A fixed point approach towards stability of delay differential equations with applications to neural networks

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CHAPTER 6

Stochastic delayed neural networks

This chapter presents stability properties of a class of stochastic delayed neural networks without impulses and a class of stochastic delayed neural networks with impulses.

In Section 6.1, we present new conditions for asymptotic stability and exponential stability of a class of stochastic recurrent neural networks with discrete and distributed time varying delays. Our approaches are based on the method using fixed point theory and the method using an appropriate integral inequality, which do not resort to any Liapunov function or Liapunov functional. Our results neither require the boundedness, monotonicity and differentiability of the activation functions nor differentiability of the time varying delays. In particular, a class of neural networks without stochastic perturbations is also considered by using the two approaches.

In Section 6.2, we consider the impulsive effects on the class of stochastic delayed recurrent neural networks that is discussed in Section 6.1. New sufficient conditions for asymptotic stability and exponential stability of the class of impulsive stochastic delayed recurrent neural networks are presented by using fixed point methods. In particular, as in Section 6.1, a class of impulsive neural networks without stochastic perturbations is also considered.

6.1 Stability of stochastic delayed neural networks

6.1.1 Introduction and main results

During the past few decades, neural networks such as Hopfield neural networks [53], Cellular neural networks [24, 25], Cohen-Grossberg neural networks [136] and bidirectional associative memory neural networks (BAM Networks) [68, 69, 70] have been well investigated since they play an important role in many areas such as combinatorial optimization, signal processing and pattern recognition.

Due to the finite switching speed of neurons and amplifiers, time delays which may lead to instability and bad performance in neural processing and signal transmission are commonly encountered in both biological and artificial neural networks. In addition, neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths [128]. Thus there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays [146]. In these circumstances the signal propagation is not instantaneous and may not be suitably modeled with discrete delays. Therefore, a more appropriate way which incorporates continuously distributed delays in neural network models has been used. Further, due to random fluctuations and probabilistic causes in the network, noises do exist in a neural network. Thus, it is necessary and rewarding to study stochastic effects to the stability property of neural networks.

Liapunov's direct method has long been viewed the main classical method of studying stability problems in many areas of stochastic delay differential equations. The success of Lyapunov's direct method depends on finding a suitable Liapunov function or Liapunov functional. However, it may be difficult to look for a good Liapunov functional for some classes of stochastic delay differential equations. Therefore, an alternative may be explored to overcome such difficulties.

It was proposed by Burton [13] and his co-workers to use fixed point methods to study the stability problem for deterministic systems. Luo [90] and Appleby [4] have applied this method to deal with the stability problems for stochastic delay differential equations, and afterwards, a great number of classes of stochastic delay differential equations are discussed by using fixed point methods, see, for example, [34, 91, 92, 117, 118]. It turns out that the fixed point method is a powerful technique in dealing with stability problems for deterministic and stochastic differential equations with delays. Moreover, it has an advantage that it can yield the existence, uniqueness and stability criteria of the considered system in one step. Chen [21, 23] has applied an appropriate integral inequality to study exponential stability of some classes of stochastic delay differential equations, and it turns out that it is a convenient way to discuss exponential stability of a system.

The aim of this section is to study a general class of stochastic neural networks by using fixed point methods and the method by employing an appropriate integral inequality. Indeed, we consider the following class of stochastic neural networks with varying discrete and distributed delays which is described by

$$\begin{aligned} dx_i(t) &= \Big[-c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau(t))) \\ &\quad + \sum_{j=1}^{n} l_{ij} \int_{t-r(t)}^{t} h_j(x_j(s))\, ds\Big]\, dt + \sum_{j=1}^{n} \sigma_{ij}(t, x_j(t), x_j(t-\tau(t)))\, dw_j(t), \end{aligned} \tag{6.1}$$

or

$$\begin{aligned} dx(t) &= \Big[-Cx(t) + Af(x(t)) + Bg(x(t-\tau(t))) + W\int_{t-r(t)}^{t} h(x(s))\, ds\Big]\, dt \\ &\quad + \sigma(t, x(t), x(t-\tau(t)))\, dw(t) \end{aligned}$$

for $i = 1, 2, 3, \cdots, n$, where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in \mathbb{R}^n$ is the state vector associated with the neurons; $C = \mathrm{diag}(c_1, c_2, \cdots, c_n) > 0$ where $c_i > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and the external stochastic perturbations; $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ and $W = (l_{ij})_{n\times n}$ represent the connection weight matrix, delayed connection weight matrix and distributed delayed connection weight matrix, respectively; f_j, g_j, h_j are activation functions, $f(x(t)) = (f_1(x(t)), f_2(x(t)), \cdots, f_n(x(t)))^T \in \mathbb{R}^n$, $g(x(t)) = (g_1(x(t)), g_2(x(t)), \cdots, g_n(x(t)))^T \in \mathbb{R}^n$, $h(x(t)) = (h_1(x(t)), h_2(x(t)), \cdots, h_n(x(t)))^T \in \mathbb{R}^n$. Moreover, $w(t) = (w_1(t), w_2(t), \cdots, w_n(t))^T \in \mathbb{R}^n$ is an n-dimensional Brownion motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$ (i.e. $\mathcal{F}_t$ = completion of $\sigma\{\omega(s) : 0 \leq s \leq t\}$) and $\sigma : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\sigma = (\sigma_{ij})_{n\times n}$ is the diffusion coefficient matrix. $\tau(t)$ and $r(t)$ denote a discrete time varying delay and the bound of a distributed time varying delay, respectively. Denote $\vartheta = \inf_{t\geq 0}\{t - \tau(t), t - r(t)\}$.

The initial condition for the system (6.1) is given by

$$x(t) = \phi(t), \qquad t \in [\vartheta, 0], \tag{6.2}$$

where $t \mapsto \phi(t) = (\phi_1(t), \phi_2(t), \cdots, \phi_n(t))^T \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$ with the norm defined as

$$\|\phi\|^p = \sup_{\vartheta \le t \le 0} \left(\mathbb{E} \sum_{i=1}^{n} |\phi_i(t)|^p \right),$$

where $\mathbb{E}$ denotes expectation with respect to the probability measure $\mathbb{P}$ and $p \ge 2$.

To obtain our main results, we suppose the following conditions are satisfied:

(A1) the delays $\tau(t), r(t)$ are continuous functions such that $t - \tau(t) \to \infty$ and $t - r(t) \to \infty$ as $t \to \infty$;

(A2) $f_j(x)$, $g_j(x)$, and $h_j(x)$ satisfy Lipschitz conditions. That is, for each $j = 1, 2, 3, \cdots, n$, there exist constants α_j, β_j, γ_j such that for every $x, y \in \mathbb{R}^n$,

$$|f_j(x) - f_j(y)| \le \alpha_j |x - y|, \quad |g_j(x) - g_j(y)| \le \beta_j |x - y|, \quad |h_j(x) - h_j(y)| \le \gamma_j |x - y|;$$

(A3) Assume that $f(0) \equiv 0$, $g(0) \equiv 0$, $h(0) \equiv 0$, $\sigma(t, 0, 0) \equiv 0$;

(A4) $\sigma(t, x, y)$ satisfies a Lipschitz condition. That is, there are nonnegative constants μ_i and ν_i such that $\forall\ i, j$,

$$(\sigma_{ij}(t, x, y) - \sigma_{ij}(t, u, v))^2 \le \mu_j (x_j - u_j)^2 + \nu_j (y_j - v_j)^2.$$

It follows from [43, 98] that under the hypotheses $(A1), (A2), (A3)$ and $(A4)$, system (6.1) with initial condition (6.2) has one unique global solution which is denoted by $x(t, \phi)$ or $x(t)$ such that $t \mapsto x(t, \phi) : [0, \infty) \to L^p(\Omega; \mathbb{R}^n)$ is adapted and continuous and $\mathbb{E}[\sup_{0 \le s \le t} \|x(s, 0, \phi)\|^p] < \infty$ for $t > 0$. Clearly, system (6.1) admits the trivial solution $x(t, 0, 0) \equiv 0$.

Definition 6.1.1. *The trivial solution of system* (6.1) *is said to be stable in pth* ($p \ge 2$) *moment if for arbitrary given $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|\phi\|^p < \delta$ yields that*

$$\mathbb{E}\|x(t, \phi)\|^p < \varepsilon, \quad t \ge 0.$$

where $\phi \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$. In particular, when $p = 2$, the trivial solution is said to be mean square stable.

Definition 6.1.2. *The trivial solution of system* (6.1) *is said to be asymptotically stable in pth* ($p \ge 2$) *moment if it is stable in pth moment and there exists a $\delta > 0$, such that $\|\phi\|^p < \delta$ implies*

$$\lim_{t \to \infty} \mathbb{E}\|x(t, \phi)\|^p = 0.$$

where $\phi \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$.

Definition 6.1.3. *The trivial solution of system* (6.1) *is said to be pth* ($p \geq 2$) *moment exponentially stable if there exists a pair of constants* $\lambda, C > 0$ *such that*

$$\mathbb{E}\|x(t,\phi)\|^p \leq C\mathbb{E}\|\phi\|^p e^{-\lambda t}, \quad t \geq 0,$$

holds for $\phi \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$. *Especially, when* $p = 2$, *we speak of exponentially stable in mean square.*

Different choices of norms can be considered on spaces of stochastic processes. The norms we choose should be such that the space under consideration is complete and the equation yields a contraction with respect to the norm. For the system (6.1) with initial condition (6.2), we consider the following two different complete spaces which are defined by using two types of norms.

Define $\mathcal{S}_\phi$ the space of all $\mathcal{F}_t$-adapted processes $\varphi(t,\omega) : [\vartheta, \infty) \times \Omega \to \mathbb{R}^n$ such that $\varphi \in C\left([\vartheta, \infty), L^p_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)\right)$. Moreover, we require $\varphi(t,\cdot) = \phi(t)$ for $t \in [\vartheta, 0]$ and $\mathbb{E}\sum_{i=1}^n |\varphi_i(t)|^p \to 0$ as $t \to \infty$, $i = 1, 2, \cdots, n$. If we define the norm

$$\|\varphi\|^p := \sup_{t \geq \vartheta} \left(\mathbb{E} \sum_{i=1}^n |\varphi_i(t)|^p \right), \tag{6.3}$$

then $\mathcal{S}_\phi$ is a complete metric space. Using a contraction mapping defined on the space $\mathcal{S}_\phi$ and applying a contraction mapping principle, we obtain our first result. Its proof is given in Subsection 6.1.2.

Theorem 6.1.4. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the function* $r(t)$ *is bounded by a constant* r ($r > 0$)*;*

(ii) *and such that*

$$\begin{aligned}\alpha \triangleq\ & 5^{p-1}\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n |a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 5^{p-1}\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n |b_{ij}|^q|\beta_j|^q\right)^{p/q} \\ & + 5^{p-1}\sum_{i=1}^n \left(\frac{r}{c_i}\right)^p\left(\sum_{j=1}^n |l_{ij}|^q|\gamma_j|^q\right)^{p/q} + 5^{p-1}n^{p-1}\sum_{i=1}^n c_i^{-p/2}\left(\mu^{p/2} + \nu^{p/2}\right) < 1,\end{aligned} \tag{6.4}$$

where $\mu = \max\{\mu_1, \mu_2, \cdots, \mu_n\}$, $\nu = \max\{\nu_1, \nu_2, \cdots, \nu_n\}$*;*

then the trivial solution of (6.1) *is pth moment asymptotically stable.*

Consider a case when both the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ. Let $\phi \in L^p_{\mathcal{F}_0}(\Omega, C([\vartheta, 0], \mathbb{R}^n))$, define $\mathcal{C}_\phi$ to be the space of all $\mathcal{F}_t$-adapted processes $\varphi(t,\omega) : [-\tau, \infty) \times \Omega \to \mathbb{R}^n$ such that $\varphi \in L^p(\Omega, C([\vartheta, \infty), \mathbb{R}^n))$. Moreover, we set $\varphi(t,\cdot) = \phi(t)$ for $t \in [\vartheta, 0]$, $\varphi(t,\cdot) = \phi(\vartheta)$ for $t \in [-\tau, \vartheta]$ (in case $-\tau < \vartheta$), and for $t \to \infty$, $\sum_{i=1}^n \mathbb{E}\sup_{t-\tau \leq s \leq t} |\varphi_i(s)|^p \to 0$. The norm on $\mathcal{C}_\phi$ is defined as

$$\|\varphi\|^p = \sup_{t \geq 0}\left[\sum_{i=1}^n \mathbb{E}\left(\sup_{t-\tau \leq s \leq t} |\varphi_i(s)|^p\right)\right], \tag{6.5}$$

then $\mathcal{C}_\phi$ is a complete metric space. Using a contraction mapping defined on the space $\mathcal{C}_\phi$ and applying a contraction mapping principle, we obtain our second result, which is proved in Subsection 6.1.3.

Theorem 6.1.5. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ $(\tau > 0)$;*

(ii) *and such that*

$$\alpha \triangleq 5^{p-1}e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 5^{p-1}e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}$$
$$+5^{p-1}\tau^p e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}$$
$$+5^{p-1}K_p n^p e^{pc\tau}q^p c^{1-p/2}(2c)^{-1}\left(\mu^{p/2}+\nu^{p/2}\right) < 1, \tag{6.6}$$

where $c = \min\{c_1, c_2, \cdots c_n\}$, $\mu = \max\{\mu_1, \mu_2, \cdots \mu_n\}$, $\nu = \max\{\nu_1, \nu_2, \cdots \nu_n\}$*;*

then the trivial solution of (6.1) *is pth moment asymptotically stable. More than that, for every* $\varepsilon > 0$*, there exists a* $\delta > 0$ *such that* $\|\phi\| < \delta$ *implies* $\sum_{i=1}^{n}\mathbb{E}\sup_{t-\tau\le s\le t}|x_i(s)|^p < \varepsilon$ *and* $\lim_{t\to\infty}\left\{\mathbb{E}\left[\sup_{t-\tau\le s\le t}\|x(s,0,\phi)\|^p\right]\right\} = 0.$

Remark 6.1.6. *In some papers, see, for example,* [89, 90, 131, 132]*, the norm for the space of stochastic process is defined as*

$$\|\varphi\|_{[0,t]} = \left[\mathbb{E}\left(\sup_{s\in[0,t]}|\varphi(s)|^2\right)\right]^{1/2}.$$

As in [90]*, in order to show $P(\mathcal{S}) \subseteq \mathcal{S}$, we need to estimate $\mathbb{E}\sup_{s\in[0,t]}|I_5(s)|^2$, where*

$$I_5(s) = \int_0^s e^{-\int_z^s h(u)\,du}\left[c(z)x(z) + e(z)x(z-\delta(z))\right]d\omega(z).$$

However, $I_5(s)$ is not a local martingale (*see Section* 1.4 *for its proof*)*. Hence, Burkholder-Davis-Gundy Inequality can not be applied directly.*

Using an appropriate integral inequality, we obtain sufficient conditions for exponential stability of (6.1) with initial condition (6.2), which is our third result. For its proof, see Subsection 6.1.4.

Theorem 6.1.7. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ;*

(ii) *and such that*

$$5^{p-1}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+5^{p-1}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q} \tag{6.7}$$
$$+5^{p-1}\left(\frac{\tau}{c}\right)^p\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}+5^{p-1}n^pc^{-p/2}(\mu^{p/2}+\nu^{p/2})<1,$$

where $c=\min\{c_1,c_2,\cdots,c_n\}$, $\mu=\max\{\mu_1,\mu_2,\cdots,\mu_n\}$, $\nu=\max\{\nu_1,\nu_2,\cdots,\nu_n\}$*;*

then the trivial solution of (6.1) *is exponentially stable in pth moment,*

Remark 6.1.8. *The stability criteria we provided in our main results are only in terms of the system parameters* c_i*,* a_{ij}*,* b_{ij}*,* l_{ij}*, etc. Hence, these criteria can usually be verified easily in applications.*

Remark 6.1.9. *Many articles, see, for example,* [116, 120] *have studied stochastic neural network* (6.1) *and special cases of* (6.1)*. However, they impose the following condition on the delays*

(H) *the discrete delay* $\tau(t)$ *is differentiable function and* $r(t)$ *in the distributed delay is nonnegative and bounded, that is, there exist constants* τ_M,ζ,r_M *such that*

$$0\le\tau(t)\le\tau_M,\quad \tau'(t)\le\zeta,\quad r(t)\le r_M. \tag{6.8}$$

In our results, condition (H) *is replaced by other assumptions, which may be satisfied when* (H) *is not.*

Theorem 6.1.7 can, for example, be applied to establish exponential stability in *pth* moment of a two dimensional stochastically perturbed Hopfield neural network with time-varying delay, the delay is bounded but not differentiable, see Example 6.1.31 for details.

Consider a case when there are no stochastic effects in the system (6.1), which then comes down to the neural network described by

$$\frac{dx_i(t)}{dt}=-c_ix_i(t)+\sum_{j=1}^{n}a_{ij}f_j(x_j(t))+\sum_{j=1}^{n}b_{ij}g_j(x_j(t-\tau(t)))+\sum_{j=1}^{n}d_{ij}\int_{t-r(t)}^{t}h_j(x_j(s))\,ds, \tag{6.9}$$
$$i=1,2,3,\cdots,n,$$

or

$$\frac{dx(t)}{dt}=-Cx(t)+Af(x(t))+Bg(x-\tau(t))+D\int_{t-r(t)}^{t}h(x(s))\,ds, \tag{6.10}$$

where $x(\cdot)=(x_1(\cdot),x_2(\cdot),\cdots,x_n(\cdot))^T$ is the neuron state vector of the transformed system (6.9).

