Light with a twist : ray aspects in singular wave and quantum optics

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Twisted light between rotating mirrors

3.1 Introduction

The possibly very rich structure of optical cavity modes is fully determined by the boundary condition that the electric field must vanish on the mirror surfaces. This implies that the wave fronts (surfaces of equal phase) of a mode that propagates inside the cavity fit on the mirror surfaces. As a result, the mirror surfaces are nodal planes of the standing wave pattern that is formed inside the cavity. The common approach to finding the modes of a paraxial optical cavity is by considering a freely propagating Gaussian beam and requiring its wave fronts to fit on the mirror surfaces [12]. This is straightforward in the standard case of a cavity with two spherical mirrors. The resulting equation can be solved to obtain the beam parameters that characterize a complete and orthonormal set of Hermite-Gaussian modes. Geometric stability comes in as the necessary and sufficient condition for a cavity to have stationary modes and the round-trip Gouy phases determine the corresponding frequency spectrum. This approach allows for generalization to the case of astigmatic mirrors, which are curved differently in two mutually perpendicular transverse directions, provided that the mirror axes are parallel. The problem of finding the paraxial modes of a cavity with non-aligned astigmatic mirrors requires far more advanced analytical [38, 34] or, as discussed in the previous chapter, algebraic techniques.

In this chapter we consider an additional, and surprisingly different, source of complexity. We study the propagation and diffraction of light inside a two-mirror cavity that is rotating at a uniform velocity about its optical axis. It remains true that the electric field vanishes on the mirror surfaces. Since cavity modes are usually defined as monochromatic solutions of the wave equation that obey the boundary conditions imposed by the mirrors, the mode concept requires special attention in this time-dependent case. As opposed to, for instance, vibration, uniform rotation is homogeneous in time, which means that all instants of time, and therefore all cavity round trips, are equivalent. In this special case, it is natural to require that modes adopt the time-dependence of the cavity so that they rotate along with the mirrors. We show that this property can be used as a defining property of rotating cavity modes. As an example, we consider the case of a rotating astigmatic two-mirror cavity. We generalize the geometric-algebraic method that we have developed in the previous chapter to derive explicit expressions of the rotating cavity modes and apply these to study some of their physical properties.

The material in this chapter is organized as follows. In the next section we discuss the perturbative approach to the paraxial approximation [45] and its generalization to the time-dependent case [46]. This helps us to ensure the consistency of our approach in that we retain all terms up to the same order of the expansion. In section 3.3 and section 3.4 we generalize the operator description of paraxial wave optics [42] to account for the time-dependence of a rotating cavity and show how modes can be defined in such a system. Explicit expressions of the rotating cavity modes are derived in section 3.5 and section 3.6, where we also discuss some of their physical properties. In section 3.7 we focus on the role of spatial symmetries in special limiting cases while section 3.8 is devoted to the orbital angular momentum in the rotating cavity modes. Explicit results for specific cases are briefly discussed in section 3.9.

3.2 Time-dependent paraxial propagation

The spatial structure of an optical beam is characterized by a vector field $\mathbf{u}(\mathbf{r}, t)$, which describes the spatial and temporal variations of the vector components of the field that are slow compared to those arising from the carrier wave. The profile $\mathbf{u}(\mathbf{r}, t)$ defines the electric field of the beam by

$$\mathbf{E}(\mathbf{r}, t) = \mathrm{Re}\left\{E_0 \mathbf{u}(\mathbf{r}, t)\, e^{ikz - i\omega t}\right\} , \tag{3.1}$$

where E_0 is an amplitude factor and $\omega = ck$ is the frequency of the carrier wave, with k the wave number and c the speed of light. In vacuum the electric field obeys the wave equation

$$\nabla^2 \mathbf{E} = \frac{1}{c^2} \frac{\partial^2 \mathbf{E}}{\partial t^2} , \tag{3.2}$$

with the additional requirement that it has a vanishing divergence

$$\nabla \cdot \mathbf{E} = 0 . \tag{3.3}$$

Essential for the paraxial approximation is that the beam has a small opening angle, which we indicate by the smallness parameter δ. Then the beam waist is of the order of the parameter γ, and the diffraction length (or Rayleigh range) is of the order of the parameter b, where

$$\frac{1}{k} = \delta\gamma = \delta^2 b . \tag{3.4}$$

So the diffraction length is much larger than the beam waist, which is much larger than the wavelength. The smallness of δ ensures that the variations of the profile $\mathbf{u}(\mathbf{r}, t)$ with the longitudinal coordinate z are slow compared to the variations with the transverse coordinates $\rho = (x, y)^{\mathrm{T}}$, which are, in turn, slow compared to the variations of the carrier wave $\exp(ikz)$ with z. By using δ as an expansion parameter, the time-independent paraxial wave equation (2.32) can be obtained directly from the wave equation (3.2) while resolving the apparent paradox that paraxial fields cannot have a completely vanishing divergence [45]. As we shall discuss in a moment, this approach allows for generalization to the time-dependent case [46].

Time dependent optical fields necessarily have spectral structure in addition to their spatial structure. The concept of a mode loses its meaning if the difference in diffraction of the frequency components becomes significant, i.e., if the diffraction due to the time dependence of the profile becomes important. Conversely, we shall show that the mode concept remains meaningful if the time scale for variation of the cavity boundaries is slower than the transit time through the focal range of the beam. This transit time is of the order of

$$a = \frac{b}{c} = \frac{1}{\omega\delta^2} \ . \tag{3.5}$$

In order to obtain the time-dependent paraxial wave equation, the profile $\mathbf{u}$ is expanded in powers of the opening angle δ

$$\mathbf{u}(\mathbf{r}, t) = \sum_{n=0}^{\infty} \delta^n \mathbf{u}^{(n)}(\mathbf{r}, t) \ . \tag{3.6}$$

Since we need to account for the (relative) order of the magnitudes of the derivatives of $\mathbf{u}$, it is convenient to introduce the scaled variables $\xi = x/\gamma$, $\eta = y/\gamma$, $\zeta = z/b$ and $\tau = t/a$. In these variables, the derivatives can be treated as being of the same order in δ. Substituting the expression (3.1) for the electric field in the wave equation (3.2) then gives

$$\left(\frac{\partial^2}{\partial\xi^2} + \frac{\partial^2}{\partial\eta^2} + 2i\frac{\partial}{\partial\zeta} + 2i\frac{\partial}{\partial\tau}\right)\mathbf{u}(\mathbf{r}, t) = \delta^2\left(\frac{\partial^2}{\partial\tau^2} - \frac{\partial^2}{\partial\zeta^2}\right)\mathbf{u}(\mathbf{r}, t) \ , \tag{3.7}$$

while the transversality condition (3.3) yields

$$\delta\left(\frac{\partial u_x}{\partial\xi} + \frac{\partial u_y}{\partial\eta}\right) = -\delta^2\frac{\partial u_z}{\partial\zeta} - iu_z \ . \tag{3.8}$$

It is natural to assume that to zeroth order the z component of $\mathbf{u}$ vanishes, and equation (3.8) shows that such a solution can be found. Then to zeroth order of the paraxial approximation the electric field lies in the transverse plane. In the special case of uniform polarization it can be written as

$$\mathbf{u}^{(0)}(\mathbf{r}, t) = \epsilon u(\mathbf{r}, t) \ , \tag{3.9}$$

where ϵ is a transverse polarization vector. The scalar profile $u(\mathbf{r}, t)$ obeys the time-dependent paraxial wave equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + 2ik\frac{\partial}{\partial z} + \frac{2ik}{c}\frac{\partial}{\partial t}\right)u(\mathbf{r}, t) = 0 \ . \tag{3.10}$$

The expansion (3.6) then shows that all even orders of the transverse components are coupled by equation (3.7) while all odd orders can be assumed to vanish. Equation (3.8) connects odd orders of the z component to the even orders of the transverse components, which implies that all even orders (including the zeroth) of the longitudinal component vanish. The first-order contribution to the profile is longitudinal and by using equation (3.8) it can be expressed in the zeroth order term

$$\delta \mathbf{u}^{(1)}(\mathbf{r}, t) = \frac{i}{k}\left(\epsilon_x \frac{\partial}{\partial x} + \epsilon_y \frac{\partial}{\partial y}\right) u(\mathbf{r}, t)\mathbf{e}_z \ , \tag{3.11}$$

where $\mathbf{e}_z$ is the unit vector in the z direction.

Up to first order of the paraxial approximation, finding the modes of a cavity with rotating mirrors requires solving the time-dependent paraxial wave equation (3.10) with the boundary condition that the electric field (3.1) vanishes at the mirror surfaces at all times. The range of validity of this time-dependent wave equation provides a natural upper limit to the rotation frequency of the mirrors. In a typical experimental set-up the diffraction length of the modes of an optical cavity is of the order of magnitude of the mirror separation, so that the period of the rotation of the mirrors can be at most comparable to the cavity round-trip time. This yields an upper bound to the rotation frequency Ω

$$\Omega \lesssim \frac{c\pi}{L} \ , \tag{3.12}$$

where L is the mirror separation and c is the speed of light. The average lifetime of a photon inside the cavity, which we leave out of our consideration here, provides a natural lower bound to the rotation frequency of the mirrors.

