Light with a twist : ray aspects in singular wave and quantum optics

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Twisted light

1.1 Introduction

In this thesis we theoretically investigate optical modes with a highly non-trivial spatial, and in some cases also spectral, structure. We introduce an algebraic method to obtain explicit expressions of the modes to all orders inside a two-mirror cavity with twisted boundary conditions and apply these to study some of their physical properties. We generalize the concept of a cavity mode to the case of a two-mirror cavity that is put into uniform rotation about its optical axis and focus on the special case of a rotating astigmatic two-mirror cavity. We extend our algebraic method to account for time-dependent mirror settings and study some optical and opto-dynamical properties of this simple, but surprisingly rich, set-up. We discuss a complete and general characterization of the parameter space underlying basis sets of paraxial optical modes and study the geometric phase shift that arises from it. This phase shift constitutes the ultimate generalization of the Gouy phase in paraxial wave optics. We show that, in free space, the concept of a rotating mode of the radiation field can be generalized beyond the paraxial regime and show that the field can be quantized in an orthonormal, but otherwise arbitrary, basis of rotating modes, thereby constructing the first exact quantum theory of rotating light.

In this first, introductory, chapter we put the material discussed in the rest of the thesis in a somewhat broader context. Twisted and rotating boundary conditions are a natural source of orbital angular momentum and vorticity in optical fields. In the next section we give a brief historical introduction to these topics and discuss some applications in various branches of modern quantum optics. The mathematical method that we develop and apply to characterize the dynamics of, mostly classical, wave fields generalizes well-established operator techniques from quantum mechanics. It is exact up to leading order of the (time-dependent) paraxial approximation and hinges upon the tight connection between wave and ray optics. Throughout the thesis we shall mostly use it in its canonical operator representation. However, both the analogy with quantum mechanics and the connection with ray optics are more conveniently discussed in the equivalent integral representation, which is the optical analogue of the path-integral formulation of quantum mechanics. This is worked out explicitly in sections 1.3 and 1.4. In the final section of this chapter, we give a detailed outline of this thesis.

1.2 Optical angular momentum

The ability of light to exert torques and forces on a material object was first recognized by Kepler. In his book *De cometis libelli tres* [1], published in 1619, he proposed that the empirical fact that a comet's tail always points away from the sun, is due to a radiative force exerted by the sun light. Initially, this proposal attracted quite some attention, especially in the context of the then ongoing debate whether light is composed of particles or should be considered a wave phenomenon. However, since various attempts to experimentally observe mechanical forces of light failed, the interest slowly dwindled [2].

When in the early 1860's Maxwell was the first to realize that light is a manifestation of the electromagnetic field [3], it became possible to study the mechanical properties of light, such as its energy, momentum and angular momentum, within the framework of classical electrodynamics. By studying the exchange of energy between a set of charged particles and the electric and magnetic fields, Poynting showed in 1884 that the energy density associated with the electromagnetic field in vacuum can be expressed as $\left(\epsilon_0|\mathbf{E}|^2 + \mu_0^{-1}|\mathbf{B}|^2\right)/2$, where $\mathbf{E}$ and $\mathbf{B}$ are respectively the electric and the magnetic field, and ϵ_0 and μ_0 are the permittivity and the permeability of vacuum [4]. From similar considerations, one may deduce that the momentum density of the electromagnetic field in vacuum can be expressed as $\mathcal{P} = \epsilon_0 \mathbf{E} \times \mathbf{B}$ so that the angular momentum associated with the electromagnetic field is given by [4]

$$\mathbf{J} = \int d_3\mathbf{r} \left(\mathbf{r} \times \mathcal{P}\right) = \epsilon_0 \int d_3\mathbf{r} \left(\mathbf{r} \times (\mathbf{E} \times \mathbf{B})\right). \tag{1.1}$$

