Light with a twist : ray aspects in singular wave and quantum optics

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Geometric phases for astigmatic optical modes of arbitrary order

6.1 Introduction

In the twenty-five years that have passed since Berry published his landmark paper [87], the geometric phase has turned out to be a very unifying concept in physics. Various phase shifts and rotation angles both in classical and quantum physics have been proven to originate from the geometry of the underlying parameter space. One of the first examples was given by Pancharatnam [88] who discovered that the phase shift due to a cyclic transformation of the polarization of an optical field is equal to half the enclosed area on the Poincaré sphere for polarization states. Other optical examples of geometric phases are the phase shift that arises from the variation of the direction of the wave vector of an optical field through a fiber [89] and the phase that is associated with the cyclic manipulation of a squeezed state of light [90]. The Gouy phase shift, which is due to the variation of the beam parameters (the beam width and the radius of curvature of the wave front) of a Gaussian optical beam, can also be interpreted geometrically [91].

In analogy with the geometric phase for polarization (or spin) states of light, van Enk has proposed a geometric phase that arises from cyclic mode transformations of paraxial optical beams carrying orbital angular momentum [92]. The special case of isotropic first-order modes is equivalent to the polarization case [93] and, as was experimentally demonstrated by Galvez et. al., the geometric phase shift acquired by a first-order mode that is transformed along a closed trajectory on the corresponding Poincaré sphere also equals half the enclosed surface on this sphere [94]. Similar experiments have been performed with second-order modes [95], in particular to show that exchange of orbital angular momentum is necessary for a non-trivial geometric phase to occur [96]. However, in the general case of isotropic modes of order N, the connection with the geometry of the $N + 1$-dimensional mode space is not at all obvious.

In this chapter, we present a complete and general analysis of the phase shift of transverse optical modes of arbitrary order when propagating through a paraxial optical set-up, thereby resolving this issue. Paraxial optical modes with different transverse mode indices (n, m) are connected by bosonic ladder operators in the spirit of the algebraic description of the quantum-mechanical harmonic oscillator and complete sets of transverse modes $|u_{nm}\rangle$ can thus be obtained from two pairs of ladder operators [17]. We show that the geometries of the subspaces of modes with fixed transverse mode numbers n and m, which are closed under mode transformations, are all carbon copies of the geometry underlying the ladder operators. We fully characterize this geometry including both the generalized beam parameters, which characterize the astigmatism and orientation of the intensity and phase patterns of a Gaussian fundamental mode, and the degrees of freedom associated with the nature and orientation of the higher-order modes. We find a dynamical and a geometric contribution to the phase shift of a mode under propagation through an optical set-up, which both have a clear significance in terms of this parameter space.

The material in this chapter is organized as follows. In the next section we briefly summarize the operator description of paraxial wave optics. We discuss its group-theoretical structure, which is essential for our ladder-operator approach, and show how paraxial ray optics emerges from it. In section 6.3 we discuss how complete basis sets of transverse modes can be obtained from two pairs of bosonic ladder operators. We discuss the transformation properties of the ladder operators, and, thereby, of the modes and characterize the ten degrees of freedom that are associated with the choice of a basis of transverse modes. Two of those degrees of freedom relate to overall phase factors of the ladder operators and, therefore, of the modes. In section 6.4, we show that the variation of these phases under propagation through a set-up originates from the variation of the other parameters. We discuss an analogy with the Aharonov-Bohm effect in quantum mechanics and show that both contributions to the phase shift are geometric in that they are fully determined by the trajectory through the parameter space. However, only the geometric contribution relates to the geometry of this space. Section 6.5 is devoted to the specific, but experimentally relevant, case of mode transformations of non-astigmatic modes. In the final section, we summarize our results and draw our conclusions.

6.2 Canonical description of paraxial optics

6.2.1 Position and propagation direction as conjugate variables

A monochromatic paraxial beam of light that propagates along the z direction is conveniently described by the complex scalar profile $u(\rho, z)$, which characterizes the spatial structure of the field beyond the structure of the carrier wave $\exp(ikz - i\omega t)$. The two-dimensional vector $\rho = (x, y)^{\mathrm{T}}$ denotes the transverse coordinates. The electric and magnetic fields of the beam can be expressed as

$$\mathbf{E}(\rho, z, t) = \mathrm{Re}\left\{E_0 \epsilon u(\rho, z) e^{ikz - i\omega t}\right\} \tag{6.1}$$

and

$$\mathbf{B}(\rho, z, t) = \mathrm{Re}\left\{\frac{E_0}{c}(\mathbf{e}_z \times \epsilon) u(\rho, z) e^{ikz - i\omega t}\right\} , \tag{6.2}$$

where E_0 is the amplitude of the field, ϵ is the transverse polarization, $\mathbf{e}_z$ is the unit vector along the propagation direction and $\omega = ck$ is the optical frequency with c the speed of light. The slowly varying amplitude $u(\rho, z)$ obeys the paraxial wave equation

$$\left(\nabla_\rho^2 + 2ik\frac{\partial}{\partial z}\right) u(\rho, z) = 0 , \tag{6.3}$$

where $\nabla_\rho^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the transverse Laplacian. Under the assumption that the transverse variation of the field appears on a much larger length scale than the wavelength, this description of paraxial wave optics is consistent with Maxwell's equations in free space [45].

The paraxial wave equation (6.3) has the form the Schrödinger equation for a free particle in two dimensions. The longitudinal coordinate z plays the role of time while the transverse coordinates $\rho = (x, y)^{\mathrm{T}}$ constitute the two-dimensional space. This analogy allows us to adopt the Dirac notation of quantum mechanics to describe the evolution of a classical wave field [42]. In the Schrödinger picture, we introduce state vectors $|u(z)\rangle$ in the Hilbert space L^2 of square-integrable transverse states of the wave field, where the z coordinate parameterizes the trajectory along which the field propagates. The states are properly normalized $\langle u(z)|u(z)\rangle = 1$ for all z and the field profile in real space can be expressed as $u(\rho, z) = \langle \rho | u(z)\rangle$. Just as in quantum mechanics, the transverse coordinates may be viewed as a hermitian vector operator $\hat{\rho} = (\hat{x}, \hat{y})^{\mathrm{T}}$ acting on the Hilbert space. The derivatives with respect to these coordinates constitute canonically conjugate operators. Rather than the conjugate transverse momentum operator $-i\partial/\partial\rho$, which has the significance of the normalized transverse momentum of the field, it is convenient to construct the propagation-direction operator by dividing the transverse momentum operator by the normalized longitudinal momentum k. Thus, we obtain the hermitian vector operator $\hat{\theta} = (\hat{\vartheta}_x, \hat{\vartheta}_y)^{\mathrm{T}} = -(i/k)(\partial/\partial x, \partial/\partial y)^{\mathrm{T}}$. The transverse position and propagation-direction operators obey the canonical commutation rules

$$[\hat{\rho}_a, k\hat{\theta}_b] = i\delta_{ab} , \tag{6.4}$$

where the indices a and b run over the x and y components. In analogy with quantum mechanics, we introduce the transverse field profile in propagation-direction representation

$$\tilde{u}(\theta, z) = \langle\theta|u(z)\rangle = \frac{k}{2\pi}\int d_2\rho\; u(\rho, z)e^{-ik\theta^{\mathrm{T}}\rho}\;, \tag{6.5}$$

which is the two-dimensional Fourier transform of $u(\rho, z)$ and characterizes the transverse propagation-direction distribution of the field.

In geometric optics, a ray of light is fully characterized in a transverse plane z by its transverse position ρ and propagation direction θ, which are usually combined in the four-dimensional ray vector $\mathfrak{z}^{\mathrm{T}} = (\rho^{\mathrm{T}}, \theta^{\mathrm{T}})$. The operator description of paraxial wave optics may be viewed as a formally quantized (wavized) description of light rays, where ρ and θ have been replaced by hermitian operators $\hat{\rho}$ and $\hat{\theta}$ that obey canonical commutation rules (6.4) and $1/k = ƛ$ plays the role of $\hbar$ [31]. These operators are conveniently combined in the ray operator $\hat{\mathfrak{z}}^{\mathrm{T}} = (\hat{\rho}^{\mathrm{T}}, \hat{\theta}^{\mathrm{T}})$. In analogy with quantum mechanics, where the expectation values of the position and momentum operators have a clear classical significance in the limit $\hbar \to 0$, a paraxial wave field reduces to a ray in the limit of geometric optics $ƛ \to 0$. Its transverse position and propagation direction in the transverse plane z are characterized by the expectation values $\langle u(z)|\hat{\rho}|u(z)\rangle$ and $\langle u(z)|\hat{\theta}|u(z)\rangle$.

6.2.2 Group-theoretical structure of paraxial wave and ray optics

Both the diffraction of a paraxial beam under free propagation, as described by the paraxial wave equation (6.3), and the transformations due to lossless optical elements can be expressed as unitary transformations $|u_{\mathrm{out}}\rangle = \hat{U}|u_{\mathrm{in}}\rangle$ on the transverse state of the field. In general, a unitary operator can be expressed as

$$\hat{U}(\{a_j\}) = e^{-i\sum_j a_j \hat{T}_j}\;, \tag{6.6}$$

where $\{a_j\}$ is a set of real parameters and $\{\hat{T}_j\}$ a set of hermitian generators, i.e., $\hat{T}_j^\dagger = T_j$. In the present case of paraxial propagation and paraxial (first-order) optical elements, the generators are quadratic forms in the transverse position and propagation-direction operators. This is exemplified by the paraxial wave equation (6.3), which in operator notation takes the following form

$$\frac{\partial}{\partial z}|u(z)\rangle = -\frac{ik}{2}\hat{\theta}^2|u(z)\rangle \tag{6.7}$$

and is formally solved by

$$|u(z)\rangle = \exp\left(-\frac{ikz\hat{\theta}^2}{2}\right)|u(0)\rangle\;. \tag{6.8}$$

This shows that that free propagation of a paraxial field is generated by $k\hat{\theta}^2/2$, which is obviously quadratic in the canonical operators. Since the ray operator $\hat{\mathfrak{z}}$ has four components, the number of squares of the operators is four while the number of mixed products is $\binom{4}{2} = 6$, which gives a total of ten quadratic forms. They are hermitian and can be chosen as

$$T_1 = \hat{x}^2 \ , \quad T_2 = \hat{y}^2 \ , \quad T_3 = \hat{x}\hat{y} \ , \quad T_4 = \frac{k}{2}\left(\hat{x}\hat{\vartheta}_x + \hat{\vartheta}_x\hat{x}\right) \ , \quad T_5 = \frac{k}{2}\left(\hat{y}\hat{\vartheta}_y + \hat{\vartheta}_y\hat{y}\right) \ ,$$
$$T_6 = k\hat{x}\hat{\vartheta}_y \ , \quad T_7 = k\hat{y}\hat{\vartheta}_x \ , \quad T_8 = k^2\hat{\vartheta}_x\hat{\vartheta}_y \ , \quad T_9 = k^2\hat{\vartheta}_x^2 \quad \text{and} \quad T_{10} = k^2\hat{\vartheta}_y^2 \ . \tag{6.9}$$

In terms of these generators, free propagation of a paraxial beam (6.8) is described by

$$|u(z)\rangle = \exp\left(-\frac{i\left(\hat{T}_9 + \hat{T}_{10}\right)z}{2k}\right)|u(0)\rangle \ . \tag{6.10}$$

The mixed product $\hat{T}_8$ appears in the generator of free propagation through an anisotropic medium, i.e., a medium in which the refractive index depends on the propagation direction θ. In that case the propagator can be expressed as $\exp(-ik\hat{\theta}^{\mathrm{T}}\mathsf{N}^{-1}\hat{\theta}z/2)$, where N is a real and symmetric matrix that characterizes the (quadratic) variation of the refractive index with the propagation direction. If the anisotropy of the refractive index is not aligned along the ϑ_x and ϑ_y directions, this transformation involves $\hat{T}_8$. A thin astigmatic lens imposes a Gaussian phase profile. The unitary transformation that describes it can be expressed as

$$|u_{\mathrm{out}}\rangle = \exp\left(-\frac{ik\rho^{\mathrm{T}}\mathsf{F}^{-1}\rho}{2}\right)|u_{\mathrm{in}}\rangle \ , \tag{6.11}$$

where F is a real and symmetric 2×2 matrix whose eigenvalues correspond to the focal lengths of the lens while the corresponding, mutually perpendicular, eigenvectors fix its orientation in the transverse plane. In the general case of an astigmatic lens that is not aligned along the x and y directions, this transformation involves the generators $\hat{T}_1$, $\hat{T}_2$ and $\hat{T}_3$. A rotation of the beam profile in the transverse plane can be represented by

$$|u_{\mathrm{rot}}\rangle = e^{-i(\hat{T}_6 - \hat{T}_7)\phi}|u\rangle \ , \tag{6.12}$$

where $\hat{T}_6 - \hat{T}_7 = -i(x\partial/\partial y - y\partial/\partial x)$ is the orbital angular momentum operator and ϕ is the rotation angle. The operators $\hat{T}_4$ and $\hat{T}_5$ generate transformations that rescale a field profile along the x and y directions respectively, i.e.,

$$u_{\mathrm{out}}(x,y,z) = \langle\rho|u_{\mathrm{out}}(z)\rangle = \langle\rho|e^{i\log(c_x)\hat{T}_4 + i\log(c_y)\hat{T}_5}|u_{\mathrm{in}}(z)\rangle = \sqrt{c_x c_y}\ u_{\mathrm{in}}(c_x x, c_y y, z) \ . \tag{6.13}$$

Physically speaking, such transformations correspond to the deformation of a field profile due to refraction at the interface between two dielectrics with different refractive indices.

From the canonical commutation relations (6.4), it follows that the commutator of any two generators (6.9) is a linear combination of the generators. In mathematical terms, the algebra of the generators is closed, which means that $[\hat{T}_k, \hat{T}_l] = i\sum_m g_{klm}\hat{T}_m$ with real structure constants g_{klm}. We shall prove that the unitary transformations (6.6) with the generators (6.9) form a ten-parameter Lie group. For reasons that will become clear this group is called the metaplectic group $Mp(4)$.