3.3 Operator description of time-dependent paraxial wave optics

3.3.1 Operators and transformations

The standard time-independent paraxial wave equation follows if we omit the time derivative in equation (3.10). This has the same structure as the Schrödinger equation for a free particle in two dimensions, with k taking the place of $m/\hbar$ and the longitudinal coordinate z playing the role of time. This analogy can be exploited by adopting the Dirac notation of quantum mechanics to describe classical light beams [42]. This naturally leads to an operator description of paraxial wave optics. Here, we show that this description can be generalized to include the time dependence of the scalar beam profile $u(\rho, z, t)$, even though the time dependence of an optical beam does not have an analogue in quantum mechanics.

We associate to the beam profile $u(\rho, z, t)$ a vector $|u(z, t)\rangle$ in the Hilbert space of paraxial modes of the radiation field

$$u(\rho, z, t) = \langle \rho | u(z, t) \rangle \ , \tag{3.13}$$

where $|\rho\rangle$ is an eigenstate of the transverse position operator $\hat{\rho} = (\hat{x}, \hat{y})^{\mathrm{T}}$. The corresponding momentum operator can be represented by $k\hat{\theta} = (k\hat{\vartheta}_x, k\hat{\vartheta}_y)^{\mathrm{T}} = -i(\partial/\partial x, \partial/\partial y)^{\mathrm{T}}$. The transformations of paraxial propagation and lossless optical elements such as thin lenses can be expressed as unitary transformations in the transverse mode space. Expressing the time-dependent paraxial wave equation (3.10) in terms of the momentum operators gives

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)|u(z,t)\rangle = -\frac{ik}{2}\hat{\theta}^2|u(z,t)\rangle \ , \tag{3.14}$$

which is formally solved by

$$|u(z,t)\rangle = \exp\left(-\frac{ikz}{2}\hat{\theta}^2\right)|u(0,t-z/c)\rangle = \hat{U}_{\mathrm{f}}(z)|u(0,t-z/c)\rangle \ , \tag{3.15}$$

where $\hat{U}_{\mathrm{f}}(z)$ denotes the unitary operator that describes free propagation of a paraxial beam. This result shows that the time-dependent paraxial wave equation (3.10) describes paraxial beam propagation while incorporating retardation effects. A thin spherical lens imposes a Gaussian phase profile. Hence, the transformation caused by such a lens can be expressed as [27]

$$|u_{\mathrm{out}}\rangle = \exp\left(-\frac{ik\hat{\rho}^2}{2f}\right)|u_{\mathrm{in}}\rangle \ , \tag{3.16}$$

where f is the focal length of the lens. The generalization of this transformation to the case of a lens that has astigmatism is given by

$$|u_{\mathrm{out}}\rangle = \exp\left(-\frac{ik\hat{\rho}^{\mathrm{T}}\mathsf{F}^{-1}\hat{\rho}}{2}\right)|u_{\mathrm{in}}\rangle = \hat{U}_{\mathrm{l}}(\mathsf{F})|u_{\mathrm{in}}\rangle \ , \tag{3.17}$$

where F is a real and symmetric 2×2 matrix. The eigenvalues of F characterize the focal lengths of the lens while the, mutually orthogonal, real eigenvectors fix its orientation in the transverse plane.

3.3.2 Rotating lenses and frequency combs

The unitary operator that rotates a scalar function about the z axis can be expressed as

$$\hat{U}_{\mathrm{rot}}(\alpha) = \exp(-i\alpha\hat{L}_z) \ , \tag{3.18}$$

where α is the rotation angle and $\hat{L}_z = \hat{\rho}\times k\hat{\theta} = -i(x\partial/\partial y - y\partial/\partial x) = -i\partial/\partial\phi$ is the z component of the orbital angular momentum operator. The inverse of this rotation is a rotation in the opposite direction, so that $\hat{U}^{\dagger}_{\mathrm{rot}}(\alpha) = \hat{U}_{\mathrm{rot}}(-\alpha)$. The transformation of a rotated lens can be expressed as

$$\hat{U}_{\mathrm{rot}}(\alpha)U_{\mathrm{l}}(\mathsf{F})\hat{U}^{\dagger}_{\mathrm{rot}}(\alpha) \ . \tag{3.19}$$

This (anti-Heisenberg) transformation property makes sense if one realizes that rotating a lens is equivalent to rotating the profile in the opposite direction, applying the lens and rotating the profile backward. The beam transformation caused by an astigmatic lens (3.17) only involves

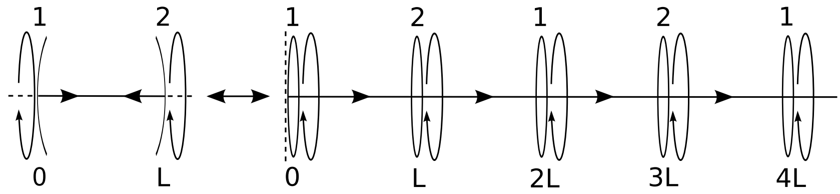

Figure 3.1: Unfolding a rotating optical cavity into an equivalent periodic lens guide. The mirrors are replaced by rotating lenses with the same focal lengths and the $z = 0$ reference plane is indicated by the dashed line.

the position operator $\hat{\rho} = (\hat{x}, \hat{y})^{\mathrm{T}}$. The anti-Heisenberg transformation of the position operator under a rotation about the z axis (3.18) can be expressed as

$$\hat{U}_{\mathrm{rot}}(\alpha)\hat{\rho}\hat{U}^{\dagger}_{\mathrm{rot}}(\alpha) = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}\hat{\rho} = \mathsf{P}^{\mathrm{T}}(\alpha)\hat{\rho}\ , \tag{3.20}$$

with $\mathsf{P}(\alpha)$ a two-dimensional rotation matrix. By using this transformation property of the position operators, the transformation of a rotated lens (3.19) can be expressed as

$$\hat{U}_{\mathrm{l}}\Big(\mathsf{P}(\alpha)\mathsf{F}\mathsf{P}^{\mathrm{T}}(\alpha)\Big)\ . \tag{3.21}$$

For a lens rotating at angular velocity Ω, the rotation angle is $\alpha = \Omega t$, so that the time-dependent beam transformation caused by the rotating lens is given by $\hat{U}_{\mathrm{l}}(\mathsf{F}(t))$ where $\mathsf{F}(t) = \mathsf{P}(\Omega t)\mathsf{F}(0)\mathsf{P}^{\mathrm{T}}(\Omega t)$. Without loss of generality we can choose the real and symmetric matrix $\mathsf{F}(t)$ diagonal at $t = 0$

$$\mathsf{F}(0) = \begin{pmatrix} f_\xi & 0 \\ 0 & f_\eta \end{pmatrix}\ . \tag{3.22}$$

By using equations (3.19-3.22) and introducing cylindrical coordinates with $x = R\cos\phi$, $y = R\sin\phi$, the time-dependent transformation of a rotating lens can be expressed as

$$\hat{U}_{\mathrm{l}}(\mathsf{F}(t)) = \exp\Big[-\frac{ikR^2}{4}\left(f_\xi^{-1} + f_\eta^{-1}\right) - \frac{ikR^2}{4}\left(f_\xi^{-1} - f_\eta^{-1}\right)\cos(2\Omega t - 2\phi)\Big]\ . \tag{3.23}$$

Using the Jacobi-Anger expansion of a plane wave in cylindrical waves: $\exp(iz\cos\phi) = \sum_{l=-\infty}^{\infty} i^l J_l(z)\exp(il\phi)$, where J_l are Bessel functions of the first kind [47], this result can be rewritten as:

$$\hat{U}(\mathsf{F}(t)) = \exp\Big[-\frac{ikR^2}{4}\left(f_\xi^{-1} + f_\eta^{-1}\right)\Big]\times \sum_l \Big\{J_l\Big(\frac{ikR^2}{4}\left(f_\xi^{-1} - f_\eta^{-1}\right)\Big)\exp\Big[-2il\Big(\Omega t + \phi + \frac{\pi}{4}\Big)\Big]\Big\}\ . \tag{3.24}$$

It follows that a rotating lens introduces frequency side bands in a monochromatic optical field at frequencies $\omega \pm 2p\Omega$ with $p \in \mathbb{Z}$ [48].