The initial condition for the system (6.9) is

$$x(t)=\phi(t),\qquad t\in[\vartheta,0], \tag{6.11}$$

where ϕ is a continuous function with the norm defined by $\|\phi\|=\sup_{\vartheta\le t\le 0}\sum_{i=1}^{n}|\phi_i(t)|$.

Assume that $(A1)-(A3)$ are satisfied, then (6.9) admits a trivial solution $x=0$. Denote by $x(t,\phi)=(x_1(t,\phi_1),\cdots,x_n(t,\phi_n))^T\in\mathbb{R}^n$ the solution of (6.9) with initial condition (6.11).

Definition 6.1.10. *For the system* (6.9) *with initial condition* (6.11)*, we have that*

(i) *the trivial solution of* (6.9) *is said to be stable if for any* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for any initial condition* $\phi \in C([\vartheta, 0], \mathbb{R}^n)$ *satisfying* $\|\phi\| < \delta$*, we have for the corresponding solution that* $\|x(t,\phi)\| < \varepsilon$ *for* $t \geq 0$*;*

(ii) *the trivial solution of* (6.9) *is said to be asymptotically stable if it is stable and for any initial condition* $\phi \in C([\vartheta, 0], \mathbb{R}^n)$ *we have for the corresponding solution that* $\lim_{t\to\infty} \|x(t,\phi)\| = 0$*;*

(iii) *the trivial solution of* (6.9) *is said to be globally exponentially stable if there exist scalars* $\lambda > 0$ *and* $C > 0$ *such that for any initial condition* $\phi \in C([\vartheta, 0], \mathbb{R}^n)$*, we have for the corresponding solution that* $\|x(t,\phi)\| \leq Ce^{-\lambda t}\|\phi\|$ *for* $t \geq 0$.

Define $\mathcal{H}_\phi = \mathcal{H}_{1\phi} \times \mathcal{H}_{2\phi} \times \cdots \times \mathcal{H}_{n\phi}$, where $\mathcal{H}_{i\phi}$ is the space consisting of continuous functions $\varphi_i(t) : [\vartheta, \infty) \to \mathbb{R}$ such that $\varphi_i(\theta) = \phi(\theta)$ for $\vartheta \leq \theta \leq 0$ and $\varphi_i(t) \to 0$ as $t \to \infty$, $i = 1, 2 \cdots, n$. For any $\varphi(t) = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t)) \in \mathcal{H}_\phi$ and $\eta(t) = (\eta_1(t), \eta_2(t), \cdots, \eta_n(t)) \in \mathcal{H}_\phi$, if we define the metric as $d(\varphi, \eta) = \sup_{t\geq\vartheta} \sum_{i=1}^n |\varphi_i(t) - \eta_i(t)|$, then $\mathcal{H}_\phi$ becomes a complete metric space.

Using a contraction mapping defined on the space H_ϕ and applying a contraction mapping principle, we obtain our fourth result, which is proved in Subsection 6.1.5.

Theorem 6.1.11. *Suppose that the assumptions* (A1)-(A3) *hold. If the following conditions are satisfied,*

(i) *the function* $r(t)$ *is bounded by a constant* r $(r > 0)$*;*

(ii) *and such that*

$$\alpha \triangleq \sum_{i=1}^n \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^n \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \sum_{i=1}^n \frac{r}{c_i} \max_{j=1,2,\cdots,n} |d_{ij}\gamma_j| < 1; \quad (6.12)$$

then the trivial solution of (6.9) *is asymptotically stable.*

Remark 6.1.12. *Theorem* 6.1.11 *is an extension and improvement of the result in Lai and Zhang* [74].

By establishing an appropriate integral inequality, we obtain sufficient conditions for exponential stability of (6.9), which is our fifth result. Its proof is given in Subsection 6.1.6.

Theorem 6.1.13. *Suppose that the assumptions* (A1)-(A3) *hold. If the following conditions are satisfied,*

(i) *the discrete delay* $\tau(t)$ *and* $r(t)$ *in the distributed delay are bounded by a constant* τ $(\tau > 0)$*;*

(ii) *and such that*

$$\frac{1}{c}\sum_{i=1}^n \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \frac{1}{c}\sum_{i=1}^n \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \frac{1}{c}\sum_{i=1}^n \tau \max_{j=1,2,\cdots,n} |d_{ij}\gamma_j| < 1, \quad (6.13)$$

where $c = \min\{c_1, c_2, \cdots, c_n\}$*;*

then the trivial solution of (6.9) *with initial condition* (6.11) *is exponentially stable.*

Remark 6.1.14. *Several exponential stability results* [77, 126, 127] *were provided for the system* (6.9)*, by constructing an appropriate Liapunov functional and employing linear matrix inequality* (LMI) *method, and their results depends on the condition that the delays are satisfied* (H)*. From our main results, we provide other assumptions. The delays in our results are required to be bounded.*

Remark 6.1.15. *From Theorem* 6.1.11 *and Theorem* 6.1.13*, we find that the terms with* f, g, h *in equation* (6.10) *can be viewed as perturbations of the stable equation* $dx(t)/dt = -Cx(t)$*. Condition* (ii) *in Theorem* 6.1.11 *and condition* (ii) *in Theorem* 6.1.13 *require the perturbation to be small relative to the stabilizing force of* C*.*

Theorem 6.1.13 can, for example, be applied to establish exponential stability of a two dimensional cellular neural network with time-varying delay, see Example 6.1.29 for details.

The rest of this section is organized as follows. In Subsection 6.1.2, we present a proof of Theorem 6.1.4. The proof of Theorem 6.1.5 is presented in Subsection 6.1.3 and the proof of Theorem 6.1.7 is given in Section 6.1.4. we present the proofs of Theorem 6.1.11 and Theorem 6.1.13 in Subsection 6.1.5 and Subsection 6.1.6, respectively. Some examples are given to illustrate our main results in Subsection 6.1.7.

6.1.2 Proof of Theorem 6.1.4

In this subsection, we prove Theorem 6.1.4. We start with some preparations.

Lemma 6.1.16. ([96, 129]) *If* $w(t) = (w_1(t), w_2(t), \cdots, w_n(t))^T$ $(t \geq 0)$ *is a* n*-dimensional Brownian motion defined on a complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, then for each* $t \geq 0$*, we have the following formula*

$$\mathbb{E}\left(\int_0^t f_i(s)\, dw_i(s) \int_0^t f_j(s)\, dw_j(s)\right) = \mathbb{E}\int_0^t f_i(s) f_j(s)\, d\langle w_i, w_i\rangle_s,$$

where $\langle w_i, w_i\rangle_s = \delta_{ij} s$ *are the cross-variations, and* δ_{ij} *is the correlation coefficient,* f_i *is adapted and* $f_i \in L^2(\Omega \times [0,t])$*,* $i, j = 1, 2, \cdots, n$*.*

If we multiply both sides of (6.1) by $e^{c_i t}$ and integrate from 0 to t, we obtain

$$\begin{aligned} x_i(t) &= e^{-c_i t} x_i(0) + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n a_{ij} f_j(x_j(s))\, ds + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n b_{ij} g_j(x_j(s-\tau(s)))\, ds \\ &\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(x_j(u))\, du\, ds \\ &\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, dw_j(s) \end{aligned} \tag{6.14}$$

for $t \geq 0$, $i = 1, 2, 3, \cdots, n$.

Lemma 6.1.17. *Define an operator by* $(Q\varphi)(t) = \phi(t)$ *for* $t \in [\vartheta, 0]$*, and for* $t \geq 0$*,* $i = 1, 2, 3, \cdots, n$*,*

$$\begin{aligned}(Q\varphi)_i(t) &= e^{-c_i t}\varphi_i(0) + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n a_{ij} f_j(\varphi_j(s))\, ds \\ &\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n b_{ij} g_j(\varphi_j(s-\tau(s)))\, ds \\ &\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(\varphi_j(u))\, du\, ds \\ &\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n \sigma_{ij}(s, \varphi_j(s), \varphi_j(s-\tau(s)))\, dw_j(s). \qquad (6.15)\end{aligned}$$

Suppose that the assumption (A1)-(A4) *holds. If conditions* (i) *and* (ii) *in Theorem* 6.1.4 *are satisfied, then* $Q : \mathcal{S}_\phi \to \mathcal{S}_\phi$ *and* Q *is a contraction mapping.*

Proof. Denote $(Q\varphi)_i(t) := J_{1i}(t) + J_{2i}(t) + J_{3i}(t) + J_{4i}(t) + J_{5i}(t)$, where

$$\begin{aligned}J_{1i}(t) &= e^{-c_i t}\varphi_i(0), \qquad J_{2i}(t) = \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n a_{ij} f_j(\varphi_j(s))\, ds, \\ J_{3i}(t) &= \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n b_{ij} g_j(\varphi_j(s-\tau(s)))\, ds, \\ J_{4i}(t) &= \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(\varphi_j(u))\, du\, ds, \\ J_{5i}(t) &= \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n \sigma_{ij}(s, \varphi_j(s), \varphi_j(s-\tau(s)))\, dw_j(s).\end{aligned}$$

Step1. From the definition of the metric space $\mathcal{S}_\phi$, we have that $\mathbb{E}\sum_{i=1}^n |\varphi_i(t)|^p < \infty$ for all $t \geq 0$, $\varphi \in \mathcal{S}_\phi$.

Step2. We prove the continuity in *pth* moment of Qx on $[0, \infty)$ for $x \in \mathcal{S}_\phi$. Let $x \in \mathcal{S}_\phi$, $t_1 \geq 0$, let $r \in \mathbb{R}$ with $|r|$ sufficiently small and $r > 0$ if $t_1 = 0$, we have

$$\begin{aligned}\mathbb{E}\sum_{i=1}^n |J_{2i}(t_1+r) - J_{2i}(t_1)|^p &= \mathbb{E}\sum_{i=1}^n \Bigg| \int_0^{t_1} \left(e^{-c_i(t_1+r-s)} - e^{-c_i(t_1-s)}\right) \sum_{j=1}^n a_{ij} f_j(x_j(s))\, ds \\ &\quad + \int_{t_1}^{t_1+r} e^{-c_i(t_1+r-s)} \sum_{j=1}^n a_{ij} f_j(x_j(s))\, ds \Bigg|^p \to 0 \quad \text{as} \quad r \to 0.\end{aligned}$$

Similarly, we have that

$$\mathbb{E}\sum_{i=1}^n |J_{3i}(t_1+r) - J_{3i}(t_1)|^p \to 0 \ \text{ as } \ r \to 0, \quad \mathbb{E}\sum_{i=1}^n |J_{4i}(t_1+r) - J_{4i}(t_1)|^p \to 0 \ \text{ as } \ r \to 0.$$

In the following, we check the continuity of $J_{5i}(t)$.

$$\begin{aligned}
&\mathbb{E}\sum_{i=1}^{n}|J_{5i}(t_1+r)-J_{5i}(t_1)|^p\\
&=\mathbb{E}\sum_{i=1}^{n}\bigg|\int_0^{t_1}\Big(e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}\Big)\sum_{j=1}^{n}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s)\\
&\qquad+\int_{t_1}^{t_1+r}e^{-c_i(t_1+r-s)}\sum_{j=1}^{n}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s)\bigg|^p\\
&\le n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\bigg|\int_0^{t_1}\Big(e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}\Big)\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s)\\
&\qquad+\int_{t_1}^{t_1+r}e^{-c_i(t_1+r-s)}\sum_{j=1}^{n}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s)\bigg|^p\\
&\le (2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\bigg|\int_0^{t_1}\Big(e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}\Big)\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s)\bigg|^p\\
&\qquad+(2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\bigg|\int_{t_1}^{t_1+r}e^{-c_i(t_1+r-s)}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s)\bigg|^p\\
&=(2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\bigg\{\mathbb{E}\bigg[\int_0^{t_1}\Big(e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}\Big)^2\sigma_{ij}^2(s,x_j(s),x_j(s-\tau(s)))\,ds\bigg]^{p/2}\\
&\qquad+\mathbb{E}\bigg[\int_{t_1}^{t_1+r}e^{-2c_i(t_1+r-s)}\sigma_{ij}^2(s,x_j(s),x_j(s-\tau(s)))\,ds\bigg]^{p/2}\bigg\}\to 0\quad\text{as}\quad r\to 0.
\end{aligned}$$

Thus, Qx is indeed continuous in pth moment on $[0,\infty)$.

Step3. We prove that $Q(\mathcal{S}_\phi)\subseteq\mathcal{S}_\phi$.

$$\mathbb{E}\sum_{i=1}^{n}|(Q\varphi)_i(t)|^p=\mathbb{E}\sum_{i=1}^{n}\bigg|\sum_{j=1}^{5}J_{ji}(t)\bigg|^p\le 5^{p-1}\sum_{j=1}^{5}\mathbb{E}\sum_{i=1}^{n}|J_{ji}(t)|^p. \tag{6.16}$$

Now, we estimate the terms on the right-hand side of the above inequality.

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|J_{2i}(t)|^p &\le \sum_{i=1}^{n}\mathbb{E}\bigg[\int_0^t e^{-\frac{c_i(t-s)}{q}}e^{-\frac{c_i(t-s)}{p}}\sum_{j=1}^{n}|a_{ij}||f_j(\varphi_j(s))|\,ds\bigg]^p\\
&\le \sum_{i=1}^{n}\mathbb{E}\bigg[\bigg(\int_0^t e^{-c_i(t-s)}\,ds\bigg)^{p/q}\int_0^t e^{-c_i(t-s)}\bigg(\sum_{j=1}^{n}|a_{ij}||f_j(\varphi_j(s))|\bigg)^p ds\bigg]\\
&\le \sum_{i=1}^{n}c_i^{-p/q}\mathbb{E}\bigg[\int_0^t e^{-c_i(t-s)}\bigg(\sum_{j=1}^{n}|a_{ij}||\alpha_j||\varphi_j(s)|\bigg)^p ds\bigg]\\
&\le \sum_{i=1}^{n}c_i^{-p/q}\bigg(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\bigg)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(s)|^p\bigg)ds.
\end{aligned}\tag{6.17}$$

Since $\varphi \in \mathcal{S}_\phi$, we have that $\lim_{t\to\infty} \mathbb{E}\sum_{i=1}^n |\varphi_i(t)|^p = 0$. Thus for any $\varepsilon > 0$, there exists $T_1 > 0$ such that $t \geq T_1$ implies $\mathbb{E}\sum_{i=1}^n |\varphi_i(t)|^p < \varepsilon$, combining with (6.17), we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^n |J_{2i}(t)|^p &\leq \sum_{i=1}^n c_i^{-p/q}\left(\sum_{j=1}^n |a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^{T_1} e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^n |\varphi_j(s)|^p\right) ds \\
&\quad +\sum_{i=1}^n c_i^{-p/q}\left(\sum_{j=1}^n |a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_{T_1}^{t} e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^n |\varphi_j(s)|^p\right) ds \\
&< \sum_{i=1}^n c_i^{-p}e^{-c_i t}(e^{c_i T_1}-1)\left(\sum_{j=1}^n |a_{ij}|^q|\alpha_j|^q\right)^{p/q}\sup_{0\leq s\leq T_1}\left[\mathbb{E}\left(\sum_{j=1}^n |\varphi_j(s)|^p\right)\right] \\
&\quad +\varepsilon\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n |a_{ij}|^q|\alpha_j|^q\right)^{p/q}.
\end{aligned}$$

Hence, from the fact that $c_i > 0$ $(i = 1, 2, \cdots, n)$, we obtain that $\mathbb{E}\sum_{i=1}^n |J_{2i}(t)|^p \to 0$ as $t \to \infty$.