By Helmholtz's theorem, the electric and magnetic field can be decomposed into the transverse radiation field and the longitudinal Coulomb field such that $\mathbf{E} = \mathbf{E}_{\|} + \mathbf{E}_{\perp}$, with $\nabla \cdot \mathbf{E}_{\perp} = 0$ and $\nabla \times \mathbf{E}_{\|} = 0$. From Maxwell's equations it follows that the longitudinal contribution to the magnetic field vanishes so that the angular momentum arising from the radiation field can be obtained from equation (1.1) by replacing the electric field by the transverse electric field $\mathbf{E}_{\perp}$. In general, it is convenient to introduce a scalar potential Φ and a vector potential $\mathbf{A}$ such that $\mathbf{E} = -\nabla\Phi - \dot{\mathbf{A}}$ and $\mathbf{B} = \nabla \times \mathbf{A}$ [4]. The scalar potential does not contribute to the radiation field so that $\mathbf{E}_{\perp} = -\dot{\mathbf{A}}_{\perp}$. Substitution of $\mathbf{E}_{\perp} = -\dot{\mathbf{A}}_{\perp}$ and $\mathbf{B} = \nabla \times \mathbf{A}$ in the expression of the angular momentum of the radiation field in vacuum yields after partial integration [5]

$$\mathbf{J}_{\text{rad}} = \mathbf{L}_{\text{rad}} + \mathbf{S}_{\text{rad}} \ , \tag{1.2}$$

with

$$\mathbf{L}_{\text{rad}} = \epsilon_0 \sum_i \int d_3\mathbf{r}\ (\dot{\mathbf{E}}_\perp)_i \, (\mathbf{r} \times \nabla) \, \mathbf{A}_i \qquad \text{and} \qquad \mathbf{S}_{\text{rad}} = \epsilon_0 \sum_i \int d_3\mathbf{r}\ \dot{\mathbf{E}}_\perp \times \mathbf{A} \ , \tag{1.3}$$

where the index i runs over the vector components of the field. The first contribution in equation (1.3) is extrinsic in that it depends on the origin of the coordinate system used. By a proper choice of the origin, it can be made to vanish. As such, it may be viewed as the wave-optical analogue of the orbital angular momentum associated with the center-of-mass motion of two bodies, one of which orbits around the other [6]. The second contribution in equation (1.3), on the other hand, is obviously intrinsic and has the flavor of spin. However, although it may be shown that it indeed takes the form of the expectation value of the spin of a spin-1 particle, its interpretation as a spin is not without severe and fundamental difficulties [5]. These difficulties originate from the fact that the photon travels at the speed of light and, therefore, by special relativity, must have zero rest mass. The spin of a massive particle may be defined as its total angular momentum in a co-moving frame but, since the photon travels at the speed of light, its co-moving frame is non-existent. As a result, its spin is ill-defined [7]. This is illustrated by the fact that, in a quantized description of the radiation field, the operators corresponding to the components of $\mathbf{S}_{\text{rad}}$ do not obey the proper commutation rules [8, 6].

Physically speaking, the intrinsic (or spin) contribution to the angular momentum of the radiation field arises from its vector nature. In a circularly polarized beam it amounts to $\hbar$ per photon. Since a linearly polarized beam contains equal contributions of the two opposite circular polarizations, it bears no net spin angular momentum. Already in 1936, it has been demonstrated experimentally that the exchange of spin angular momentum between a circularly polarized beam of light and a birefringent crystal through which it propagates, gives rise to a torque on the crystal [9]. The extrinsic (or orbital) contribution to the angular momentum, on the other hand, arises from the phase structure of the field. Although optical forces arising from transverse phase gradients had been observed in optical tweezers [10], it was not before 1992 that it was realized that optical beams bearing orbital angular momentum can easily be produced and manipulated in experimental set-ups with laser beams [11]. In the standard case of a Laguerre-Gaussian beam [12], the orbital angular momentum arises from an optical vortex on the beam axis. Optical vortices are point singularities of the phase of the radiation field and give rise to helical wave fronts, which characterize a circular rather than a linear distribution of the transverse momentum [13]. During the past decades, the physics of optical vortices has been studied widely in the field of singular optics [14, 15]. In addition to vorticity, also general astigmatism contributes to the orbital angular momentum in optical fields [16, 17]. General astigmatism arises when the transverse intensity and phase distributions of an optical beam are anisotropic and non-aligned [18]. It gives rise to tumbling of the beam under free propagation [17]. The orbital angular momentum per photon in optical beams with vortices and/or general astigmatism can be significantly larger than the spin angular momentum per photon in a circularly polarized beam. Under realistic experimental conditions, values of $10\hbar$ per photon can be achieved easily. Perhaps the most natural source of optical orbital angular momentum is physical rotation of a transverse field pattern [19, 20]. However, under typical circumstances, this rotational contribution is very small compared to the orbital angular momentum due to the transverse structure of a beam.