3.4 Modes in a rotating cavity

3.4.1 Lens guide picture

In order to describe the evolution of a profile vector $|u(z,t)\rangle$ inside a rotating cavity it is convenient to unfold the cavity into an equivalent lens guide. As illustrated in figure 3.1, the mirrors are replaced by lenses with the same focal lengths. Rather than describing the bouncing back and forth inside the cavity we describe the propagation along the axis of the lens guide, with coordinate z. The dashed line on the left of the first lens in figure 3.1 indicates the transverse reference plane of the lens guide, which is positioned at $z = 0$. The profile in any transverse plane of the lens guide is connected to the profile in the input plane by a unitary transformation. Just as in equation (3.15), this time-dependent connection involves retardation, as described by

$$|u(z,t)\rangle = \hat{U}(z,t)|u(0,t-z/c)\rangle \ . \tag{3.25}$$

The unitary operator $\hat{U}(z,t)$ can be constructed by successive application of the transformations of the optical elements and free propagation that are in between the reference plane and the z plane in the correct order. We need only two different transformation operators for the lenses in the lens guide, which we denote for simplicity as $\hat{U}_1(t)$ and $\hat{U}_2(t)$. For the lens guide that corresponds to a cavity rotating at the uniform angular velocity Ω, these time-dependent operators are given by equation (3.19) with rotation angle $\alpha = \Omega t$, so that

$$\hat{U}_i(t) = \hat{U}_{\mathrm{rot}}(\Omega t)\hat{U}_1(\mathsf{F}_i(0))\hat{U}^{\dagger}_{\mathrm{rot}}(\Omega t) \ , \tag{3.26}$$

with $i = 1,2$ labeling the two lens types. Since the orientations of the rotating lenses depend on time, retardation effects must be included. As an example we give the operator that connects the profile vectors in transverse planes that are separated by one period of the lens guide

$$\hat{U}(2L,t) = \hat{U}_{\mathrm{f}}(L)\hat{U}_2(t-L/c)\hat{U}_{\mathrm{f}}(L)\hat{U}_1(t-2L/c) \ . \tag{3.27}$$

Obviously, all lenses that correspond to the same mirror of the cavity have the same orientation at any instant of time. Nevertheless, as a result of the finiteness of the speed of light, the orientation of two lenses that correspond to the same mirror of the cavity is perceived differently by a light pulse that propagates through the lens guide.

3.4.2 Rotating modes

Cavity modes are resonant field distributions inside an optical cavity. In a stationary cavity, a field pattern is resonant only if it repeats itself after each round trip through the cavity. The natural generalization of this mode criterion to the rotating case is by requiring that the field pattern in the corresponding lens guide is the same in every period, at a single given instant of time. This implies that the mode vector $|u(0,t)\rangle$ in the reference plane for a given value of

t repeats itself after one period $2L$ up to a phase factor

$$|u(2L,t)\rangle = e^{-i\chi}|u(0,t)\rangle \ . \tag{3.28}$$

The phase χ generalizes the Gouy phase for the round trip in a stationary cavity. Since the lens guide is rotating at a uniform velocity Ω, this mode criterion (3.28) can be obeyed only if the mode pattern rotates along with the lenses, so that the time dependence of a mode vector must be determined by

$$|u(z,t)\rangle = \hat{U}_{\mathrm{rot}}(\Omega t)|v(z)\rangle \ . \tag{3.29}$$

The homogeneous time dependence of this profile (3.29) can be eliminated by introducing the z dependent profile

$$|v(z)\rangle = |u(z,0)\rangle \ , \tag{3.30}$$

which has the significance of the profile vector in the co-rotating frame. By combining the relation (3.29) with equations (3.25-3.27), we find that the propagation of a beam profile in this frame is governed by the general relation

$$|v(z)\rangle = \hat{U}(z,0)\hat{U}_{\mathrm{rot}}(-\Omega z/c)|v(0)\rangle \ , \tag{3.31}$$

so that the product $\hat{U}(z,0)\hat{U}_{\mathrm{rot}}(-\Omega z/c)$ can be viewed as the operator for free propagation in the co-rotating frame. In the special case of the propagation over one period of the lens guide, we find

$$|v(2L)\rangle = \hat{U}_{\mathrm{rt}}\hat{U}^{\dagger}_{\mathrm{rot}}(\Omega t)|u(0,t)\rangle = \hat{U}_{\mathrm{rt}}|v(0)\rangle \ , \tag{3.32}$$

where $\hat{U}_{\mathrm{rt}}$ is given by the expression

$$\hat{U}_{\mathrm{rt}} = \hat{U}_{\mathrm{f}}(L)\hat{U}_{\mathrm{rot}}(-\Omega L/c)\hat{U}_{2}(0)\hat{U}_{\mathrm{f}}(L)\hat{U}_{\mathrm{rot}}(-\Omega L/c)\hat{U}_{1}(0) \tag{3.33}$$

and has the significance of the transformation operator over a single period of the lens guide in the co-rotating frame. Notice that the operator for free evolution $\hat{U}_{\mathrm{f}}$ is denoted as a function of length while the lens operators $\hat{U}_i$ are denoted as a function of time and the rotation operator $\hat{U}_{\mathrm{rot}}$ is denoted as a function of angle.

Now the mode criterion (3.28) in the reference plane $z = 0$ in the rotating frame is obeyed by the eigenvectors of the round-trip operator $\hat{U}_{\mathrm{rt}}$. Once these mode vectors are determined, we can use the propagation equation (3.15) and the time dependence (3.29) to obtain the shape of the modes at other time instants, and at any position within a period of the lens guide. The eigenvalues, which are specified by the phase angles χ, determine the resonance frequencies of the modes. In the next section we shall indicate how the modes can be obtained explicitly from a ladder-operator method.

3.5 Ray matrices and ladder operators

3.5.1 Time-dependent ray matrices

The transverse spatial structure of paraxial modes in cavities with spherical mirrors is known to be similar to the spatial structure of the stationary states of a two-dimensional quantum harmonic oscillator [12]. Complete sets of modes can be generated by using bosonic ladder operators [33]. In chapter 2, we have derived a ladder operator method to find explicit expressions of the paraxial modes of an optical cavity that has general astigmatism. These ladder operators are conveniently expressed in terms of the eigenvectors of the ray matrix for one period in the lens guide, or, equivalently, for one round trip in the cavity. Here we generalize this approach to account for the time dependence that arises from the rotation of the cavity. In this case, also the ray matrices depend on time.

In geometric paraxial optics, a light ray is specified by its position $\rho = (x, y)^T$ and its propagation direction $\theta = \partial\rho/\partial z$ in the transverse plane z. They can conveniently be combined into a four-dimensional column vector

$$\mathfrak{r}(z) = \begin{pmatrix} \rho(z) \\ \theta(z) \end{pmatrix} . \tag{3.34}$$

The linear transformation of a ray $\mathfrak{r}$ through a paraxial astigmatic optical system can be described by a product of 4×4 ray matrices, which generalize the well-known $ABCD$ matrices to the case of two independent transverse dimensions [12]. Here, we shall construct the ray matrix $M(z)$ that describes the transformation of a ray through the lens guide from the reference plane at $z = 0$ to the transverse plane z. In wave optics, the position of a light beam is the expectation value of the operator $\hat{\rho}$, while its propagation direction is the expectation value of $\hat{\theta}$, which is the ratio of the transverse and the longitudinal momentum. This is confirmed by the fact that the Heisenberg evolution of the operator vector $\hat{\mathfrak{r}} = (\hat{\rho}, \hat{\theta})^T$ reproduces the ray matrix, as exemplified by the general identity

$$\hat{U}^\dagger(z) \begin{pmatrix} \hat{\rho} \\ \hat{\theta} \end{pmatrix} \hat{U}(z) = M(z) \begin{pmatrix} \hat{\rho} \\ \hat{\theta} \end{pmatrix} . \tag{3.35}$$

The propagation operator $\hat{U}$ acts on the operator nature of $\hat{\rho}$ and $\hat{\theta}$, while the matrix M acts on the four-dimensional ray vector. This relation may be viewed as the optical analogue of the Ehrenfest theorem in quantum mechanics [49]. Note that the commutation relations for the components of the position and propagation-direction operators take the form

$$[\hat{\rho}_x, \hat{\theta}_x] = [\hat{\rho}_y, \hat{\theta}_y] = i\lambda\!\!\!^{-} = i/k . \tag{3.36}$$

The ray matrix $M(z)$ for propagation from the plane $z = 0$ to the plane z of the lens guide can be constructed as the product of the ray matrices for the regions of free propagation and the lenses in between these planes in the right order. The ray matrix for free propagation is described by

$$\hat{U}_f^\dagger(z) \begin{pmatrix} \hat{\rho} \\ \hat{\theta} \end{pmatrix} \hat{U}_f(z) = \begin{pmatrix} \mathsf{1} & z\mathsf{1} \\ \mathsf{0} & \mathsf{1} \end{pmatrix} \begin{pmatrix} \hat{\rho} \\ \hat{\theta} \end{pmatrix} = M_f(z) \begin{pmatrix} \hat{\rho} \\ \hat{\theta} \end{pmatrix} , \tag{3.37}$$

where $\mathsf{1}$ and $\mathsf{0}$ are the 2×2 unit and zero matrices respectively. The transformation for a thin astigmatic lens can be expressed as

$$\hat{U}_l^\dagger(\mathsf{F}) \begin{pmatrix} \hat{\rho} \\ \hat{\theta} \end{pmatrix} \hat{U}_l(\mathsf{F}) = \begin{pmatrix} \mathsf{1} & \mathsf{0} \\ -\mathsf{F}^{-1} & \mathsf{1} \end{pmatrix} \begin{pmatrix} \hat{\rho} \\ \hat{\theta} \end{pmatrix} = M_l(\mathsf{F}) \begin{pmatrix} \hat{\rho} \\ \hat{\theta} \end{pmatrix} . \tag{3.38}$$