With the similar computation as (6.17), we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^n |J_{3i}(t)|^p &\leq \sum_{i=1}^n c_i^{-p/q}\left(\sum_{j=1}^n |b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^{t} e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^n |\varphi_j(s-\tau(s)))|^p\right) ds \\
\mathbb{E}\sum_{i=1}^n |J_{4i}(t)|^p &\leq \sum_{i=1}^n c_i^{-p/q}\left(\sum_{j=1}^n |l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^{t} e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^n \left|\int_{s-r(s)}^{s}\varphi_j(u)\,du\right|^p\right] ds \\
&\leq \sum_{i=1}^n \left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^n |l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^{t} e^{-c_i(t-s)}\int_{s-r(s)}^{s}\mathbb{E}\left[\sum_{j=1}^n |\varphi_j(u)|^p\right] du\,ds.
\end{aligned} \tag{6.18}$$

Using Lemma 6.1.16, we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^n |J_{5i}(t)|^p &= \sum_{i=1}^n \mathbb{E}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n \sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,dw_j(s)\right|^p \\
&\leq n^{p-1}\sum_{i=1}^n\sum_{j=1}^n \mathbb{E}\left\{\left[\int_0^t e^{-c_i(t-s)}|\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))|\,dw_j(s)\right]^2\right\}^{p/2} \\
&= n^{p-1}\sum_{i=1}^n\sum_{j=1}^n \mathbb{E}\left[\int_0^t e^{-2c_i(t-s)}\sigma_{ij}^2(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,ds\right]^{p/2} \\
&\leq n^{p-1}\sum_{i=1}^n\sum_{j=1}^n \mathbb{E}\left[\int_0^t e^{-2c_i(t-s)}\left(\mu_j\varphi_j^2(s)+\nu_j\varphi_j^2(s-\tau(s))\right)\,ds\right]^{p/2}
\end{aligned}$$

$$
\begin{aligned}
&\le n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\Bigg[\bigg(\int_0^t e^{-2c_i(t-s)}\mu_j\varphi_j^2(s)\,ds\bigg)^{p/2}\\
&\quad+\bigg(\int_0^t e^{-2c_i(t-s)}\nu_j\varphi_j^2(s-\tau(s))\,ds\bigg)^{p/2}\Bigg]\\
&\le n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\Bigg[\bigg(\int_0^t e^{-2c_i(t-s)}\,ds\bigg)^{p/2-1}\int_0^t e^{-2c_i(t-s)}\mu_j^{p/2}|\varphi_j(s)|^p\,ds\Bigg]\\
&\quad+n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\Bigg\{\bigg(\int_0^t e^{-2c_i(t-s)}\,ds\bigg)^{p/2-1}\int_0^t e^{-2c_i(t-s)}\nu_j^{p/2}|\varphi_j(s-\tau(s))|^p\,ds\Bigg\}\\
&\le n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\Bigg[\mu^{p/2}\int_0^t e^{-2c_i(t-s)}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(s)|^p\bigg)ds\\
&\quad+\nu^{p/2}\int_0^t e^{-2c_i(t-s)}E\bigg(\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\bigg)ds\Bigg]\\
&\le n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\Bigg[\mu^{p/2}\int_0^t e^{-c_i(t-s)}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(s)|^p\bigg)ds\\
&\quad+\nu^{p/2}\int_0^t e^{-c_i(t-s)}E\bigg(\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\bigg)ds\Bigg]. \qquad (6.19)
\end{aligned}
$$

Since $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|^p\to 0$, $t-\tau(t)\to\infty$ and $t-r(t)\to\infty$ as $t\to\infty$, for each $\varepsilon>0$, there exists $T_2>0$ such that $t\ge T_2$ implies $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t-\tau(s))|^p<\varepsilon$ and $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t-r(t))|^p<\varepsilon$. From (6.18), we obtain that

$$
\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t)|^p &\le \sum_{i=1}^{n}c_i^{-p/q}\bigg(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\bigg)^{p/q}\int_0^{T_2}e^{-c_i(t-s)}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\bigg)ds\\
&\quad+\sum_{i=1}^{n}c_i^{-p/q}\bigg(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\bigg)^{p/q}\int_{T_2}^{t}e^{-c_i(t-s)}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\bigg)ds\\
&< \sum_{i=1}^{n}\bigg(\frac{1}{c_i}\bigg)^{p/q}e^{-c_it}\int_0^{T_2}e^{c_is}\,ds\bigg(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\bigg)^{p/q}\\
&\quad\times\sup_{\vartheta\le s\le T_2}\bigg\{\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\bigg)\bigg\}+\varepsilon\sum_{i=1}^{n}c_i^{-p}\bigg(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\bigg)^{p/q}\\
\mathbb{E}\sum_{i=1}^{n}|J_{4i}(t)|^p &\le \sum_{i=1}^{n}\bigg(\frac{r}{c_i}\bigg)^{p/q}\bigg(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\bigg)^{p/q}\int_0^{T_2}e^{-c_i(t-s)}\int_{s-r(s)}^{s}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(u)|^p\bigg)du\,ds\\
&\quad+\sum_{i=1}^{n}\bigg(\frac{r}{c_i}\bigg)^{p/q}\bigg(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\bigg)^{p/q}\int_{T_2}^{t}e^{-c_i(t-s)}\int_{s-r(s)}^{s}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(u)|^p\bigg)du\,ds
\end{aligned}
$$

$$< \sum_{i=1}^{n} re^{-c_i t}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\sup_{\vartheta\le u\le T_2}\left\{\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)|^p\right)\right\}\frac{(e^{c_iT_2}-1)}{c_i}$$
$$+\sum_{i=1}^{n}\frac{\varepsilon r}{c_i}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}.$$

Further, from (6.19), we obtain

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|J_{5i}(t)|^p \;\; &\le \;\; n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\Bigg[\mu^{p/2}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds \\
&\quad +\nu^{p/2}\int_0^t e^{-c_i(t-s)}E\left(\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\right)ds\Bigg] \\
&< \;\; n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\Bigg\{\mu^{p/2}\sup_{0\le s\le T_2}\left[\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)\right] \\
&\quad +\nu^{p/2}\sup_{\vartheta\le s\le T_2}\left[E\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)\right]\Bigg\}\frac{e^{-c_it}\left(e^{c_iT_2}-1\right)}{c_i} \\
&\quad +n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left(\frac{\varepsilon(\mu^{p/2}+\nu^{p/2})}{c_i}\right).
\end{aligned}$$

Hence, let $t\to\infty$, from the fact that $c_i>0$ $(i=1,2,\cdots,n)$, we obtain that

$$\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t)|^p\to 0,\quad \mathbb{E}\sum_{i=1}^{n}|J_{4i}(t)|^p\to 0,\quad \text{and}\quad \mathbb{E}\sum_{i=1}^{n}|J_{5i}(t)|^p\to 0.$$

Thus, combining with (6.16), we obtain that $\mathbb{E}\sum_{i=1}^{n}|(Q\varphi)_i(t)|^p\to 0$ as $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|^p\to 0$. Therefore, $Q:\mathcal{S}_\phi\to\mathcal{S}_\phi$.

Step4. We prove that Q is a contraction mapping. For any $\varphi,\psi\in\mathcal{S}_\phi$, from (6.17)-(6.19), we obtain

$$\begin{aligned}
&\sup_{s\ge\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}|Q\varphi_i(s)-Q\psi_i(s)|^p\right\} \\
&\le 4^{p-1}\sup_{s\ge\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}a_{ij}\Big(f_j(x_j(u))-f_j(y_j(u))\Big)du\right|^p\right\} \\
&\quad +4^{p-1}\sup_{s\ge\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}b_{ij}\Big(g_j(x_j(u-\tau(u)))-g_j(y_j(u-\tau(u)))\Big)du\right|^p\right\} \\
&\quad +4^{p-1}\sup_{s\ge\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}\Big(h_j(\varphi_j(v))-h_j(\psi_j(v))\Big)dv\,du\right|^p\right\}
\end{aligned}$$

$$
\begin{aligned}
&+4^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}\Big(\sigma_{ij}(s,x_j(s),x_j(u-\tau(u)))\right.\right.\\
&\qquad\left.\left.-\sigma_{ij}(s,y_j(s),y_j(s-\tau(u)))\Big)\,dw_j(u)\right|^p\right\}\\
&\leq 4^{p-1}\left\{\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\right.\\
&\quad\left.+\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}+n^{p-1}\sum_{i=1}^{n}c_i^{-p/2}\big(\mu^{p/2}+\nu^{p/2}\big)\right\}\\
&\quad\times\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{j=1}^{n}|\varphi_j(s)-\psi_j(s)|^p\right\}=\alpha\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{j=1}^{n}|\varphi_j(s)-\psi_j(s)|^p\right\}.
\end{aligned}
$$

From (6.4), we obtain that $Q:\mathcal{S}_\phi\to\mathcal{S}_\phi$ is a contraction mapping. □

We are now ready to prove Theorem 6.1.4.

Proof. From Lemma 6.1.17, by a contraction mapping principle, we obtain that Q has a unique fixed point $x(t)$, which is a solution of (6.1) with $x(t)=\phi(t)$ as $t\in[\vartheta,0]$ and $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p\to 0$ as $t\to\infty$.

Now, we prove that the trivial solution of (6.1) is *pth* moment stable. Let $\varepsilon>0$ be given and choose $\delta>0$ ($\delta<\varepsilon$) such that $5^{p-1}\delta<(1-\alpha)\varepsilon$.

If $x(t)=(x_1(t),x_2(t),\cdots,x_n(t))^T$ is a solution of (6.1) with the initial condition satisfying $\mathbb{E}\sum_{i=1}^{n}|\phi_i(t)|^p<\delta$, then $x(t)=(Qx)(t)$ defined in (6.15). We claim that $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p<\varepsilon$ for all $t\geq 0$. Notice that $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p<\varepsilon$ for $t\in[\vartheta,0]$, we suppose that there exists $t^*>0$ such that $\mathbb{E}\sum_{i=1}^{n}|x_i(t^*)|^p=\varepsilon$ and $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p<\varepsilon$ for $\vartheta\leq t<t^*$, then it follows from (6.4), we obtain that

$$
\begin{aligned}
&\mathbb{E}\sum_{i=1}^{n}|x_i(t^*)|^p\\
&\leq 5^{p-1}\mathbb{E}\sum_{i=1}^{n}e^{-pc_it^*}|x_i(0)|^p\\
&\quad+5^{p-1}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^{t^*}e^{-c_i(t^*-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s)|^p\right)ds\\
&\quad+5^{p-1}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^{t^*}e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s-\tau(s)))|^p\right)ds\\
&\quad+5^{p-1}\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^{t^*}e^{-c_i(t^*-s)}\int_{s-r(s)}^{s}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(u)|^p\right)du\,ds
\end{aligned}
$$

$$
\begin{aligned}
&+5^{p-1}n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\Bigg[\mu^{p/2}\int_0^{t^*}e^{-c_i(t^*-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s)|^p\right)ds\\
&\quad+\nu^{p/2}\int_0^{t^*}e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s-\tau(s))|^p\right)ds\Bigg]\\
&\le\Bigg[5^{p-1}\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+5^{p-1}\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\
&\quad+5^{p-1}\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^p\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}+5^{p-1}n^{p-1}\sum_{i=1}^{n}c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)\Bigg]\varepsilon+5^{p-1}\delta\\
&<(1-\alpha)\varepsilon+\alpha\varepsilon=\varepsilon,
\end{aligned}
$$

which is a contradiction. Therefore, the trivial solution of (6.1) is asymptotically stable in *pth* moment. □

Corollary 6.1.18. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the function $r(t)$ is bounded by a constant r ($r > 0$),*

(ii) *and such that*

$$
\begin{aligned}
&5\sum_{i=1}^{n}c_i^{-2}\left(\sum_{j=1}^{n}a_{ij}^2\alpha_j^2\right)+5\sum_{i=1}^{n}c_i^{-2}\left(\sum_{j=1}^{n}b_{ij}^2\beta_j^2\right)+5\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^2\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)\\
&+20n\sum_{i=1}^{n}c_i^{-1}(\mu+\nu)<1,
\end{aligned}
$$

where c,μ,ν are defined as in Theorem 6.1.4,

then the trivial solution of (6.1) is asymptotically stable in mean square.

Consider the stochastic neural networks without distributed delays

$$
\begin{aligned}
dx_i(t) &= \left[-c_ix_i(t)+\sum_{j=1}^{n}a_{ij}f_j(x_j(t))+\sum_{j=1}^{n}b_{ij}g_j(x_j(t-\tau(t)))\right]dt\\
&\quad+\sum_{j=1}^{n}\sigma_{ij}(t,x_j(t),x_j(t-\tau(t)))\,dw_j(t) \qquad (6.20)
\end{aligned}
$$

for $i=1,2,3,\cdots,n$.

Corollary 6.1.19. *Suppose that the assumptions* (A1)-(A4) *hold. The trivial solution of (6.20) is asymptotically stable in pth moment if the following inequality holds,*

$$
\begin{aligned}
&4^{p-1}\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+4^{p-1}\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\
&+4^{p-1}n^{p-1}\sum_{i=1}^{n}c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)<1, \qquad (6.21)
\end{aligned}
$$

where μ,ν are defined as in Theorem 6.1.4. Note that the discrete delay $\tau(t)$ can be unbounded.

Remark 6.1.20. *Condition* (A4) *can be relaxed. In fact, if* $p = 2$*, then*

$$(\mathrm{A4}') \quad \forall i, \quad \sum_{j=1}^{n} (\sigma_{ij}(t,x,y) - \sigma_{ij}(t,u,v))^2 \leq \sum_{j=1}^{n} \mu_j (x_j - u_j)^2 + \nu_j (y_j - v_j)^2 \tag{6.22}$$

is sufficient, as can be easily observed from the proof of Theorem 6.1.4. If $p \geq 2$*, then* (A4) *can also be replaced by* $(A4')$*, but the factor* n^{p-1} *in front of the last term in* (6.4) *has to be repalced by* $n^{(3p/2)-2}$*. This can be seen from the proof of Theorem 6.1.4 with the aid of a few more application of the Hölder inequality.*

6.1.3 Proof of Theorem 6.1.5

In this subsection, we prove Theorem 6.1.5. We start with some preparations.

Lemma 6.1.21. *Define an operator by* $(P\varphi)(t) = \phi(t)$ *for* $t \in [-\tau, 0]$*, and for* $t \geq 0$*,* $(P\varphi)(t)$ *is defined as the right hand side of* (6.15)*. If the conditions* (i) *and* (ii) *in Theorem 6.1.5 are satisfied, then* $P : \mathcal{C}_\phi \to \mathcal{C}_\phi$ *is a contraction mapping.*

Proof. Observe that all terms at the right hand side of (6.15) have continuous paths, almost surely. Now, we prove that $P(\mathcal{C}_\phi) \subseteq \mathcal{C}_\phi$.

$$\sum_{i=1}^{n} \mathbb{E}\left[\sup_{t-\tau \leq s \leq t} |(P\varphi)_i(s)|^p\right] = \sum_{i=1}^{n} \mathbb{E}\left[\sup_{t-\tau \leq s \leq t} \left|\sum_{j=1}^{5} J_{ji}(s)\right|^p\right] \leq 5^{p-1} \sum_{j=1}^{5}\sum_{i=1}^{n} \mathbb{E}\left[\sup_{t-\tau \leq s \leq t} |J_{ji}(s)|^p\right].$$

Estimating the terms on the right-hand side of the above inequality. Let $c = \min\{c_1, c_2, c_3, \cdots, c_n\}$, and let q be such that $1/p + 1/q = 1$,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n} \sup_{t-\tau \leq s \leq t} |J_{2i}(s)|^p\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n} \sup_{t-\tau \leq s \leq t} \left|\int_0^s e^{-c_i(s-u)} \sum_{j=1}^{n} a_{ij} f_j(\varphi_j(u))\, du\right|^p\right] \\
&\leq c^{-p/q} \mathbb{E}\left\{\sum_{i=1}^{n} \sup_{t-\tau \leq s \leq t} \left[\int_0^s e^{-c(s-u)} \left(\sum_{j=1}^{n} |a_{ij}| \alpha_j |\varphi_j(u)|\right)^p du\right]\right\} \\
&\leq c^{-p/q} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^q |\alpha_j|^q\right)^{p/q} \mathbb{E}\left\{\sup_{t-\tau \leq s \leq t} \left[\int_0^s e^{-c(s-u)} \left(\sum_{j=1}^{n} |\varphi_j(u)|^p\right) du\right]\right\} \\
&\leq c^{-p/q} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^q |\alpha_j|^q\right)^{p/q} \sum_{j=1}^{n} \mathbb{E}\left\{\sup_{t-\tau \leq s \leq t} \left[\int_0^s e^{-c(s-u)} |\varphi_j(u)|^p\, du\right]\right\} \\
&\leq c^{-p/q} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^q |\alpha_j|^q\right)^{p/q} \sum_{j=1}^{n} \mathbb{E}\left\{\sup_{t-\tau \leq s \leq t} \left[\int_0^s e^{-c(s-u)} \left(\sup_{u-\tau \leq v \leq u} |\varphi_j(v)|^p\right) du\right]\right\} \\
&\leq e^{c\tau} c^{-p/q} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^q |\alpha_j|^q\right)^{p/q} \sum_{j=1}^{n} \mathbb{E}\left[\int_0^t e^{-c(t-u)} \left(\sup_{u-\tau \leq v \leq u} |\varphi_j(v)|^p\right) du\right] \qquad (6.23)
\end{aligned}$$

Since $\sum_{j=1}^n \mathbb{E}\sup_{t-\tau\le s\le t}|\varphi_j(s)|^p \to 0$ as $t\to\infty$, then for any $\varepsilon>0$, there exists $T_1\ge 0$ such that $t\ge T_1$ implies

$$\sum_{j=1}^n \mathbb{E}\left(\sup_{t-\tau\le s\le t}|\varphi_j(s)|^p\right) < \varepsilon,$$

which yields that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t e^{-c(t-u)}\left(\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\right)du\right]\\
&=\int_0^{T_1} e^{-c(t-u)}\mathbb{E}\left(\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\right)du+\int_{T_1}^t e^{-c(t-u)}\mathbb{E}\left(\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\right)du\\
&\le \int_0^{T_1} e^{-c(t-u)}\left(\sup_{\vartheta\le v\le T_1}|\varphi_j(v)|^p\right)du+\frac{\varepsilon}{c}.
\end{aligned}$$

