode indices, for instance, $|q|$, in favor of the frequency $\omega = \omega_\nu$. For a fixed value of ω, the scalar mode functions (7.33) can be expressed as

$$H_\lambda(\mathbf{r};\omega) = \left(\frac{1}{2\pi}\right) J_l(\kappa R) e^{il\phi} e^{\pm \frac{iz}{c}\sqrt{\omega^2+2l\omega\Omega+l^2\Omega^2-c^2\kappa^2}} , \tag{7.44}$$

where $\lambda = (\kappa, l, \pm, \omega)$ and the + and − signs correspond to fields that propagate in the positive and negative z directions respectively. The corresponding vectorial modes $\mathbf{H}_\mu$ can be obtained by applying equations (7.38) and (7.39). In terms of these mode functions, the expansion (7.42) reduces to

$$\mathbf{V}_\nu(\rho, z) = \sum_\sigma \int_0^\infty \kappa d\kappa \sum_l \sum_\pm \langle \mathbf{H}_\mu | \mathbf{V}_\nu \rangle \mathbf{H}_\mu(\mathbf{r}) , \tag{7.45}$$

where the summation over ± denotes a summation over the two propagation directions along the z axis. If we limit the discussion to fields for which the expansion (7.44) only involves components with a fixed sign of q, the coefficients of the expansion (7.45) can be obtained from integration in the transverse plane

$$\langle \mathbf{H}_\nu | \mathbf{V}_\mu \rangle = \int_0^\infty R dR \int_0^{2\pi} d\phi \left(\mathbf{H}_\nu(R,\phi,z)\right)^* \cdot \mathbf{V}_\mu(R,\phi,z) . \tag{7.46}$$

This result shows that, for a given value of the frequency in the rotating frame ω and a given propagation direction along the z axis, the spatial dependence of a transverse vectorial mode is fully determined by the field pattern in a single transverse plane. In the more general case of monochromatic fields that contain components that propagate in both directions along the z axis, the field can be separated in two parts that propagate in opposite directions along the z axis. In that case, the analogous expressions can be derived for each of these two parts.

7.3.4 Rotating modes in the paraxial approximation

The expansion (7.45) of a set of monochromatic vectorial modes $\mathbf{V}_\mu$ in the basis of TE and TM modes corresponding to monochromatic scalar Bessel modes $H_\lambda(\mathbf{r})$ establishes the connection with the paraxial description discussed in chapter 3 in a very natural way. Essential to the paraxial approximation is the assumption that the field propagates mainly along a well-defined direction, so that the wave-vector components transverse to the dominant propagation direction are small compared to the length of the wave vector. In the case of the scalar monochromatic Bessel modes (7.44), this implies that $c^2\kappa^2 << \omega^2$. In the terminology of section 3.2, the ratio $c\kappa/\omega$ can be used as a smallness parameter δ. Analogous to the discussion in chapter 3, we also assume that $\Omega \sim \delta^2\omega$, which is a slowly-varying envelope approximation. Then, by expanding the square root in the argument of the exponent in equation (7.44) up to first order in powers of δ, the monochromatic scalar Bessel modes (7.44) reduce to

$$H_\lambda(\rho, z;\omega) \simeq \exp\left(\pm\frac{i\omega z}{c}\left(1 + \frac{l\Omega}{\omega} - \frac{c^2\kappa^2}{2\omega^2}\right)\right) H_\lambda(\rho, 0;\omega) , \tag{7.47}$$