The ray matrix of a rotation about the z axis follows from the identity

$$\hat{U}^{\dagger}_{\text{rot}}(\alpha)\begin{pmatrix} \hat{\rho} \\ \hat{\theta} \end{pmatrix}\hat{U}_{\text{rot}}(\alpha) = \begin{pmatrix} \mathsf{P}(\alpha) & 0 \\ 0 & \mathsf{P}(\alpha) \end{pmatrix}\begin{pmatrix} \hat{\rho} \\ \hat{\theta} \end{pmatrix} = M_{\text{rot}}(\alpha)\begin{pmatrix} \hat{\rho} \\ \hat{\theta} \end{pmatrix} . \tag{3.39}$$

The identities (3.35) and (3.38) remain valid for rotating lenses, which makes both the operators $\hat{U}$ and the ray matrices M depend on time. The transformation of a time-dependent ray in the reference plane to another transverse plane z is given by

$$\boldsymbol{z}(z, t) = M(z, t)\,\boldsymbol{z}(0, t - z/c)) \tag{3.40}$$

in analogy to equation (3.25). A co-rotating incident ray in the reference plane must give a co-rotating ray everywhere in the lens guide, and the ray matrices in the rotating frame become independent of time. In complete analogy to equation (3.32), this implies that the transformation of a ray in the rotating frame over one period from the reference plane is given by the round-trip ray matrix

$$M_{\text{rt}} = M_{\text{f}}(L)M_{\text{rot}}(-\Omega L/c)M_2(0)M_{\text{f}}(L)M_{\text{rot}}(-\Omega L/c)M_1(0) , \tag{3.41}$$

with $M_1(0)$ and $M_2(0)$ the ray matrices for the lenses 1 and 2 at time 0.

Any ray matrix that describes the transformation of a (sequence of) lossless optical elements obeys the following identity

$$M^{\text{T}}GM = G , \qquad \text{where} \qquad G = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} . \tag{3.42}$$

This property generalizes the requirement that the determinant of a ray matrix must be equal to 1 to optical systems that have two independent transverse dimensions. It is easy to show that the ray matrices that we have used obey this identity (3.42). The product of matrices that obey equation (3.42) obeys it as well and in mathematical terms the set of 4×4 matrices that obey this identity forms the real symplectic group $\text{Sp}(4, \mathbb{R})$. Both the underlying algebra and the physics of such linear phase space transformations have been studied in detail, see, for instance, reference [39].

3.5.2 Ladder operators in reference plane

The similarity between Hermite-Gaussian modes of a cavity with spherical mirrors and harmonic-oscillator eigenstates can be traced back to the fact that in the paraxial limit the Heisenberg evolution (3.35) of the position and propagation-direction operators $\hat{\rho}$ and $\hat{\theta}$ is linear, so that ladder operators, which are linear in these operators, preserve their general shape under propagation and optical elements. In chapter 2 we have demonstrated that the ladder operators that generate the modes of a cavity with non-parallel astigmatic mirrors are determined by the eigenvectors of the round-trip ray matrix. In the rotating frame, the relation between the propagation operators and the ray matrix for a round trip is basically the same as for a stationary one, and equations (3.33) and (3.41) are obviously analogous. This allows us to apply the same technique to obtain expressions for the modes of the rotating cavity. In order to define the ladder operators we shall need the eigenvectors and eigenvalues of the ray matrix equation (3.41). Stability of the rotating cavity requires that the eigenvalues are unitary, and since the matrix $M_{\rm rt}$ is real, this implies that its eigenvectors come in two pairs μ_1, μ_1^* and μ_2, μ_2^* that are each other's complex conjugate. The eigenvalue relations are written as

$$M_{\rm rt}\mu_1 = e^{i\chi_1}\mu_1 \quad \text{and} \quad M_{\rm rt}\mu_2 = e^{i\chi_2}\mu_2 \ . \tag{3.43}$$

From the general property (3.42) of ray matrices one directly obtains the generalized (symplectic) orthogonality properties

$$\mu_1^{\rm T} G \mu_2 = \mu_1^\dagger G \mu_2 = 0 \ , \tag{3.44}$$

while the eigenvectors can be normalized in order to obey the identities

$$\mu_1^\dagger G \mu_1 = \mu_2^\dagger G \mu_2 = 2i \ . \tag{3.45}$$

We shall now prove that the ladder operators that define the shape of the modes in the reference plane $z = 0$ at time 0 are easily expressed in terms of the eigenvectors μ_1 and μ_2 of the ray matrix $M_{\rm rt}$. Following the approach discussed in the previous chapter, we introduce two lowering operators

$$\hat{a}_i = \sqrt{\frac{k}{2}}\mu_i G \begin{pmatrix} \hat{\rho} \\ \hat{\theta} \end{pmatrix} \ , \tag{3.46}$$

where $i = 1, 2$. From the generalized orthonormality properties (3.44) and (3.45) of the eigenvectors μ_i combined with the canonical commutation rules (3.36) it follows that the ladder operators obey the bosonic commutation rules

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij} \ . \tag{3.47}$$

Any set of ladder operators that obey these commutation relations defines a complete and orthonormal set of transverse modes according to

$$|v_{nm}\rangle = \frac{1}{\sqrt{n!m!}} \left(\hat{a}_1^\dagger\right)^n \left(\hat{a}_2^\dagger\right)^m |v_{00}\rangle \ . \tag{3.48}$$

Apart from an overall phase factor, the fundamental mode (or ground state in the terminology of quantum mechanics) $|v_{00}\rangle$ is determined by the requirement that $\hat{a}_1|v_{00}\rangle = \hat{a}_2|v_{00}\rangle = 0$. The bosonic ladder operators obviously determine a complete and orthogonal set of modes in the reference plane $z = 0$. A more explicit expression of the fundamental mode will be given below. In chapter 5 we shall derive analytical expressions of the higher-order modes.

3.5.3 Ladder operators in arbitrary transverse plane

The eigenvectors $\mu_i(0) = \mu_i$ refer to the transformation from the reference plane at $z = 0$ to the plane $z = 2L$, in the rotating frame. We also need the modes $|v_{mn}(z)\rangle$, and therefore the eigenvectors $\mu_i(z)$ in an arbitrary transverse plane z in the lens guide, in the rotating frame. The basic time-dependent transformation of a ray is given by equation (3.40), so that

$$\mu_i(z) = M(z,0)M_{\text{rot}}(-\Omega z/c)\mu_i(0) \tag{3.49}$$

in analogy to equation (3.31) for the beam profile propagation in the rotating frame. In the special case of propagation over one period, we should take $z = 2L$. Then the ray transformation in (3.49) is M_{rt}, which gives

$$\mu_i(2L) = e^{i\chi_i}\mu_i(0)\ . \tag{3.50}$$

For notational convenience we separate the four-dimensional eigenvectors in their two-dimensional subvectors as

$$\mu_i(z) = \begin{pmatrix} r_i(z) \\ t_i(z) \end{pmatrix}\ . \tag{3.51}$$

Then we compose two 2×2 matrices out of the column vectors $r_i(z)$ and $t_i(z)$, by the definition

$$\mathsf{R}(z) \equiv (r_1(z), r_2(z)) \qquad \text{and} \qquad \mathsf{T}(z) \equiv (t_1(z), t_2(z))\ . \tag{3.52}$$

The relations (3.44) and (3.45) can be summarized as

$$\mathsf{R}^{\mathrm{T}}\mathsf{T} - \mathsf{T}^{\mathrm{T}}\mathsf{R} = 0 \qquad \text{and} \qquad \mathsf{R}^{\dagger}\mathsf{T} - \mathsf{T}^{\dagger}\mathsf{R} = 2i\mathsf{1} \tag{3.53}$$

in all transverse planes z.