Then combining with (6.23), we obatin that $\mathbb{E}\sum_{i=1}^n\sup_{t-\tau\le s\le t}|J_{2i}(s)|^p\to 0$ as $t\to\infty$. Similarly, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^n\sup_{t-\tau\le s\le t}|J_{3i}(s)|^p\right]\\
&\le c^{-p/q}\sum_{i=1}^n\left(\sum_{j=1}^n|b_{ij}|^q|\beta_j|^q\right)^{p/q}\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}\left(\sum_{j=1}^n|\varphi_j(u-\tau(u))|^p\right)du\right]\right\}\\
&\le c^{-p/q}\sum_{i=1}^n\left(\sum_{j=1}^n|b_{ij}|^q|\beta_j|^q\right)^{p/q}\sum_{j=1}^n\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}|\varphi_j(u-\tau(u))|^p\,du\right]\right\}\\
&\le c^{-p/q}\sum_{i=1}^n\left(\sum_{j=1}^n|b_{ij}|^q|\beta_j|^q\right)^{p/q}\sum_{j=1}^n\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\,du\right]\right\}\\
&\le e^{c\tau}c^{-p/q}\sum_{i=1}^n\left(\sum_{j=1}^n|b_{ij}|^q|\beta_j|^q\right)^{p/q}\sum_{j=1}^n\mathbb{E}\left[\int_0^t e^{-c(t-u)}\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\,du\right]
\end{aligned}\tag{6.24}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^n\sup_{t-\tau\le s\le t}|J_{4i}(s)|^p\right]\\
&\le c^{-p/q}\sum_{i=1}^n\left(\sum_{j=1}^n|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}\sum_{j=1}^n\left|\int_{u-r(u)}^u\varphi_j(v)\,dv\right|^p du\right]\right\}\\
&\le c^{-p/q}\sum_{i=1}^n\left(\sum_{j=1}^n|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\sum_{j=1}^n\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}\left|\int_{u-r(u)}^u\varphi_j(v)\,dv\right|^p du\right]\right\}\\
&\le \tau^p c^{-p/q}\sum_{i=1}^n\left(\sum_{j=1}^n|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\sum_{j=1}^n\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\,du\right]\right\}\\
&\le \tau^p e^{c\tau}c^{-p/q}\sum_{i=1}^n\left(\sum_{j=1}^n|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\sum_{j=1}^n\mathbb{E}\left[\int_0^t e^{-c(t-u)}\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\,du\right].
\end{aligned}\tag{6.25}$$

Let $\mu = \max\{\mu_1, \mu_2, \cdots, \mu_n\}$, $\nu = \max\{\nu_1, \nu_2, \cdots, \nu_n\}$. Due to the fact that

$$\left| \int_0^s e^{-c_i(s-u)} \sigma_{ij}(u, \varphi_j(u), \varphi_j(u-\tau(u)))\, dw_j(u) \right|^p$$

is a submartingale and the supremum of submartingale is also a submartingale, using Doob's inequality for positive submartingale, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n} \sup_{t-\tau\le s\le t} |J_{5i}(s)|^p\right] \\
&\le n^{p-1} \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\left[\sup_{t-\tau\le s\le t}\left| \int_0^s e^{-c_i(s-u)} \sigma_{ij}(u, \varphi_j(u), \varphi_j(u-\tau(u)))\, dw_j(u) \right|^p\right] \\
&\le n^{p-1} \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\sup_{t-\tau\le r\le t}\left| \int_0^s e^{-c(r-u)} \sigma_{ij}(u, \varphi_j(u), \varphi_j(u-\tau(u)))\, dw_j(u) \right|^p\right]\right\} \\
&\le n^{p-1} \sum_{i=1}^{n}\sum_{j=1}^{n} q^p \sup_{t-\tau\le s\le t}\left\{\mathbb{E}\left[\sup_{t-\tau\le r\le t}\left| \int_0^s e^{-c(r-u)} \sigma_{ij}(u, \varphi_j(u), \varphi_j(u-\tau(u)))\, dw_j(u) \right|^p\right]\right\} \\
&\le n^{p-1} e^{pc\tau} \sum_{i=1}^{n}\sum_{j=1}^{n} q^p \sup_{t-\tau\le s\le t}\left[\mathbb{E}\left| \int_0^s e^{-c(t-u)} \sigma_{ij}(u, \varphi_j(u), \varphi_j(u-\tau(u)))\, dw_j(u) \right|^p\right] \\
&\le K_p n^{p-1} e^{pc\tau} \sum_{i=1}^{n}\sum_{j=1}^{n} q^p \sup_{t-\tau\le s\le t}\left[\mathbb{E}\left(\int_0^s e^{-2c(t-u)} \sigma_{ij}^2(u, \varphi_j(u), \varphi_j(u-\tau(u)))\, du \right)^{p/2}\right] \\
&\le K_p n^{p-1} e^{pc\tau} \sum_{i=1}^{n}\sum_{j=1}^{n} q^p 2^{p/2-1} \sup_{t-\tau\le s\le t} \mathbb{E}\left[\left(\int_0^s e^{-2c(t-u)} \left(\mu_j \varphi_j^2(u)\right) du \right)^{p/2}\right] \\
&\quad + K_p n^{p-1} e^{pc\tau} \sum_{i=1}^{n}\sum_{j=1}^{n} q^p 2^{p/2-1} \sup_{t-\tau\le s\le t}\left[\mathbb{E}\left(\int_0^s e^{-2c(t-u)} \left(\nu_j \varphi_j^2(u-\tau(u))\right) du \right)^{p/2}\right] \\
&\le K_p n^{p-1} e^{pc\tau} \sum_{i=1}^{n}\sum_{j=1}^{n} q^p 2^{p/2-1} \sup_{t-\tau\le s\le t}\left\{\mathbb{E}\left[\left(\int_0^s e^{-2c(t-u)}\, du \right)^{p/2-1}\right.\right. \\
&\qquad \left.\left.\times\left(\int_0^s e^{-2c(t-u)} \mu_j^{p/2} |\varphi_j(u)|^p\, du + \int_0^s e^{-2c(t-u)} \nu_j^{p/2} |\varphi_j(u-\tau(u))|^p\, du \right)\right]\right\} \\
&\le K_p n^p e^{pc\tau} q^p c^{1-p/2} (\mu^{p/2} + \nu^{p/2}) \int_0^t e^{-2c(t-u)} \sum_{j=1}^{n} \mathbb{E}\left[\sup_{u-\tau\le v\le u} |\varphi_j(v)|^p\, du\right].
\end{aligned} \tag{6.26}$$

Using the similar arguments as for the term (6.23) and combining with (6.24), (6.25) and (6.26), we obtain that $\sum_{i=1}^{n} \mathbb{E}\left[\sup_{t-\tau\le s\le t} |(P\varphi)_i(s)|^p\right] \to 0$ as $t \to \infty$. Thus, $P(\mathcal{C}_\phi) \subseteq \mathcal{C}_\phi$.

Finally, we prove that P is a contraction mapping. For any $\varphi, \psi \in \mathcal{C}_\phi$, from (6.23)-(6.26), we obtain that

$$\begin{aligned}
&\sup_{t\geq 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}|(P\varphi)_i(s)-(P\psi)_i(s)|^p\right]\right\}\\
&\leq 4^{p-1}\sup_{t\geq 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}a_{ij}\Big(f_j(\varphi_j(u))-f_j(\psi_j(u))\Big)\,du\right|^p\right]\right\}\\
&\quad+4^{p-1}\sup_{t\geq 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}b_{ij}\right.\right.\right.\\
&\qquad\left.\left.\left.\times\Big(g_j(\varphi_j(u-\tau(u)))-g_j(\psi_j(u-\tau(u)))\Big)\,du\right|^p\right]\right\}\\
&\quad+4^{p-1}\sup_{t\geq 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}\Big(h_j(\varphi_j(v))-h_j(\psi_j(v))\Big)\,dv\,du\right|^p\right]\right\}\\
&\quad+4^{p-1}\sup_{t\geq 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}\left|\int_0^s e^{-c_i(s-u)}\right.\right.\right.\\
&\qquad\left.\left.\left.\times\sum_{j=1}^{n}[\sigma_{ij}(u,\varphi_j(u),\varphi_j(u-\tau(u)))-\sigma_{ij}(u,\psi_j(u),\psi_j(u-\tau(u)))]\,dw_j(u)\right|^p\right]\right\}\\
&\leq 4^{p-1}\left\{e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\right.\\
&\quad\left.+\tau^p e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}+K_p n^p e^{pc\tau}q^p c^{1-p/2}(2c)^{-1}\left(\mu^{p/2}+\nu^{p/2}\right)\right\}\\
&\qquad\times\sup_{t\geq 0}\sum_{j=1}^{n}\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|\varphi_j(s)-\psi_j(s)|^p\right]=\alpha\sup_{t\geq 0}\sum_{j=1}^{n}\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|\varphi_j(s)-\psi_j(s)|^p\right].
\end{aligned}$$

From (6.6), we obtain that $P:\mathcal{C}_\phi\to\mathcal{C}_\phi$ is a contraction mapping. □

We are now ready to prove Theorem 6.1.5

Proof. From Lemma 6.1.21, by a contraction mapping principle, we obtain that P has a unique fixed point $x(t)$, which is a solution of (6.1) with $x(t)=\phi(t)$ as $t\in[\vartheta,0]$ and $\sum_{i=1}^{n}\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|x_i(s)|^p\right]\to 0$ as $t\to\infty$.

We prove that the trivial solution of (6.1) is *pth* moment stable. Let $\varepsilon>0$ be given, we suppose that there exists $t^*>0$ such that

$$\begin{aligned}
\sum_{i=1}^{n}\mathbb{E}\left[\sup_{t^*-\tau\leq s\leq t^*}|x_i(s)|^p\right] &= \varepsilon,\\
\sum_{i=1}^{n}\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|x_i(s)|^p\right] &< \varepsilon \quad\text{for}\quad \vartheta\leq \mathrm{t}<\mathrm{t}^*,
\end{aligned}$$

choose $\delta>0$ ($\delta<\varepsilon$) satisfying

$$5^{p-1}e^{-pct^*}\delta<(1-\alpha)\varepsilon. \tag{6.27}$$

If $x(t) = (x_1(t), x_2(t), \cdots x_n(t))^T$ is a solution of (6.1) with the initial condtion satisfying $\|\phi\|^p < \delta$, then $x(t) = (Px)(t)$ defined in (6.15). We claim that $\|x\|^p < \varepsilon$ for all $t \geq 0$. It follows from (6.4) and (6.27), we obtain that

$$
\begin{aligned}
&\sum_{i=1}^{n} \mathbb{E}\left[\sup_{t^*-\tau\leq s\leq t^*} |x_i(s)|^p\right] \\
&\leq 5^{p-1}\sum_{j=1}^{5}\sum_{i=1}^{n} \mathbb{E}\left[\sup_{t^*-\tau\leq s\leq t^*} |J_{ji}(s)|^p\right] \\
&\leq 5^{p-1}e^{-pct^*}\delta + 5^{p-1}\Bigg\{e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q} \\
&\qquad +\tau^p e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q} + K_p n^p e^{pc\tau} q^p c^{1-p/2}(2c)^{-1}\left(\mu^{p/2}+\nu^{p/2}\right)\Bigg\}\varepsilon \\
&< (1-\alpha)\varepsilon + \alpha\varepsilon = \varepsilon,
\end{aligned}
$$

which is a contradiction. Thus, the proof follows. □

6.1.4 Proof of Theorem 6.1.7

In this subsection, we prove Theorem 6.1.7. We start with a lemma presenting an integral inequality lemma.

Lemma 6.1.22. *Consider $c, \tau > 0$, positive constants $\lambda_1, \lambda_2, \lambda_3$ and a function $y : [-\tau, \infty) \to [0, \infty)$. If $\lambda_1 + \lambda_2 + \tau\lambda_3 < c$ and the following inequality holds,*

$$
y(t) \leq \begin{cases} y_0 e^{-ct} + \lambda_1 \int_0^t e^{-c(t-s)} y(s)\, ds + \lambda_2 \int_0^t e^{-c(t-s)} y(s-\tau(s))\, ds \\ \qquad +\lambda_3 \int_0^t e^{-c(t-s)} \int_{s-r(s)}^{s} y(u)\, du\, ds \qquad t \geq 0, \\ y_0 e^{-ct}, \quad t \in [-\tau, 0], \end{cases} \tag{6.28}
$$

then we have $y(t) \leq y_0 e^{-\gamma t} (t \geq -\tau)$, where γ is a positive root of the transcendental equation $\frac{1}{c-\gamma}\left(\lambda_1 + e^{\gamma\tau}\lambda_2 + \frac{e^{\gamma\tau}-1}{\gamma}\lambda_3\right) = 1$.

Proof. Let $F(\gamma) = \frac{1}{c-\gamma}\left(\lambda_1 + e^{\gamma\tau}\lambda_2 + \frac{e^{\gamma\tau}-1}{\gamma}\lambda_3\right) - 1$. We have $F(0)F(c^-) < 0$, that is, there exists a positive constant $\gamma \in (0, c)$ such that $F(\gamma) = 0$. For any $\varepsilon > 0$, let

$$C_\varepsilon = \varepsilon + y_0.$$

To prove the lemma, we claim that (6.28) implies

$$y(t) \leq C_\varepsilon e^{-\gamma t}, \quad t \geq -\tau. \tag{6.29}$$

It is easily shown that (6.29) holds for $t \in [-\tau, 0]$. Assume that there exists $t_1^* > 0$ such that

$$y(t) < C_\varepsilon e^{-\gamma t}, \quad t \in [-\tau, t_1^*), \qquad y(t_1^*) = C_\varepsilon e^{-\gamma t_1^*}. \tag{6.30}$$

Combining with (6.28), we have

$$
\begin{aligned}
y(t_1^*) &\le y_0 e^{-ct_1^*} + \lambda_1 \int_0^{t_1^*} e^{-c(t_1^*-s)} y(s)\, ds + \lambda_2 \int_0^{t_1^*} e^{-c(t_1^*-s)} y(s-\tau(s))\, ds \\
&\quad + \lambda_3 \int_0^{t_1^*} e^{-c(t_1^*-s)} \int_{s-r(s)}^{s} y(u)\, du\, ds \\
&< y_0 e^{-ct_1^*} + C_\varepsilon \lambda_1 \int_0^{t_1^*} e^{-c(t_1^*-s)} e^{-\gamma s}\, ds + C_\varepsilon \lambda_2 \int_0^{t_1^*} e^{-c(t_1^*-s)} e^{-\gamma(s-\tau(s))}\, ds \\
&\quad + C_\varepsilon \lambda_3 \int_0^{t_1^*} e^{-c(t_1^*-s)} \int_{s-r(s)}^{s} e^{-\gamma u}\, du\, ds \\
&= \left[y_0 - \frac{C_\varepsilon}{c-\gamma} \left(\lambda_1 + e^{\gamma\tau}\lambda_2 + \frac{e^{\gamma\tau}-1}{\gamma}\lambda_3 \right) \right] e^{-ct_1^*} \\
&\quad + \frac{C_\varepsilon}{c-\gamma} \left(\lambda_1 + e^{\gamma\tau}\lambda_2 + \frac{e^{\gamma\tau}-1}{\gamma}\lambda_3 \right) e^{-\gamma t_1^*}.
\end{aligned}
$$

From the definition of C_ε, we have

$$
y_0 - \frac{C_\varepsilon}{c-\gamma} \left(\lambda_1 + e^{\gamma\tau}\lambda_2 + \frac{e^{\gamma\tau}-1}{\gamma}\lambda_3 \right) = y_0 - C_\varepsilon < 0.
$$

Then, together with the definition of γ, we obtain that $y(t_1^*) < C_\varepsilon e^{-\gamma t_1^*}$, which contradicts (6.30), so (6.29) holds. As $\varepsilon > 0$ is arbitrarily small, in view of (6.29), it follows that $y(t) \le y_0 e^{-\gamma t}$ for $t \ge -\tau$. □

Proof. For the representation (6.14), using (6.17)-(6.19), we obtain that

$$
\begin{aligned}
&E \sum_{i=1}^{n} |x_i(t)|^p \\
&\le 5^{p-1} e^{-ct} \sum_{i=1}^{n} \mathbb{E}|\phi_i(0)|^p \\
&\quad + 5^{p-1} c^{-p/q} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^q |\alpha_j|^q \right)^{p/q} \int_0^t e^{-c(t-s)} \mathbb{E}\left(\sum_{j=1}^{n} |x_j(s)|^p \right) ds \\
&\quad + 5^{p-1} c^{-p/q} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |b_{ij}|^q |\beta_j|^q \right)^{p/q} \int_0^t e^{-c(t-s)} \mathbb{E}\left[\sum_{j=1}^{n} |x_j(s-\tau(s)))|^p \right] ds \\
&\quad + 5^{p-1} \left(\frac{\tau}{c} \right)^{p/q} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |l_{ij}|^q |\gamma_j|^q \right)^{p/q} \int_0^t e^{-c(t-s)} \int_{s-r(s)}^{s} \mathbb{E}\left[\sum_{j=1}^{n} |x_j(u)|^p \right] du\, ds \\
&\quad + 5^{p-1} n^p c^{1-p/2} \Bigg\{ \mu^{p/2} \int_0^t e^{-c(t-s)} \mathbb{E}\left[\sum_{j=1}^{n} |x_j(s)|^p \right] ds \\
&\qquad + \nu^{p/2} \int_0^t e^{-c(t-s)} \mathbb{E}\left[\sum_{j=1}^{n} |x_j(s-\tau(s))|^p \right] ds \Bigg\}.
\end{aligned}
$$

Hence, by using Lemma 6.1.22 and (6.7), we obtain that the trivial solution of (6.1) is exponentially stable in *pth* moment. □

Corollary 6.1.23. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ ($\tau > 0$),*

(ii) *and such that*

$$5c^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}a_{ij}^2\alpha_j^2+5c^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n}b_{ij}^2\beta_j^2+5c^{-2}\tau^2\sum_{i=1}^{n}\sum_{j=1}^{n}l_{ij}^2\gamma_j^2+20n^2c^{-1}(\mu+\nu)<1,$$

where c, μ, ν are defined as in Theorem 6.1.4,

then the trivial solution of (6.1) is exponentially stable in mean square.