Since the states $|u(z)\rangle$ are normalized, the expectation values $\langle u(z)|\hat{\rho}|u(z)\rangle$ and $\langle u(z)|\hat{\theta}|u(z)\rangle$ have the significance of the average transverse position and the average propagation direction of the field. A special property of the unitary transformations in equation (6.6) with the quadratic generators given by (6.9), is that the Heisenberg transformation $\hat{U}^{\dagger}\,\hat{\mathscr{z}}\,\hat{U}$ of the vector operator $\hat{\mathscr{z}}^{\mathrm{T}} = (\hat{\rho}^{\mathrm{T}}, \hat{\theta}^{\mathrm{T}})$ is linear, so that it can be expressed as

$$\hat{U}^{\dagger}(\{a_j\})\,\hat{\mathscr{z}}\,\hat{U}(\{a_j\}) = M(\{a_j\})\,\hat{\mathscr{z}} \;, \tag{6.14}$$

where $M(\{a_j\})$ is the 4×4 ray matrix that describes the transformation of a ray $\mathscr{z}^{\mathrm{T}} = (\rho^{\mathrm{T}}, \theta^{\mathrm{T}})$ under the optical element that is described by the state-space operator $\hat{U}(\{a_j\})$. The defining properties of the position and momentum operators, i.e., that they are hermitian and obey canonical commutation rules (6.4), are preserved under this unitary Heisenberg transformation. It follows that $M(\{\alpha_j\})$ is real and obeys the identity

$$M^{\mathrm{T}}(\{a_j\})GM(\{a_j\}) = G \qquad \text{with} \qquad G = \begin{pmatrix} \mathsf{0} & \mathsf{1} \\ -\mathsf{1} & \mathsf{0} \end{pmatrix} , \tag{6.15}$$

where $\mathsf{0}$ and $\mathsf{1}$ denote the 2×2 zero and unit matrices respectively, so that G is a 4×4 matrix. This identity (6.14) ensures that the operator expectation values $\langle u(z)|\,\hat{\mathscr{z}}\,|u(z)\rangle$ of the transverse position and propagation direction transform as a ray, i.e., trace out the path of a ray when the field propagates through an optical set-up. This shows how paraxial ray optics emerges from paraxial wave optics and, as such, the identity (6.14) may be viewed as an optical analogue of the Ehrenfest theorem in quantum mechanics [49]. The manifold of rays $\mathscr{z}$ constitutes a phase space in the mathematical sense. The real and linear transformations on this manifold that obey the relation (6.15), or, equivalently, preserve the canonical commutation rules (6.4), are ray matrices. The product of two ray matrices is again a ray matrix so that ray matrices form a group. The group of real 4×4 ray matrices, which preserve the bilinear form $\mathscr{z}^{\mathrm{T}}G\mathscr{s}$, where $\mathscr{z}$ and $\mathscr{s}$ are ray vectors, is called the symplectic group $Sp(4,\mathbb{R})$. The term symplectic, which is a syllable-by-syllable translation of the Latin "complex" to Ancient Greek and literally means "braided together", refers to the fact that a phase space is a joint space of position and propagation direction (momentum). The 4×4 ray matrices in $Sp(4,\mathbb{R})$ emerge from a set of unitary state-space transformations, which, as one may prove from equation (6.14), constitute a group under operator multiplication. As was mentioned already, this group is called the metaplectic group $Mp(4)$. For real rays $\mathscr{z}, \mathscr{s} \in \mathbb{R}^4$, the products $\mathscr{z}^{\mathrm{T}}G\mathscr{z}$ and $\mathscr{s}^{\mathrm{T}}G\mathscr{s}$ vanish. The product $\mathscr{z}^{\mathrm{T}}G\mathscr{s}$ does not vanish and is obviously conserved under paraxial propagation and optical elements. It is called the Lagrange invariant [29, 97] and has the significance of the phase-space extent of a pair of rays $\mathscr{z}$ and $\mathscr{s}$. Conservation of this quantity is an optical analogue of Liouville theorem in statistical mechanics.

The commutators of the quadratic generators $\hat{T}_j$ and the position and propagation-direction operators are linear in these operators, so that we can write

$$-i[\hat{T}_j, \hat{\mathscr{z}}] = J_j\hat{\mathscr{z}} \;, \tag{6.16}$$

where the 4×4 matrices J_j are real. Explicit expressions of these matrices are given in appendix 6.A. Applying equation (6.14) to infinitesimal transformations immediately shows that the ray matrix corresponding to the unitary state-space operator in equation (6.6) is given by

$$M(\{\alpha_j\}) = e^{-\sum_j \alpha_j J_j} \ . \tag{6.17}$$

Equation (6.16) provides a general relationship between the generators $\{\hat{T}_j\}$ of the unitary state-space transformations (6.6) and the generators $\{J_j\}$ of the corresponding ray matrices (6.17). By applying equation (6.15) to infinitesimal transformations, one finds that the generators obey $J_j^{\mathrm{T}} G + G J_j = 0$. Moreover, from equation (6.16) one may prove that

$$[[\hat{T}_i, \hat{T}_j], \hat{\boldsymbol{z}}] = [J_i, J_j] \hat{\boldsymbol{z}} \ . \tag{6.18}$$

Using the Lie algebra $[\hat{T}_k, \hat{T}_l] = i \sum_m g_{klm} \hat{T}_m$ we find that $[J_k, J_l] = -\sum_m g_{klm} J_m$. This proves that the metaplectic and symplectic groups are homomorphic, i.e., for every $\hat{U} \in Mp(4)$ there is a corresponding $M \in Sp(4, \mathbb{R})$. The reverse of this statement is not true; a ray matrix M fixes a corresponding transformation $\hat{U}$ up to an overall phase. The homomorphism is an isomorphism up to this phase.

By using equation (6.15) and the expressions of the unitary transformations (6.10), (6.11), (6.12) and (6.13) or, equivalently, the relation between (6.16) the sets of generators $\{\hat{T}_j\}$ and $\{J_j\}$ and the definition (6.17) of the ray matrices, one finds the 4×4 ray matrices that describe propagation, a thin lens, a rotation in the transverse plane and the rescaling of a beam profile due to refraction at the interface between two dielectrics. These ray matrices, some of which have been given explicitly in sections 2.2 and 3.5, generalize the well-known ABCD matrices to the case of two independent transverse degrees of freedom [12].

The group-theoretical structure that we have discussed in this section can easily be generalized to the case of D spatial dimensions. In that case there are $2D$ canonical operators. These give rise to $2D + \binom{2D}{2} = 2D^2 + D$ linearly independent quadratic forms, which generate state-space transformations that constitute the metaplectic group $Mp(2D)$. The corresponding ray matrices obey the $2D-$dimensional generalization of equation (6.15) and form the corresponding symplectic group $Sp(2D, \mathbb{R})$. In case of a single transverse dimension, the three hermitian quadratic forms can be chosen as x^2, $k(\hat{x}\hat{\vartheta}_x + \hat{\vartheta}_x\hat{x})/2$ and $k^2\hat{\vartheta}_x^2$. In the analogous case of the quantum-mechanical description of a particle in three dimensions, the number of quadratic forms is twenty-one.

6.3 Basis sets of paraxial modes

6.3.1 Ladder operators

As a result of the quadratic nature of the generators (6.9), a, possibly astigmatic, Gaussian beam profile at the $z = 0$ input plane of a paraxial optical set-up will retain its Gaussian shape in all other transverse planes z. This is the general structure of a transverse fundamental mode. Complete sets of higher-order transverse modes that preserve their general shape under paraxial propagation and paraxial optical elements can be obtained by repeated application of bosonic raising operators $\hat{a}_p^\dagger(0)$ in the $z = 0$ plane [33]. In the present case of two transverse dimensions, we need two independent raising operators so that $p = 1, 2$. Both the raising operators and the corresponding lowering operators $\hat{a}_p(0)$ are linear in the transverse position and propagation-direction operators $\hat{\rho}$ and $\hat{\theta}$. Their transformation property under unitary transformations $\in Mp(4)$ follows from the requirement that acting with a transformed ladder operator on a transformed state must be equivalent to transforming the raised or lowered state, i.e.,

$$\hat{a}_{\mathrm{out}}^{(\dagger)}|u_{\mathrm{out}}\rangle = \hat{a}_{\mathrm{out}}^{(\dagger)}\hat{U}|u_{\mathrm{in}}\rangle = \hat{U}\hat{a}_{\mathrm{in}}^{(\dagger)}|u_{\mathrm{in}}\rangle \; . \tag{6.19}$$

In view of the unitarity of $\hat{U}$, this requires that

$$\hat{a}_{\mathrm{out}}^{(\dagger)} = \hat{U}\hat{a}_{\mathrm{in}}^{(\dagger)}\hat{U}^\dagger \; . \tag{6.20}$$

Since the generators (6.9) are quadratic in the position and propagation-direction operators, the ladder operators preserve their general structure and remain linear in these operators under this transformation (6.20). Moreover, their bosonic nature is preserved so that they obey the commutation rules

$$[\hat{a}_p(z), \hat{a}_q^\dagger(z)] = \delta_{pq} \tag{6.21}$$

in all transverse planes z of the optical set-up if (and only if) they obey bosonic commutation rules in the $z = 0$ plane. When the fundamental Gaussian mode $|u_{00}(z)\rangle$ is chosen such that the lowering operators give zero when acting upon it, i.e., $\hat{a}_1(z)|u_{00}(z)\rangle = \hat{a}_2(z)|u_{00}(z)\rangle = 0$, the commutation rules (6.21) guarantee that the modes

$$|u_{nm}(z)\rangle = \frac{1}{\sqrt{n!m!}} \left(\hat{a}_1^\dagger(z)\right)^n \left(\hat{a}_2^\dagger(z)\right)^m |u_{00}(z)\rangle \; , \tag{6.22}$$

form a complete set in all transverse planes z. For a given optical system, the complete set of modes is thus fully characterized by the choice of the two bosonic ladder operators $\hat{a}_p(0)$ in the reference plane $z = 0$.

In chapter 2, we have shown that, in the special case of an astigmatic two mirror-cavity, the ladder operators, and thereby the cavity modes, can be directly obtained as the eigenvectors of the ray matrix for one round trip inside the cavity. In the present case of an open system, we are free to choose the parameters that specify the ladder operators in the $z = 0$ input plane. A convenient way to do this is to choose an arbitrary ray matrix $M_0 \in Sp(4, \mathbb{R})$. This ray matrix can be chosen independent of the properties of the optical system, and of the ray matrices that describe the transformations of its elements. However, as we shall see, a necessary and sufficient restriction is that M_0 has four eigenvectors μ for which the matrix element $\mu^\dagger G \mu$ does not vanish. It is obvious that this matrix element is purely imaginary so that the eigenvectors must be complex. Since M_0 is real, this implies that for each eigenvector μ_p also μ_p^* is one of the eigenvectors so that the eigenvectors come in two complex conjugate pairs, obeying the eigenvalue relations $M_0\mu_p = \lambda_p\mu_p$ and $M_0\mu_p^* = \lambda_p^*\mu_p^*$, with $p = 1, 2$. Without loss of generality we can assume that the matrix elements $\mu_p^\dagger G\mu_p$ are positive imaginary. Then we can write

$$\mu_p^\dagger G\mu_p = 2i \qquad \text{and} \qquad \mu_p^{\mathrm{T}} G\mu_p = 0\ , \tag{6.23}$$

where $p = 1, 2$. The first relation can be assured by proper normalization of the eigenvectors, whereas the second follows from the antisymmetry of G. By taking matrix elements of the symplectic identity $M_0^{\mathrm{T}} GM_0 = G$, we find the relations

$$\lambda_p^*\lambda_q\mu_p^\dagger G\mu_q = \mu_p^\dagger G\mu_q \qquad \text{and} \qquad \lambda_p\lambda_q\mu_p^{\mathrm{T}} G\mu_q = \mu_p^{\mathrm{T}} G\mu_q\ . \tag{6.24}$$

Assuming that the two eigenvalues λ_1 and λ_2 are different, we conclude that

$$\mu_1^\dagger G\mu_2 = 0 \qquad \text{and} \qquad \mu_1^{\mathrm{T}} G\mu_2 = 0\ . \tag{6.25}$$

When the eigenvalues are degenerate, i.e., $\lambda_1 = \lambda_2$, one can find infinitely many pairs of linearly independent vectors μ_1 and μ_2 that obey these symplectic orthonormality properties. Following the approach discussed in chapter 2, we now specify the ladder operators in the $z = 0$ input plane by the expressions

$$\hat{a}_p(0) = \sqrt{\frac{k}{2}}\mu_p^{\mathrm{T}} G\,\hat{\boldsymbol{z}} \qquad \text{and} \qquad \hat{a}_p^\dagger(0) = \sqrt{\frac{k}{2}}\mu_p^\dagger G\,\hat{\boldsymbol{z}}\ . \tag{6.26}$$

The symplectic orthonormality properties (6.23) and (6.25) of the eigenvectors μ_p and μ_p^* ensure that the ladder operators in the input plane obey bosonic commutation relations (6.21). From the general transformation property of the ladder operators (6.20), combined with the Ehrenfest relation (6.14) between $\hat{U}$ and M, one may show that the ladder operators in other transverse planes z are given by the same expressions (6.26) when μ_p is replaced by $\mu_p(z) = M(z)\mu_p$. Here, $M(z)$ is the ray matrix that describes the transformation of ray from the $z = 0$ input plane to the transverse plane z. It can be constructed by multiplying the ray matrices that describe the optical elements of which the set-up consists and free propagation between them in proper order. The fact that the properties (6.23) and (6.25) are conserved under symplectic transformations $\in Sp(4, \mathbb{R})$ confirms that the ladder operators remain bosonic in all transverse planes of the set-up.

Since the modes are fully characterized by the choice of two complex vectors μ_p, we expect that the expectation values of physically relevant operators can be expressed in terms of these vectors. The average transverse position and momentum of the beam trace out the path of a ray. This implies that the expectation values $\langle u_{nm}|\hat{\rho}|u_{nm}\rangle$ and $\langle u_{nm}|\hat{\theta}|u_{nm}\rangle$ vanish. In appendix 6.B we prove, however, that the expectation values of the generators $\hat{T}_j$ are, in general, different from zero and can be expressed as

$$\langle u_{nm}|\hat{T}_j|u_{nm}\rangle = \frac{1}{2}\left\{\left(n + \frac{1}{2}\right)\mu_1^\dagger GJ_j\mu_1 + \left(m + \frac{1}{2}\right)\mu_2^\dagger GJ_j\mu_2\right\}\ . \tag{6.27}$$

This result generalizes the expression (2.82) of the orbital angular momentum in twisted cavity modes that we derived in chapter 2. It is noteworthy that these properties of the modes are fully characterized by the generators J_j and the complex ray vectors μ_p, which both have a clear geometric-optical significance.

Finally, it is worthwhile to notice that the results of this section remain valid when the number of (transverse) dimensions is different. In particular, the same method gives explicit expressions for complete orthogonal sets of time-dependent wave functions that solve the Schrödinger equation of a free particle in three-dimensional space.