where the + and − signs again correspond to fields that propagate in the positive and negative z directions. Since $\nabla_\rho^2 H_\lambda = -\kappa^2 H_\lambda$ and $\hat{L}_z H_\lambda = l H_\lambda$, the exponential term in equation (7.47) takes the form of the paraxial propagator in the rotating frame (3.31), acting on the transverse Bessel mode function $H_\lambda(\rho, 0)$, z being replaced by $-z$ for modes propagating in the negative z direction. This shows that the paraxial Bessel modes (7.47) are exact solutions of the paraxial wave equation in a rotating frame (5.4). The longitudinal components of the TM modes are of the order of δ smaller than the transverse components and, in leading order of the paraxial approximation, both the TE and the TM modes corresponding to the scalar mode functions (7.47) are polarized in the transverse plane. Moreover, the transverse variation of the polarization is slow compared to that of the transverse beam profile as characterized by $H_\lambda(\rho, z)$ so that, up to first order in δ, the transverse polarization of the Bessel modes can be chosen independent of the spatial mode indices. In the paraxial approximation, a vectorial Bessel mode $\mathbf{G}_\mu$ thus reduces to $\epsilon_\sigma H_\lambda$, where σ labels two linearly independent transverse polarization states and λ is a set of spatial mode indices. Analogous to the discussion above, the paraxial Bessel modes $\epsilon_\sigma H_\lambda$ constitute a complete basis set of paraxial modes. An arbitrary (set of) paraxial modes $\epsilon_\tau V_\lambda$, where τ labels the polarization states, can be expanded in this basis. In the case of the rotating cavity modes that we have described in chapters 3 and 5, the paraxial mode functions are given by $V_\lambda = v_{nm} \exp(ikz)$ with $\lambda = (n, m, k)$ the mode profiles in the rotating frame v_{nm} given by equation (3.58). The uniform polarization ϵ can be chosen independently of the spatial indices λ.

Notice that, analogous to the description in section 3.2, this approach is perturbative in that it allows for obtaining higher-order corrections by taking higher-order powers of δ into account. However, the spatial and polarization degrees of freedom are decoupled only in lowest non-vanishing order of the paraxial approximation.

7.4 Quantization

7.4.1 Normal variables for a rotating field

As discussed in the previous section, the vectorial Bessel mode functions accommodate the transformation from the rotating to the stationary frame and vice versa. In order to derive expressions of the normal variables associated with the Bessel-mode components of the field in a rotating frame, we substitute the expansion (7.41) in the expression (7.19) of the rotating field in the stationary frame and obtain

$$\mathbf{A}(\mathbf{r}, t) = \sum_\sigma \int_0^\infty \kappa d\kappa \sum_l \int_{-\infty}^\infty dq \, \langle \mathbf{G}_\mu | \mathbf{C} \rangle e^{-il\Omega t} \mathbf{G}_\mu(\mathbf{r}) \tag{7.48}$$

for the real vector potential in the stationary frame. Its time derivative can be expressed as

$$\dot{\mathbf{A}}(\mathbf{r}, t) = \sum_\sigma \int_0^\infty \kappa d\kappa \sum_l \int_{-\infty}^\infty dq \, \Big(\langle \mathbf{G}_\mu | \dot{\mathbf{C}}(t) \rangle - il\Omega \langle \mathbf{G}_\mu | \mathbf{C}(t) \rangle \Big) e^{-il\Omega t} \mathbf{G}_\mu(\mathbf{r}) \, . \tag{7.49}$$

From the definitions (7.9) and (7.10), we find that the normal variables corresponding to the Bessel-mode components of the rotating field are given by

$$a_\mu(t) = \left(\frac{\epsilon_0}{2\hbar c \sqrt{\kappa^2 + q^2}} \right)^{1/2} e^{-il\Omega t} \Big(i \langle \mathbf{G}_\mu | \dot{\mathbf{C}}(t) \rangle + \omega_{\mu^*} \langle \mathbf{G}_\mu | \mathbf{C}(t) \rangle \Big) \tag{7.50}$$

and

$$a_{\mu^*}(t) = \left(\frac{\epsilon_0}{2\hbar c \sqrt{\kappa^2 + q^2}} \right)^{1/2} e^{il\Omega t} \Big(i \langle \mathbf{G}_{\mu^*} | \dot{\mathbf{C}}(t) \rangle + \omega_{\mu} \langle \mathbf{G}_{\mu^*} | \mathbf{C}(t) \rangle \Big) \tag{7.51}$$

where $c\sqrt{\kappa^2 + q^2}$ arises as the frequency of the Bessel modes in the stationary frame and ω_{μ^*} and ω_μ are given by equation (7.36). With the normal variables in equations (7.50) and (7.51), the field in the stationary frame and the corresponding momentum take the form of equations (7.12) and (7.13), the mode functions $\mathbf{F}_\lambda$ being replaced by the vectorial Bessel modes $\mathbf{G}_\mu$.