In the eighteen years that have passed since the first experiments were performed, optical orbital angular momentum has played a central role in various branches of modern quantum optics [21]. As opposed to the space of polarization states, which is inherently two-dimensional, the space of optical orbital angular momentum states is infinite-dimensional. Since in 2001, quantum entanglement in the orbital angular momentum of photons was first demonstrated experimentally [22], this infinite-dimensional nature has offered a whole range of interesting possibilities and challenges see, for instance, reference [23]. Also in the field of optical tweezers, the orbital angular momentum has been used to manipulate small particles [24]. Recently, it has been shown theoretically that the orbital angular momentum in a Laguerre-Gaussian beam can be sufficiently large to trap and cool the rotational degree of freedom of a mirror [25]. This suggests possible application of optical orbital angular momentum in the rapidly developing field of (cavity) opto-mechanics.

1.3 Classical optics and quantum mechanics

Long before the days of Maxwell, and even before the debate whether light consists of particles or should be considered a wave was settled in favor of the wave description by Young's famous double-slit experiments in 1801, the propagation and diffraction of waves was pretty well-understood. In 1678, Huygens first formulated his principle that every point on a wave front acts as a source of spherical waves. The wave front at a distant location is the envelope of these spherical waves. It took until 1690 before Huygens published the principle in his book *Traité de la lumière* [26]. Between 1815 and 1819, the wave theory of light was significantly refined by Fresnel, who, in the spirit of Young's double-slit experiment, added the notion of interference to what is nowadays called the Huygens-Fresnel principle. For monochromatic complex scalar waves $E(\mathbf{r}, t) = E(\mathbf{r}) \exp(-i\omega t)$ it can be expressed as [27]

$$E(\rho, z) = \frac{2\pi k}{i} \int d_2\rho_0 \, \frac{\exp\left(ik|\mathbf{r} - \mathbf{r}_0|\right)}{|\mathbf{r} - \mathbf{r}_0|} E(\rho_0, z_0) \cos\theta \,, \tag{1.4}$$

where $\rho = (x, y)^{\mathrm{T}}$ is the transverse position vector, $k = \omega/c$ is the wave number and θ is the angle between the position vector $\mathbf{r}$ and the normal to the wave front in the z_0 plane. The Huygens-Fresnel integral (1.4) characterizes the complex spatial field $E(\rho, z)$ in some transverse plane z as a coherent superposition of spherical waves emanating from point sources in the plane z_0. The amplitudes and relative phases of the spherical waves are given by the transverse field distribution $E(\rho_0, z_0)$ in the plane z_0. The additional obliquity factor $\cos\theta$ is related to the fact that only spherical waves that propagate away from the sources are taken into consideration [28]. It gives the spherical waves an angular profile. In its original form, the Huygens principle explains reflection and refraction of light at an interface. The modified Huygens-Fresnel principle as described by equation (1.4) also describes phenomena arising from interference and diffraction. At the time it was formulated, the Huygens principle was more of brilliant but somewhat qualitative guess rather than a formal and mathematically rigorous statement. However, in 1882 Kirchhoff derived the Huygens-Fresnel integral directly from Maxwell's equations [4]. In hindsight, the Huygens-Fresnel integral (1.4) may be considered as the first example of a path integral in physics.

The fundamental principle that underlies the ray-optical description of light is the principle of least time, first formulated by Fermat in 1662. This principle states that a ray of light optimizes the optical path length between two points in space and plays a role analogous to that of the principle of least action in classical mechanics. Since the speed of light is determined by the optical density of the medium, as characterized by the refractive index $n(x, y, z)$, the velocity of a ray of light is not an independent dynamical variable. It follows that a ray of light can be fully characterized by its three spatial coordinates as a function of some parameter, which we choose to be the z coordinate. In that case, the optical path length of a ray can be expressed as

$$L = \int_{z_1}^{z_2} dz\, n(x, y, z) \sqrt{1 + x'^2 + y'^2}\ , \tag{1.5}$$

where z_1 and z_2 are the z coordinates of the begin and end points of the ray and where $x' = \partial x/\partial z$ and $y' = \partial y/\partial z$. Since the path length plays the role of the action, the argument of the integral in equation (1.5) is the Lagrangian $\mathcal{L}$ that describes the propagation of optical rays through a medium characterized by the refractive index $n(x, y, z)$. With the corresponding momenta, which are defined as $\partial\mathcal{L}/\partial x'$ and $\partial\mathcal{L}/\partial y'$, this naturally leads to a canonical formulation of geometric optics [29].