The dependence of the ladder operators on z in the rotating frame is determined by the requirement that when acting on a rotating solution of the time-dependent paraxial wave equation, they must produce another solution. In view of equation (3.31), this requirement takes the form

$$\hat{a}_i(z) = \hat{U}(z,0)U_{\text{rot}}(-\Omega z/c)\hat{a}_i(0)U^{\dagger}_{\text{rot}}(-\Omega z/c)\hat{U}^{\dagger}(z,0)\ . \tag{3.54}$$

In the right-hand sides of this equation the propagation operators $\hat{U}$ act on the operators $\hat{\rho}$ and $\hat{\theta}$. In accordance with the general Ehrenfest relation (3.35), and the relation (3.42), this gives rise to a product $GM^{-1} = M^{\mathrm{T}}G$ when we substitute the expression (3.46) for the lowering operator. This leads to the conclusion that the z dependent lowering operator obeys the relation

$$\hat{a}_i(z) = \sqrt{\frac{k}{2}}\mu_i(z)G\begin{pmatrix} \hat{\rho} \\ \hat{\theta} \end{pmatrix} = \sqrt{\frac{k}{2}}(r_i(z)\hat{\theta} - t_i(z)\hat{\rho}) \tag{3.55}$$

for all values of z.

3.6 Structure of the modes

3.6.1 Algebraic expressions of the modes

The fundamental mode $|v_{00}(z)\rangle$ in the rotating frame obeys the requirement that the lowering operators $\hat{a}_i(z)$ give zero when acting on it for all transverse planes z in the lens guide. An explicit analytical expression of the normalized mode function as it propagates through the lens guide can be found after a slight generalization of our results for a stationary cavity discussed in chapter 2. There we have shown that the fundamental mode can be expressed in terms of the z dependent eigenvectors, which give rise to the 2×2 matrices $\mathsf{R}(z)$ and $\mathsf{T}(z)$. For rotating cavities, the same result applies in the rotating frame, where the time dependence disappears. In the rotating frame, the beam profile of the fundamental mode is given by the general Gaussian expression

$$v_{00}(R, z) = \langle \rho | v_{00}(z) \rangle = \sqrt{\frac{k}{\pi \det \mathsf{R}(z)}} \exp\left(\frac{ik\rho^{\mathrm{T}} \mathsf{T}(z) \mathsf{R}^{-1}(z) \rho}{2} \right) . \tag{3.56}$$

From the properties (3.53) of the matrices R and T it follows that the matrix $\mathsf{S} = -i\mathsf{T}\mathsf{R}^{-1}$ is symmetric. In the intervals between the lenses, the corresponding time-dependent mode $|u_{00}(z,t)\rangle = \hat{U}_{\mathrm{rot}}(\Omega t)|v_{00}(z)\rangle$ obeys the time-dependent paraxial wave equation (3.10), and the input-output relation for $|u_{00}(z,t)\rangle$ across a lens of type 1 or 2 corresponds to the lens operator $\hat{U}_1(t)$ or $\hat{U}_2(t)$ as in equation (3.17).

The periodicity (3.50) of the eigenvectors μ_i ensures that the matrix $\mathsf{S}(z) = -i\mathsf{T}(z)\mathsf{R}^{-1}(z)$ is periodic with period $2L$. Moreover, the determinant of R picks up a phase factor after one period, according to the identity

$$\det \mathsf{R}(2L) = e^{i(\chi_1 + \chi_2)} \det \mathsf{R}(0) . \tag{3.57}$$

As a result, the fundamental mode (3.56) picks up a phase factor $\exp(-i(\chi_1 + \chi_2)/2)$ after a period of the lens guide, or over a cavity round trip.

The higher-order modes $|v_{nm}(z)\rangle$ in the rotating frame are obtained from the fundamental mode by using the z dependent version of (3.48)

$$|v_{nm}(z)\rangle = \frac{1}{\sqrt{n!m!}} \left(\hat{a}_1^\dagger(z) \right)^n \left(\hat{a}_2^\dagger(z) \right)^m |v_{00}(z)\rangle . \tag{3.58}$$

The periodicity (3.50) of the eigenvector is reflected in a similar periodicity of the lowering operator, in the form

$$\hat{a}_i(z + 2L) = e^{i\chi_i} \hat{a}_i(z) , \tag{3.59}$$

which in turn will give rise to a periodicity of the modes in the rotating frame $|v_{nm}(z)\rangle$. From this equation we find that the raising operator gets an additional phase $\exp(-i\chi_i)$ after a round trip. The phase factor picked up by the mode $|v_{nm}\rangle$ (or by the time-dependent mode $|u_{nm}\rangle$) is therefore specified by the relation

$$|v_{nm}(2L)\rangle = e^{-i\chi_1(n+1/2) - i\chi_2(m+1/2)} |v_{nm}(0)\rangle . \tag{3.60}$$

The resonant wave numbers of the modes follow from the requirement that the complex electric field

$$\mathbf{E}_{nm}(\rho, z, t) = E_0 \epsilon u_{nm}(\rho, z, t)\, e^{ikz - i\omega t} \tag{3.61}$$

is periodic over a round trip. This implies that the wave number k of the transverse modes must obey the identity

$$2kL - \chi_1\left(n + \frac{1}{2}\right) - \chi_2\left(m + \frac{1}{2}\right) = 2\pi q \tag{3.62}$$

where $q \in \mathbb{Z}$ is the longitudinal mode index. Note that the round-trip Gouy phases χ_1 and χ_2, and thereby the resonant wavelengths are affected by the rotation. This is obvious since they arise from the eigenvalues of the round-trip ray matrix M_{rt}, which according to equation (3.41) contains the angular velocity Ω.

3.6.2 Spectral structure

Just as in equation (3.29), the time-dependent mode as viewed from the (non-rotating) laboratory frame follows from the mode $|v_{nm}(z)\rangle$ by a simple rotation, so that:

$$|u_{nm}(z, t)\rangle = \hat{U}_{\text{rot}}(\Omega t)|v_{nm}(z)\rangle\ . \tag{3.63}$$

The mode function $u_{nm}(\rho, z, t) = \langle\rho|u_{nm}(z, t)\rangle$ is a co-rotating solution of the time-dependent paraxial wave equation (3.10). Since this mode function depends on time, the electric field (3.61) is no longer monochromatic. The spectral structure follows directly from the polar expansion

$$u_{nm}(\rho, z, t) = u_{nm}(R, \phi - \Omega t, z) = \sum_l g_{nml}(R, z) e^{il(\phi - \Omega t)}\ . \tag{3.64}$$

From this result it is clear that the rotation of a mode converts the l^{th} Fourier component along the azimuthal angle ϕ of the spatial distribution to a frequency component $\omega + l\Omega$. Since the fundamental Gaussian mode (3.56) is even under inversion in the transverse plane $\rho \to -\rho$, the expansion (3.64) for u_{00} contains only even values of l, so that the fundamental mode only contains side bands at frequencies $\omega + 2p\Omega$ with $p \in \mathbb{Z}$ and $\omega = ck$. The ladder operators $\hat{a}_i$ are obviously odd under this inversion, so that the modes $|u_{nm}\rangle$ with even values of $n + m$ only contain the even sidebands $\omega + 2p\Omega$, while the modes with odd values of $n + m$ only contain the odd sidebands $\omega + (2p + 1)\Omega$. The separation between neighboring sidebands is always equal to 2Ω, which reflects that the cavity returns to an equivalent orientation after a rotation over an angle of 180°.

3.6.3 The cavity field

We have unfolded a cavity with rotating mirrors into a lens guide with rotating lenses and described a method to obtain expressions of the transverse modes that are reproduced after each period of the lens guide. In order to obtain an expression of the electric field inside the cavity, the lens-guide modes must be folded back into the cavity

$$\mathbf{E}_{\text{cav}}(\mathbf{r}, t) = \text{Re}\Big\{ - i\epsilon E_0 \left[u(\rho, z, t) e^{ikz} - u(\rho, 2L - z, t) e^{-ikz} \right] e^{-i\omega t} \Big\} \tag{3.65}$$

for $0 < z < L$. In the transverse planes near the two mirrors the two terms between the square brackets differ by phase factors $\exp(-ik\rho^{\mathrm{T}} \mathsf{F}_{1,2}^{-1}(t)\rho/2)$. For $z \simeq 0$ and $z \simeq L$ the electric field can be expressed as

$$\mathbf{E}_{\text{cav}}(\mathbf{r}, t) = 2\text{Re}\left\{ \epsilon E_0 f_{1,2}(\rho, t) \sin\left(kz \mp \frac{k\rho^{\mathrm{T}} \mathsf{F}_{1,2}^{-1} \rho}{4} \right) e^{-i\omega t} \right\} , \tag{3.66}$$

where the − and + signs apply near mirror 1 and 2 respectively and $f_{1,2}(\rho, t)$ is the profile in the imaginary plane halfway the lenses in the lens guide picture. The sine term in equation (3.65) is the natural generalization of a standing wave to a field with transverse spatial structure. It shows that the electric field vanishes on the mirror surfaces even if the mode profiles halfway the mirrors $f_{1,2}(\rho, t)$ have phase structure so that the wave fronts of the field (3.65) do not fit on the mirror surfaces.