Corollary 6.1.24. *Let $p \geq 2$. Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ ($\tau > 0$),*

(ii) *and such that*

$$4^{p-1}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+4^{p-1}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}$$
$$+4^{p-1}n^pc^{-p/2}(\mu^{p/2}+\nu^{p/2})<1,$$

where c, μ, ν are defined as in Theorem 6.1.4,

then the trivial solution of (6.20) is exponentially stable in pth moment.

6.1.5 Proof of Theorem 6.1.11

In this subsection, we prove Theorem 6.1.11. We start with some preparations.

Multiply both sides of (6.9) by e^{c_it} and integrate from 0 to t, we obtain that for $t \geq 0$,

$$\begin{aligned} x_i(t) &= e^{-c_it}x_i(0)+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}a_{ij}g_j(x_j(s))\,ds+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}b_{ij}g_j(x_j(s-\tau(s)))\,ds \\ &\quad+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}d_{ij}\int_{s-r(s)}^{s}g_j(x_j(u))\,du\,ds, \quad i=1,2,3,\cdots,n. \end{aligned} \tag{6.31}$$

Lemma 6.1.25. *Define an operator by $(Px)(\theta)=\phi(\theta)$ for $\vartheta \leq \theta \leq 0$, and for $t \geq 0$,*

$$\begin{aligned} (Px)_i(t) &= e^{-c_it}x_i(0)+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}a_{ij}g_j(x_j(s))\,ds \\ &\quad+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}b_{ij}g_j(x_j(s-\tau(s)))\,ds \\ &\quad+\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}d_{ij}\int_{s-r(s)}^{s}g_j(x_j(u))\,du\,ds := \sum_{i=1}^{4}I_i(t). \end{aligned} \tag{6.32}$$

If the conditions (i) *and* (i) *in Theorem 6.1.11 are satisfied, then $P : \mathcal{H}_\phi \to \mathcal{H}_\phi$ and P is a contraction mapping.*

Proof. First, we prove that $P\mathcal{H}_\phi \subseteq \mathcal{H}_\phi$. In view of (6.32), we have that, for fixed time $t_1 \geq 0$, it is easy to check that $\lim_{r\to 0}[(Px)_i(t_1+r)-(Px)_i(t_1)] = 0$. Thus, P is continuous on $[0,\infty)$. Note that $(Px)_i(\theta) = \phi(\theta)$ for $\vartheta \leq \theta \leq 0$, we obtain that P is indeed continuous on $[\vartheta,\infty)$.

Next, we prove that $\lim_{t\to\infty}(Px)_i(t) = 0$ for $x_i(t) \in \mathcal{H}_{i\phi}$. Since $x_i(t) \in \mathcal{H}_{i\phi}$, we have that $\lim_{t\to\infty} x_i(t) = 0$. Then for any $\varepsilon > 0$, there exists $T_i > 0$ such that $s \geq T_i$ implies $|x_i(s)| < \varepsilon$. Choose $T = \max_{i=1,2,\cdots,n}\{T_i\}$, combining with condition (A2),

$$\begin{aligned}
|I_2(t)| &= \left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n a_{ij}f_j(x_j(s))\,ds\right| \\
&\leq \int_0^T e^{-c_i(t-s)}\sum_{j=1}^n |a_{ij}k_j||x_j(s)|\,ds + \int_T^t e^{-c_i(t-s)}\sum_{j=1}^n |a_{ij}\alpha_j||x_j(s)|\,ds \\
&\leq \sum_{j=1}^n |a_{ij}\alpha_j| \sup_{0\leq s\leq T}|x_j(s)|\int_0^T e^{-c_i(t-s)}\,ds + \varepsilon\sum_{j=1}^n |a_{ij}\alpha_j|\int_T^t e^{-c_i(t-s)}\,ds \\
&\leq e^{-c_i t}\sum_{j=1}^n |a_{ij}\alpha_j| \sup_{0\leq s\leq T}|x_j(s)|\int_0^T e^{-c_i s}\,ds + \frac{\varepsilon}{c_i}\sum_{j=1}^n |a_{ij}\alpha_j|. \qquad (6.33)
\end{aligned}$$

From the fact that $c_i > 0$ $(i = 1,2,\cdots,n)$ and estimate (6.33), we have that $I_2(t) \to 0$ as $t \to \infty$.

Since $x_i(t) \to 0$ and $t - \tau(t) \to \infty$ as $t \to \infty$, for each $\varepsilon > 0$, there exists $T_i' > 0$ such that $s \geq T_i'$ implies $|x_i(s-\tau(s))| < \varepsilon$ for $i = 1,2,\cdots,n$. Choose $T' = \max_{i=1,2,\cdots n}\{T_i'\}$, we obtain

$$\begin{aligned}
|I_3(t)| &= \left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n b_{ij}g_j(x_j(s-\tau(s)))\,ds\right| \\
&\leq \int_0^{T'} e^{-c_i(t-s)}\sum_{j=1}^n |b_{ij}\beta_j||x_j(s-\tau(s))|\,ds + \int_{T'}^t e^{-c_i(t-s)}\sum_{j=1}^n |b_{ij}k_j||x_j(s-\tau(s))|\,ds \\
&\leq e^{-c_i t}\sum_{j=1}^n |b_{ij}\beta_j| \sup_{\vartheta\leq s\leq T'}|x_j(s)|\int_0^{T'} e^{c_i s}\,ds + \frac{\varepsilon}{c_i}\sum_{j=1}^n |b_{ij}\beta_j|. \qquad (6.34)
\end{aligned}$$

From the fact that $c_i > 0$ $(i = 1,2,\cdots,n)$ and estimate (6.34), we have that $I_3(t) \to 0$ as $t \to \infty$.

Since $x_i(t) \to 0$ and $t - r(t) \to \infty$ as $t \to \infty$, for each $\varepsilon > 0$, there exists $T_i^* > 0$ such that $s \geq T_i^*$ implies $|x_i(s-r(s))| < \varepsilon$ for $i = 1,2,\cdots,n$. Choose $T^* = \max_{i=1,2,\cdots n}\{T_i^*\}$, we obtain

$$\begin{aligned}
|I_4(t)| &= \left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n d_{ij}\int_{s-r(s)}^s h_j(x_j(u))\,du\,ds\right| \\
&\leq \int_0^{T^*} e^{-c_i(t-s)}\sum_{j=1}^n |d_{ij}\gamma_j|\int_{s-r(s)}^s |x_j(u)|\,du\,ds + \varepsilon r\int_{T^*}^t e^{-c_i(t-s)}\sum_{j=1}^n |d_{ij}\gamma_j|\,ds \\
&\leq r\sum_{j=1}^n |d_{ij}\gamma_j| \sup_{\vartheta\leq u\leq T^*}|x_j(u)|\int_0^{T^*} e^{-c_i(t-s)}\,ds + \frac{\varepsilon r}{c_i}\sum_{j=1}^n |d_{ij}\gamma_j|. \qquad (6.35)
\end{aligned}$$

From the fact that $c_i > 0 (i = 1, 2, \cdots, n)$ and estimate (6.35), we have that $I_4(t) \to 0$ as $t \to \infty$. From the above estimate, we conclude that $\lim_{t\to\infty}(Px)_i)(t) = 0$ for $x_i(t) \in \mathcal{H}_{i\phi}$. Therefore, $P : \mathcal{H}_\phi \to \mathcal{H}_\phi$.

Now, we prove that P is a contraction mapping. For any $x, y \in \mathcal{H}_\phi$, from (6.33) and (6.35), we obtain that

$$
\begin{aligned}
&\sum_{i=1}^{n} |(Px)_i(t) - (Py)_i(t)| \\
&\quad \le \sum_{i=1}^{n} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} |x_j(s) - y_j(s)|\, ds \\
&\qquad + \sum_{i=1}^{n} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} |x_j(s-\tau(s)) - y_j(s-\tau(s))|\, ds \\
&\qquad + \sum_{i=1}^{n} \max_{j=1,2,\cdots,n} |d_{ij}\gamma_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} \int_{s-r(s)}^{s} |x_j(u) - y_j(u)|\, du\, ds \\
&\quad \le \sum_{i=1}^{n} \left\{ \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \frac{r}{c_i} \max_{j=1,2,\cdots,n} |d_{ij}\gamma_j| \right\} \\
&\qquad \times \sup_{\vartheta \le s \le t} \sum_{j=1}^{n} |x_j(s) - y_j(s)| = \alpha \sup_{\vartheta \le s \le t} \sum_{j=1}^{n} |x_j(s) - y_j(s)|.
\end{aligned}
$$

Hence, we obtain that P is a contraction mapping. □

We are now ready to prove Theorem 6.1.11.

Proof. Let P be defined as in Lemma 6.1.25, by a contraction mapping principle, P has a unique fixed point $x \in \mathcal{H}_\phi$ with $x(\theta) = \phi(\theta)$ on $\vartheta \le \theta \le 0$ and $x(t) \to 0$ as $t \to \infty$.

To obtain asymptotic stability, it remains to prove that the trivial solution $x = 0$ of (6.9) is stable. For any $\varepsilon > 0$, choose $\sigma > 0$ and $\sigma < \varepsilon$ satisfying the condition $\sigma + \varepsilon\alpha < \varepsilon$.

If $x(t, s, \phi) = (x_1(t, s, \phi), x_2(t, s, \phi), \cdots, x_n(t, s, \phi))$ is the solution of (6.9) with the initial condition $\|\phi\| < \sigma$, then we claim that $\|x(t, s, \phi)\| < \varepsilon$ for all $t \ge 0$. Indeed, we suppose that there exists $t^* > 0$ such that

$$
\sum_{i=1}^{n} |x_i(t^*; s, \phi)| = \varepsilon, \qquad \text{and} \qquad \sum_{i=1}^{n} |x_i(t; s, \phi)| < \varepsilon \quad \text{for} \quad 0 \le t < t^*. \tag{6.36}
$$

From (6.12) and (6.31), we obtain

$$
\begin{aligned}
\sum_{i=1}^{n} |x_i(t^*; s, \phi)| \;\le\; & \sum_{i=1}^{n} \Bigg[|e^{-c_i t^*} x_i(0)| + \int_0^{t^*} e^{-c_i(t^*-s)} \sum_{j=1}^{n} |a_{ij} f_j(x_j(s))|\, ds \\
& + \int_0^{t^*} e^{-c_i(t^*-s)} \sum_{j=1}^{n} |b_{ij} g_j(x_j(s-\tau(s)))|\, ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \int_0^{t^*} e^{-c_i(t^*-s)} \sum_{j=1}^{n} |d_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))| \, du \, ds \Bigg] \\
&< \quad \sigma + \varepsilon \sum_{i=1}^{n} \left(\frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \frac{r}{c_i} \max_{j=1,2,\cdots,n} |d_{ij}\gamma_j| \right) \\
&\leq \quad \sigma + \varepsilon\alpha < \varepsilon,
\end{aligned}
$$

which contradicts (6.36). Therefore, $\|x(t, s, \phi)\| < \varepsilon$ for all $t \geq 0$. This completes the proof. □

Let $d_{ij} \equiv 0$ for $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$. The system (6.9) is then reduced to

$$
\frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t - \tau(t))), \tag{6.37}
$$

which is the description of a cellular neural network with time-varying delays. Following the result of Theorem 6.1.11, we have the following corollary.

Corollary 6.1.26. *Suppose that the assumptions* (A1)-(A3) *hold. If the following condition is satisfied,*

$$
\sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| < 1, \tag{6.38}
$$

then the trivial solution of (6.37) *is asymptotically stable.*

Remark 6.1.27. *Note that the delay in Corollary* 6.1.26 *can be unbounded. Lai and Zhang* [74] *studied the asymptotic stability* (6.37) *as well. However, the additional condition*

$$
\max_{i=1,2,\cdots n} \left[\frac{1}{c_i} \sum_{j=1}^{n} |a_{ij}k_j| + \frac{1}{c_i} \sum_{j=1}^{n} |b_{ij}k_j| \right] < \frac{1}{\sqrt{n}} \tag{6.39}
$$

is needed in Theorem 4.1 *of* [74]. *It is clear that Corollary* 6.1.26 *is an improvement of the result in* [74].

6.1.6 Proof of Theorem 6.1.13

Proof. From the represention (6.31), we obtain that

$$
\begin{aligned}
\sum_{i=1}^{n} |x_i(t)| \quad \leq \quad & e^{-ct} \sum_{i=1}^{n} |x_i(0)| + \sum_{i=1}^{n} \max_{j=1,2,\cdots n} \{|a_{ij}k_j|\} \int_0^t e^{-c(t-s)} \sum_{j=1}^{n} |x_j(s)| \, ds \\
& + \sum_{i=1}^{n} \max_{j=1,2,\cdots n} \{|b_{ij}k_j|\} \int_0^t e^{-c(t-s)} \sum_{j=1}^{n} |x_j(s - \tau(s))| \, ds \\
& + \sum_{i=1}^{n} \max_{j=1,2,\cdots n} \{|d_{ij}k_j|\} \int_0^t e^{-c(t-s)} \sum_{j=1}^{n} \int_{s-r(s)}^{s} |x_j(u)| \, du \, ds.
\end{aligned}
$$

Combining with Lemma 6.1.22, we obtain that the trivial solution of (6.9) with initial condition (6.11) is exponentially stable. □

For the cellular neural network (6.37), we have the following result.

Corollary 6.1.28. *Suppose that the assumptions* (A1)-(A3) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ,*

(ii) *and such that*

$$\sum_{i=1}^{n} \max_{j=1,2,\cdots,n} |a_{ij}k_j| + \sum_{i=1}^{n} \max_{j=1,2,\cdots,n} |b_{ij}k_j| < c, \quad c = \min\{c_1, c_2, \cdots c_1\},$$

then the trivial solution of (6.37) *with initial condition* (6.11) *is exponentially stable.*

6.1.7 Examples

Example 6.1.29. *Consider the following two-dimensional cellular neural network*

$$\frac{dx(t)}{dt} = -Cx(t) + Ag(x(t)) + Bg(x - \tau(t)),$$

where

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 6/7 & 3/7 \\ -1/7 & -1/7 \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 6/7 & 2/7 \\ 3/7 & 1/7 \end{pmatrix}.$$

The activation function is described by $g_i(x) = \frac{|x+1|-|x-1|}{2}$ for $i = 1, 2$. The time-varying delay $\tau(t)$ is continuous and $|\tau(t)| \le \tau$, where τ is a constant.

It is clear that $\alpha_i = \beta_i = 1$ for $i = 1, 2$. We check the condition (6.38) in Corollary 6.1.26,

$$\sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |a_{ij}\alpha_j| + \sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |b_{ij}\beta_j| \le \frac{1}{3} \times \left(\frac{6}{7} + \frac{1}{7} + \frac{6}{7} + \frac{3}{7}\right) = \frac{16}{21} < 1.$$

Hence, by Corollary 6.1.26, the trivial solution $x = 0$ of this cellular neural network is asymptotically stable.

However, the condition (6.39) becomes

$$\max_{i=1,2} \left\{ \frac{1}{c_i} \sum_{j=1}^{2} |a_{ij}\alpha_j| + \frac{1}{c_i} \sum_{j=1}^{2} |b_{ij}\beta_j| \right\} = \frac{17}{21} > \frac{1}{\sqrt{2}}.$$

Hence, Theorem 4.1 of [74] is not applicable.

Example 6.1.30. *Consider a two-dimensional stochastic recurrent neural network with time-varying delays*

$$\begin{aligned} dx(t) &= -\begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} dt + \begin{pmatrix} 2 & 0.4 \\ 0.6 & 1 \end{pmatrix}\begin{pmatrix} 0.2\tanh(x_1(t)) \\ 0.2\tanh(x_2(t)) \end{pmatrix} dt \\ &\quad + \begin{pmatrix} -0.8 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0.2\tanh(x_1(t-\tau_1(t))) \\ 0.2\tanh(x_2(t-\tau_2(t))) \end{pmatrix} dt \\ &\quad + \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \int_{t-r(t)}^{t} 0.2\tanh(x_1(s))\,ds \\ \int_{t-r(t)}^{t} 0.2\tanh(x_2(s))\,ds) \end{pmatrix} dt \\ &\quad +\sigma(t, x(t), x(t-\tau(t)))\,dw(t), \end{aligned} \tag{6.40}$$

where $\tau_1(t), \tau_2(t), r(t)$ *are continuous functions such that* $t-\tau(t) \to \infty$ *as* $t \to \infty$ *and* $|r(t)| \leq 1$, $\sigma : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ *satisfies*

$$trace\left[\sigma^T(t,x,y)\sigma(t,x,y)\right] \leq 0.003(x_1^2 + x_2^2 + y_1^2 + y_2^2),$$

and $w(t)$ *is a two dimensional Brownian motion.*

We suppose $p = 2$, and take $\mu_i = \nu_i = 0.003$ for $i = 1, 2$, by simple computation, we have $\alpha_i = 0.2$ for $i = 1, 2$, $c = \min\{c_1, c_2\} = 5$, $\mu = \nu = 0.003$. From Corollary 6.1.18, we have that

$$\begin{aligned} &5\sum_{i=1}^{2} c_i^{-2}\left(\sum_{j=1}^{2} a_{ij}^2\alpha_j^2\right) + 5\sum_{i=1}^{2} c_i^{-2}\left(\sum_{j=1}^{2} b_{ij}^2\alpha_j^2\right) + 5\sum_{i=1}^{2}\left(\frac{\tau}{c_i}\right)^2\left(\sum_{j=1}^{2} l_{ij}^2\alpha_j^2\right) \\ &+20 \times 2 \times \sum_{i=1}^{2} c_i^{-1}(\mu + \nu) < 0.256 < 1. \end{aligned}$$

Then the trivial solution of (6.40) is mean square asymptotically stable.