In many optical set-ups, the light propagates along a well-defined direction so that paraxial approximations (from the Ancient Greek $\pi\alpha\rho\alpha$, which literally means alongside of) are justified. In mathematical terms, the assumption that the light mainly propagates along the z axis implies that $x', y' << 1$. In case of paraxial propagation through vacuum ($n = 1$), the Lagrangian arising from the path length (1.5) can be approximated by

$$\mathcal{L} = 1 + \frac{1}{2}\left(x'^2 + y'^2\right) . \tag{1.6}$$

The corresponding momenta are given by $\vartheta_x = \partial\mathcal{L}/\partial x' = x'$ and $\vartheta_y = \partial\mathcal{L}/\partial y' = y'$ and correspond to direction angles measured with respect to the z axis. Following the standard construction of the Feynman path integral [30], $\hbar$ being replaced by $\lambda\!\!\!^{-} = \lambda/2\pi = 1/k$, the path integral corresponding to the Lagrangian in equation (1.6) can be expressed as

$$E(\rho, z) = \frac{2\pi k e^{ik(z-z_0)}}{i(z - z_0)} \int d_2\rho_0\ \exp\left(\frac{ik\left((x - x_0)^2 + (y - y_0)^2\right)}{2(z - z_0)}\right) E(\rho_0, z_0)\ , \tag{1.7}$$

which clearly is a paraxial approximation of the Huygens-Fresnel integral (1.4) [12]. If we write the field as the product $E(\rho, z) = u(\rho, z)\exp(ikz)$ of a spatial profile $u(\rho, z)$ and a carrier wave $\exp(ikz)$, it follows that (1.7) corresponds to the general solution of the paraxial wave equation

$$\left(\nabla_\rho^2 + 2ik\frac{\partial}{\partial z}\right)u(\rho, z) = 0 , \tag{1.8}$$

where $\nabla_\rho^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the transverse Laplacian. The paraxial wave equation (1.8) takes the form of the Schrödinger equation for a free particle in two dimensions, $\lambda\!\!\!^{-} = 1/k$ playing the role of $\hbar/m$ with m the mass of the particle. It describes the spatial evolution of the profile $u(\rho, z)$ of a paraxial beam, which characterizes its large-scale spatial structure, and plays a central role in this thesis.

It is noteworthy that, in the present context of monochromatic scalar waves, the paraxial approximation plays a role similar to that of the non-relativistic approximation in quantum mechanics [31]. In the optical case, the exact wave equation reduces to the paraxial wave equation for fields that mainly propagate along a well-defined direction while in quantum mechanics, the Klein-Gordon equation, which describes a massive scalar field, reduces to the ordinary Schrödinger equation for fields that only contain mainly time-like components.

1.4 First-order optics

An interesting property of the paraxial Lagrangian (1.6) is that the evolution of the transverse coordinates $\rho = (x, y)^{\mathrm{T}}$ and the corresponding momenta $\theta = (\vartheta_x, \vartheta_x)^{\mathrm{T}} = \partial\rho/\partial z$, which are conveniently combined in a four-dimensional ray vector $\mathfrak{r}^{\mathrm{T}} = (\rho^{\mathrm{T}}, \theta^{\mathrm{T}})$, is linear. The solution of the Euler-Lagrange equation deriving from the Lagrangian (1.6) can be expressed as

$$\mathfrak{r}(z) = \begin{pmatrix} \mathsf{1} & \mathsf{1}z \\ \mathsf{0} & \mathsf{1} \end{pmatrix} \mathfrak{r}(0) , \tag{1.9}$$