6.3.2 Degrees of freedom in fixing a set of modes

We have shown that there is a one-to-one correspondence between the defining properties of a ray matrix, i.e., that it is real and obeys the identity (6.15), and the properties (6.23) and (6.25) of the complex eigenvectors μ_p that ensure that the ladder operators (6.26) are bosonic. This implies that all different basis sets of complex vectors μ_p that obey these identities must be related by symplectic transformations, i.e., each of these sets can be written as $\{M\mu_p\} \cup \{M\mu_p^*\}$, with $M \in Sp(4,\mathbb{R})$ and $\{\mu_p\} \cup \{\mu_p^*\}$ the set of complex eigenvectors of a specific ray matrix $M_0 \in Sp(4,\mathbb{R})$. Since $\{M\mu_p\} \cup \{M\mu_p^*\}$ constitutes the set of eigenvectors of MM_0M^{-1}, it follows that the freedom in choosing a set of complex vectors that generate two pairs of bosonic ladder operators (6.26) is equivalent to the freedom of choosing a ray matrix $M \in Sp(4,\mathbb{R})$. As a result, the number of independent parameters associated with this choice is equal to the number of generators of $Sp(4,\mathbb{R})$, which is ten. In order to give a physical interpretation of these degrees of freedom, we follow the characterization discussed in chapter 5 and decompose the complex ray vectors into two-dimensional subvectors so that $\mu_p^{\mathrm{T}}(z) = (r_p^{\mathrm{T}}(z), t_p^{\mathrm{T}}(z))$. In terms of these subvectors, the ladder operators take the following form

$$\hat{a}_p(z) = \sqrt{\frac{k}{2}}(r_p^{\mathrm{T}}(z)\hat{\theta} - t_p^{\mathrm{T}}(z)\hat{\rho}) \qquad \text{and} \qquad \hat{a}_p^{\dagger}(z) = \sqrt{\frac{k}{2}}(r_p^{\dagger}(z)\hat{\theta} - t_p^{\dagger}(z)\hat{\rho}) \,, \tag{6.28}$$

where $p = 1, 2$. An explicit expression of the Gaussian fundamental mode can be given if we combine the two-dimensional column vectors r_p and t_p into

$$\mathsf{R}(z) = (r_1(z), r_2(z)) \qquad \text{and} \qquad \mathsf{T}(z) = (t_1(z), t_2(z)) \,. \tag{6.29}$$

The objects R and T take the form of 2×2 matrices, but since r_p and t_p are transverse vectors, R and T do not transform as such under ray-space transformations $\in Sp(4,\mathbb{R})$ nor under transformations on the transverse plane. The symplectic orthonormality properties (6.23) and (6.25) of the vectors μ_p can be expressed as

$$\mathsf{R}^{\dagger}(z)\mathsf{T}(z) - \mathsf{T}^{\dagger}(z)\mathsf{R}(z) = 2i\mathbb{1} \qquad \text{and} \qquad \mathsf{R}^{\mathrm{T}}(z)\mathsf{T}(z) - \mathsf{T}^{\mathrm{T}}(z)\mathsf{R}(z) = 0 \,, \tag{6.30}$$

and hold for all values of z. Now, the fundamental transverse mode in plane z can be written as

$$u_{00}(\rho, z) = \sqrt{\frac{k}{\pi \det \mathsf{R}(z)}} \exp\left(-\frac{k\rho^{\mathrm{T}}\mathsf{S}(z)\rho}{2}\right) \,, \tag{6.31}$$

where $\mathsf{S} = -i\mathsf{T}\mathsf{R}^{-1}$. As opposed to R and T, S is a 2×2 matrix in the transverse plane and transforms accordingly. It can be checked directly that acting upon $|u_{00}(z)\rangle$ with the lowering operators $\hat{a}_1(z)$ and $\hat{a}_2(z)$ gives zero. The fundamental mode (6.31) is properly normalized and has been constructed such that it solves the paraxial wave equation (6.3) under free propagation. Moreover, one may check that it transforms properly under the transformations of optical elements. The second relation in equation (6.30) guarantees that S is symmetric. This is obvious when we multiply the relation from the left with $(\mathsf{R}^{\mathrm{T}})^{-1}$, and from the right with R^{-1}. The real and imaginary parts S_{r} and S_{i} of S respectively characterize the astigmatism of the intensity and phase patterns. The real part can be written as $\mathsf{S}_{\mathrm{r}} = (-i\mathsf{T}\mathsf{R}^{-1} + i(\mathsf{R}^{\dagger})^{-1}\mathsf{T}^{\dagger})/2$. With the first relation in equation (6.30) this shows that $\mathsf{R}\mathsf{S}_{\mathrm{r}}\mathsf{R}^{\dagger} = 1$. This leads to the identity

$$\mathsf{R}\mathsf{R}^{\dagger} = \mathsf{S}_{\mathrm{r}}^{-1} , \tag{6.32}$$

which shows that S_{r} is positive definite. As a result, the curves of constant intensity in the transverse plane are ellipses. Moreover, the fundamental mode is square-integrable. Depending on the sign of $\det \mathsf{S}_{\mathrm{i}}(z)$ the curves of constant phase in the transverse plane are ellipses, hyperbolas or parallel straight lines. Under free propagation, S is a slowly varying smooth function of z. Optical elements, on the other hand, may instantaneously modify the astigmatism. The astigmatism of both the intensity and the phase patterns is characterized by two widths in mutually perpendicular directions and one angle that specifies the orientation of the curves of constant intensity or phase. The total number of degrees of freedom that specify the astigmatism, and, thereby, the matrix symmetric S, is thus equal to six.

Two of the remaining four degrees of freedom are related to the nature and orientation of the higher-order mode patterns. From equation (6.32), we find that R can be expressed as $\mathsf{S}_{\mathrm{r}}^{-1/2}\sigma^{\mathrm{T}}$, where σ is a unitary 2×2 matrix. Notice that S_{r} is real and positive so that $\mathsf{S}_{\mathrm{r}}^{-1/2}$ is well-defined. It is illuminating to rewrite the complex ray vectors μ_1 and μ_2 as

$$\begin{pmatrix} \mu_1 & \mu_2 \end{pmatrix} = \begin{pmatrix} \mathsf{R} \\ \mathsf{T} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\mathsf{S}_{\mathrm{i}} & 1 \end{pmatrix} \begin{pmatrix} \mathsf{S}_{\mathrm{r}}^{-1/2} & 0 \\ 0 & \mathsf{S}_{\mathrm{r}}^{1/2} \end{pmatrix} \begin{pmatrix} \sigma^{\mathrm{T}} & 0 \\ 0 & \sigma^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} \tilde{\mu}_x & \tilde{\mu}_y \end{pmatrix} , \tag{6.33}$$

where $\tilde{\mu}_x = (1, 0, i, 0)^{\mathrm{T}}$ and $\tilde{\mu}_y = (0, 1, 0, i)^{\mathrm{T}}$ are the complex ray vectors that correspond to the ladder operators that generate the stationary states of an isotropic harmonic oscillator in two dimensions. The first matrix in the second right-hand-side of this expression (6.33) is the ray matrix that describes the transformation of a thin astigmatic lens. It imposes the elliptical or hyperbolic wave front of the optical modes on the harmonic oscillator functions. The second matrix has the form of the ray matrix that describes the deformation of a mode due to refraction. It rescales the modes along two mutually perpendicular transverse directions and accounts for the astigmatism of the intensity patterns. The third matrix involves the complex matrix σ and obeys the generalization of equation (6.15) to complex matrices. Since it is complex, however, it is not a ray matrix $\in Sp(4, \mathbb{R})$. In order to clarify its significance, we rewrite equation (6.33) in terms of the ladder operators, which are conveniently combined in the vector operator $(\hat{a}_1, \hat{a}_2)^{\mathrm{T}}$. By using the definition if the ladder operators (6.26) and the

Ehrenfest relation (6.14), the transformation in equation (6.33) can be expressed as

$$\begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} = \sqrt{\frac{k}{2}}(\mathsf{R}^{\mathrm{T}}\hat{\theta} - \mathsf{T}^{\mathrm{T}}\hat{\rho}) =$$
$$-i\sqrt{\frac{k}{2}}\sigma \exp\left(-\frac{ik\rho^{\mathrm{T}}\mathsf{S}_i\rho}{2}\right)\left(\mathsf{S}_r^{1/2}\hat{\rho} + i\mathsf{S}_r^{-1/2}\hat{\theta}\right)\exp\left(\frac{ik\rho^{\mathrm{T}}\mathsf{S}_i\rho}{2}\right) . \qquad (6.34)$$

The linear combination of the position and momentum operators between the brackets takes the form of the lowering-operator vector for an isotropic harmonic oscillator in two dimensions. Again, the 2×2 matrix S_r accounts for the astigmatism of the intensity patterns by rescaling the ladder operators and, therefore, the modes they generate. The exponential terms take the form of the mode-space transformation for a thin astigmatic lens and impose the curved wave fronts. From right to left, the lowering operators (6.34) as well as the corresponding raising operators, first remove the curved wave front, then modify the mode patterns and eventually restore the wave front again. The 2×2 matrix σ is a unitary transformation in the space of the lowering operators $\hat{a}_1$ and $\hat{a}_2$ and transforms accordingly. It arises from the $U(2)$ symmetry of the isotropic harmonic oscillator in two dimensions and accounts for the fact that any, properly normalized, linear combination of bosonic lowering operators yields another bosonic lowering operator. Up to overall phases, to which we come in a moment, this transformation can be parameterized as $\hat{a}_1 \to \eta_1\hat{a}_1 + \eta_2\hat{a}_2$ and $\hat{a}_2 \to -\eta_1^*\hat{a}_1 + \eta_2^*\hat{a}_2$ with $|\eta_1|^2 + |\eta_2|^2 = 1$. The two obvious degrees of freedom that are associated with the spinor $\eta = (\eta_1, \eta_2)^{\mathrm{T}}$ are the relative amplitude and the relative phase of its components. Analogous to the Poincaré sphere for polarization states (or the Bloch sphere for spin-1/2 states), they can be mapped onto a sphere. For reasons that will become clear, this sphere is called the Hermite-Laguerre sphere [17]. Since η_1 and η_2 are spinor components in a linear rather than a circular basis, this mapping takes the following form

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{\frac{i\varphi}{2}}\cos\frac{\vartheta}{2} + e^{-i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \\ -ie^{\frac{i\varphi}{2}}\cos\frac{\vartheta}{2} + ie^{-i\frac{\varphi}{2}}\sin\frac{\vartheta}{2} \end{pmatrix} , \qquad (6.35)$$

where ϑ and φ are the polar and azimuthal angles on the sphere. The mapping is such that the north pole ($\vartheta = 0$) corresponds to ladder operators that generate astigmatic Laguerre-Gaussian modes with positive helicity. The south pole ($\vartheta = \pi$) corresponds to Laguerre-Gaussian modes with the opposite helicity while the equator ($\vartheta = \pi/2$) corresponds to Hermite-Gaussian modes. Other values of the polar angle ϑ correspond to generalized Gaussian modes [44]. The azimuth angle φ determines the transverse orientation of the higher-order mode patterns. Since paraxial optical modes are invariant under rotations over π in the transverse plane, the mapping in equation (6.35) is such that a rotation over φ on the sphere corresponds to a rotation of the mode pattern over $\phi = \varphi/2$.

The unitary matrix that describes the ladder operator transformation corresponding to the spinor η is constructed as

$$\sigma_0(\eta) = \begin{pmatrix} \eta_1 & \eta_2 \\ -\eta_2^* & \eta_1^* \end{pmatrix} , \qquad (6.36)$$

where the second row is fixed up to a phase factor by the requirement that σ_0 must be unitary. With this convention, the two rows of sigma correspond to antipodal points on the Hermite-Laguerre sphere. Completely fixing the matrix $\sigma \in U(2)$, however, requires four independent degrees of freedom. The remaining two, which are not incorporated in η, are phase factors. Any matrix $\sigma \in U(2)$ can be written as

$$\sigma = \begin{pmatrix} e^{i\chi_1} & 0 \\ 0 & e^{i\chi_2} \end{pmatrix} \sigma_0(\eta) \ . \tag{6.37}$$

The phase factors $\exp(i\chi_p)$ correspond to overall phases of the vectors μ_p and, therefore, of the ladder operators (6.26). The vectors μ_p can be written as

$$\mu_p = e^{i\chi_p} \nu_p(\mathsf{S}, \eta) \ , \tag{6.38}$$

where $p = 1, 2$ and $\nu_p(\mathsf{S}, \eta)$ is completely determined by S and η according equation (6.33), σ being replaced by $\sigma_0(\eta)$. Although the vectors ν_1 and ν_2 obey symplectic orthonormality conditions (6.23) and are, therefore, not independent, the phases χ_1 and χ_2 are independent. From equation (6.37) and the fact that $\mathsf{R} = \mathsf{S}_\mathrm{r}^{-1/2} \sigma^\mathrm{T}$ it is clear that the argument of $\det \mathsf{R}$ is equal to $\chi_1 + \chi_2$ so that the overall phase of the fundamental mode (6.31) is given by $-(\chi_1 + \chi_2)/2$. The overall phases of the two raising operators are respectively $-\chi_1$ and $-\chi_2$, so that the phase factors in the higher order modes $|u_{nm}(z)\rangle$ are given by $\exp(-i\chi_{nm})$ with

$$\chi_{nm} = \left(n + \frac{1}{2}\right)\chi_1 + \left(m + \frac{1}{2}\right)\chi_2 \ . \tag{6.39}$$

In a single transverse plane, such overall phase factors do not modify the physical properties of the mode pattern. The evolution of these phase under propagation and optical elements, however, can be measured interferometrically.