7.4.2 Normal variables in the rotating frame

In case of a complete set of rotating modes $\{\mathbf{V}_\mu\}$, it is more natural to describe the dynamics of the radiation field in terms of a set of normal variables that characterize the amplitudes and corresponding momenta in these rotating modes. In this section, we show that it is possible to introduce such variables and derive the corresponding Hamiltonian. The expressions in equations (7.50) and (7.51) suggest to introduce normal variables for the Bessel-mode components in the rotating frame as

$$c_\mu(t) = \left(\frac{\epsilon_0}{2\hbar c \sqrt{\kappa^2 + q^2}} \right)^{1/2} \Big(i \langle \mathbf{G}_\mu | \dot{\mathbf{C}}(t) \rangle + \omega_{\nu^*} \langle \mathbf{G}_\mu | \mathbf{C}(t) \rangle \Big) \tag{7.52}$$

and

$$c_{\mu^*}(t) = \left(\frac{\epsilon_0}{2\hbar c \sqrt{\kappa^2 + q^2}} \right)^{1/2} \Big(i \langle \mathbf{G}_{\mu^*} | \dot{\mathbf{C}}(t) \rangle + \omega_{\mu} \langle \mathbf{G}_{\nu^*} | \mathbf{C}(t) \rangle \Big) . \tag{7.53}$$

Notice that, although their shape is very similar to that of normal variables in a stationary frame, both the anti-symmetric way in which the frequencies ω_ν and ω_{ν^*} appear and the square-root factor, which involves the frequency of the Bessel mode in a stationary frame, are signatures of the fact that these are normal variables in a non-inertial frame. They are related to the normal variables (7.50) and (7.51) in the stationary frame by the unitary transformations

$$a_\mu(t) = e^{-il\Omega t} c_\mu(t) \qquad \text{and} \qquad a_{\mu^*}(t) = e^{il\Omega t} c_{\mu^*}(t) . \tag{7.54}$$

The field in the rotating frame and its derivative can be expressed as

$$\mathbf{C}(\mathbf{r}, t) = \sum_\sigma \int_0^\infty \kappa d\kappa \sum_l \int_{-\infty}^\infty dq \left(\frac{\hbar}{2\epsilon_0 c \sqrt{\kappa^2 + q^2}} \right)^{1/2} \Big(c_\mu(t) \mathbf{G}_\mu(\mathbf{r}) + c_\mu^*(t) \mathbf{G}_{\mu^*}(\mathbf{r}) \Big) \tag{7.55}$$

and

$$\dot{\mathbf{C}}(\mathbf{r},t) = -i\sum_{\sigma}\int_0^{\infty}\kappa d\kappa \sum_l \int_{-\infty}^{\infty} dq \left(\frac{\hbar}{2\epsilon_0 c\sqrt{\kappa^2+q^2}}\right)^{1/2} \times \\ \omega_\mu\Big(c_\mu(t)\mathbf{G}_\mu(\mathbf{r}) - c^*_\mu(t)\mathbf{G}_{\mu^*}(\mathbf{r})\Big)\,, \tag{7.56}$$

where we have used that $\omega_{\mu^*} + \omega_\mu = 2c\sqrt{\kappa^2+q^2}$ and $\omega_{\mu^*} - \omega_\mu = 2l\Omega$. Since the $\mathbf{G}_\mu$ and $\mathbf{G}_{\mu^*}$ obey the wave equation in the rotating frame at the frequencies ω_μ and $-\omega_\mu$, this result is consistent with the fact that $\mathbf{C}(\mathbf{r},t)$ obeys equation (7.23). By using equation (7.24), we find that

$$\mathbf{\Pi_C}(\mathbf{r},t) = -i\sum_{\sigma}\int_0^{\infty}\kappa d\kappa \sum_l \int_{-\infty}^{\infty} dq \left(\frac{\hbar\epsilon_0 c\sqrt{\kappa^2+q^2}}{2}\right)^{1/2} \Big(c_\mu(t)\mathbf{G}_\mu(\mathbf{r}) - c^*_\mu(t)\mathbf{G}_{\mu^*}(\mathbf{r})\Big)\,. \tag{7.57}$$