where $\mathsf{0}$ and $\mathsf{1}$ are the 2×2 zero and unit matrices. The fact that this transformation can be represented by a 4×4 matrix is not a unique property of paraxial propagation through vacuum. It is well-known that, in leading order of the paraxial approximation, the transformations due to various lossless optical elements such as thin lenses and mirrors can also be represented by real 4×4 matrices acting on a ray vector $\mathfrak{r}$ [12]. In the special case of isotropic elements, all four 2×2 submatrices of the 4×4 ray matrix are proportional to the 2×2 unit matrix. It follows that the transformation of the two transverse components $(x, \vartheta_x)^{\mathrm{T}}$ and $(y, \vartheta_y)^{\mathrm{T}}$ of a ray $\mathfrak{r}$ can be described by the same reduced 2×2 ray matrix. Such 2×2 ray matrices are called *ABCD* matrices [12]. Free propagation as described by equation (1.9) is obviously isotropic. It is an example of a transformation that can be described by an *ABCD* matrix. A ray that lies in a plane through the optical axis of the element through which it passes can be characterized by its distance to the optical axis $R = |\rho|$ and the corresponding direction angle $\vartheta = \partial\rho/\partial z$. One may show easily that also in this case the transformation due to an isotropic optical element can be represented by an *ABCD* matrix. If the two-dimensional vector that characterizes such a ray is denoted $\boldsymbol{\imath} = (R, \vartheta)^{\mathrm{T}}$, this transformation can be expressed as

$$\boldsymbol{\imath}_{\text{out}} = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \boldsymbol{\imath}_{\text{in}} \ , \tag{1.10}$$

with $A, B, C, D \in \mathbb{R}$. The transverse position R and the propagation direction ϑ constitute a pair of canonically conjugate variables. From the fact that this canonical structure is preserved under the lossless transformation in equation (1.10), it follows that a physical *ABCD* matrix must have a unit determinant so that $AD - BC = 1$. The transformation of a sequence of optical elements can be constructed as the product of the ray matrices describing each of the elements and, since the determinant of a matrix product equals the product of the determinants of the matrices, it follows that the determinant of any *ABCD* matrix that describes a lossless isotropic optical set-up is equal to 1.

The optical path length between a point R_1 in the input plane and a point R_2 in the output plane of an isotropic optical set-up that is described by an *ABCD* matrix can be expressed as [12]

$$L(R_1, R_2) = L_0 + \frac{1}{2B}\left(AR_1^2 - 2R_1R_2 + DR_2^2\right) \ , \tag{1.11}$$

where L_0 is the path length along the optical axis of the set-up. The corresponding path integral, with $\lambda\!\!\!^{-} = 1/k$ again playing the role of $\hbar$, takes the following form [32, 12]

$$E(\rho_2, z_2) = \frac{2\pi k e^{ikL_0}}{iB} \int d_2\rho_1 \ \exp\left(\frac{ik\left(AR_1^2 - 2R_1R_2 + DR_2^2\right)}{2B}\right) E(\rho_1, z_1) \ . \tag{1.12}$$

This obviously reduces to equation (1.7) in case of free propagation $A = D = 1$ and $B = L_0 = z_2 - z_1$. The expression in equation (1.7) shows explicitly that the propagation of a wave through an optical set-up is fully determined by the geometric-optical characteristics of the optical set-up. This description of wave propagation is geometric in that it derives from the Fermat principle, which has a clear geometric significance.

It is well-known from textbook quantum mechanics that the path-integral description is exact in the case of first-order systems, the non-relativistic free particle and the harmonic oscillator being the simplest examples [30]. Since Gaussian integrals can be solved exactly, it follows that the evolution of Gaussian wave packets under the integral transformation for a first-order system can be calculated analytically. Less well-known is that the path-integral description is also exact for infinitely many complete sets of excited states, which, analogous to the case of the harmonic oscillator, can be obtained from pairs of bosonic ladder operators. In the optical context, such excited states have the significance of higher-order transverse modes [33]. Two very well-known examples are the Hermite-Gaussian and Laguerre-Gaussian modes, which are of crucial importance in experiments with laser beams [12].