The astigmatism of the modes, as characterized by the 2×2 matrix S, can be modified in any desired way by appropriate combinations of the optical elements that we have discussed in section 6.2. As will be discussed in section 6.5, the degrees of freedom associated with the spinor η can be manipulated by mode convertors and image rotators. Although we shall see that variation of the phase factors $\exp(i\chi_p)$ is, in general, unavoidable when the other parameters are modified, we show here that it is possible to construct a ray matrix $\in Sp(4, \mathbb{R})$ that solely changes these phase factors. Such a ray matrix is defined by the requirement that

$$M_\chi(\{\chi_p\}) \begin{pmatrix} \mu_1 & \mu_2 & \mu_1^* & \mu_2^* \end{pmatrix} = \begin{pmatrix} e^{i\chi_1}\mu_1 & e^{i\chi_2}\mu_2 & e^{-i\chi_1}\mu_1^* & e^{-i\chi_2}\mu_2^* \end{pmatrix} \ , \tag{6.40}$$

so that the vectors μ_p and μ_p^* are eigenvectors of M_χ. The corresponding eigenvalues are unitary. In terms of R and T this relation can be expressed as

$$M_\chi(\{\chi_p\}) \begin{pmatrix} \mathsf{R} & \mathsf{R}^* \\ \mathsf{T} & \mathsf{T}^* \end{pmatrix} = \begin{pmatrix} \mathsf{R} & \mathsf{R}^* \\ \mathsf{T} & \mathsf{T}^* \end{pmatrix} \begin{pmatrix} \mathsf{C} & 0 \\ 0 & \mathsf{C}^* \end{pmatrix} , \tag{6.41}$$

where

$$\mathsf{C} = \begin{pmatrix} e^{i\chi_1} & 0 \\ 0 & e^{i\chi_2} \end{pmatrix} . \tag{6.42}$$

By using that

$$\begin{pmatrix} \mathsf{R} & \mathsf{R}^* \\ \mathsf{T} & \mathsf{T}^* \end{pmatrix}^{-1} = \frac{1}{2i} \begin{pmatrix} -\mathsf{T}^\dagger & \mathsf{R}^\dagger \\ \mathsf{T}^\mathrm{T} & -\mathsf{R}^\mathrm{T} \end{pmatrix}, \tag{6.43}$$

which follows directly from the identities in equation (6.30), we find that M_χ can be expressed as

$$M_\chi(\{\chi_p\}) = \frac{1}{2i} \begin{pmatrix} \mathsf{R} & \mathsf{R}^* \\ \mathsf{T} & \mathsf{T}^* \end{pmatrix} \begin{pmatrix} \mathsf{C} & 0 \\ 0 & \mathsf{C}^* \end{pmatrix} \begin{pmatrix} -\mathsf{T}^\dagger & \mathsf{R}^\dagger \\ \mathsf{T}^\mathrm{T} & -\mathsf{R}^\mathrm{T} \end{pmatrix} = \\ \frac{1}{2i} \begin{pmatrix} -\mathsf{R}\mathsf{C}\mathsf{T}^\dagger + \mathsf{R}^*\mathsf{C}^*\mathsf{T}^\mathrm{T} & \mathsf{R}\mathsf{C}\mathsf{R}^\dagger - \mathsf{R}^*\mathsf{C}^*\mathsf{R}^\mathrm{T} \\ -\mathsf{T}\mathsf{C}\mathsf{T}^\dagger + \mathsf{T}^*\mathsf{C}^*\mathsf{T}^\mathrm{T} & \mathsf{T}\mathsf{C}\mathsf{R}^\dagger - \mathsf{T}^*\mathsf{C}^*\mathsf{R}^\mathrm{T} \end{pmatrix} \tag{6.44}$$

This ray matrix adds overall phases $\exp(\pm i\chi_p)$ to the eigenvectors μ_p and μ_p^*. It is real and one may check that it obeys the identity (6.15) so that it is a physical ray matrix $\in Sp(4, \mathbb{R})$.

In this section, we have argued that the number of degrees of freedom associated with the choice of two pairs of ladder operators that generate a basis set of modes in a transverse plane z is equal to number of generators of $Sp(4, \mathbb{R})$, which is ten. We have shown that six of those are related to the astigmatism of the modes as characterized by a the complex and symmetric 2×2 matrix S. Two of the other four are angles on the Hermite-Laguerre sphere that characterize a spinor η, which determines the nature and orientation of the higher-order modes. The remaining two are overall phases of the ladder operators. All these degrees of freedom can be manipulated in any desired way by choosing a suitable ray matrix $\in Sp(4, \mathbb{R})$.

6.3.3 Gouy phase

In the limiting case of non-astigmatic modes that propagate through an isotropic optical system the 2×2 matrix S is a symmetric matrix with degenerate eigenvalues so that it can be considered a scalar $s = s_\mathrm{r} + is_\mathrm{i}$. If we choose $\sigma_0 = 1$, the higher-order modes are Hermite-Gaussian. In that case, the complex ray vectors are given by $\mu_1 = (r, 0, t, 0)^\mathrm{T}$ and $\mu_2 = (0, r, 0, t)^\mathrm{T}$, with $r, t \in \mathbb{C}$. The symplectic normalization condition (6.23) implies that $r^*t - t^*r = 2i$. The real part s_r of $s = -it/r$ determines the beam width $w = \sqrt{2/(ks_r)}$ of the fundamental mode while the imaginary part s_i fixes the radius of curvature of its wave fronts according to $R = 1/s_\mathrm{i}$. Under free propagation over a distance z, the vectors μ_1 and μ_2 transform according to

$$\mu_1(z) = \begin{pmatrix} r + zt \\ 0 \\ t \\ 0 \end{pmatrix} \quad \text{and} \quad \mu_2(z) = \begin{pmatrix} 0 \\ r + zt \\ 0 \\ t \end{pmatrix}. \tag{6.45}$$

The parameters r, t and s remain scalar and free propagation does not introduce an overall phase difference between μ_1 and μ_2 so that η, or, equivalently σ_0, is independent of z. Without loss of generality we can choose $z = 0$ to coincide with the focal plane of the mode, which implies that $s \in \mathbb{R}$ so that $r^* t = -t^* r = i$. Since s_r, and, therefore, $\mathsf{R} = \sigma_0 s_\mathrm{r}$ cannot pick up a phase, we find that

$$\chi(z) - \chi(0) = \arg\left(\frac{r + zt}{r}\right) = \arctan\left(\frac{tz}{r}\right) = \arctan\left(\frac{z}{z_\mathrm{R}}\right) , \tag{6.46}$$

where $z_\mathrm{R} = ir/t$ is the Rayleigh range. This is the well-known Gouy phase for a Gaussian mode [12]. Since the vectors μ_1 and μ_2 pick up an overall phase $\chi(z)$, the raising operators pick up a phase $-\chi(z)$. The phase shift of the higher-order modes (6.22) is then given by $\exp(-i(n + m + 1)\chi)$ and depends on the total mode number $N = n + m$ only. As a result of this degeneracy, the same expression holds in the non-astigmatic case with $\sigma_0 \neq 1$. In that case, it is still true that the components of η are independent of z.

Generalization to astigmatic modes is straightforward only if the modes have simple astigmatism and if the orientation of the higher-order mode patterns is aligned along the astigmatism of the fundamental mode. In that case, the vectors μ_p pick up different Gouy phases and the components of η are independent of z. As will be discussed in section 6.5, this is not true in the case of non-astigmatic modes that propagate through an optical set-up with simple astigmatism. In the more general case of modes with general astigmatism that propagate through an arbitrary set-up of paraxial optical elements, the z dependence of S depends on η and vice versa [17]. In this case no simple expressions of the Gouy phases can be derived. The phase in equation (6.39) may be viewed as the ultimate generalization of the Gouy phase within paraxial wave optics.

6.4 The geometric interpretation of the variation of the phases χ_{nm}

6.4.1 Evolution of the phases χ_{nm}

In this section we show that variation of the phase differences χ_p between μ_p and ν_p (6.38) is, in general, unavoidable under (a sequence of) mode transformations that modify the degrees of freedom associated with S and η. From the discussion in the previous section it is clear that the generalized Gouy phases were defined such that they vary only under transformations that involve free propagation. However, for later purposes, it is convenient to formulate the description of mode transformations that give rise to phase shifts in a slightly more general way.

Suppose that the unitary state-space transformation that describes (a part of) a trajectory through the parameter space is given by $\hat{U}(\zeta) = \exp(-i\hat{T}\zeta)$, where $\hat{T}$ is a (linear combination of the) generator(s) defined in equation (6.9) and ζ is a real parameter that parameterizes the trajectory. In this case, the ζ dependent ladder operators (6.20) obey the anti-Heisenberg equation of motion

$$\left[\hat{a}^{(\dagger)}(\zeta), \hat{T}\right] = -i\frac{\partial \hat{a}^{(\dagger)}}{\partial \zeta} . \tag{6.47}$$

In terms of the complex ray vectors $\mu_p(\zeta)$ and the ray matrix $M(\zeta) = \exp(-J\zeta)$ that corresponds to $\hat{U}(\zeta)$ according to relation (6.14), this equation of motion takes the form of a symplectic Schrödinger equation and can be expressed as

$$\frac{\partial \mu_p}{\partial \zeta} = -J\mu_p(\zeta) \ . \tag{6.48}$$

Substitution of $\mu_p(\zeta) = \exp(i\chi_p)\nu_p(\zeta)$ yields after dividing by $\exp(i\chi_p)$

$$i\frac{\partial \chi_p}{\partial \zeta}\nu_p(\zeta) + \frac{\partial \nu_p}{\partial \zeta} = -J\nu_p(\zeta) \ . \tag{6.49}$$

By multiplying from the left with $\nu_p^\dagger G$, using the normalization condition $\nu_p^\dagger G \nu_p = 2i$ and rearranging the terms we find that

$$\frac{\partial \chi_p}{\partial \zeta} = \frac{1}{2}\left\{\nu_p^\dagger G J \nu_p + \nu_p^\dagger G \frac{\partial \nu_p}{\partial \zeta}\right\} \ . \tag{6.50}$$

The generator J represents a conserved quantity. Hence, the first term between the curly brackets does not depend on the parameter ζ and the above equation (6.50) can be integrated to obtain

$$\chi_p(\zeta) = \frac{1}{2}\left\{\left(\nu_p^\dagger G J \nu_p\right)\zeta + \int_0^\zeta d\zeta' \nu_p^\dagger G \frac{\partial \nu_p}{\partial \zeta'}\right\} \ . \tag{6.51}$$

The first term between the curly brackets constitutes a dynamical contribution to the phase shift and arises from the fact that J corresponds to a constant of motion. The second term, on the other hand, relates to the geometry of the complex ray space and is the natural generalization of Berry's geometric phase to this case. In the next section, we derive an equivalent expression from which the geometric significance of the phase shifts (6.51) is more obvious.

6.4.2 Analogy with the Aharonov-Bohm effect

In quantum mechanics, it is well-known that the coupling of a particle with charge q to the magnetic vector potential $\mathbf{A}(\mathbf{r})$ gives rise to a measurable phase shift $(q/\hbar)\int_C \mathbf{A}\cdot d\mathbf{r}$ of the wave function when the particle moves along a trajectory $C = \mathbf{r}(t)$. This effect occurs even when the magnetic field $\mathbf{B} = \nabla \times \mathbf{A}$ vanishes everywhere along the trajectory and is known as the Aharonov-Bohm effect [98].

The physical properties that are associated with the wave function that describes a particle in quantum mechanics are not affected by the transformation $\psi(\mathbf{r}, t) \to \exp(i\phi(\mathbf{r}))\psi(\mathbf{r}, t)$. The Schrödinger equation is obviously not invariant under this local $U(1)$ gauge transformation. When gauge invariance of the Schrödinger equation is imposed, the vector potential $\mathbf{A}(\mathbf{r})$ arises as the corresponding gauge field. In this picture, the Aharonov-Bohm phase is due to the coupling to a gauge field, the conserved charge q being the coupling constant. As such it is a direct consequence of the $U(1)$ gauge invariance of quantum electrodynamics.

In this section, we point out an analogy between the generalized Gouy phase and the Aharonov-Bohm effect. This gives some new insights in the nature and origin of this geometric phase and allows us to derive an expression of the phase (6.51) from which its origin in the geometry of the underlying parameter space is obvious. It is convenient to combine the parameters that characterize the eight degrees of freedom that are associated with the matrix S and the spinor η into a vector $\vec{\mathcal{R}} = (\mathcal{R}_1, \mathcal{R}_2, ...)^{\mathrm{T}}$. The corresponding differential operator, which is a vector in the eight-dimensional parameter space, is defined as $\vec{\nabla}_{\vec{\mathcal{R}}} = (\partial/\partial\mathcal{R}_1, \partial/\partial\mathcal{R}_2, ...)^{\mathrm{T}}$.

The physical properties, for example those in equation (6.27), of the transverse mode fields (6.22), which are generated by the ladder operators constructed from the vectors μ_p, are not affected by transformations of the type

$$\mu_p \to e^{i\psi_p(\vec{R})}\mu_p \ , \tag{6.52}$$

where $p = 1, 2$. This symmetry property can be thought of as local $U(1) \otimes U(1)$ gauge invariance. The ray matrix $\in Sp(4, \mathbb{R})$ that describes such gauge transformations (6.52) figures in equation (6.44). As shown in appendix 6.C, the two corresponding real generators J_{χ_p} can be constructed from the eigenvectors μ_p. The vector μ_1 is an eigenvector of J_{χ_1} with eigenvalue $-i$. Since J_{χ_1} is real, the complex conjugate vector μ_1^* is an eigenvector of J_{χ_p} with eigenvalue i. Moreover, $J_{\chi_1}\mu_2 = J_{\chi_1}\mu_2^* = 0$. Similarly, μ_2 and μ_2^* are eigenvectors of J_{χ_2} with eigenvalues $-i$ and i and $J_{\chi_2}\mu_1 = J_{\chi_2}\mu_1^* = 0$. Since invariance under the gauge transformation (6.52) is a local and continuous symmetry, it gives rise to conserved Noether charges. The gauge transformations are generated by two different generators, hence there are two Noether charges, which can be expressed as $v_p^\dagger G J_{\chi_p} v_p/2 = 1$, where the factor $1/2$ arises from the fact that a symplectic vector space is a joint space of position and momentum and where we have used that $J_{\chi_p} v_p = -i$ and $v_p^\dagger G v_p = 2i$. In appendix 6.C, we prove that the corresponding state-space generators $\hat{T}_{\chi_p}$ can be expressed as $\left(\hat{a}_p^\dagger \hat{a}_p + \hat{a}_p \hat{a}_p^\dagger\right)/2$ so that the charges of a mode (6.22) are given by $\langle u_{nm}|\hat{T}_{\chi_1}|u_{nm}\rangle = (n + 1/2)$ and $\langle u_{nm}|\hat{T}_{\chi_2}|u_{nm}\rangle = (m + 1/2)$. Since the gauge transformation in equation (6.44) is constructed from the eigenvectors μ_p, it varies throughout the parameters space. As a result, the generators $\hat{T}_{\chi_p}$ can be constructed only locally and vary through the parameter space according to the ladder-operator transformation in equation (6.20). However, since the modes also vary, it follows that Noether charges $(n + 1/2)$ and $(m + 1/2)$ of the modes $|u_{nm}\rangle$ are globally conserved.