The hamiltonian in the rotating frame (7.26) can be expressed as

$$H_{\text{rot}} = \frac{1}{2}\sum_{\sigma}\int_0^{\infty}\kappa d\kappa \sum_l \int_{-\infty}^{\infty} dq\; \hbar\omega_\mu\Big(c^*_\mu c_\mu + c_\mu c^*_\mu\Big)\,. \tag{7.58}$$

The form of this Hamiltonian confirms that the harmonic structure of the dynamics of the modes survives in the rotating frame. The classical dynamics of the Bessel modes in the rotating frame is described the Hamilton equations with the Hamiltonian (7.58) and with the field and corresponding momentum as specified by equations (7.56) and (7.57).

From the expansion in equation (7.41), it follows that the normal variables that characterize the amplitude and momentum in a complete and orthonormal set of rotating modes $\{\mathbf{V}_\nu\}$ may be defined as properly normalized linear combinations of the normal variables for the Bessel modes, i.e.,

$$v_\nu(t) = \sum_{\sigma}\int_0^{\infty}\kappa d\kappa \sum_l \int_{-\infty}^{\infty} dq\; \langle \mathbf{V}_\nu|\mathbf{G}_\mu\rangle c_\mu(t) \tag{7.59}$$

and

$$v_{\nu^*}(t) = \sum_{\sigma}\int_0^{\infty}\kappa d\kappa \sum_l \int_{-\infty}^{\infty} dq\; \langle \mathbf{V}_{\nu^*}|\mathbf{G}_{\mu^*}\rangle c_{\mu^*}(t)\,, \tag{7.60}$$

where $\mu = (\kappa, l, q, \sigma)$. By using the definitions (7.52) and (7.53) of c_μ and c_{μ^*} and the fact that the matrix element $\langle \mathbf{V}_\nu|\mathbf{G}_\mu\rangle$ differs from zero only when $\omega_\nu^2 = \omega_\mu^2$ while the matrix element $\langle \mathbf{V}_{\nu^*}|\mathbf{G}_{\mu^*}\rangle$ differs from zero only when $\omega_{\nu^*}^2 = \omega_{\mu^*}^2$, we find that

$$v_\nu(t) = \sum_{\sigma}\int_0^{\infty}\kappa d\kappa \sum_l \int_{-\infty}^{\infty} dq \left(\frac{\epsilon_0}{2\hbar(\omega_\nu + l\Omega)}\right)^{1/2} \times \\ \langle \mathbf{V}_\nu|\mathbf{G}_\mu\rangle\Big(i\langle \mathbf{G}_\nu|\dot{\mathbf{C}}(t)\rangle + (\omega_\nu + 2l\Omega)\langle \mathbf{G}_\mu|\mathbf{C}(t)\rangle\Big) \tag{7.61}$$

and

$$v_{\nu^*}(t) = \sum_\sigma \int_0^\infty \kappa d\kappa \sum_l \int_{-\infty}^\infty dq \left(\frac{\epsilon_0}{2\hbar(\omega_\nu - l\Omega)} \right)^{1/2} \times \\ \langle \mathbf{V}_{\nu^*} | \mathbf{G}_{\mu^*} \rangle \Big(i \langle \mathbf{G}_{\nu^*} | \dot{\mathbf{C}}(t) \rangle + (\omega_\nu - 2l\Omega) \langle \mathbf{G}_{\mu^*} | \mathbf{C}(t) \rangle \Big) \,. \quad (7.62)$$