3.7 Spatial symmetries

As discussed in the previous chapter (section 2.5), the lens guide corresponding to a stationary two-mirror cavity has inversion symmetry in the imaginary planes halfway the lenses. Rotation breaks this symmetry and no spatial symmetries remain in the case of a rotating cavity with general astigmatism. This is different in the case of a rotating cavity with simple astigmatism. In this section, we give a more formal description of the inversion symmetry of a stationary cavity as well as of the spatial symmetries associated with simple astigmatism. We show that both survive in a modified fashion in the case of a rotating cavity with simple astigmatism.

3.7.1 Inversion symmetry of a stationary cavity

We consider inversion $z_s + z \to z_s - z$ of the lens guide with respect to a given transverse plane z_s. As indicated in figure 3.2, the propagation direction of a ray $\theta = \partial\rho/\partial z$ picks up a minus sign under $z_s + z \to z_s - z$ so that the transformation of a ray $\boldsymbol{z}^{\mathrm{T}} = (\rho^{\mathrm{T}}, \theta^{\mathrm{T}})$ under this inversion is given by

$$\begin{pmatrix} \rho \\ \theta \end{pmatrix} \to N \begin{pmatrix} \rho \\ \theta \end{pmatrix} \qquad \text{with} \qquad N = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \begin{pmatrix} \mathsf{1} & \mathsf{0} \\ \mathsf{0} & -\mathsf{1} \end{pmatrix} . \tag{3.67}$$

As a result, the inverted transformation corresponding to a ray matrix M takes the following modified form

$$N M^{-1} N^{\mathrm{T}} = M . \tag{3.68}$$

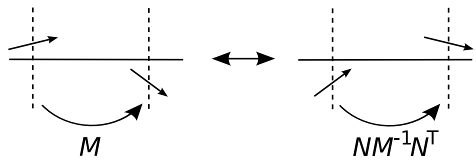

Figure 3.2: Inversion of an optical set-up. When both a ray trajectory and the elements of which the set-up consists are inverted with respect to some transverse plane z_s, the angles ϑ_x and ϑ_y, which specify the propagation direction of a ray, change sign. This is accounted for by the matrix N as defined in equation (3.67).

If the lens guide has inversion symmetry with respect to the plane $z = z_\mathrm{s}$, the ray matrix $M_\mathrm{rt}(z_\mathrm{s})$ that describes the transformation of a round trip starting from this plane must be equal to the corresponding inverted transformation, so that

$$NM_\mathrm{rt}^{-1}(z_\mathrm{s})N^\mathrm{T} = M_\mathrm{rt}(z_\mathrm{s}) \; . \tag{3.69}$$

In terms of the eigenvectors μ_i of the round-trip ray matrix $M_\mathrm{rt}(z_\mathrm{s})$, this symmetry property implies that

$$M_\mathrm{rt}(z_\mathrm{s})\mu_i(z_\mathrm{s}) = NM_\mathrm{rt}^{-1}(z_\mathrm{s})N^\mathrm{T}\mu_i(z_\mathrm{s}) = \lambda_i\mu_i(z_\mathrm{s}) \; , \tag{3.70}$$

where $\lambda_i = \exp(-i\chi_i)$ are the corresponding unitary eigenvalues. Using that $\mu_i^*(z_\mathrm{s})$ is an eigenvector of $M_\mathrm{rt}^{-1}(z_\mathrm{s})$ with eigenvalue λ_i, we find that inversion symmetry implies that

$$N\mu_i(z_\mathrm{s}) = \mu_i^*(z_\mathrm{s}) \; . \tag{3.71}$$

It follows that $r_i(z_\mathrm{s})$ and $\mathsf{R}(z_\mathrm{s})$ are real while $t_i(z_\mathrm{s})$ and $\mathsf{T}(z_\mathrm{s})$ are purely imaginary. As a result, both the fundamental mode (3.56) and the ladder operators (3.46) are real in the symmetry plane so that all modes are real in that plane.

Rotation obviously breaks the inversion symmetry in the transverse planes halfway the lenses and the round-trip ray matrix in the co-rotating frame (3.41) does not obey the identity (3.69). As a result the mode profiles halfway the lenses $f_{1,2}(\rho, t)$ are not real in general and contribute to the phase structure of the cavity field close to the mirrors (3.66).

3.7.2 Simple astigmatism

The lens guide corresponding to a stationary two-mirror cavity with simple astigmatism has symmetry directions parallel to the mirror axes in all transverse planes. These symmetry directions arise from the invariance of the cavity under reflections in the planes through the mirror axes and the cavity axis. In terms of the round-trip ray matrix $M(z)$, this symmetry property can be expressed as

$$QM_{\mathrm{rt}}(z)Q^{\mathrm{T}} = M_{\mathrm{rt}}(z) \qquad \text{with} \qquad Q = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \equiv \begin{pmatrix} \mathsf{Q} & 0 \\ 0 & \mathsf{Q} \end{pmatrix} . \tag{3.72}$$

The transformation Q describes a reflection in the xz plane. Similarly, $-Q$ describes a reflection in yz plane. In terms of the eigenvectors, this invariance (3.72) implies that $Q\mu_i(z) = \mu_i(z)$ so that $\mathsf{QR} = \mathsf{R}$ and $\mathsf{QT} = \mathsf{T}$. It follows that $u(\rho, z) = u(\mathsf{Q}\rho, z)$ so that the phase and intensity patterns of all the modes are aligned along the symmetry directions.

One may show easily that a stationary or rotating cavity with general astigmatism as well as a rotating cavity with simple astigmatism lacks the symmetry described by equation (3.72).

3.7.3 Rotating cavities with simple astigmatism

Rotation breaks both the inversion symmetry of a stationary cavity (3.69) and the spatial symmetries arising from simple astigmatism (3.72). As a result, no spatial symmetries remain in the case of a rotating cavity with general astigmatism. In the special case of a rotating cavity with simple astigmatism, however, one may prove explicitly that both symmetries survive when combined with inversion of the rotation direction $\Omega \to -\Omega$. These statements can be expressed as

$$QM_{\mathrm{rt}}(z;\Omega)Q^{\mathrm{T}} = M_{\mathrm{rt}}(z;-\Omega) , \tag{3.73}$$

and

$$NM_{\mathrm{rt}}^{-1}(z_{\mathrm{s}};\Omega)N^{\mathrm{T}} = M_{\mathrm{rt}}(z_{\mathrm{s}};-\Omega) , \tag{3.74}$$

where z_{s} is an inversion-symmetry plane and $M_{\mathrm{rt}}(z;-\Omega)$ is the ray matrix that describes the transformation of a round trip starting from a transverse plane z. The ray matrix $M_{\mathrm{rt}}(z;-\Omega)$ depends on the rotation frequency Ω according to equation (3.41). Combining equations (3.73) and (3.74) yields

$$QNM_{\mathrm{rt}}^{-1}(z;\Omega)(QN)^{\mathrm{T}} = M_{\mathrm{rt}}(z;\Omega) . \tag{3.75}$$

In terms of the eigenvectors $\mu_i(z;\Omega)$ this implies that

$$QN\mu_i(z;\Omega) = \mu_i^*(z;\Omega) , \tag{3.76}$$

from which we conclude that $\mathsf{QR}(z;\Omega) = \mathsf{R}^*(z;\Omega)$ and $\mathsf{QT}(z;\Omega) = -\mathsf{T}^*(z;\Omega)$ so that $v(\mathsf{Q}\rho, z) = v^*(\rho, z)$. As a result, the intensity patterns $|u(\rho, z, t)|^2$ of the modes of a rotating cavity with simple astigmatism are aligned along the mirror axes at all times, while the phase patterns are not.

3.8 Orbital angular momentum

In section 2.5.3, we have shown that the twisted boundary conditions that are imposed by a pair of non-parallel astigmatic mirrors induce orbital angular momentum in the modes of such a cavity. In the present case of a rotating cavity, we expect the twisted mode propagation in the rotating frame (3.31) combined with non-isotropic boundary conditions to give rise to orbital angular momentum as well. In this section we analyze the orbital angular momentum in rotating cavity modes. We discuss the contribution due to their phase structure and show that the physical rotation of the mode patterns contributes only in higher-order of δ. As a result this contribution is significant only in special cases where the orbital angular momentum due to the mode structure vanishes [19, 20].