1.5 Thesis outline

In this thesis, we study the spatial structure and physical properties of higher-order optical modes that have a twisted nature due to the presence of astigmatism and optical vortices. Our characterization of such twisted states of light involves pairs of bosonic ladder operators that generate a basis set of optical modes. Although the ladder operators act in the wave-optical domain, we shall demonstrate that their transformation under paraxial propagation and optical elements can be expressed in terms of the ray matrix that also describes the transformation of a ray. The ladder operators generate a complete set of higher-order mode patterns that exactly solve the Huygens-Fresnel integral (1.12) for an arbitrary first-order system. In regions of free propagation, the modes obey the paraxial wave equation (1.8). As opposed to the Huygens-Fresnel integral, which cannot easily be generalized to the case of set-ups with non-isotropic optical elements, the ladder operator-method allows for direct generalization to the case of transverse modes with astigmatism. Although the method keeps its elegance and simplicity, the spatial patterns of astigmatic higher-order modes display a very rich structure that gives rise to vorticity and orbital angular momentum. In the first two chapters, we apply the ladder-operator method to study the mode structure and the physical properties that arise from it in the presence of twisted and rotating boundary conditions.

- In chapter 2 we show that the paraxial modes of a geometrically stable two-mirror cavity with general astigmatism, i.e., a cavity that consists of two non-aligned astigmatic mirrors, can be obtained from pairs of bosonic ladder operators. From the transformation property of the ladder operators it follows that the ladder operators that generate the cavity modes can be constructed from the eigenvectors of the round-trip ray matrix that describes the transformation of a ray after one round trip through the cavity. The eigenvalues determine the frequency spectrum of the cavity. As a result of the twisted nature of the astigmatic boundary conditions, the spatial structure of the cavity modes becomes twisted as well. This twist induces vorticity and orbital angular momentum in the cavity modes.

- In chapter 3 we generalize the concept of an optical cavity mode to the case of a cavity in uniform rotation. We generalize the ladder-operator method developed in the second chapter to account for the time dependence of a rotating cavity and obtain explicit expressions of the rotating cavity modes. These are applied to study some of their physical properties including the rotationally induced orbital angular momentum.

Although relatively simple in terms of the number of degrees of freedom involved, a rotating astigmatic two-mirror cavity turns out to be a surprisingly rich dynamical system. Chapters 4 and 5 are devoted to specific dynamical properties of a rotating cavity and its modes.

- In chapter 4 we show that rotation affects the focusing properties of the mirrors of an astigmatic two-mirror cavity in such a way that the cavity can both be stabilized and destabilized by rotation. As such it bears some similarity with both the Paul trap and the gyroscope. We study the rotationally induced transition from a stable to an unstable geometry and vice versa in terms of the structure of and the orbital angular momentum in the rotating cavity modes.

- In chapter 5 we show that optical vortices appear in the modes of an astigmatic two-mirror cavity when it is put into rotation about its optical axis. We study some physical properties of the emerging vortex pattern. We make a comparison with rotationally induced vortices in material systems and discuss explicit results for a specific case. In section 5.5, we discuss limitations of possible experimental realizations of an optical set-up that captures the essential features of the rotating astigmatic cavities that we study in chapters 3, 4 and 5.

Since the transformations of the ladder operators can be expressed in terms of a ray matrix, which has a clear geometric significance, it follows that also the ladder-operator method is geometric in that it relates to the principle of Fermat.

- In chapter 6 we focus on such geometrical aspects. We study the geometry of the parameter space underlying the pairs of bosonic ladder operators and the geometric phase shifts that it gives rise to. Such phase shifts constitute the ultimate generalization of the Gouy phase in paraxial wave optics. We recover both the ordinary Gouy phase shift and the geometric phase that arises from cyclic transformations of optical beams bearing orbital angular momentum as limiting cases. We discuss an analogy with the Aharonov-Bohm effect in quantum eletrodynamics that reveals some deep insights in the nature and origin of this geometric phase.

Finally, the last chapter completes the discussion of rotating light.

- In chapter 7 we show that the exact wave equation, which derives without approximations from Maxwell's equations, allows for solutions that are monochromatic in a rotating frame. Since, in complex notation, monochromatic fields are separable in space and time, it follows that these solutions are stationary in a rotating frame. As a result, both the polarization and the spatial patterns of the vector components of the corresponding fields rotate uniformly in a stationary frame. We discuss the quantization of the radiation field in an orthonormal but otherwise arbitrary basis of such rotating modes. We derive the equations of motion for light in a rotating frame and show that quantization in the rotating frame is consistent with quantization in the stationary frame. We discuss the paraxial counterpart of the exact theory and indicate how a quantum-optical description of the rotating cavity modes, as introduced in chapter 3, can be obtained.