Notice that the normal variables (7.61) and (7.62) reduce to the ordinary normal variables in the stationary frame in the absence of rotation, i.e., for $\Omega = 0$. Using the completeness of the vectorial Bessel modes (7.40), the definitions in equations (7.59) and (7.60) can be inverted to obtain

$$c_\mu(t) = \sum_\nu \langle \mathbf{G}_\mu | \mathbf{V}_\nu \rangle v_\nu(t) \qquad \text{and} \qquad c_{\mu^*}(t) = \sum_\nu \langle \mathbf{G}_{\mu^*} | \mathbf{V}_{\nu^*} \rangle v_{\nu^*}(t) \,, \quad (7.63)$$

where we have assumed that $\{\mathbf{V}_\nu\}$ is a discrete set of modes. Again using that the matrix elements $\langle \mathbf{G}_\mu | \mathbf{V}_\nu \rangle$ differ from 0 only when $\omega_\mu^2 = \omega_\nu^2$, the Hamiltonian (7.58) can be expressed as

$$H_{\text{rot}} = \frac{1}{2} \sum_\nu \hbar\omega_\nu \Big(v_\nu^* v_\nu + v_\nu v_\nu^* \Big) \,. \quad (7.64)$$

Thus, we have obtained a complete description of the classical dynamics of the radiation field in terms of normal variables for an orthonormal but otherwise arbitrary set of rotating modes $\{\mathbf{V}_\nu\}$.

7.4.3 Canonical quantization

In the stationary frame, quantization is performed by replacing the real field $\mathbf{A}$ and canonical momentum $\mathbf{\Pi_A}$ by hermitian operators that obey canonical commutation rules (7.16). The normal variables a_μ, as specified by equation (7.50), and their complex conjugates a_μ^* become bosonic annihilation and creation operators. The field operator in the stationary frame takes the form of equation (7.12) when the modes $\mathbf{F}_\lambda$ are replaced by the Bessel modes $\mathbf{G}_\nu$ and the normal variables are replaced by the creation and annihilation operators. The quantum evolution of the rotating field operators is governed by the Heisenberg equation of motion. The Hamiltonian takes the form of equation (7.15) when the normal variables are replaced by the creation and annihilation operators that correspond to the normal variables defined in equation (7.50) and (7.51).

Quantization in the rotating frame involves replacing the field in the rotating frame $\mathbf{C}(\mathbf{r}, t)$ and the corresponding momentum $\mathbf{\Pi}_\mathrm{C}(\mathbf{r}, t)$ (7.24) by vector operators $\hat{\mathbf{C}}(\mathbf{r}, t)$ and $\hat{\mathbf{\Pi}}_\mathrm{C}(\mathbf{r}, t)$ whose components obey canonical commutation rules

$$\Big[\hat{C}_i(\mathbf{r}), \hat{\Pi}_{\mathbf{C}j}(\mathbf{r}') \Big] = i\hbar \delta_\perp(\mathbf{r} - \mathbf{r}') \delta_{ij} \,, \quad (7.65)$$

where the indices i and j run over the vector components. The other, independent, commutators of the components of $\hat{\mathbf{C}}$ and $\hat{\mathbf{\Pi}}_\mathrm{C}$ vanish. From the definition (7.19) of the complex field in the rotating frame and the expression (7.25) of the corresponding momentum, it is evident that quantization in the rotating frame is consistent with quantization in the stationary frame. Substitution of the expansions (7.55) and (7.57) in the canonical commutation rules in the rotating frame (7.65) shows that also the normal variables in the rotating frame become bosonic creation and annihilation operators. This is in obvious agreement with the transformations in equation (7.54). Since the transformation described by the equations (7.59) and (7.60) is a properly normalized unitary transformation in the space of the normal variables, it follows that the same is true for the normal variables that describe the dynamics of the field in terms of the amplitudes and momenta of the rotating modes $\mathbf{V}_\mu$. The quantum dynamics in the rotating frame is described by the Heisenberg equation of motion with the Hamiltonian in equation (7.58) or, equivalently, (7.64) when the normal variables are replaced by creation and annihilation operators.

Notice, that since the transformation in equation (7.54), and also the transformations in equations (7.59) and (7.60), are properly normalized unitary transformations, the vacuum as perceived from the rotating frame is the same as that perceived from the stationary frame (7.18).