In terms of $\vec{\mathcal{R}}$ and $\vec{\nabla}_{\vec{\mathcal{R}}}$, the equations of motion of the vectors μ_p (6.48) can be rewritten as

$$\left(\vec{\nabla}_{\vec{\mathcal{R}}}\mu_p\right) \cdot \frac{\partial\vec{\mathcal{R}}}{\partial\zeta} = -J\mu_p \ . \tag{6.53}$$

These equations are obviously not invariant under the gauge transformations (6.52). Imposing gauge invariance yields the modified equations of motion

$$\left(\left(\vec{\nabla}_{\vec{\mathcal{R}}} + i\vec{A}_p\right) v_p\right) \cdot \frac{\partial\vec{\mathcal{R}}}{\partial\zeta} = -Jv_p \ , \tag{6.54}$$

where the gauge fields $\vec{A}_p$ are vector fields in the parameter space of $\vec{\mathcal{R}}$ that are defined by their transformation property under the gauge transformations (6.52)

$$\vec{A}_p \to \vec{A}_p - \vec{\nabla}_{\vec{\mathcal{R}}}\psi_p \ . \tag{6.55}$$

With these transformation properties, the equation of motion (6.54) is manifestly invariant under the gauge transformations (6.52). The general solution of this equation (6.54) can be expressed as

$$\nu_p = \mu_p e^{-i \int_C \vec{A}_p \cdot d\vec{\mathcal{R}}} \,, \tag{6.56}$$

where C is a trajectory $\vec{\mathcal{R}}(\zeta)$ and μ_p solves the equation of motion without the gauge field (6.53). In full analogy with the Aharonov-Bohm effect, this shows that the phase difference between μ_p and ν_p is due to the fact that the latter is coupled to the gauge field $\vec{A}_p$. Since we have defined the vectors μ_p so as to include the appropriate geometric-phase factor while they are not coupled to the gauge fields, the coupling of ν_p to the gauge fields removes the geometric phase rather than introducing it. The geometric origin of the phases is evident in that they are determined only by the trajectory C and do not depend on the velocity $\partial\vec{\mathcal{R}}/\partial\zeta$. By using equation (6.54) they can be expressed as

$$\chi_p = \int_C \vec{A}_p \cdot d\vec{\mathcal{R}} = \frac{1}{2}\int_0^\zeta d\zeta' \left\{ \nu_p G J \nu_p + \left(\nu_p^\dagger G \vec{\nabla}_{\mathcal{R}} \nu_p\right) \cdot \frac{\partial \vec{R}}{\partial \zeta} \right\} \,, \tag{6.57}$$

which is in obvious agreement with equation (6.51).

In analogy with the Aharonov-Bohm effect, the Noether charges $\nu_p^\dagger G J_{\chi_p} \nu_p / 2 = 1$ determine the strength of the coupling of the vectors μ_p to the gauge fields $\vec{A}_p$. This is consistent with the fact that the vectors ν_p pick up phases χ_p. The Noether charges of the modes (6.22), however, are equal to $n + 1/2$ and $m + 1/2$ and depend on the mode numbers n and m. As a result, the modes $|u_{nm}\rangle$ couple differently to the (corresponding state-space) gauge fields and, therefore, experience different phase shifts. This is in obvious agreement with equation (6.39).

The Noether currents $(\nu_p^\dagger G J_{\chi_p} \nu_p/2)\partial\vec{\mathcal{R}}/\partial\zeta = \partial\vec{\mathcal{R}}/\partial\zeta$ are uniform throughout the parameters space of $\vec{\mathcal{R}}$. It follows that the "physical" fields or Berry curvatures $F_{\alpha\beta} = \partial_\alpha (A_p)_\beta - \partial_\beta (A_p)_\alpha$, where the indices α and β run over the parameter-space vector components, are constant so that the gauge fields $\vec{A}_p(\vec{\mathcal{R}})$ cannot possess any non-trivial dynamics. Attributing the generalized Gouy phases χ_p to coupling to gauge fields $\vec{A}_p$, which do not have any dynamical properties in their own rights, may seem a bit tautological. On the other hand, the analysis discussed here shows that the structure that underlies the generalized Gouy phase shifts (6.39) is that of a gauge theory. In this picture, the appearance of phase shifts under propagation through an optical set-up is the unavoidable consequence of the $U(1)\times U(1)$ gauge invariance of the dynamics of paraxial optical modes, or, equivalently, of the fact that the mode charges $n + 1/2$ and $m + 1/2$ are conserved under state-space transformations $\in Mp(4, \mathbb{R})$.

The connection between the gauge invariance as discussed here does not depend on the specific structure of the symplectic vector space. Our results as well as the Aharonov-Bohm effect indicate that there is a more general connection between local gauge invariance and the appearance of geometric phases, see, for instance, reference [99].

6.5 Geometric phases for non-astigmatic modes

6.5.1 Ray matrices on the Hermite-Laguerre sphere

A particulary interesting limiting case of the geometric phases that we discuss in this chapter, are the phase shifts due to mode conversions on the Hermite-Laguerre sphere, each point on which characterizes a basis set of higher-order modes. We focus on non-astigmatic modes in their focal planes so that S can be considered a real scalar $s \in \mathbb{R}$. We shall construct ray matrices and corresponding state-space operators that solely modify the degrees of freedom associated with the Hermite-laguerre sphere and study the geometric phases arising from such transformations.

The azimuth angle φ on the Hermite-Laguerre sphere specifies the orientation in the transverse plane of the set of higher-order modes. It can be modified by the rotation operator $\exp(-i\varphi\hat{s}_3)$, where $\hat{s}_3 = k(\hat{x}\hat{\vartheta}_y - \hat{y}\hat{\vartheta}_x)/2 = (\hat{T}_6 - \hat{T}_7)/2$ is the corresponding generator. The factor $1/2$ accounts for the fact that a rotation angle φ on the Hermite-Laguerre sphere corresponds to a $\phi = \varphi/2$ in the transverse plane. For reasons that will become clear, the ray matrix that describes a rotation in a plane parallel to an equatorial plane on the Hermite-Laguerre sphere is denoted M_3. It takes the following form

$$M_3(\varphi) = e^{-\varphi\Sigma_3} = \begin{pmatrix} \mathsf{P}\left(\frac{\varphi}{2}\right) & 0 \\ 0 & \mathsf{P}\left(\frac{\varphi}{2}\right) \end{pmatrix} = \begin{pmatrix} \cos\left(\frac{\varphi}{2}\right) & -\sin\left(\frac{\varphi}{2}\right) & 0 & 0 \\ \sin\left(\frac{\varphi}{2}\right) & \cos\left(\frac{\varphi}{2}\right) & 0 & 0 \\ 0 & 0 & \cos\left(\frac{\varphi}{2}\right) & -\sin\left(\frac{\varphi}{2}\right) \\ 0 & 0 & \cos\left(\frac{\varphi}{2}\right) & \sin\left(\frac{\varphi}{2}\right) \end{pmatrix}, \quad (6.58)$$

where $\Sigma_3 = (J_6 - J_7)/2$ is the corresponding ray-space generator and $\mathsf{P} \in SO(2)$ is a 2×2 rotation matrix. When this ray matrix acts on an arbitrary pair of complex ray vectors μ_p that obey the identities (6.23) and (6.25), the matrix S transforms according to $\mathsf{S} \to \mathsf{P}\mathsf{S}\mathsf{P}^{\mathrm{T}}$. In the present case of scalar S, this transformation only modifies the orientation of the mode patterns and does not affect S.

Another class of transformations that solely act upon the Hermite-Laguerre sphere are those that describe mode converters. Mode converters consists of a pair of astigmatic or cylindrical lenses [100]. The distance between the lenses and their radii of curvature are chosen such that the Gouy phase shift introduces a phase difference ϑ between the eigenvectors μ_1 and μ_2 of the transformation of the mode converter. If the input and output plane of the mode converter are chosen such that they respectively coincide with focal planes of the incident and outgoing modes and if the modes are matched to the mirrors so that S is scalar and equal to 1 in appropriate units determined by the mirrors, the eigenvectors of the mode converter are given by $\tilde{\mu}_1 = (1, 0, i, 0)^{\mathrm{T}}$ and $\tilde{\mu_2} = (0, 1, 0, i)^{\mathrm{T}}$ and their complex conjugates. The ray matrix that describes the transformation that introduces a phase difference ϑ between

$\tilde{\mu}_1$ and $\tilde{\mu}_2$ can then be constructed as

$$M_1(\vartheta) = e^{-\vartheta \Sigma_1} = \begin{pmatrix} \cos\left(\frac{\vartheta}{2}\right) & 0 & \sin\left(\frac{\vartheta}{2}\right) & 0 \\ 0 & \cos\left(\frac{\vartheta}{2}\right) & 0 & -\sin\left(\frac{\vartheta}{2}\right) \\ -\sin\left(\frac{\vartheta}{2}\right) & 0 & \cos\left(\frac{\vartheta}{2}\right) & 0 \\ 0 & \sin\left(\frac{\vartheta}{2}\right) & 0 & \cos\left(\frac{\vartheta}{2}\right) \end{pmatrix}, \tag{6.59}$$

where $\Sigma_1 = k(J_1 - J_2)/4 + (J_9 - J_{10})/(4k)$ is the corresponding generator. Again, one may prove easily that this transformation does not affect the astigmatic degrees of freedom if S is scalar and equal to 1. The corresponding state-space generator is given by $\hat{s}_1 = k(\hat{T}_1 - \hat{T}_2)/4 + (\hat{T}_9 - \hat{T}_{10})/(4k) = k(\hat{x}^2 - \hat{y}^2 + \hat{\vartheta}_x^2 - \hat{\vartheta}_y^2)/4$.

So far, we have constructed two of the three ray matrices that only modify the nature and orientation of the higher-order modes. The third corresponds to a mode converter in a basis that is rotated over $\pi/4$ in the transverse plane, or, equivalently over $\pi/2$ in the equatorial plane of the Hermite-Laguerre sphere. The ray matrix that describes such a transformation can be obtained as

$$M_2(\vartheta) = M_3(\pi/4) M_1(\vartheta) M_3^{-1}(\pi/4) = e^{-\vartheta \Sigma_2} = \begin{pmatrix} \cos\left(\frac{\vartheta}{2}\right) & 0 & 0 & \sin\left(\frac{\vartheta}{2}\right) \\ 0 & \cos\left(\frac{\vartheta}{2}\right) & \sin\left(\frac{\vartheta}{2}\right) & 0 \\ 0 & -\sin\left(\frac{\vartheta}{2}\right) & \cos\left(\frac{\vartheta}{2}\right) & 0 \\ -\sin\left(\frac{\vartheta}{2}\right) & 0 & 0 & \cos\left(\frac{\vartheta}{2}\right) \end{pmatrix}, \tag{6.60}$$

where $\Sigma_2 = kJ_3/2 + J_8/(2k)$ is the corresponding generator. The corresponding state-space generator is given by $\hat{s}_2 = k\hat{T}_3/2 + \hat{T}_8/(2k) = k(xy + \hat{\vartheta}_x \hat{\vartheta}_y)/2$. Since M_3 and M_1 do not affect the astigmatic degrees of freedom if S is scalar and equal to 1, it follows that the same is true for M_3.

By using the canonical commutation relations (6.4) and the definitions of the generators $\hat{s}_1$, $\hat{s}_2$ and $\hat{s}_3$ in terms of the canonical operators, one may easily show that the generators obey an $SU(2)$ algebra

$$[\hat{s}_1, \hat{s}_2] = i\hat{s}_3 \tag{6.61}$$

and cyclic permutations. The ray-space generators obey

$$[\Sigma_1, \Sigma_2] = \Sigma_3 \tag{6.62}$$

so that the matrices $i\Sigma_1$, $i\Sigma_2$ and $i\Sigma_3$ also constitute an $SU(2)$ algebra. Thus we have obtained both a metaplectic and a symplectic realization of an $SU(2)$ algebra. This proves the well-known fact that $SU(2)$ is a subgroup of $Mp(4)$ and, therefore, of $Sp(4, \mathbb{R})$.

6.5.2 Spinor transformations

Since the generators Σ_1, Σ_2 and Σ_3 constitute an $SU(2)$ algebra, an arbitrary pair of ray vectors μ_p on the Hermite-Laguerre sphere can be expressed as a linear combination of the eigenvectors of one of these generators. In analogy with section 6.3, where we introduced the components of η as the coefficients of the expansion of an arbitrary bosonic lowering operator in terms of the two lowering operators for a harmonic oscillator in two dimensions, we can write an arbitrary pair of complex ray vectors on the Hermite-Laguerre sphere as

$$\mu_1 = \eta_1 \tilde{\mu}_1 + \eta_2 \tilde{\mu}_2 \qquad \text{and} \qquad \mu_2 = -\eta_2^* \tilde{\mu}_1 + \eta_1^* \tilde{\mu}_2 \,, \tag{6.63}$$

where $\tilde{\mu}_1 = (1, 0, i, 0)^{\mathrm{T}}$ and $\tilde{\mu}_2 = (1, 0, i, 0)^{\mathrm{T}}$ are eigenvectors of Σ_1 with eigenvalues $-i$ and i respectively, (they are also eigenvectors of $J_{\mathrm{HO}} = k(J_1 + J_2)/4 + (J_9 + J_{10})/(4k)$ with degenerate eigenvalues i). Notice that, analogous to the construction in section 6.3 and the mapping in equation (6.35), the components of η are spinor components in a linear rather than in a circular basis. The symplectic orthogonality properties (6.23) and (6.25) require that the expansions in equation (6.63) do not involve the complex conjugate vectors μ_p^*. Moreover, they ensure normalization of η such that $|\eta_1|^2 + |\eta_2|^2 = 1$.

Since the ray matrices M_1, M_2 and M_3, as defined in equations (6.59), (6.60) and (6.58), only modify the degrees of freedom associated with η, these transformations can be expressed in the two-dimensional spinor space. In particular, the transformation described by M_3 (6.58) can be expressed as

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \to \begin{pmatrix} \cos\left(\frac{\varphi}{2}\right) & -\sin\left(\frac{\varphi}{2}\right) \\ \sin\left(\frac{\varphi}{2}\right) & \cos\left(\frac{\varphi}{2}\right) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = e^{-i\varphi\tau_3/2} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} , \tag{6.64}$$

where

$$\tau_3 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{6.65}$$

is the corresponding generator. Similarly, the transformations (6.59) and (6.60) of mode converters can be rewritten in terms of the spinor components as

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \to \begin{pmatrix} e^{\frac{i\vartheta}{2}} & 0 \\ 0 & e^{\frac{-i\vartheta}{2}} \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = e^{-i\vartheta\tau_1/2} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \tag{6.66}$$

and

$$\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} \to \begin{pmatrix} \cos\left(\frac{\vartheta}{2}\right) & i\sin\left(\frac{\vartheta}{2}\right) \\ i\sin\left(\frac{\vartheta}{2}\right) & \cos\left(\frac{\vartheta}{2}\right) \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = e^{-i\varphi\tau_2/2} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} , \tag{6.67}$$

where the corresponding spinor generators are given by

$$\tau_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{and} \qquad \tau_2 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} . \tag{6.68}$$

As a result of the fact that we have the defined the spinor components with respect to the eigenvectors of Σ_1 rather than of Σ_3, the spinor generators τ_1, τ_2 and τ_3 take the form of Pauli matrices in a rotated basis. They also form an $SU(2)$ algebra, i.e., $[\tau_1, \tau_2] = i\tau_3$ and cyclic permutations. This algebra is closed and the matrix transformations of η on the Hermite-Laguerre sphere that are generated by τ_1, τ_2 and τ_3 are analogues of Jones matrices in polarization optics.