Several chapters in this thesis are based on material that has been (or will be) published elsewhere. Although all of them have been rewritten, I have tried to keep them independently readable. As a result, there is some overlap, in particular between the three chapters on rotating cavities. Physics-oriented readers may at first read chapters 4 and 5, in which the emphasis is on the physical phenomena rather than on the mathematical method used, and go back to the relatively heavy mathematics in chapter 2 and 3 at a later stage. Mathematically oriented readers, on the other hand, may consider first reading chapter 6, in which the mathematics underlying the ladder-operator method that is crucial to this thesis is discussed in its most general and, as a result, abstract form.

The notation used in this thesis has been harmonized as much as possible without sacrificing intuition. Generally speaking, vectors in three-dimensional space are set in a bold font while vectors in the transverse plane are denoted with small Greek letters. Both two and four-dimensional ray vectors are denoted in a script font while vectors in other (parameter) spaces are denoted with arrows above the symbol. The bra-ket notation of quantum mechanics is used to denote vectors in the Hilbert space of transverse states of classical light. Quantum states of the radiation field are indicated with bra and ket vectors with round brackets. All operators are denoted with a hat above the symbol. Matrices acting on the transverse spatial coordinates (or momenta) are set in a sans serif font while ray matrices, which act on either two- or four-dimensional rays are set in the standard roman font. These and other notational conventions used in this thesis are listed in table 1.1.

Symbol	Meaning
Coordinates	
(x, y, z)	Cartesian coordinates
$(R\cos\phi, R\sin\phi, z)$	Cylindrical coordinates
$(r\sin\vartheta\cos\varphi, r\sin\vartheta\sin\varphi, r\cos\vartheta)$	Spherical coordinates
Vectors	
$\mathbf{r} = (x, y, z)^{\mathrm{T}}$	Position in three dimensions
$\mathbf{k} = (k_x, k_y, k_z)^{\mathrm{T}}$	Wave vector in three dimensions
$\rho = (x, y)^{\mathrm{T}}$	Transverse position
$\theta = (\vartheta_x, \vartheta_y)^{\mathrm{T}}$	Transverse propagation direction
$\epsilon = (\epsilon_x, \epsilon_y)^{\mathrm{T}}$	Transverse polarization
$\mathscr{r}, \mathscr{s} = (\rho, \theta)^{\mathrm{T}}$	Two- and four-dimensional real ray vectors
μ, ν	Normalized complex (eigen)rays
$\vec{A}, \vec{R}$	Vectors in a parameter space
$\lvert u\rangle, \lvert v\rangle$	Transverse states of classical light
$\lvert ...)$	Quantum state of light
Fields	
$\mathbf{E}, \mathbf{B}$	Electric and magnetic fields
$\mathbf{A}$	Vector potential
$\mathbf{C}$	Vector potential in a rotating frame
$\mathbf{F}, \mathbf{G}$	Vectorial mode functions
$\mathbf{V}$	Vectorial mode function in a rotating frame
Operators	
$\hat{a}_{1,2}^{(\dagger)}, \hat{b}_{x,y}^{(\dagger)}$	Raising and lowering operators
$\hat{a}_{\lambda}^{(\dagger)}, \hat{c}_{\mu}^{(\dagger)}, \hat{v}_{\nu}^{(\dagger)}$	Creation and annihilation operators (chapter 7)
Generalized beam parameters	
$r_{1,2}$ and $t_{1,2}$	Scalar coefficients of a ladder operator
R and T	2×2 coefficient matrices of a vector of ladder operators
$\mathsf{S} = \mathsf{V}^{-1}$	2×2 matrices that characterize the astigmatism
$\chi_{1,2}$	Generalized Gouy phases
η, ξ	Spinors on the Hermite-Laguerre sphere
Beam profiles	
u, v	Transverse profile of a paraxial beam
$\tilde{u}$	Transverse Fourier transform of u

Table 1.1: List of Symbols