7.5 Summary, conclusion and outlook

In this chapter we have presented the first exact quantum-optical description of rotating light, or, equivalently, quantized the radiation field in an orthonormal but otherwise arbitrary basis of rotating modes $\{\mathbf{V}_\mu\}$. Rotating modes are defined as divergence free (7.22) monochromatic solutions of the wave equation in a rotating frame (7.23). In complex notation, these fields are separable in space and time so that the corresponding physical fields are stationary in the rotating frame. As a result, they rotate uniformly in a stationary frame. We have shown that the set of vectorial Bessel modes both with transverse electric (TE) and transverse magnetic (TM) polarization are exact eigenstates of $\hat{J}_z$ and, therefore, of the rotation operator $\exp(-i\Omega t\hat{J}_z)$. It follows that the fields corresponding to these modes only pick up a frequency shift under the transformation from a stationary to a rotating frame. As a result, the Bessel-mode fields are monochromatic in both frames. As the Bessel modes are monochromatic in the stationary frame, the free radiation field can be quantized in this basis in the usual way. Since they are also monochromatic in the rotating frame, an arbitrary rotating mode $\mathbf{V}_\mu$, which is monochromatic in the rotating frame, can be expanded in the subset of Bessel modes that have the same frequency in the rotating frame. The simple transformation property of Bessel modes to the stationary frame naturally leads to an expression of the field operator corresponding to the rotating mode as a linear combination of the field operators for the Bessel modes in the stationary frame. Alternatively, the field can be quantized directly in the rotating frame. We have shown that this is equivalent to quantization in the stationary frame.

The approach discussed in this chapter is particulary suited to describe the quantum dynamics of a set of modes that solve rotating boundary conditions, such as the rotating cavity modes discussed in chapters 3 and 5. In that respect it is complementary to the approach discussed in reference [107], where approximate rotating solutions in free space are constructed from stationary ones. As opposed to reference [107], the theory presented here is exact and does not require paraxial and/or slowly-varying-envelope approximations. On the other hand, the approach in reference [107] is more flexible in that it allows for a quantum description of fields with a rotating polarization and/or a rotating mode pattern whereas the work discussed here only concerns uniformly rotating fields.

The method discussed in this chapter concerns quantization of the free radiation field in the Coulomb gauge. As a result, its validity is restricted to energy scales where vacuum fluctuations in full quantum electrodynamics (e.g. electron-positron pair creation) are negligible. A special property of the transformation to a rotating frame that we have applied in this chapter is that it does not affect the vacuum state of the radiation field. From a relativistic point-of-view, other definitions of the transformation to a rotating frame may be more natural [109, 110]. These lead to a different definition of the vacuum in the rotating frame [111]. The transformation to a rotating frame that we have used here is fundamentally different from the transformation to the co-moving frame of an orbiting observer. Also in that case the vacuum is perceived differently, which may be understood as an example of the Unruh effect [112].

The scalar Bessel beams G_λ that we have studied in section 7.3, were first proposed some twenty years ago [113, 114] and have been investigated in detail both theoretically and experimentally, see, for instance, reference [115] for a recent review. The vectorial Bessel beams $\mathbf{G}_\mu$ are less well-known but have also been studied before [6, 116]. Since the production of Bessel beams in experiments is well-established, it should be possible to construct the rotating fields that we have discussed in this chapter as a superposition of their rotational-Doppler shifted components. Production of quantum coherent superpositions of such modes is probably far more involved.

An interesting application of the theory discussed in this chapter would be to study the quantum interference of two single-photon fields that have the same spectral and spatial structure in a given transverse plane of their own co-rotating frames but rotate at different frequencies and, possibly, in opposite directions. From the results of section 7.3, it is clear that rotation has strong and distinct effects on the spectral and spatial structure of the modes. As a result, the probability of photon bunching in a quantum-interferometric set-up, which is essentially determined by the spatial and spectral overlap of the two modes, depends strongly on the two rotation frequencies.