If $\tau(t)$ is bounded, from Corollary 6.1.23, we obtain that

$$5c^{-2}\sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}^2\alpha_j^2 + 5c^{-2}\sum_{i=1}^{2}\sum_{j=1}^{n} b_{ij}^2\alpha_j^2 + 5c^{-2}\tau^2\sum_{i=1}^{2}\sum_{j=1}^{2} l_{ij}^2\alpha_j^2 + 20 \times 4c^{-1}(\mu + \nu) < 0.298.$$

Hence, the trivial solution of (6.40) is mean square exponentially stable.

Example 6.1.31. *Consider a two-dimensional stochastically perturbed HNN with time-varying delays,*

$$dx(t) = [-Cx(t) + Af(x(t)) + Bg(x_\tau(t))]\,dt + \sigma(t, x(t), x_\tau(t))\,dw(t), \tag{6.41}$$

where $f_i(x) = \frac{1}{5}\arctan x$, $g_i(x) = \frac{1}{5}\tanh x = \frac{1}{5}(e^x - e^{-x})/(e^{x_i} + e^{-x})$, $i = 1, 2$, $\tau(t) = \frac{1}{2}\sin t + \frac{1}{2}$,

$$C = \begin{pmatrix} 5 & 0 \\ 0 & 4.5 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 0.4 \\ 0.6 & 1 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} -0.8 & 2 \\ 1 & 4 \end{pmatrix}.$$

In this example, let $p = 3$, *take* $\alpha_j = 0.2$, $\beta_j = 0.2$, $j = 1, 2$, $\sigma : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ *satisfies*

$$\sigma_{i1}(t,x,y)^2 \leq 0.01(x_1^2 + y_1^2) \quad \text{and} \quad \sigma_{i2}(t,x,y)^2 \leq 0.01(x_2^2 + y_2^2), \quad i = 1, 2,$$

and $w(t)$ *is a two dimensional Brownian motion.*

Note that the exponential stability of (6.41) has been studied in Sun and Cao [120] by employing the method of variation of parameter, inequality technique and stochastic analysis.

Now, we check the condition in Corollary 6.1.24,

$$4^{p-1}c^{-(1+p/q)}\sum_{i=1}^{2}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+4^{p-1}c^{-(1+p/q)}\sum_{i=1}^{2}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}$$
$$+4^{p-1}2^p c^{-p/2}(\mu^{p/2}+\nu^{p/2})<0.18<1.$$

From Corollary 6.1.24, the trivial solution of (6.41) is exponentially stable.

6.2 Stability of stochastic delayed neural networks with impulses

6.2.1 Introduction and main results

Besides delay and stochastic effects, impulsive effects are also likely to exist in the neural networks systems, which could stabilize or destabilize the systems. Therefore, it is of interest to take delay effects, stochastic effects and impulsive effects into account in investigations of the dynamical behavior of neural networks.

In this section, we apply fixed point methods to study asymptotic stability and exponential stability of a class of stochastic delayed neural networks with impulsive effects, which is described by

$$\begin{cases} dx_i(t)=\Big[-c_ix_i(t)+\sum_{j=1}^{n}a_{ij}f_j(x_j(t))+\sum_{j=1}^{n}b_{ij}g_j(x_j(t-\tau(t))) \\ \qquad +\sum_{j=1}^{n}l_{ij}\int_{t-r(t)}^{t}h_j(x_j(s))\,ds\Big]\,dt \\ \qquad +\sum_{j=1}^{n}\sigma_{ij}(t,x_j(t),x_j(t-\tau(t)))\,dw_j(t), \quad t\neq t_k \\ \\ \Delta x_i(t_k)=I_{ik}(x_i(t_k)), \quad t=t_k, \quad k=1,2,3,\cdots \end{cases} \tag{6.42}$$

or

$$\begin{cases} dx(t)=\Big[-Cx(t)+Af(x(t))+Bg(x(t-\tau(t)))+W\int_{t-r(t)}^{t}h(x(s))\,ds\Big]\,dt \\ \qquad +\sigma(t,x(t),x(t-\tau(t)))\,dw(t), \quad t\neq t_k \\ \\ \Delta x(t_k)=I_k(x(t_k)), \quad t=t_k, \quad k=1,2,3,\cdots \end{cases}$$

$i=1,2,3,\cdots,n$, where $x(t)=(x_1(t),x_2(t),\cdots x_n(t))^T\in\mathbb{R}^n$ is the state vector associated with the neurons; $C=\mathrm{diag}(c_1,c_2,\cdots,c_n)>0$ where $c_i>0$ represents the rate with which the ith unit will reset its ponential to the resting state in isolation when disconnected from the network and the external stochastic perturbations; $A=(a_{ij})_{n\times n}$, $B=(b_{ij})_{n\times n}$ and $W=(l_{ij})_{n\times n}$ represent the connection weight matrix, delayed connection weight matrix and distributed delayed connection weight matrix, respectively; f_j, g_j, h_j are activation functions, $f(x(t))=(f_1(x(t)),f_2(x(t)),\cdots f_n(x(t)))^T\in\mathbb{R}^n$, $g(x(t))=(g_1(x(t)),g_2(x(t)),\cdots,g_n(x(t)))^T\in\mathbb{R}^n$, $h(x(t))=(h_1(x(t)),h_2(x(t)),\cdots,h_n(x(t)))^T\in\mathbb{R}^n$, $w(t)=(w_1(t),w_2(t),\cdots,w_n(t))^T\in\mathbb{R}^n$ is an n-dimensional Brownion motion defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ with natural complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$ (i.e. $\mathcal{F}_t$ = completion of $\sigma\{w(s) : 0 \leq s \leq t\}$) and $\sigma : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\sigma = (\sigma_{ij})_{n\times n}$ is the diffusion coefficient matrix. $\Delta x_i(t_k) = I_{ik}(x_i(t_k)) = x_i(t_k^+) - x_i(t_k^-)$ is the impulse at moment t_k, and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = +\infty$, $x_i(t_k^+)$ and $x_i(t_k^-)$ stand for the right-hand and left-hand limit of $x_i(t)$ at $t = t_k$, respectively. $I_{ik}(x_i(t_k))$ shows the abrupt change of $x_i(t)$ at the impulsive moment t_k and $I_{ik}(\cdot) \in C\left(L^p_{\mathcal{F}_t}(\Omega;\mathbb{R}^n), L^p_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)\right)$. $\tau(t)$ and $r(t)$ denote a discrete time varying delay and the bound of a distributed time varying delay, respectively. Denote $\vartheta = \inf_{t\geq 0}\{t - \tau(t), t - r(t)\}$.

The initial condition for the system (6.42) is given by

$$x(t) = \phi(t), \qquad t \in [\vartheta, 0], \tag{6.43}$$

where $t \mapsto \phi(t) = (\phi_1(t), \phi_2(t), \cdots, \phi_n(t))^T \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)\right)$ with the norm is defined as

$$\|\phi\|^p = \sup_{\vartheta\leq s\leq 0}\left(\mathbb{E}\sum_{i=1}^n |\phi_i(s)|^p\right),$$

where $\mathbb{E}$ denotes expectation with respect to the probability measure $\mathbb{P}$ and $p \geq 2$.

To obtain our main results, we suppose the following conditions are satisfied:

(A1) the delays $\tau(t), r(t)$ are continuous functions such that $t - \tau(t) \to \infty$ and $t - r(t) \to \infty$ as $t \to \infty$;

(A2) $f_i(x)$, $g_i(x)$, and $h_i(x)$ satisfy Lipschitz condition. That is, for each $i = 1, 2, 3, \cdots, n$, there exist constants α_i, β_i, γ_i such that for every $x, y \in \mathbb{R}^n$,

$$|f_j(x) - f_i(y)| \leq \alpha_i|x - y|, \quad |g_i(x) - g_i(y)| \leq \beta_i|x - y|, \quad |h_j(x) - h_i(y)| \leq \gamma_i|x - y|;$$

(A3) there exists nonegative constants p_{ik} such that for any $x, y \in \mathbb{R}^n$,

$$|I_{ik}(x) - I_{ik}(y)| \leq p_{ik}|x - y|,\ i = 1, 2, \cdots, n,\ k = 1, 2, 3, \cdots;$$

(A4) assume that $f(0) \equiv 0$, $g(0) \equiv 0$, $h(0) \equiv 0$, $\sigma(t, 0, 0) \equiv 0$, $I_{ik}(0) \equiv 0$, $i = 1, 2, \cdots, n$, $k = 1, 2, 3, \cdots$;

(A5) $\sigma(t, x, y)$ satisfies a Lipschitz condition. That is, there are nonnegative constants μ_i and ν_i such that $\forall\ i, j$,

$$(\sigma_{ij}(t, x, y) - \sigma_{ij}(t, u, v))^2 \leq \mu_j(x_j - u_j)^2 + \nu_j(y_j - v_j)^2.$$

The solution $x(t) := x(t, \phi)$ of the system (6.42) is, for the time t, a piecewise continuous vector-valued function with the first kind discontinuity at the points t_k $(k = 1, 2, \cdots)$, where it is left continuous, i.e.,

$$x_i(t_k^-) = x_i(t_k), \qquad x_i(t_k^+) = x_i(t_k) + I_{ik}(x_i(t_k)), \qquad i = 1, 2, \cdots n, \qquad k = 1, 2, \cdots.$$

Define $\mathcal{S}_\phi$ the space of all $\mathcal{F}_t$-adapted processes $\varphi(t, w) : [\vartheta, \infty) \times \Omega \to \mathbb{R}^n$ such that $\varphi : [\vartheta, \infty) \mapsto L^p_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ is continuous on $t \neq t_k$ $(k = 1, 2, \cdots)$, $\lim_{t\to t_k^-} \varphi(t, \cdot)$ and $\lim_{t\to t_k^+} \varphi(t, \cdot)$ exist, and

$\lim_{t\to t_k^-}\varphi(t,\cdot)=\varphi(t_k,\cdot)$ for $k=1,2,\cdots$. Moreover, we set $\varphi(t,\cdot)=\phi(t)$ for $t\in[\vartheta,0]$ and $\mathbb{E}\left(\sum_{i=1}^n|\varphi_i(t)|^p\right)\to 0$ as $t\to\infty$, $i=1,2,\cdots,n$. If we define the metric as the form

$$\|\varphi\|^p := \sup_{t\geq\vartheta}\left(\mathbb{E}\sum_{i=1}^n|\varphi_i(t)|^p\right), \tag{6.44}$$

then $\mathcal{S}_\phi$ is a complete metric space with respect to the norm (6.44). Using the contraction mapping defined on the space $\mathcal{S}_\phi$ and applying a contraction mapping principle, we obtain our first result, which is proved in Subsection 6.2.2.

Theorem 6.2.1. *Suppose that the assumptions* (A1)-(A5) *hold. If the following conditions are satisfied,*

(i) *the function $r(t)$ is bounded by a constant r ($r>0$);*

(ii) *there exist constants p_i ($i=1,2,\cdots,n$) such that $p_{ik}\leq p_i(t_k-t_{k-1})$, $k=1,2,\cdots$;*

(iii) *and such that*

$$\begin{aligned}\alpha \triangleq\ & 6^{p-1}\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+6^{p-1}\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\ &+6^{p-1}\sum_{i=1}^n\left(\frac{r}{c_i}\right)^p\left(\sum_{j=1}^n|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\\ &+6^{p-1}n^{p-1}\sum_{i=1}^n c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)+6^{p-1}c^{-1}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}<1,\end{aligned}\tag{6.45}$$

where $c=\min\{c_1,c_2,\cdots,c_n\}$, $\mu=\max\{\mu_1,\mu_2,\cdots,\mu_n\}$, $\nu=\max\{\nu_1,\nu_2,\cdots,\nu_n\}$,

then the trivial solution of (6.42) *is pth moment asymptotically stable.*

Define $\mathcal{C}_\phi$ the space of all $\mathcal{F}_t$-adapted processes $\varphi(t,\omega):[\vartheta,\infty)\times\Omega\to\mathbb{R}^n$ such that $\varphi:[\vartheta,\infty)\mapsto L^p_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ is continuous on $t\neq t_k$ ($k=1,2,\cdots$), $\lim_{t\to t_k^-}\varphi(t,\cdot)$ and $\lim_{t\to t_k^+}\varphi(t,\cdot)$ exist, and $\lim_{t\to t_k^-}\varphi(t,\cdot)=\varphi(t_k,\cdot)$ for $k=1,2,\cdots$. Moreover, we set $\varphi(t,\cdot)=\phi(t)$ for $t\in[\vartheta,0]$ and $e^{\lambda t}\mathbb{E}\left(\sum_{i=1}^n|\varphi_i(t)|^p\right)\to 0$ as $t\to\infty$, $\lambda<\min\{c_1,c_2,\cdots,c_n\}$, $i=1,2,\cdots,n$. Then $\mathcal{C}_\phi$ is a complete metric space with respect to the norm (6.44). Using a contraction mapping defined on the space $\mathcal{C}_\phi$ and applying a contraction mapping principle, we obtain our second result. For its proof, see Subsection 6.2.3.

Theorem 6.2.2. *Suppose that the assumptions* (A1)-(A5) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ ($\tau>0$);*

(ii) *there exist constants p_i ($i=1,2,\cdots,n$) such that $p_{ik}\leq p_i(t_k-t_{k-1})$, $k=1,2,\cdots$;*

(iii) *and such that*

$$\begin{aligned}\alpha \triangleq\ & 6^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 6^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\ & +6^{p-1}\sum_{i=1}^{n}\left(\frac{\tau}{c_i}\right)^p\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\\ & +6^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right) + 6^{p-1}c^{-1}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\} < 1, \quad (6.46)\end{aligned}$$

where $c = \min\{c_1, c_2, \cdots, c_n\}$, $\mu = \max\{\mu_1, \mu_2, \cdots, \mu_n\}$, $\nu = \max\{\nu_1, \nu_2, \cdots, \nu_n\}$;

then the trivial solution of (6.42) *is pth moment exponentially stable.*

Remark 6.2.3. *In Theorem 6.2.2, both the discrete delay $\tau(t)$ and distributed delay $r(t)$ are required to be bounded, while the discrete delay $\tau(t)$ in Theorem 6.2.1 can be unbounded. It is clear that the conditions in Theorem 6.2.1 and Theorem 6.2.2 do not require the differentiability of delays. In addition, condition* (A2) *implies that the activation functions discussed in this section may be unbounded, non-monotonic and non-differentiable.*

Remark 6.2.4. *The system* (6.42) *is quite general and it includes several well-known neural network models as its special cases, see, for example, the models in* [54, 74, 78, 83, 116, 120, 129, 142]. *Sakthivel et al.* [116] *has considered asymptotic stability in mean square of the system* (6.42) *with linear impulsive effects, by employing Liapunov functional method and using linear matrix inequality optimization approach. However, the time varying delays in* [116] *should satisfy*

$$(\mathrm{H}_1) \quad 0 \le h_1 \le \tau(t) \le h_2, \qquad\qquad \tau'(t) \le \mu,$$

where h_1, h_2 are constants, the distributed delay $r(t)$ is bounded, $0 \le r(t) \le \bar{r}$, $\bar{r}$ is a constant. In our results, the condition (H_1) *is replaced by other assumptions, and the assumptions in Theorem 6.2.1 and Theorem 6.2.2 may be satisfied if* (H_1) *is not.*

Remark 6.2.5. *In this section, our approach is based on fixed point methods, and in one step, a fixed point argument can yield the existence and stability criteria of the considered system. However, when using Liapunov's direct method, one must independently verify that a solution exists. The stability criteria we provided in our main results are only in terms of the system parameters c_i, a_{ij}, b_{ij}, l_{ij}, p_i etc. Hence, these criteria can be verified easily in applications.*

Consider the a when there are no stochastic perturbations on the system (6.42), the stochastic neural networks become usual neural network which can be described as

$$\begin{cases} \frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau(t))) \\ \qquad + \sum_{j=1}^{n} l_{ij}\int_{t-r(t)}^{t} h_j(x_j(s))\,ds, \quad t \ne t_k \\ \Delta x_i(t_k) = I_{ik}(x_i(t_k)), \quad t = t_k, \quad k = 1, 2, 3, \cdots \end{cases} \qquad (6.47)$$

or

$$\begin{cases} \frac{dx(t)}{dt} = -Cx(t) + Af(x(t)) + Bg(x-\tau(t)) + D\int_{t-r(t)}^{t} h(x(s))\,ds, \quad t \ne t_k \\ \Delta x(t_k) = I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, 3, \cdots \end{cases}$$

for $i = 1, 2, 3, \cdots, n$, where $x(\cdot) = (x_1(\cdot), x_2(\cdot), \cdots, x_n(\cdot))^T$ is the neuron state vector of the transformed system (6.47).