The leading-order contribution to the orbital angular momentum per unit length in a monochromatic paraxial field is given by equation (2.80). The obvious extension of this result to the present case of polychromatic modes involves a summation over the frequency components $\omega + l\Omega$ with $l \in \mathbb{Z}$. Substitution of the spectral expansion (3.64) then gives the following expression for the orbital angular momentum in the rotating cavity modes (3.63)

$$\mathcal{L}_{nm} = \sum_{l=-\infty}^{\infty} \frac{\epsilon_0 |E_0|^2}{2(\omega + l\Omega)} \int_0^{2\pi} d\phi \int_0^{\infty} R dR \; l |g_{nml}(R, z)|^2 \; . \tag{3.77}$$

Using that the rotation frequency Ω is of the order of δ^2 smaller than the optical frequency ω, the leading-order contribution to the orbital angular momentum can be expressed as

$$\mathcal{L}_{nm}^{(0)} = \frac{\epsilon_0 |E_0|^2}{2\omega} \sum_{l=-\infty}^{\infty} \int_0^{2\pi} d\phi \int_0^{\infty} R dR \; l |g_{nml}(R, z)|^2 = N\hbar \langle v_{nm}(z) | \hat{\rho} \times k\hat{\theta} | v_{nm}(z) \rangle \; , \tag{3.78}$$

where the superscript (0) refers to the fact that this contribution is of order 0 in Ω/ω. This contribution obviously has the significance of the orbital angular momentum in the co-rotating frame and it is due to the spatial phase and intensity structure of the modes. Its dependence on the rotation frequency Ω can be traced back to fact that the round-trip ray matrix in the rotating frame (3.41) depends on Ω so that the same is true for the modes $|v_{nm}(z)\rangle$. In analogy with equation (2.81), this result (3.78) can be expressed in terms of the eigenvectors of the round-trip ray matrix (3.41)

$$\mathcal{L}_{nm}^{(0)} = N\hbar \left\{ \left(n + \frac{1}{2} \right) \mathrm{Re}(r_1^*(z) \times t_1(z)) + \left(m + \frac{1}{2} \right) \mathrm{Re}(r_2^*(z) \times t_2(z)) \right\} \; , \tag{3.79}$$

where $N = \epsilon_0 |E_0|^2 / (2\hbar\omega)$ is the number of photons per unit length and $r_i(z)$ and $t_i(z)$ are related to the eigenvectors $\mu_i(z)$ of the round-trip ray matrix (3.41) by equations (3.51) and (3.49). Since free space is isotropic, $\mathcal{L}_{nm}^{(0)}$ is conserved under propagation from one mirror to the other. In the special case of a rotating cavity with simple astigmatism, the modified inversion symmetry (3.75) requires that the orbital angular momentum takes the same value $0 < z < L$ and $L < z < 2L$ so that the modes do not exert a torque on the mirrors. This is not true if the cavity has general astigmatism. In that case both the astigmatism and the rotational deformation of the modes contribute to the orbital angular momentum, which takes different values in the two intervals of the lens guide. As a result, rotating modes with general astigmatism do exert a torque on the mirrors. However, also in the general-astigmatic case, there is net orbital angular momentum in the rotating cavity field. In this respect, rotating cavity modes with general astigmatism are different from their stationary counterparts.

The next-to-leading-order contribution to the orbital angular momentum is of the order of $\Omega/\omega \sim \delta^2$ smaller than the leading order term (3.78). It is non-negligible only in specific cases where the orbital angular momentum due to the spatial structure of the modes vanishes, for instance, for reasons of symmetry. It can be expressed as

$$\mathcal{L}^{(1)}_{nm} = -\frac{\epsilon_0|E_0|^2\Omega}{2\omega^2}\sum_{l=-\infty}^{\infty}\int_0^{2\pi} d\phi \int_0^{\infty} RdR\ l^2|g_{nml}(R,z)|^2\ . \tag{3.80}$$

This contribution is due to the fact that the mode patterns $u_{nm}(z,t)$ (3.64) in the laboratory frame rotate as a function of time. This general expression of the orbital angular momentum in a paraxial mode that does not possess orbital angular momentum in the co-rotating frame, is a slight generalization of the results in references [19] and [20]. A naive interpretation of the fact the rotational contribution to the orbital angular momentum is always negative would be that a light field has a negative moment of inertia. However, even in free space rotation has significant effects on the spatial and spectral structure of a light field so that it does not at all resemble a rigid body. As a result the concept of a moment of inertia is ill-defined in this wave-mechanical context [19, 20].

3.9 Examples

We illustrate some of the physical properties of rotating cavity modes by investigating two specific examples. We discuss the spatial structure of the intensity patterns of rotating cavity modes both in case of simple and in case of general astigmatism. The spectral structure of simple-astigmatic rotating cavity modes is discussed as well. Typical examples of the dependence of the orbital angular momentum on the rotation frequency are discussed in the next chapter, while the rotational deformation of the phase structure of optical cavity modes is discussed in detail in chapter 5.

3.9.1 Rotating simple astigmatism

The simplest realization of a uniformly rotating cavity consists of a stationary spherical and a rotating astigmatic (or cylindrical) mirror. In the absence of rotation the modes of such a cavity are astigmatic Hermite-Gaussian modes [12]. A cavity of this type has transverse symmetry directions parallel to the mirror axes along which the modes scale differently. Typical examples of the intensity patterns of astigmatic Hermite-Gaussian modes in the immediate neighborhood of the mirrors are shown in the upper windows in figure 3.3. Notice that the

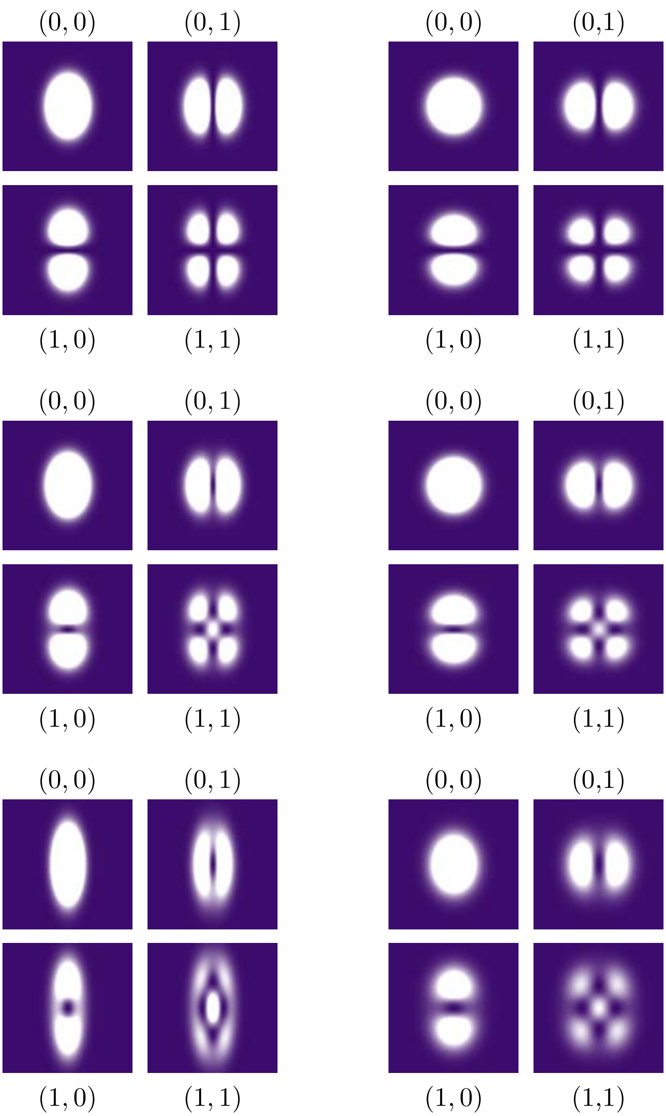

Figure 3.3: Intensity patterns of the (0, 0), (0, 1), (1, 0) and (1, 1) modes of the cavity between a stationary spherical and a rotating astigmatic mirror for different values of the rotation frequency Ω. The plots in the left column show the intensity patterns near the spherical mirror while the plots in right column show the intensity pattern near the astigmatic mirror. The radius of curvature of the spherical mirror is $4L$, where L is the mirror separation. The radius of curvature of the astigmatic mirror in the horizontal direction of the plot is equal to 2L while its radius of curvature in the vertical direction is 20L. From the top to the bottom the rotation frequency is increased from $\Omega = 0$ to $\Omega = c\pi/(30L)$ and $\Omega = c\pi/(6L)$.

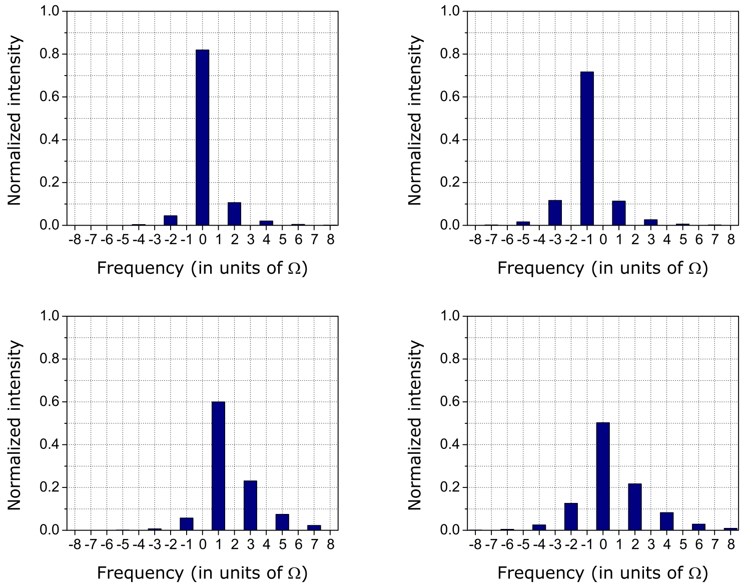

Figure 3.4: Spectral structure of the $(0, 0)$, $(0, 1)$, $(1, 0)$ and $(1, 1)$ modes of the cavity between a stationary spherical and rotating astigmatic mirror. The radius of curvature of the spherical mirror is equal to $4L$, where L is the mirror separation and the radii of curvature of the astigmatic mirror are equal to $2L$ and $20L$ respectively. The rotation frequency is equal to $\Omega = c\pi/(6L)$.

astigmatism of the intensity patterns is most pronounced on the spherical mirror. This is due to the fact that the astigmatism of a mirror is visible in the intensity pattern of the reflected beam only after free propagation over some distance.