Since the complex vectors $\tilde{\mu}_1$ and $\tilde{\mu}_2$ are eigenvectors of Σ_1 with eigenvalues $-i$ and i respectively, the spinor corresponding to a point (ϕ, θ) on the Hermite-Laguerre sphere can be expressed as

$$\eta(\phi, \theta) = (-i)^{1/2} e^{-i(\phi+\pi/2)\tau_3} e^{-i\theta\tau_1} e^{i\pi\tau_2/2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} , \tag{6.69}$$

where the factor $(-i)^{1/2} = \exp(-i\pi/4)$ is introduced to make this identity consistent with equation (6.35).

6.5.3 Mode-space transformations

In dimensionless notation, the lowering operators corresponding to the complex ray vectors $\tilde{\mu}_p$ can be expressed as

$$\hat{b}_x = \sqrt{\frac{k}{2}}(\hat{x} + i\hat{\vartheta}_x) \qquad \text{and} \qquad \hat{b}_y = \sqrt{\frac{k}{2}}(\hat{y} + i\hat{\vartheta}_y) . \tag{6.70}$$

The corresponding raising operators $\hat{b}_x^\dagger$ and $\hat{b}_y^\dagger$ generate the set of harmonic-oscillator states in two dimensions $|v_{nm}\rangle$ according to equation (6.22), the raising operators being replaced by the harmonic-oscillator raising operators. This set corresponds to $\eta = (1, 0)^{\mathrm{T}}$, which is on the equator of the Hermite-Laguerre sphere. The antipodal point $\eta = (0, 1)^{\mathrm{T}}$ gives rise to the same set of modes $|v_{mn}\rangle$, the mode indices being interchanged. The modes corresponding to an arbitrary point on the Hermite-Laguerre sphere can be expanded as

$$|u_{nm}(\eta)\rangle = \frac{1}{\sqrt{n!m!}} \left(\eta_1^* \hat{b}_x^\dagger + \eta_2^* \hat{b}_y^\dagger\right)^n \left(-\eta_2 \hat{b}_x^\dagger + \eta_1 \hat{b}_y^\dagger\right)^m |v_{00}\rangle . \tag{6.71}$$

By using that $[\hat{b}_x^\dagger, \hat{b}_y^\dagger] = 0$, this can be rewritten as

$$|u_{nm}(\eta)\rangle = \sum_{p=0}^{n} \sum_{q=0}^{m} \sqrt{\frac{(n+m-p-q)!(p+q)!}{n!m!}} \binom{n}{p} \binom{m}{q} \times \\ (\eta_1^*)^{n-p} (\eta_2^*)^p (-\eta_2)^{m-q} (\eta_1)^q |v_{(n+m-p-q)(p+q)}\rangle , \tag{6.72}$$

which expresses the transformed state $|u_{nm}(\eta)\rangle$ as an expansion in two-dimensional harmonic-oscillator states of the same order $N = n+m$. Conversely, this result shows that the subspaces of modes of fixed order $N = n + m$ are closed under transformations (mode conversions) on the Hermite-Laguerre sphere.

In general, the subspace of modes of fixed order N is an $N + 1$-dimensional subspace of the Hilbert space of transverse states of the field. The unitary transformations on this subspace form the group $SU(N + 1)$. Only in the special case of first order modes, the most general unitary transformation is equivalent to the $SU(2)$ transformation that figures

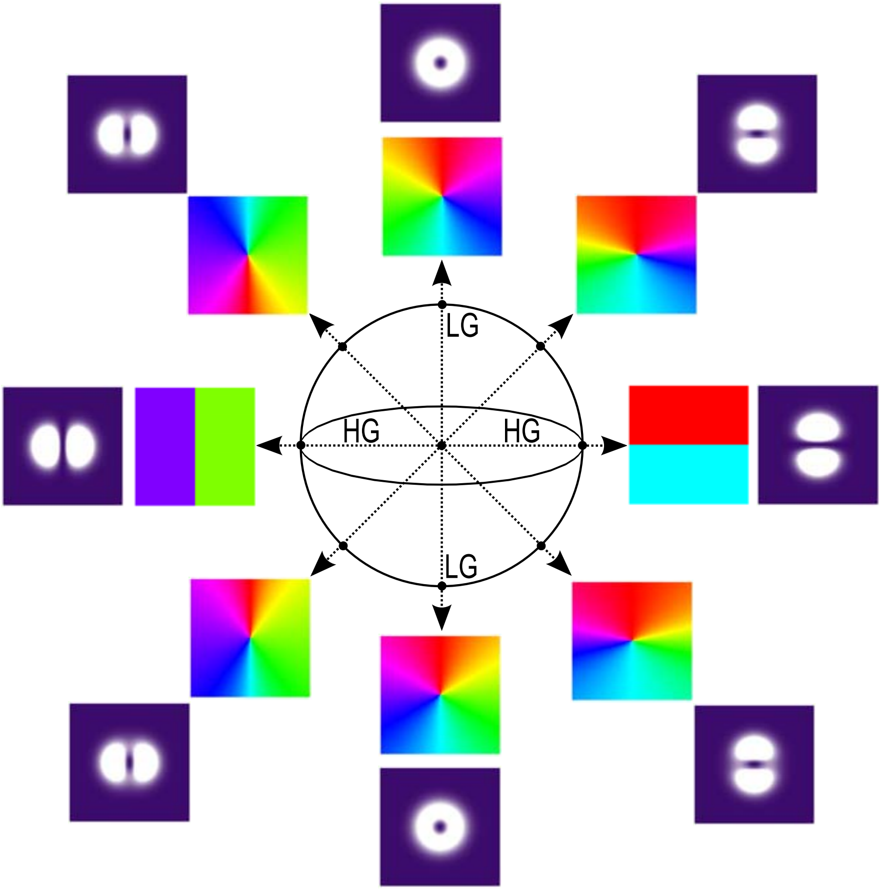


Figure 6.1: Intensity and false-color phase patterns of the modes that lie in the $\varphi = 0$ plane of the Hermite-Laguerre, or, equivalently, Poincaré sphere for the non-astigmatic first-order modes $|u_{01}\rangle$. The north and a south poles ($\vartheta = 0, \pi$) respectively correspond to Laguerre-Gaussian modes with $l = n - m = -1$ and $l = m - n = 1$. On both poles $p = \min(n, m) = 0$. The modes on the equator ($\vartheta = \pi/2$) are Hermite-Gaussian while modes for intermediate values of ϑ are generalized Gaussian modes. The color coding in the phase patterns is such that the color changes in a continuous fashion from red via yellow, green, blue and purple back to red when the phase changes from 0 to 2π.

in equation (6.72). It follows that the subspace of first-order modes is isomorphic to the Hermite-Laguerre sphere for the ladder operators. This sphere, which is an analogue of the Poincaré sphere for polarization states [93], as well as the intensity and phase structure of some of the modes that lie on it, is plotted in figure 6.1. In the general case of $N > 1$, $SU(2)$ is a subgroup of the group $SU(N+1)$ of unitary transformations on the subspace of modes of fixed order N. This accounts for the fact that only specific transformations on the Hilbert space of transverse states of the field can be achieved by mode converters and image rotators. In case of $N > 1$, the transformation in equation (6.72) gives rise to a sphere for each combination (n, m) of the transverse mode numbers. Since $|u_{nm}(\eta)\rangle$ and $|u_{mn}(\eta)\rangle$ correspond to antipodal points on the same sphere, it follows that, depending on the parity of N, only $(N+2)/2$ (for even N) or $(N+1)/2$ (for odd N) of these spheres are independent, i.e., not related by rotations over π. All of these spheres are isomorphic to the Hermite-Laguerre sphere for the ladder operators. Since, in general, the modes on a given sphere cannot be expressed as a linear combination of the modes on the poles, it follows that, for $N > 1$, these spheres are not Poincaré spheres in the strict sense. The two spheres for second-order modes, as well as the intensity and phase patterns of some of the modes that lie on them, are plotted in figure 6.2.

The mode-transformation in equation (6.72), together with the matrix representation of the spinor transformation that we have discussed above, provides a matrix description of beam transformations of non-astigmatic optical modes of arbitrary order. It generalizes the description discussed in references [101, 102], which applies to first order modes.

By inverting the relations in equation (6.70) and their hermitian conjugates, the position and propagation-direction operators can be expressed in terms of the ladder operators. Using this result, the state-space generators can be written as $\hat{s}_1 = (\hat{b}_x^\dagger \hat{b}_x - \hat{b}_y^\dagger \hat{b}_y)/2$, $\hat{s}_2 = (\hat{b}_x^\dagger \hat{b}_y + \hat{b}_x \hat{b}_y^\dagger)/2$ and $\hat{s}_3 = (\hat{b}_x^\dagger \hat{b}_y - \hat{b}_x \hat{b}_y^\dagger)/(2i)$, which is a Schwinger representation of the $SU(2)$ algebra. Here, the $SU(2)$ algebra (6.61) is ensured by the boson commutation relations (6.21). This representation provides a complete and closed description of the modes and transformations on the Hermite-Laguerre sphere in terms of the ladder operators.

6.5.4 Geometric phases and the Aharonov-Bohm analogy

The spinor η, as defined by equation (6.35), is completely determined by the azimuthal and polar angles on the Hermite-Laguerre sphere. The reverse of this statement is not true; choosing a point on the Hermite-Laguerre sphere fixes a properly normalized spinor $\xi^\dagger \xi = 1$ up to an overall phase factor so that $\xi = \exp(i\chi)\eta(\phi, \xi)$ with $\chi \in \mathbb{R}$. In the limiting case of transformations on the Hermite-Laguerre sphere, it follows from equation (6.63), or from the equivalent expansion in terms of the lowering operators in equation (6.70), that the two raising operators $a_1^\dagger$ and $\hat{a}_2^\dagger$ pick up equal but opposite phases $-\chi$ and χ respectively. The modes $|u_{nm}(\eta)\rangle$ (6.71) pick up a phases $\exp(-i\chi_{nm})$ with

$$\chi_{nm} = (n - m)\chi \; . \tag{6.73}$$

Such phases do not modify the physical properties of the modes but their variation under (a sequence of) transformations on the Hermite-Laguerre sphere can be measured interferometrically. Analogous to the discussion in section 6.4, we shall show that the variation χ has a geometric interpretation in terms of the Hermite-Laguerre sphere. We consider (a sequence of) state-space transformations that only modify the degrees of freedom associated with the nature and orientation of the higher-order modes. The evolution of the ladder operators under such transformations is described by the anti-Heisenberg equation of motion (6.47) when $\hat{T}$ is replaced by a generator $\hat{s}/2$, which is a linear combination of $\hat{s}_1$, $\hat{s}_2$ and $\hat{s}_3$, and ζ parameterizes a trajectory on the Hermite-Laguerre sphere. The factor $1/2$ in the generator is introduced for notational convenience. In terms of a spinor ξ, the equation of motion (6.48) takes the following form

$$\frac{\partial \xi}{\partial \zeta} = -\frac{i\tau\xi(\zeta)}{2} , \tag{6.74}$$

where τ is the spinor generator that corresponds to $\hat{s}$. It is a linear combination of τ_1, τ_2 and τ_3. The spinor ξ picks up the appropriate phase factor. Substitution of $\xi = \exp(i\chi)\eta$ gives

$$i\eta\frac{\partial \chi}{\partial \zeta} + \frac{\partial \eta}{\partial \zeta} = -\frac{i\tau\eta}{2} . \tag{6.75}$$

Using that $\eta^\dagger\eta = 1$, this result can be rewritten as

$$\frac{\partial \chi}{\partial \zeta} = i\eta^\dagger\frac{\partial \eta}{\partial \zeta} - \frac{\eta^\dagger\tau\eta}{2} . \tag{6.76}$$

The generator τ represents a constant of motion so that this result can be integrated to yield

$$\chi(\zeta) = -\frac{(\eta^\dagger\tau\eta)\zeta}{2} + i\int_0^\zeta d\zeta'\eta^\dagger\frac{\partial \eta}{\partial \zeta'} . \tag{6.77}$$

This result can also be obtained directly from substitution of the complex ray vectors μ_p, as defined by equation (6.63), in the general expression of the geometric phase shift (6.51). The first term in equation (6.51) arises from the fact that $\tau/2$ represents a conserved quantity. The second term constitutes the well-known geometric phase shift that is experienced by a spinor when it is transported along a trajectory on the Hermite-Laguerre sphere. Analogous to the discussion in section 6.4, both contributions are geometric in that they are fully determined by the trajectory on the Hermite-Laguerre sphere but only the second relates to the geometry of the Hermite-Laguerre sphere. It is natural to use spherical coordinates $\vec{\mathcal{R}} = (r\sin(\vartheta)\cos(\varphi), r\sin(\vartheta)\sin(\varphi), r\cos(\vartheta))^{\mathrm{T}}$ to parameterize points on the Hermite-Laguerre sphere. For a closed trajectory that consists of geodesics, the first contribution in equation (6.77) vanishes [103]. Then, the phase shift (6.77) can be rewritten as

$$\chi = i\int_0^z dz'\eta^\dagger\frac{\partial \eta}{\partial z'} = i\int_0^z dz'\eta^\dagger\left(\vec{\nabla}_{\vec{\mathcal{R}}}\eta\right)\cdot\frac{\partial \vec{\mathcal{R}}}{\partial z'} = i\oint_C \eta^\dagger\left(\vec{\nabla}_{\vec{\mathcal{R}}}\eta\right)\cdot d\vec{\mathcal{R}} , \tag{6.78}$$