The initial condition for the system (6.47) is

$$x(t) = \phi(t), \qquad t \in [\vartheta, 0], \tag{6.48}$$

where ϕ is a continuous function with the norm defined by $\|\phi\| = \sum_{i=1}^{n} \sup_{\vartheta \le s \le 0} |\phi_i(s)|$. Define $\mathcal{H}_\phi = \mathcal{H}_{1\phi} \times \mathcal{H}_{2\phi} \times \cdots \mathcal{H}_{n\phi}$, where $\mathcal{H}_{i\phi}$ is the space consisting of continuous functions $\varphi_i(t) : \mathbb{R}^+ \to \mathbb{R}$ such that $\varphi_i(t)$ is continuous on $t \neq t_k$ $(k = 1, 2, \cdots)$, $\lim_{t \to t_k^-} \varphi_i(t)$ and $\lim_{t \to t_k^+} \varphi_i(t)$ exist, and $\lim_{t \to t_k^-} \varphi_i(t) = \varphi_i(t_k)$. Moreover, we set $\varphi_i(\theta) = \phi(\theta)$ for $\vartheta \le \theta \le 0$ and $\varphi_i(t) \to 0$ as $t \to \infty$, $i = 1, 2 \cdots n$. For any $\varphi(t), \eta(t) \in \mathcal{H}_\phi$, if we define the metric as

$$d(\varphi, \eta) = \sum_{i=1}^{n} \sup_{t \ge \vartheta} |\varphi_i(t) - \eta_i(t)|, \tag{6.49}$$

then $\mathcal{H}_\phi$ is a complete metric space with respect to the norm (6.49). Using a contraction mapping defined on the space H_ϕ and applying a contraction mapping principle, we obtain our third result, which is proved in Subsection 6.2.4.

Theorem 6.2.6. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the function $r(t)$ is bounded by a constant r $(r > 0)$;*

(ii) *there exist constants p_i $(i = 1, 2, \cdots, n)$ such that $p_{ik} \le p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;*

(iii) *and such that*

$$\begin{aligned} \alpha \quad &\triangleq \quad \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| \\ &\quad + \sum_{i=1}^{n} \frac{r}{c_i} \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| + \max_{i=1,2,\cdots,n} \left\{ \frac{p_i}{c_i} \right\} < 1; \end{aligned} \tag{6.50}$$

then the trivial solution of (6.47) *is asymptotically stable.*

Define $\mathcal{B}_\phi = \mathcal{B}_{1\phi} \times \mathcal{B}_{2\phi} \times \cdots \mathcal{B}_{n\phi}$, where $\mathcal{B}_{i\phi}$ is the space consisting of continuous functions $\varphi_i(t) : \mathbb{R}^+ \to \mathbb{R}$ such that $\varphi_i(t)$ is continuous on $t \neq t_k$ $(k = 1, 2, \cdots)$, $\lim_{t \to t_k^-} \varphi_i(t)$ and $\lim_{t \to t_k^+} \varphi_i(t)$ exist, and $\lim_{t \to t_k^-} \varphi_i(t) = \varphi_i(t_k)$. Moreover, we set $\varphi_i(\theta) = \phi(\theta)$ for $\vartheta \le \theta \le 0$ and $e^{\lambda t}\varphi_i(t) \to 0$ as $t \to \infty$, where $\lambda < \min\{c_1, c_2, \cdots, c_n\}$, $i = 1, 2 \cdots n$. Then $\mathcal{B}_\phi$ is a complete metric space with respect to the metric (6.49). Using a contraction mapping defined on the space $\mathcal{B}_\phi$ and applying a contraction mapping principle, we obtain our fourth result, which is proved in Subsection 6.2.5.

Theorem 6.2.7. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ $(\tau > 0)$;*

(ii) *there exist constants p_i $(i = 1, 2, \cdots n)$ such that $p_{ik} \le p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;*

(iii) *and such that*

$$\alpha \triangleq \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \sum_{i=1}^{n} \frac{\tau}{c_i} \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| + \max_{i=1,2,\cdots,n} \left\{ \frac{p_i}{c_i} \right\} < 1; \quad (6.51)$$

then the trivial solution of (6.47) *is exponentially stable.*

Remark 6.2.8. *Zhang et al.* [142, 143] *have investigated exponential stability and asymptotic stability of a class of impulsive cellular neural networks by using fixed point methods, which is a special case of the system* (6.47)*. Our results in Theorem* 6.2.6 *and Theorem* 6.2.7 *improve and extend the results in* [142, 143] (*see Remark* 6.2.15 *and Remark* 6.2.17 *for more information*).

The rest of this section is organized as follows. The proofs of Theorem 6.2.1 and Theorem 6.2.2 are presented in Subsection 6.2.2 and Subsection 6.2.3, respectively. The proofs of Theorem 6.2.6 and Theorem 6.2.7 are provided in Subsection 6.2.4 and Subsection 6.2.5, respectively. Some examples are given to illustrate our main results in Subsection 6.2.6.

6.2.2 Proof of Theorem 6.2.1

In this subsection, we prove Theorem 6.2.1. We start with some preparations.

Multiply both sides of (6.42) by $e^{c_i t}$, we obtain that for $t \neq t_k$, $i = 1, 2, 3, \cdots, n$,

$$d(e^{c_i t} x_i(t)) = e^{c_i t} \left[\sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau(t))) + \sum_{j=1}^{n} l_{ij} \int_{t-r(t)}^{t} h_j(x_j(u))\, du \right] dt + e^{c_i t} \sum_{j=1}^{n} \sigma_{ij}(t, x_j(t), x_j(t-\tau(t)))\, dw_j(t). \quad (6.52)$$

Integrate (6.52) from $t_{k-1} + \varepsilon$ ($\varepsilon > 0$) to $t \in (t_{k-1}, t_k)$ ($k = 1, 2, \cdots$), we obtain that

$$e^{c_i t} x_i(t) = e^{c_i (t_{k-1}+\varepsilon)} x_i(t_{k-1}+\varepsilon) + \int_{t_{k-1}+\varepsilon}^{t} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, dw_j(s) + \int_{t_{k-1}+\varepsilon}^{t} e^{c_i s} \left[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\, du \right] ds. \quad (6.53)$$

Let $\varepsilon \to 0$ in (6.53), for $t \in (t_{k-1}, t_k)$ $(k = 1, 2, \cdots)$, we obtain that

$$\begin{aligned} e^{c_i t} x_i(t) &= e^{c_i t_{k-1}} x_i(t_{k-1}^+) + \int_{t_{k-1}}^{t} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, dw_j(s) \\ &\quad + \int_{t_{k-1}}^{t} e^{c_i s} \Bigg[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) \\ &\quad + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\, du \Bigg] ds. \end{aligned} \tag{6.54}$$

Set $t = t_k - \varepsilon$ $(\varepsilon > 0)$ in (6.54), we obtain that

$$\begin{aligned} e^{c_i (t_k - \varepsilon)} x_i(t_k - \varepsilon) &= e^{c_i t_{k-1}} x_i(t_{k-1}^+) + \int_{t_{k-1}}^{t_k - \varepsilon} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, dw_j(s) \\ &\quad + \int_{t_{k-1}}^{t_k - \varepsilon} e^{c_i s} \Bigg[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) \\ &\quad + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\, du \Bigg] ds. \end{aligned} \tag{6.55}$$

Let $\varepsilon \to 0$ in (6.55), we obtain that

$$\begin{aligned} e^{c_i t_k} x_i(t_k^-) &= e^{c_i t_{k-1}} x_i(t_{k-1}^+) + \int_{t_{k-1}}^{t_k} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, dw_j(s) \\ &\quad + \int_{t_{k-1}}^{t_k} e^{c_i s} \Bigg[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) \\ &\quad + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\, du \Bigg] ds. \end{aligned} \tag{6.56}$$

Note that $x_i(t_k) = x_i(t_k^-)$, from (6.54) and (6.56), we obtain that for $t \in (t_{k-1}, t_k]$ $(k = 1, 2, \cdots)$,

$$\begin{aligned} e^{c_i t} x_i(t) &= e^{c_i t_{k-1}} x_i(t_{k-1}^+) + \int_{t_{k-1}}^{t} e^{c_i s} \Bigg[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) \\ &\quad + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\, du \Bigg] ds \\ &\quad + \int_{t_{k-1}}^{t} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, dw_j(s) \\ &= e^{c_i t_{k-1}} x_i(t_{k-1}) + \int_{t_{k-1}}^{t} e^{c_i s} \Bigg[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) \\ &\quad + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\, du \Bigg] ds \\ &\quad + \int_{t_{k-1}}^{t} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, dw_j(s) + e^{c_i t_{k-1}} I_{i(k-1)}(x_i(t_{k-1})). \end{aligned}$$

Hence, we obtain that

$$
\begin{aligned}
e^{c_i t_{k-1}} x_i(t_{k-1}) &= e^{c_i t_{k-2}} x_i(t_{k-1}) + \int_{t_{k-2}}^{t_{k-1}} e^{c_i s} \left[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) \right. \\
&\quad \left. + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\, du \right] ds \\
&\quad + \int_{t_{k-2}}^{t_{k-1}} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, dw_j(s) + e^{c_i t_{k-2}} I_{i(k-2)}(x_i(t_{k-2})) \\
&\vdots \\
&\vdots \\
e^{c_i t_2} x_i(t_2) &= e^{c_i t_1} x_i(t_1) + \int_{t_1}^{t_2} e^{c_i s} \left[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) \right. \\
&\quad \left. + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\, du \right] ds \\
&\quad + \int_{t_1}^{t_2} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, dw_j(s) + e^{c_i t_1} I_{i1}(x_i(t_1)) \\
e^{c_i t_1} x_i(t_1) &= \phi_i(0) + \int_{0}^{t_1} e^{c_i s} \left[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) \right. \\
&\quad \left. + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\, du \right] ds \\
&\quad + \int_{0}^{t_1} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, dw_j(s),
\end{aligned}
$$

which yields that for $t > 0$,

$$
\begin{aligned}
x_i(t) &= e^{-c_i t} \phi_i(0) + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} a_{ij} f_j(x_j(s))\, ds \\
&\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s)))\, ds \\
&\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\, du\, ds \\
&\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, dw_j(s) + \sum_{0<t_k<t} e^{-c_i(t-t_k)} I_{ik}(x_i(t_k)).
\end{aligned}
$$

Lemma 6.2.9. *Define an operator by* $(Q\varphi)(t) = \phi(t)$ *for* $t \in [\vartheta, 0]$, *and for* $t \geq 0$,

$i = 1, 2, 3, \cdots, n,$

$$
\begin{aligned}
&(Q\varphi)_i(t) \qquad (6.57)\\
&= e^{-c_i t}\phi_i(0) + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n a_{ij} f_j(\varphi_j(s))\, ds + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n b_{ij} g_j(\varphi_j(s-\tau(s)))\, ds\\
&+ \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(\varphi_j(u))\, du\, ds\\
&+ \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n \sigma_{ij}(s, \varphi_j(s), \varphi_j(s-\tau(s)))\, dw_j(s) + \sum_{0<t_k<t} e^{-c_i(t-t_k)} I_{ik}(\varphi_i(t_k)).
\end{aligned}
$$

Suppose that the assumptions (A1)-(A5) *hold. If the conditions* (i)-(iii) *in Theorem* 6.2.1 *are satisfied, then* $Q : \mathcal{S}_\phi \to \mathcal{S}_\phi$ *and* Q *is a contraction mapping.*

Proof. Denote $(Q\varphi)_i(t) := J_{1i}(t) + J_{2i}(t) + J_{3i}(t) + J_{4i}(t) + J_{5i}(t) + J_{6i}(t)$, where

$$
\begin{aligned}
J_{1i}(t) &= e^{-c_i t}\varphi_i(0), \qquad J_{2i}(t) = \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n a_{ij} f_j(\varphi_j(s))\, ds,\\
J_{3i}(t) &= \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n b_{ij} g_j(\varphi_j(s-\tau(s)))\, ds,\\
J_{4i}(t) &= \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(\varphi_j(u))\, du\, ds,\\
J_{5i}(t) &= \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n \sigma_{ij}(s, \varphi_j(s), \varphi_j(s-\tau(s)))\, dw_j(s),\\
J_{6i}(t) &= \sum_{0<t_k<t} e^{-c_i(t-t_k)} P_{ik}(x_i(t_k)).
\end{aligned}
$$

Step1. From the definition of the metric space $\mathcal{S}_\phi$, we have that $\mathbb{E}\sum_{i=1}^n |\varphi_i(t)|^p < \infty$ for all $t \geq 0$, $\varphi \in \mathcal{S}_\phi$.

Step2. We prove the continuity in *pth* moment of Qx on $[0, \infty) \setminus \{t_1, t_2, \cdots\}$ for $x \in \mathcal{S}_\phi$ and left continuity and existence of a right limit at each t_k $(k = 1, 2, \cdots)$. It is clear that $(Q\varphi)_i(t)$ is continuous on $[\vartheta, 0]$. For a fixed time $t > 0$, it is easy to check that $J_{1i}(t), J_{2i}(t), J_{3i}(t), J_{4i}(t), J_{5i}(t), J_{6i}(t)$ are continuous in *pth* moment on the fixed time $t \neq t_k$ $(k = 1, 2, \cdots)$. Hence, $(Q\varphi_i)(t)$ is continuous in *pth* moment on the fixed time $t \neq t_k$ $(k = 1, 2, \cdots)$. On the other hand, as $t = t_k$, it is easy to check that $J_{1i}(t), J_{2i}(t), J_{3i}(t), J_{4i}(t), J_{5i}(t)$ are continuous in *pth* moment on the fixed time $t = t_k$ $(k = 1, 2, \cdots)$. In the following, we check *pth* moment left continuity of $J_{6i}(t)$ on $t = t_k$ $(k = 1, 2, \cdots)$. Let $r < 0$ be small enough,

$$
\begin{aligned}
&\mathbb{E}\sum_{i=1}^n |J_{6i}(t_k + r) - J_{6i}(t_k)|^p\\
&= \mathbb{E}\sum_{i=1}^n \left| \sum_{0<t_m<t_k+r} e^{-c_i(t_k+r-t_m)} I_{im}(\varphi_i(t_m)) - \sum_{0<t_m<t_k} e^{-c_i(t_k-t_m)} I_{im}(\varphi_i(t_m)) \right|^p\\
&\leq \mathbb{E}\sum_{i=1}^n \left| \left(e^{-c_i(t_k+r)} - e^{-c_i t_k}\right) \sum_{0<t_m<t_k} e^{c_i t_m} I_{im}(\varphi_i(t_m)) \right|^p,
\end{aligned}
$$

which implies that $\lim_{r\to 0^-}\mathbb{E}\sum_{i=1}^n |J_{6i}(t_k+r)-J_{6i}(t_k)|^p = 0$. Let $r>0$ be small enough,

$$\begin{aligned}
&\mathbb{E}\sum_{i=1}^n |J_{6i}(t_k+r)-J_{6i}(t_k)|^p \\
&= \mathbb{E}\sum_{i=1}^n \left| \sum_{0<t_m<t_k+r} e^{-c_i(t_k+r-t_m)} I_{im}(\varphi_i(t_m)) - \sum_{0<t_m<t_k} e^{-c_i(t_k-t_m)} I_{im}(\varphi_i(t_m)) \right|^p \\
&= \mathbb{E}\sum_{i=1}^n \left| e^{-c_i(t_k+r)} \left[\sum_{0<t_m<t_k} e^{c_i t_m} I_{im}(\varphi_i(t_m)) + e^{c_i t_k} I_{ik}(\varphi_i(t_k)) \right] \right. \\
&\qquad \left. - e^{-c_i t_k} \sum_{0<t_m<t_k} e^{c_i t_m} I_{im}(\varphi_i(t_m)) \right|^p \\
&= \mathbb{E}\sum_{i=1}^n \left| \left(e^{-c_i(t_k+r)} - e^{-c_i t_k}\right) \sum_{0<t_m<t_k} e^{c_i t_m} I_{im}(\varphi_i(t_m)) + e^{-c_i r} I_{ik}(\varphi_i(t_k)) \right|^p,
\end{aligned}$$

which implies that $\lim_{r\to 0^+}\mathbb{E}\sum_{i=1}^n |J_{6i}(t_k+r)-J_{6i}(t_k)|^p = \mathbb{E}\sum_{i=1}^n |I_{ik}(\varphi_i(t_k))|^p$.

Based on the above discussion, we obtain that $(Q\varphi)_i(t) : [\vartheta,\infty) \to L^p_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ is continuous in pth moment on $t \neq t_k$ $(k=1,2,\cdots)$, and for $t=t_k$ $(k=1,2,\cdots)$, $\lim_{t\to t_k^+}(Q\varphi)_i(t)$ and $\lim_{t\to t_k^-}(Q\varphi)_i(t)$ exist. Furthermore, we also obtain that $\lim_{t\to t_k^-}(Q\varphi)_i(t) = (Q\varphi)_i(t_k) \neq \lim_{t\to t_k^+}(Q\varphi)_i(t)$.