If the astigmatic mirror is put into rotation the mode structure changes significantly. This is shown in the other two windows of figure 3.3. As a result of the rotation, the cavity is no longer invariant under reflection in the planes through the mirror axes and the cavity axis. In the special case of simple astigmatism, it is invariant under reflection in these planes combined with inversion of the rotation direction. As a result, the intensity patterns of the modes are still aligned along the mirror axes but the phase distributions are not. Rotation also breaks the inversion symmetry of the corresponding lens guide in the imaginary planes halfway the mirrors. As a result, the higher-order modes are no longer Hermite-Gaussian modes but generalized Gaussian modes with a nature in between Hermite- and Laguerre-Gaussian modes [44]. As a result, phase singularities (optical vortices) appear, which are best visible in the center of the $(0, 1)$ and $(1, 0)$ modes in figure 3.3.

The spectral structure of the rotating modes is illustrated in figure 3.4. These spectra show that the modes are confined spectrally and confirm that they only have odd or even frequency components depending on the parity of the total mode number $n + m$. Due to the reflection symmetry in the planes through the mirror axes and the cavity axis, the orbital-angular-momentum spectrum of cavity modes with simple astigmatism is symmetric, i.e., $g_l(\rho) = g_{-l}(\rho)$ in the absence of rotation. Rotation breaks this symmetry, as is confirmed by the spectra in figure 3.4.

3.9.2 Rotating general astigmatism

The mode structure becomes significantly more complex if the cavity has general astigmatism, which is the case if it consists of two non-aligned astigmatic mirrors. Such a cavity does not have reflection symmetry planes through the optical axis. In the stationary case, the corresponding lens guide does have inversion symmetry in the imaginary plane halfway the lenses so that the higher-order modes close to the mirrors have the nature of astigmatic Hermite-Gaussian modes. Typical examples of the modes of a stationary astigmatic cavity with general astigmatism are shown in the upper window of figure 3.5.

At first sight one might guess that physical rotation of the mirrors effectively modifies their relative orientation so that it can help to reduce the effect of general astigmatism. This is not the case. The effect of rotation of the mirrors is essentially different from the effect of general astigmatism. This is illustrated in the lower window of figure 3.5. The rotation frequency is chosen such that the rotation angle after one round trip is equal but opposite to the angle between the orientations of the two mirrors. Putting the mirrors into physical rotation breaks the inversion symmetry so that the modes are no longer Hermite-Gaussian but generalized Gaussian modes that have general astigmatism. As a result, again, vortices appear.

3.10 Discussion and conclusion

In this chapter we have derived an algebraic method to obtain explicit expressions of the paraxial modes of an astigmatic optical cavity that is put into uniform rotation about its optical axis. Uniform rotation is homogeneous in time so that the explicit time-dependence of a rotating cavity can be eliminated by a transformation to the co-rotating frame. Its paraxial modes can then be obtained as solutions of the time-dependent paraxial wave equation (3.10) that are stationary in the rotating frame, i.e., rotate along with the mirrors in the laboratory frame. Up to first order of the paraxial approximation, the boundary condition that the electric field vanishes on the mirror surfaces is not affected by the transformation to the co-rotating frame. Mixing of the electric and magnetic fields is a second-order, relativistic, effect. The rotating cavity modes are thus obtained as stationary solutions in the co-rotating frame that vanish on the mirror surfaces. The regime of validity of the time-dependent paraxial wave equation provides a natural upper limit for the rotation frequency (3.12).

The method that we have used to derive expressions of the rotating cavity modes generalizes the ladder-operator method that we have introduced in the previous chapter to this time-dependent case. It involves two pairs of bosonic ladder operators that generate a complete and orthogonal set of modes in the rotating frame (3.48). The transformation of the ladder operators from a reference plane in the co-rotating frame to an arbitrary transverse plane (3.54) can be expressed in terms of the 4×4 ray matrix that describes the linear transformation of a ray through the same system. As a result, the ladder operators that generate the cavity modes can be constructed from the eigenvectors of the ray matrix for a round trip in the co-rotating frame (3.41). Just as in the case of a cavity with stationary mirrors, geometric stability turns out to be the necessary and sufficient condition for the rotating cavity to have modes. The time-dependent expressions of the the modes $|u(z,t)\rangle$ in an external observer's frame can be obtained from the corresponding modes in the rotating frame $|v(z)\rangle$ by using equation (3.29).

In the rotating frame, the ray and wave dynamics is modified even though the ray matrices do not depend on time. In section 3.7, we have studied how rotation modifies the symmetry properties of a cavity and its modes while rotational effects on their orbital angular momentum were discussed in section 3.8. In section 3.9 we have shown how the mode structure is affected by rotation for different values of the rotation frequency and that the modes remain spectrally confined as well. In the last part of section 3.9 we have studied the interplay between general astigmatism and rotation. In both cases the cavity no longer has inversion symmetry so that the higher order modes are generalized Gaussian modes that have a nature in between Hermite-Gaussian and Laguerre-Gaussian modes with optical vortices.

The mode criterion that we have formulated in this chapter hinges upon the homogeneous time dependence of a rotating cavity and cannot be generalized to cavities with mirrors that rotate at different frequencies. In principle one can define the period of such a system by considering the number of round trips that is needed for both mirrors to return to positions that are equivalent to their initial positions. Once this period is determined, the method that we have developed here can be applied to find its modes, provided that the cavity is geometrically stable at all times. Preliminary numerical calculations, however, suggest that geometric stability is a heavy requirement in this, non-homogeneous, case.

The set-up that we have discussed in this chapter is hard to realize experimentally. In section 5.5 we shall discuss several possible routes towards experimental realization of a set-up that captures the essential optical properties a rotating astigmatic two-mirror cavity. Moreover, the methods that we have developed here provide a much more general framework to cope with propagation and retardation in optical set-ups that have elements with time-dependent settings. The only restriction is that the time-dependent paraxial approximation, which we have formulated in section 3.2, is justified.

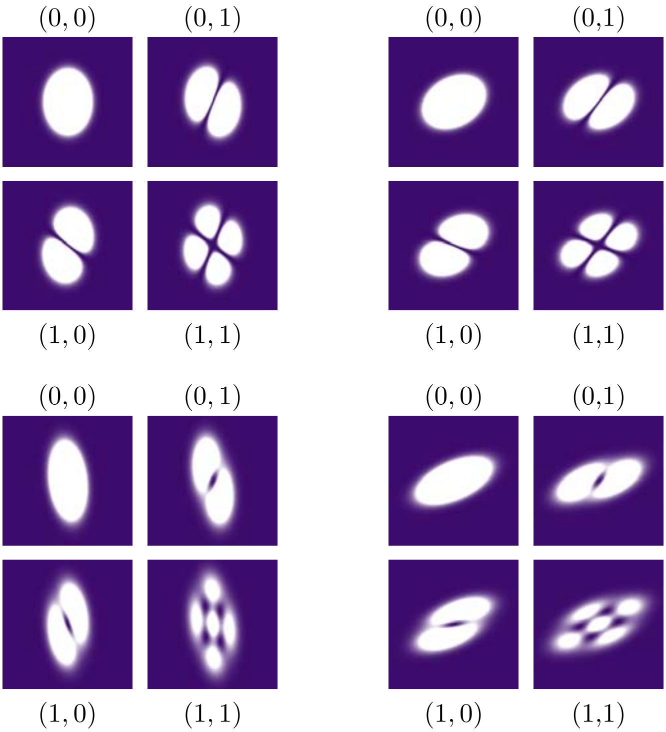

Figure 3.5: Modes of an optical cavity between two identical but non-aligned rotating astigmatic mirrors for different rotation frequencies. The mirrors have radii of curvature that are equal to $2L$ and $20L$. The axes of the right mirror coincide with the horizontal and vertical directions of the plots while the axes of the left mirror are rotated over an angle $-\pi/3$. From the top to the bottom the rotation frequency is increased from $\Omega = 0$ to $\Omega = c\pi/(6L)$. The latter frequency is chosen such that the angle over which the mirrors are rotated after each round trip is equal but opposite to the angle between the orientations of the two mirrors.