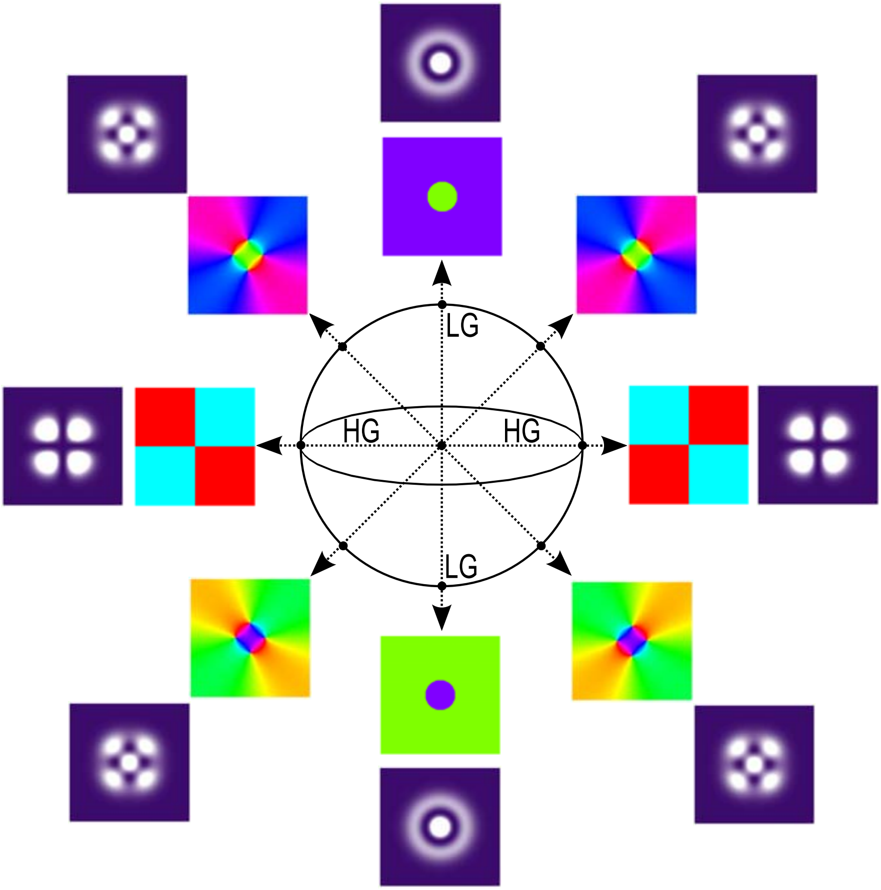
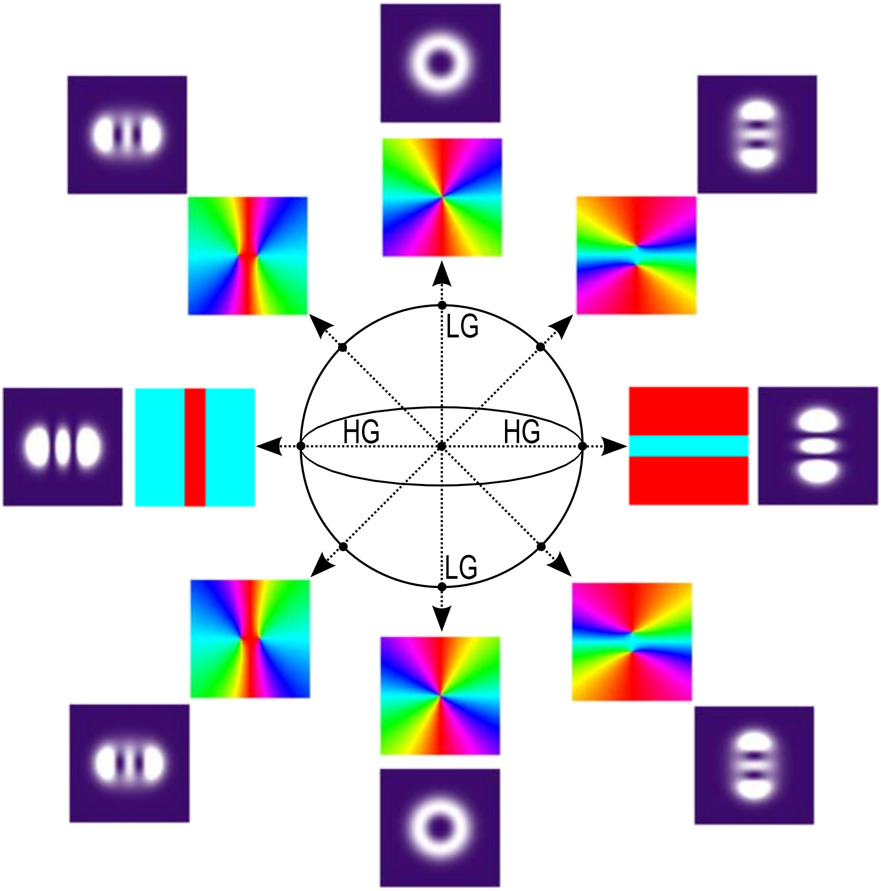


Figure 6.2: Intensity and false-color phase patterns of the modes that lie in the $\varphi = 0$ plane of two the Hermite-Laguerre spheres for non-astigmatic second-order modes $|u_{11}\rangle$ (this page) and $|u_{02}\rangle$ (next page). In the figure on this page, the north and south poles ($\vartheta = 0, \pi$) respectively correspond to Laguerre-Gaussian modes with $l = n - m = m - n = 0$ and $p = \min(n, m) = 0$ while the modes on the equator ($\vartheta = \pi/2$) are Hermite-Gaussian. The intermediate modes are generalized Gaussian modes. In the figure on the next page the north and south poles ($\vartheta = 0, \pi$) respectively correspond to a Laguerre-Gaussian mode



(continued) with $l = n - m = -2$ and $l = m - n = 2$. In both cases $p = \min(n, m) = 1$. Again, the modes on the equator ($\vartheta = \pi/2$) are Hermite-Gaussian while generalized Gaussian modes appear for intermediate values of ϑ. The color coding in the phase patterns is such that the color changes in a continuous fashion from red via yellow, green, blue and purple back to red when the phase changes from 0 to 2π. Both spheres are carbon-copies of the Hermite-Laguerre sphere on which every point characterizes two pairs of bosonic ladder operators.

where $\vec{\nabla}_{\vec{\mathcal{R}}}$ is the gradient in spherical coordinates and $C = \vec{\mathcal{R}}(z)$ is a closed trajectory on the Hermite-Laguerre sphere. In the Aharonov-Bohm picture, this phase shift is due to the coupling of η to a gauge field that arises from the $U(1)$ gauge invariance of the spinor dynamics. Comparison with equation (6.56) shows that this gauge field is given by

$$\vec{A} = \eta^{\dagger} \left(\vec{\nabla}_{\vec{\mathcal{R}}} \eta \right) . \tag{6.79}$$

By using the gradient in spherical coordinates and equation (6.35) $\vec{A}$ can be written as

$$\vec{A} = \frac{i \cot \vartheta}{2r} . \tag{6.80}$$

The corresponding "magnetic" field or Berry curvature is given by

$$\vec{B} = \vec{\nabla}_{\vec{\mathcal{R}}} \times \vec{A} = -\frac{i}{2r^2} \tag{6.81}$$

and is homogeneous on the Hermite-Laguerre sphere. It may be viewed as the field of a monopole located at the center of the Hermite-Laguerre sphere. By the virtue of Stokes' theorem, the geometric phase can be expressed as

$$\chi = i \oint_C \eta^{\dagger} \left(\vec{\nabla}_{\vec{\mathcal{R}}} \eta \right) \cdot d\vec{\mathcal{R}} = i \oint_C \vec{A} \cdot d\vec{\mathcal{R}} = i \oint_S \vec{B} \cdot d\vec{\mathcal{S}} = \frac{1}{2} \oint_S d\Omega = \frac{1}{2} \Omega , \tag{6.82}$$

where $\mathcal{S}$ is the enclosed surface on the Hermite-Laguerre sphere and Ω is the solid angle. This result establishes the well-known connection between the geometric phase acquired by a spinor that is transported along a closed trajectory on the Hermite-Laguerre sphere and the enclosed solid angle Ω on the sphere. Since we have defined the phase picked up by the higher-order modes as $\exp(-i\chi_{nm})$ with $\chi_{nm} = (n-m)\chi$, the result in equation (6.82) has the opposite sign of the analogous result for the standard case in which a spinor with positive helicity picks up a phase shift $\exp(i\chi)$.

The phase shift the modes $|v_{nm}\rangle$, as given by equation (6.73), depends only on the difference between the mode numbers n and m. In the Aharonov-Bohm picture, $n - m$ has the significance of the topological charge of a non-astigmatic mode $|v_{nm}\rangle$ and determines the strength of the coupling to the (corresponding state-space) gauge field. For modes with equal mode numbers $n = m$, the topological charge vanishes so that they do not couple to the gauge field and, therefore, do not experience a phase shift. The orbital angular momentum in non-astigmatic modes $|v_{nm}\rangle$ can be expressed as $(n-m)\cos\vartheta$ [17] and is proportional to their topological charge. It follows that in the case of a non-astigmatic mode, the exchange of orbital angular momentum between the mode and the set-up through which it propagates is necessary for a non-trivial geometric phase to occur [96, 95].

In this section, we have studied the geometric phase that arises from (cyclic) transformations on the Hermite-Laguerre sphere for higher-order modes. We have constructed ray matrices that solely modify the nature and orientation of the higher-order modes and derived the corresponding spinor and mode-space transformations. In terms of the spinor η the phase shift due to a (cyclic) transformation takes the familiar form of the geometric phase for a spinor. In experimental realizations, mode converters consist of pairs of astigmatic lenses in which the degrees of freedom associated with S are employed to achieve mode conversion [100]. As a result, there will be an additional contribution to the phase shift of the modes. This can be compensated for by measuring the interference between fields that have passed the same sequence of mode converters and image rotators but with different relative orientations [94].

6.6 Concluding remarks

We have explored the parameter space that is associated with the choice of a complete and orthonormal set of paraxial optical modes in the transverse plane. Modes are defined as solutions of the paraxial wave equation (6.3) that are fully characterized by a set of mode parameters whose variation through a paraxial optical set-up is described by the 4×4 ray matrix $M(z)$, which describes the transformation of a ray $r = (\rho, \theta)^{\mathrm{T}}$ from the $z = 0$ input plane of the set-up to the transverse plane z. Complete sets of transverse modes can be obtained from two pairs of bosonic ladder operators. The ladder operators are fully specified by two complex ray vectors μ_p with $p = 1, 2$, which characterize the mode parameters. Their variation through an optical set-up, and, thereby, the variation of the ladder operators, can conveniently be expressed in terms of $M(z)$. We have argued that there is a one-to-one correspondence between the algebraic properties of the ladder operators and the defining properties of a physical ray matrix $\in Sp(4, \mathbb{R})$, i.e., that it is real and obeys the identity (6.15). It follows that all sets of modes can be expressed in terms of two pairs of ladder operators and, moreover, that the freedom in choosing a set of modes is equivalent to the choice of an arbitrary ray matrix $M_0 \in Sp(4, \mathbb{R})$. Since $Sp(4, \mathbb{R})$ is a ten-parameter Lie group, the number of free parameters associated with this choice is equal to ten. A possible physical characterization of these degrees of freedom involves a symmetric 2×2 matrix S, which characterizes the astigmatism of the phase and intensity patterns of the fundamental mode, and a spinor η, which specifies the nature and orientation of the higher-order modes. The matrix S is fully specified by six parameters while characterization of η requires two independent parameters, which can be mapped on a Poincaré sphere. The remaining two degrees of freedom are overall phases of the ladder operators. They do not modify the physical properties of the modes in a given transverse plane z. Their variation through an optical set-up, however, gives rise to a generalized Gouy phase shift of the modes, which can be measured interferometrically. We have shown that both contributions to the variation of the overall phases through an optical set-up, as described by equation (6.51), are geometric in that they are fully determined by the trajectory $\vec{\mathcal{R}}(z)$ and do not depend on the velocity $\partial\vec{\mathcal{R}}/\partial z$. However, only the second contribution in equation (6.51) relates to the geometry of the parameter space. In the specific case of a closed trajectory on the Hermite-Laguerre sphere for non-astigmatic optical modes, the phase shifts of the two raising operators are equal but opposite. In full analogy with the Pancharatnam phase for polarization states, they are equal to half the enclosed surface on the sphere.

It is noteworthy that the overall phases χ_p of the vectors μ_p are in general only unambiguously defined in case of a closed trajectory. In particular, in the propagation-direction representation, the astigmatism of the fundamental mode $\tilde{u}_{00}(\theta, z)$ is fully specified by the symmetric matrix $\mathsf{V} = \mathsf{S}^{-1}$. Analogous to the discussion in section 6.3, the remaining degrees of freedom can be characterized by a unitary 2×2 matrix υ, which is defined such that $\mathsf{T} = \mathsf{V}_\mathrm{r}^{-1/2}\upsilon^\mathrm{T}$. It follows that υ and σ are related by $\sigma = -i\upsilon\mathsf{V}_\mathrm{r}^{-1/2}\mathsf{V}\mathsf{S}_\mathrm{r}^{1/2}$. In general $\det\left(\mathsf{V}_\mathrm{r}^{-1/2}\mathsf{V}\mathsf{S}_\mathrm{r}^{1/2}\right) \neq 1$ so that defining $\sigma = \mathsf{C}\sigma_0$ and $\upsilon = \mathsf{C}'\upsilon_0$ such that σ_0 and υ_0 have unit determinants, requires different phase matrices $\mathsf{C} \neq \mathsf{C}'$. The phase shift along a closed trajectory, however, does not depend on the phase convention used. In the limiting case of transformations of non-astigmatic modes in their focal planes, i.e., when S and V can be considered real scalars, the phases are also unambiguously defined along an open trajectory. All results presented in this chapter are, of course, independent of the phase convention that is chosen.

We have shown that the symplectic group of ladder-operator transformations $Sp(4, \mathbb{R})$ corresponds to the metaplectic group $Mp(4)$ of unitary transformations on the Hilbert space of state vectors $|u\rangle$. The metaplectic group constitutes a subgroup of the set of all possible unitary transformations. This accounts for the fact that only specific linear combinations of paraxial optical modes are modes as well, i.e., are fully characterized by a set of parameters whose variation through a paraxial optical set-up is fully described by the ray matrix $M(z)$. Each combination (n, m) of the transverse mode indices gives rise to a subspace of the Hilbert space of transverse states of the field, which is closed under metaplectic transformations. The geometries of these subspaces are all carbon copies of the geometry of the symplectic manifold underlying the ladder operators. In the limiting case of mode conversions of non-astigmatic modes, the metaplectic group reduces to $SU(2)$ and all those subspaces become spheres, which are all carbon copies of the Hermite-Laguerre sphere for the ladder operators.

We have pointed out an analogy between the Aharonov-Bohm effect in quantum electrodynamics and the generalized Gouy effect in classical wave mechanics. This reveals deep insights in the geometric origin of the latter. The physical properties of the modes (6.22) that are generated by two pairs of ladder operators are not affected by the $U(1)\otimes U(1)$ gauge transformation described by equation (6.52), or, equivalently (6.44). Imposing gauge invariance of the equations of motion (6.47) or (6.48), gives rise to two gauge fields $\vec{A}_p$ in the parameter space. Analogous to the Aharonov-Bohm effect, the geometric phase shift of the ladder operators through an optical set-up is due to the coupling these gauge fields. The raising and corresponding lowering operators have pairwise equal but opposite topological charges and experience opposite phase shifts. The topological charges of the modes $|u_{nn}\rangle$, i.e., the Noether charges that arise from the gauge invariance of the description of their propagation through an optical set-up, are given by $n + 1/2$ and $m + 1/2$ and depend on the mode numbers. As a result, the modes $|u_{nm}\rangle$ couple differently to the gauge fields and experience different phase shifts given by equation (6.39). Notice that the above-mentioned subspaces of modes with transverse mode indices n and m are all uniquely characterized by their coupling to the two (state-space) gauge fields. In the specific case of transformations on the Hermite-Laguerre sphere for higher-order modes, the phase shifts of the two lowering operators are equal but opposite. In that case, the phase shift of the modes is given by equation (6.73). In the Aharonov-Bohm picture, the variation of this phase is due to the coupling of the spinor η to a single gauge field $\vec{A}$ that arises from the $U(1)$ gauge invariance of the spinor dynamics. The topological charge of the modes $|v_{nm}\rangle$ on the Hermite-Laguerre sphere is equal $n-m$ and the "magnetic" field (Berry curvature) due to the gauge field is uniform on the Hermite-Laguerre sphere. It may be viewed as the field of a monopole located at the center of the sphere.

Although we have focused on the optical case, the mathematical structure that underlies the ladder-operator method and the phase shifts that arise from the geometry underlying the ladder operators are more general. The ray space (ρ, θ) is a phase space in the mathematical sense and the operator description of paraxial wave optics that we have discussed in section 6.2 may be viewed as a formally quantized (wavized) description of rays. Although the interpretation is different, all this is in full analogy with the quantization of classical mechanics to obtain quantum mechanics. As a result, the methods and results of this chapter can be applied to the quantum-mechanical description of wave packets. The only restriction for the ladder-operator approach to apply is that the state-space generators (or Hamiltonian in the quantum language) are quadratic in the canonical operators. The methods and results in this chapter have been formulated such that it is evident how they can be generalized to account for more independent spatial dimensions. In the general case of D dimensions, the number of generators of $Mp(2D)$ and $Sp(2D, \mathbb{R})$ is equal to $2D^2 + D$, $D^2 + D$ of which are associated with a $D \times D$ symmetric matrix that generalizes S. The remaining D^2 parameters specify a unitary matrix $\in U(D)$, which generalizes σ, and corresponds to the choice of D orthonormal $D-$component spinors and D overall phase factors. The variation of the phases under propagation (evolution) have a geometric interpretation in terms of the other degrees of freedom.

Appendices

6.A The ray-space generators J_j

In this appendix we give explicit expressions of the ray-space generators J_j. They are defined by equation (6.16) and correspond to the state-space generators $\hat{T}_j$ as defined in equation (6.9). They are given by

$$J_1 = \frac{2}{k}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J_2 = \frac{2}{k}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad J_3 = \frac{1}{k}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$
$$J_4 = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J_5 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$
$$J_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J_7 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$
$$J_8 = k\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J_9 = 2k\begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad J_{10} = 2k\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.83}$$

6.B Expectation values of the generators $\hat{T}_j$

This appendix is devoted to a proof of equation (6.27), which expresses the expectation values $\langle u_{nm}|\hat{T}_j|u_{nm}\rangle$ of the generators $\hat{T}_j$ in equation (6.9) in terms of the corresponding ray-space generators $\hat{J}_j$ as defined by equation (6.16). We prove this by mathematical induction. The special cases $\langle u_{00}|\hat{T}_j|u_{00}\rangle$ involve Gaussian standard integrals and can be proven explicitly. A formal proof by mathematical induction thus requires showing that the identity (6.27) holds for modes $|u_{n+1m}\rangle$ and $|u_{nm+1}\rangle$ if it holds for $|u_{nm}\rangle$. In order to prove this, we notice that

$$\langle u_{n+1m}|\hat{T}_j|u_{n+1m}\rangle = \frac{1}{n+1}\langle u_{nm}|\hat{a}_1\hat{T}_j\hat{a}_1^\dagger|u_{nm}\rangle\ . \tag{6.84}$$

Using that

$$\left[\hat{T}_j, \hat{a}_p\right] = \sqrt{\frac{k}{2}}\left(\mu_p^{\mathrm{T}}G\hat{T}_j\hat{\boldsymbol{z}} - \mu_p^{\mathrm{T}}G\,\hat{\boldsymbol{z}}\,\hat{T}_j\right) = \sqrt{\frac{k}{2}}\mu_p^{\mathrm{T}}G\left[\hat{T}_j, \hat{\boldsymbol{z}}\right] = i\sqrt{\frac{k}{2}}\mu_p^{\mathrm{T}}GJ_j\hat{\boldsymbol{z}}\ , \tag{6.85}$$

this can be rewritten as

$$\left(\frac{1}{n+1}\right)\langle u_{nm}|\left(\hat{T}_j\hat{a}_1 - i\sqrt{\frac{k}{2}}\mu_1^{\mathrm{T}}GJ_j\,\hat{\mathscr{e}}\right)\hat{a}_1^{\dagger}|u_{nm}\rangle =$$
$$\langle u_{nm}|\hat{T}_j|u_{nm}\rangle - \left(\frac{i}{n+1}\right)\sqrt{\frac{k}{2}}\mu_1^{\mathrm{T}}GJ_j\langle u_{nm}|\,\hat{\mathscr{e}}\,\hat{a}_1^{\dagger}|u_{nm}\rangle\ . \tag{6.86}$$

The analogous result may be derived for $|u_{nm+1}\rangle$ and proving equation (6.27) thus boils down to proving that

$$-\left(\frac{i}{n+1}\right)\sqrt{\frac{k}{2}}\mu_p^{\mathrm{T}}GJ_j\langle u_{nm}|\,\hat{\mathscr{e}}\,\hat{a}_p^{\dagger}|u_{nm}\rangle = \frac{1}{2}\mu_p^{\dagger}GJ_j\mu_p = \frac{1}{2}\left(\mu_p^{\dagger}GJ_j\mu_p\right)^{\mathrm{T}} = \frac{1}{2}\mu_p^{T}GJ_j\mu_p^{*}\ , \tag{6.87}$$

where we used that $G^{\mathrm{T}} = -G$ and that $J^{\mathrm{T}}G = -GJ$. This expression can be rewritten as

$$\langle u_{nm}|\,\hat{\mathscr{e}}\,\hat{a}_p^{\dagger}|u_{nm}\rangle = i(n+1)\sqrt{\frac{1}{2k}}\mu_p^{*}\ , \tag{6.88}$$

which we also prove by mathematical induction. Again, the special case of $|u_{00}\rangle$ can be checked explicitly. In order to prove that it is true for $|u_{n+1m}\rangle$ and $|u_{nm+1}\rangle$, we use that

$$\left[\,\hat{\mathscr{e}}\,,\hat{a}_p^{\dagger}\right] = \hat{\mathscr{e}}\left(\sqrt{\frac{k}{2}}\mu_p^{\dagger}G\,\hat{\mathscr{e}}\right) - \left(\sqrt{\frac{k}{2}}\mu_p^{\dagger}G\,\hat{\mathscr{e}}\right)\hat{\mathscr{e}} = \sqrt{\frac{k}{2}}\left[\,\hat{\mathscr{e}}\,, r_p^{*}\hat{\theta} - t_p^{*}\hat{\rho}\right] = i\sqrt{\frac{1}{2k}}\mu_p^{*} \tag{6.89}$$

and find

$$\langle u_{n+1m}|\,\hat{\mathscr{e}}\,\hat{a}_1^{\dagger}|u_{n+1m}\rangle = \left(\frac{1}{n+1}\right)\langle u_{nm}|\hat{a}_1\,\hat{\mathscr{e}}\,\hat{a}_1^{\dagger}\hat{a}_1^{\dagger}|u_{nm}\rangle =$$
$$\left(\frac{1}{n+1}\right)\langle u_{nm}|\hat{a}_1\left(\hat{a}_1^{\dagger}\,\hat{\mathscr{e}} + i\sqrt{\frac{1}{2k}}\mu_1^{*}\right)\hat{a}_1^{\dagger}|u_{nm}\rangle = \langle u_{nm}|\,\hat{\mathscr{e}}\,\hat{a}_1^{\dagger}|u_{nm}\rangle + i\sqrt{\frac{1}{2k}}\mu_1^{*}\ . \tag{6.90}$$

The analogous result may be derived for $|u_{nm+1}\rangle$. This completes the proof of equation (6.88) and, thereby, of equation (6.27).

6.C Mode-space operators corresponding to the Noether charges

In this appendix we construct both the ray-space and the corresponding state-space generators of the $U(1)\otimes U(1)$ gauge transformations. The ray matrix that describes such transformations is given by equation (6.44). To first order in the phases χ_1 and χ_2 the matrix C (6.42) is given by

$$\mathsf{C} = \begin{pmatrix} 1+i\chi_1 & 0 \\ 0 & 1+i\chi_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \chi_1\begin{pmatrix} i & 0 \\ 0 & 0 \end{pmatrix} + \chi_2\begin{pmatrix} 0 & 0 \\ 0 & i \end{pmatrix}\ . \tag{6.91}$$

Substitution in equation (6.44) then gives

$$M_{\chi}(\{\chi_p\}) = 1 + \frac{\chi_1}{2}\begin{pmatrix} -r_1t_1^{\dagger} - r_1^{*}t_1^{\mathrm{T}} & r_1r_1^{\dagger} + r_1^{*}r_1^{\mathrm{T}} \\ -t_1t_1^{\dagger} - t_1^{*}t_1^{\mathrm{T}} & t_1r_1^{\dagger} + t_1^{*}r_1^{\mathrm{T}} \end{pmatrix} +$$
$$\frac{\chi_2}{2}\begin{pmatrix} -r_2t_2^{\dagger} - r_2^{*}t_2^{\mathrm{T}} & r_2r_2^{\dagger} + r_2^{*}r_2^{\mathrm{T}} \\ -t_2t_2^{\dagger} - t_2^{*}t_2^{\mathrm{T}} & t_2r_2^{\dagger} + t_2^{*}r_2^{\mathrm{T}} \end{pmatrix}\ , \tag{6.92}$$

where $r_1 t_1^\dagger = r_1 \otimes t_1^\dagger$ etcetera are direct vector products. From $M_\chi(\{\chi_p\}) = \exp\left(-\chi_p J_{\chi_p}\right) \simeq 1 - \chi_p J_{\chi_p}$, we find that

$$J_{\chi_p} = \frac{1}{2}\begin{pmatrix} r_p t_p^\dagger + r_p^* t_p^\mathrm{T} & -r_p r_p^\dagger - r_p^* r_p^\mathrm{T} \\ t_p t_p^\dagger + t_p^* t_p^\mathrm{T} & -t_p r_p^\dagger - t_p^* r_p^\mathrm{T} \end{pmatrix} \tag{6.93}$$

where $p = 1, 2$. These generators are 4×4 matrices in the ray space. By carefully inspecting the form of the direct products and the structure of the generators J_j as given in appendix 6.A we find that

$$\hat{T}_{\chi_p} = -\frac{k}{4}\Big\{r_p^\mathrm{T}\hat{\theta}\hat{\rho}^\mathrm{T} t_p^* + r_p^\dagger\hat{\theta}\hat{\rho}^\mathrm{T} t_p^\mathrm{T} - r_p^\mathrm{T}\hat{\theta}\hat{\theta}^\mathrm{T} r_p^* - r_p^\dagger\hat{\theta}\hat{\theta}^\mathrm{T} r_p^\mathrm{T} + t_p^\mathrm{T}\hat{\rho}\hat{\theta}^\mathrm{T} r_p^* + t_p^\dagger\hat{\rho}\hat{\theta}^\mathrm{T} r_p^\mathrm{T} - t_p^\mathrm{T}\hat{\rho}\hat{\rho}^\mathrm{T} t_p^* - t_p^\dagger\hat{\rho}\hat{\rho}^\mathrm{T} t_p^\mathrm{T}\Big\} \,. \tag{6.94}$$

This can be rewritten as

$$\hat{T}_{\chi_p} = -\frac{k}{4}\left\{\begin{pmatrix}-t_p^\dagger & r_p^\dagger\end{pmatrix}\begin{pmatrix}\hat{\rho}\hat{\rho}^\mathrm{T} & \hat{\rho}\hat{\theta}^\mathrm{T} \\ \hat{\theta}\hat{\rho}^\mathrm{T} & \hat{\theta}\hat{\theta}^\mathrm{T}\end{pmatrix}\begin{pmatrix} t_p \\ -r_p\end{pmatrix} + \begin{pmatrix}-t_p^\mathrm{T} & r_p^\mathrm{T}\end{pmatrix}\begin{pmatrix}\hat{\rho}\hat{\rho}^\mathrm{T} & \hat{\rho}\hat{\theta}^\mathrm{T} \\ \hat{\theta}\hat{\rho}^\mathrm{T} & \hat{\theta}\hat{\theta}^\mathrm{T}\end{pmatrix}\begin{pmatrix} t_p^* \\ -r_p^*\end{pmatrix}\right\} \,, \tag{6.95}$$

which equals

$$\hat{T}_{\chi_p} = \frac{k}{4}\left\{\mu_p^\dagger G\,\hat{\imath}\;\; \hat{\imath}^\mathrm{T} G\mu_p + \mu_p^\mathrm{T} G\,\hat{\imath}\;\; \hat{\imath}^\mathrm{T} G\mu_p^*\right\} = \frac{k}{2}\left\{\mu_p^\dagger G\,\hat{\imath}\;\; \mu_p^\mathrm{T} G\,\hat{\imath} + \mu_p^\mathrm{T} G\,\hat{\imath}\;\; \mu_p^\dagger G\,\hat{\imath}\right\} = \frac{1}{2}\left(\hat{a}_p^\dagger \hat{a}_p + \hat{a}_p \hat{a}_p^\dagger\right) \,, \tag{6.96}$$

where, we used that $\hat{\imath}^\mathrm{T} G\mu_p$ is scalar so that $\hat{\imath}^\mathrm{T} G\mu_p = (\hat{\imath}^\mathrm{T} G\mu_p)^\mathrm{T} = -\mu_p^\mathrm{T} G\,\hat{\imath}$.