Step3. We prove that $Q(\mathcal{S}_\phi) \subseteq \mathcal{S}_\phi$. From (6.57),

$$\mathbb{E}\sum_{i=1}^n |Q\varphi_i(t)|^p = \mathbb{E}\sum_{i=1}^n \left|\sum_{j=1}^6 J_{ji}(t)\right|^p \le 6^{p-1}\sum_{j=1}^6 \mathbb{E}\left(\sum_{i=1}^n |J_{ji}(t)|^p\right). \tag{6.58}$$

Now, we estimate the right-hand terms of (6.58). From (A3), we know that $|I_{ik}(x_i(t_k))| \le p_{ik}|x_i(t_k)|$, combining with the condition (ii), we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^n |J_{6i}(t)|^p &\le \mathbb{E}\sum_{i=1}^n \left[\sum_{0<t_k<t} e^{-c_i(t-t_k)} p_{ik}|\varphi_i(t_k)|\right]^p \\
&\le \mathbb{E}\sum_{i=1}^n \left[p_i \sum_{0<t_k<t} e^{-c_i(t-t_k)} |\varphi_i(t_k)|(t_k-t_{k-1})\right]^p \\
&\le \mathbb{E}\sum_{i=1}^n \left[p_i \int_0^t e^{-c_i(t-s)} |\varphi_i(s)|\,ds\right]^p \\
&\le \mathbb{E}\sum_{i=1}^n p_i^p \left(\int_0^t e^{-c_i(t-s)}\,ds\right)^{p/q} \int_0^t e^{-c_i(t-s)} |\varphi_i(s)|^p\,ds \\
&\le \max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\} \int_0^t e^{-c(t-s)}\mathbb{E}\left(\sum_{i=1}^n |\varphi_i(s)|^p\right)ds.
\end{aligned} \tag{6.59}$$

Since $\mathbb{E}\sum_{i=1}^n |\varphi_i(t)| \to 0$, $t-\tau(t)\to\infty$ and $t-r(t)\to\infty$ as $t\to\infty$. Thus, from (6.58), (6.59) and combining with (6.17), (6.18) and (6.19), we obtain that $\mathbb{E}\sum_{i=1}^n |Q\varphi_i(t)|^p \to 0$ as

$\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|^p \to 0$. Therefore, $Q: \mathcal{S}_\phi \to \mathcal{S}_\phi$.

Step4. We prove that Q is a contraction mapping. For any $\varphi, \psi \in \mathcal{S}_\phi$, from (6.17), (6.18), (6.19), (6.58) and (6.59), we obtain

$$\begin{aligned}
&\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}|Q\varphi_i(s)-Q\psi_i(s)|^p\right\}\\
&\leq 5^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}a_{ij}\Big(f_j(x_j(u))-f_j(y_j(u))\Big)\,du\right|^p\right\}\\
&\quad+5^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}b_{ij}\Big(g_j(x_j(u-\tau(u)))-g_j(y_j(u-\tau(u)))\Big)\,du\right|^p\right\}\\
&\quad+5^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}\Big(h_j(\varphi_j(v))-h_j(\psi_j(v))\Big)\,dv\,du\right|^p\right\}\\
&\quad+5^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}\Big(\sigma_{ij}(s,x_j(s),x_j(u-\tau(u)))\right.\right.\\
&\qquad\left.\left.-\sigma_{ij}(s,y_j(s),y_j(s-\tau(u)))\Big)\,dw_j(u)\right|^p\right\}\\
&\quad+5^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\sum_{0<t_k<t}e^{-c_i(t-t_k)}\left(I_{ik}(\varphi_i(t_k))-I_{ik}(\psi_i(t_k))\right)\right|^p\right\}\\
&\leq 5^{p-1}\left\{\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\right.\\
&\quad+\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^p\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}+n^{p-1}\sum_{i=1}^{n}c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)\\
&\quad\left.+\frac{1}{c}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}\right\}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{j=1}^{n}|\varphi_j(s)-\psi_j(s)|^p\right\}=\alpha\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{j=1}^{n}|\varphi_j(s)-\psi_j(s)|^p\right\}.
\end{aligned}$$

From (6.45), we obtain that $Q: \mathcal{S}_\phi \to \mathcal{S}_\phi$ is a contraction mapping. □

We are now ready to prove Theorem 6.2.1.

Proof. From Lemma 6.2.9, by a contraction mapping principle, we obtain that Q has a unique fixed point $x(t)$, which is a solution of (6.42) with $x(t) = \phi(t)$ as $t \in [\vartheta, 0]$ and $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p \to 0$ as $t \to \infty$.

Now, we prove that the trivial solution of (6.42) is *pth* moment stable. From (6.45), For any $\varepsilon > 0$, we choose $\delta > 0$ ($\delta < \varepsilon$) such that $6^{p-1}\delta < (1-\alpha)\varepsilon$.

If $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T$ is a solution of (6.42) with the initial condtion satisfying $\mathbb{E}\sum_{i=1}^{n}|\phi_i(t)|^p < \delta$, then $x(t) = (Qx)(t)$ defined in (6.57). We claim that $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p < \varepsilon$ for all $t \geq 0$. Notice that $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p < \varepsilon$ for $t \in [\vartheta, 0]$, we suppose that there exists $t^* > 0$ such that $\mathbb{E}\sum_{i=1}^n |x_i(t^*)|^p = \varepsilon$ and $\mathbb{E}\sum_{i=1}^n |x_i(t)|^p < \varepsilon$ for $-\tau \le t < t^*$, then it follows from (6.45), we obtain that

$$
\begin{aligned}
&\mathbb{E}\sum_{i=1}^{n}|x_i(t^*)|^p \\
&\le 6^{p-1}\mathbb{E}\sum_{i=1}^{n} e^{-pc_it^*}|x_i(0)|^p \\
&\quad +6^{p-1}\sum_{i=1}^{n} c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^{t^*} e^{-c_i(t^*-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s)|^p\right)ds \\
&\quad +6^{p-1}\sum_{i=1}^{n} c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^{t^*} e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s-\tau(s)))|^p\right)ds \\
&\quad +6^{p-1}\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^{t^*} e^{-c_i(t^*-s)}\int_{s-r(s)}^{s}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(u)|^p\right)du\,ds \\
&\quad +6^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{1-p/2}\left[\mu^{p/2}\int_0^{t^*} e^{-c_i(t^*-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s)|^p\right)ds\right. \\
&\qquad \left.+\nu^{p/2}\int_0^{t^*} e^{-c_i(t-s)}E\left(\sum_{j=1}^{n}|x_j(s-\tau(s))|^p\right)ds\right] \\
&\quad +6^{p-1}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}\int_0^{t^*} e^{-c(t^*-s)}\mathbb{E}\left(\sum_{i=1}^{n}|x_i(s)|^p\right)ds \\
&\le 6^{p-1}\delta+\left[6^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+6^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\right. \\
&\quad +6^{p-1}\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}+6^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right) \\
&\quad \left.+\frac{1}{c}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}\right]\varepsilon < (1-\alpha)\varepsilon+\alpha\varepsilon=\varepsilon,
\end{aligned}
$$

which is a contradiction. Therefore, the trivial solution of (6.42) is asymptotically stable in *pth* moment. □

Let $l_{ij} \equiv 0$, the system (6.42) is reduced to

$$
\begin{cases}
dx(t) = [-Cx(t) + Af(x(t)) + Bg(x(t-\tau(t)))]\,dt \\
\qquad\qquad +\sigma(t, x(t), x(t-\tau(t)))\,dw(t), \quad t \ne t_k \\
\\
\Delta x(t_k) = I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, 3, \cdots.
\end{cases}
\tag{6.60}
$$

which is a description of a stochastically perturbed Hopfield neural networks with time-varying delays.

Corollary 6.2.10. *Suppose that the assumptions* (A1)-(A5) *hold. If the following conditions are satisfied,*

(i) *there exist constants* p_i *(*$i = 1, 2, \cdots, n$*) such that* $p_{ik} \le p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;

(ii) *and such that*

$$5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}$$
$$+5^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right) + 5^{p-1}c^{-1}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\} < 1,$$

where $c = \min\{c_1, c_2, \cdots, c_n\}$, $\mu = \max\{\mu_1, \mu_2, \cdots, \mu_n\}$, $\nu = \max\{\nu_1, \nu_2, \cdots, \nu_n\}$*;*

then the trivial solution of (6.60) *is pth moment asymptotically stable.*

Remark 6.2.11. *Note that the delay* $\tau(t)$ *in Corollary* 6.2.10 *can be unbounded.*

6.2.3 Proof of Theorem 6.2.2

Define an operator $(Q\varphi)(t) = \phi(t)$ for $t \in [\vartheta, 0]$ and for $t \ge 0$, $(Q\varphi)(t)$ is defined as the right hand side of (6.57). Following the proof of Theorem 6.2.1, we find that to show Theorem 6.2.2, we only need to prove that $e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|(Q\varphi)_i(t)|^p \to 0$ as $t \to \infty$. It follows from (6.57) that

$$e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|(Q\varphi)_i(t)|^p = e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}\left|\sum_{j=1}^{6}J_{ji}(t)\right|^p \le 6^{p-1}e^{\lambda t}\sum_{j=1}^{6}\mathbb{E}\left(\sum_{i=1}^{n}|J_{ji}(t)|^p\right). \tag{6.61}$$

Now, we estimate the right-hand terms of (6.61). First, by using Hölder's inequality,

$$\begin{aligned}
e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{2i}(t)|^p &= e^{\lambda t}\sum_{i=1}^{n}\mathbb{E}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}a_{ij}f_j(\varphi_j(s))\,ds\right|^p \qquad (6.62)\\
&\le e^{\lambda t}e^{\lambda t}\sum_{i=1}^{n}\mathbb{E}\left[\int_0^t e^{-\frac{c_i(t-s)}{q}}e^{-\frac{c_i(t-s)}{p}}\sum_{j=1}^{n}|a_{ij}||f_j(\varphi_j(s))|\,ds\right]^p\\
&\le e^{\lambda t}\sum_{i=1}^{n}\mathbb{E}\left\{\left[\int_0^t e^{-c_i(t-s)}\,ds\right]^{p/q}\int_0^t e^{-c_i(t-s)}\left[\sum_{j=1}^{n}|a_{ij}||f_j(\varphi_j(s))|\right]^p ds\right\}\\
&\le e^{\lambda t}\sum_{i=1}^{n}c_i^{-p/q}\mathbb{E}\left\{\int_0^t e^{-c_i(t-s)}\left[\sum_{j=1}^{n}|a_{ij}||\alpha_j||\varphi_j(s)|\right]^p ds\right\}\\
&\le e^{\lambda t}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds\\
&= \sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^t e^{(\lambda-c_i)(t-s)}e^{\lambda s}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds.
\end{aligned}$$

With a similar computation to (6.62), we obtain that

$$
\begin{aligned}
&e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t)|^p \qquad (6.63)\\
&\le e^{\lambda t}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\right]ds\\
&\le e^{\lambda \tau}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda(s-\tau(s))}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\right]ds.
\end{aligned}
$$

$$
\begin{aligned}
&e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{4i}(t)|^p \qquad (6.64)\\
&\le e^{\lambda t}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}\left|\int_{s-r(s)}^{s}\varphi_j(u)\,du\right|^p\right]ds\\
&\le e^{\lambda t}\sum_{i=1}^{n}\left(\frac{\tau}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\int_{s-r(s)}^{s}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)|^p\right)du\,ds\\
&\le e^{\lambda \tau}\sum_{i=1}^{n}\left(\frac{\tau}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^t e^{-(c_i-\lambda)(t-s)}\int_{s-r(s)}^{s}e^{\lambda u}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)|^p\right)du\,ds.
\end{aligned}
$$

Using Lemma 6.1.16 and Hölder's inequality, we obtain that

$$
\begin{aligned}
&e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{5i}(t)|^p \qquad (6.65)\\
&= e^{\lambda t}\sum_{i=1}^{n}\mathbb{E}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,dw_j(s)\right|^p\\
&\le e^{\lambda t}n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left\{\left[\int_0^t e^{-c_i(t-s)}|\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))|\,dw_j(s)\right]^2\right\}^{p/2}\\
&= e^{\lambda t}n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\int_0^t e^{-2c_i(t-s)}\sigma_{ij}^2(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,ds\right]^{p/2}\\
&\le e^{\lambda t}n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\int_0^t e^{-2c_i(t-s)}\left(\mu_j\varphi_j^2(s)+\nu_j\varphi_j^2(s-\tau(s))\right)ds\right]^{p/2}\\
&\le e^{\lambda t}n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\left(\int_0^t e^{-2c_i(t-s)}\mu_j\varphi_j^2(s)\,ds\right)^{p/2}\right.\\
&\qquad\left.+\left(\int_0^t e^{-2c_i(t-s)}\nu_j\varphi_j^2(s-\tau(s))\,ds\right)^{p/2}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\le e^{\lambda t} n^{p-1} 2^{p/2-1} \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\Bigg[\bigg(\int_0^t e^{-2c_i(t-s)}\,ds\bigg)^{p/2-1}\int_0^t e^{-2c_i(t-s)}\mu_j^{p/2}|\varphi_j(s)|^p\,ds\Bigg]\\
&\quad + e^{\lambda t} n^{p-1} 2^{p/2-1} \sum_{i=1}^{n}\sum_{j=1}^{n} \mathbb{E}\Bigg[\bigg(\int_0^t e^{-2c_i(t-s)}\,ds\bigg)^{p/2-1}\int_0^t e^{-2c_i(t-s)}\nu_j^{p/2}|\varphi_j(s-\tau(s))|^p\,ds\Bigg]\\
&\le e^{\lambda t} n^{p-1} \sum_{i=1}^{n} c_i^{1-p/2}\Bigg[\mu^{p/2}\int_0^t e^{-c_i(t-s)}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(s)|^p\bigg)\,ds\Bigg]\\
&\quad + e^{\lambda t} n^{p-1} \sum_{i=1}^{n} c_i^{1-p/2}\Bigg[\nu^{p/2}\int_0^t e^{-c_i(t-s)}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\bigg)\,ds\Bigg]\\
&\le n^{p-1} \sum_{i=1}^{n} c_i^{1-p/2}\Bigg[\mu^{p/2}\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda s}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(s)|^p\bigg)\,ds\Bigg]\\
&\quad + e^{\lambda\tau} n^{p-1} \sum_{i=1}^{n} c_i^{1-p/2}\Bigg[\nu^{p/2}\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda(s-\tau(s))}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\bigg)\,ds\Bigg].
\end{aligned}
$$

Further, from (A3), we know that $|I_{ik}(x_i(t_k))| \le p_{ik}|x_i(t_k)|$ for $i = 1, 2, \cdots, n$, $k = 1, 2, \cdots$. Combining with the condition that $p_{ik} \le p_i(t_k - t_{k-1})$, we obtain that

$$
\begin{aligned}
e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{6i}(t)|^p &\le e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}\Bigg[\sum_{0<t_k<t} e^{-c_i(t-t_k)}p_{ik}|\varphi_i(t_k)|\Bigg]^p\\
&\le e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}\Bigg[p_i\sum_{0<t_k<t} e^{-c_i(t-t_k)}|\varphi_i(t_k)|(t_k-t_{k-1})\Bigg]^p\\
&\le e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}\Bigg[p_i\int_0^t e^{-c_i(t-s)}|\varphi_i(s)|\,ds\Bigg]^p\\
&\le e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}p_i^p\bigg(\int_0^t e^{-c_i(t-s)}\,ds\bigg)^{p/q}\int_0^t e^{-c_i(t-s)}|\varphi_i(s)|^p\,ds\\
&\le \max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}\int_0^t e^{-(c-\lambda)(t-s)}e^{\lambda s}\mathbb{E}\bigg(\sum_{i=1}^{n}|\varphi_i(s)|^p\bigg)\,ds. \qquad (6.66)
\end{aligned}
$$

Since $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)| \to 0$, $t-\tau(t)\to\infty$ and $t-r(t)\to\infty$ as $t\to\infty$. Thus, from (6.61) to (6.66), we obtain that $e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|(Q\varphi)_i(t)|^p \to 0$ as $e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|^p \to 0$. Hence, combining the proof of Theorem 6.2.1, there exists a unique fixed point $\varphi(\cdot)$ of Q in $\mathcal{C}_\phi$, which is a solution of the system (6.42) such that $e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|^p \to 0$ as $t\to\infty$. This completes the proof.

Corollary 6.2.12. *Suppose that the assumptions* (A1)-(A5) *hold. Assume that*

(i) *the discrete delay $\tau(t)$ is bounded by a constant τ ($\tau > 0$);*

(ii) *there exist constants p_i ($i = 1, 2, \cdots, n$) such that $p_{ik} \le p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;*

(iii) *and such that*

$$5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}$$
$$+5^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right) + 5^{p-1}c^{-1}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\} < 1,$$

where $c = \min\{c_1, c_2, \cdots, c_n\}$, $\mu = \max\{\mu_1, \mu_2, \cdots, \mu_n\}$, $\nu = \max\{\nu_1, \nu_2, \cdots, \nu_n\}$, *then the trivial solution of* (6.60) *is pth moment exponentially stable.*

6.2.4 Proof of Theorem 6.2.6

In this subsection, we prove Theorem 6.2.6. We start with some preparations.

Using similar computations as in Subsection 6.2.2, we obtain that for $t \geq 0$, the system (6.47) is equivalent to

$$\begin{aligned} x_i(t) &= e^{-c_i t}x_i(0) + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} a_{ij}g_j(x_j(s))\,ds + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} b_{ij}g_j(x_j(s-\tau(s)))\,ds \\ &\quad + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} l_{ij}\int_{s-r(s)}^{s} g_j(x_j(u))\,du\,ds + \sum_{0<t_k<t} e^{-c_i(t-t_k)}I_{ik}(x_i(t_k)), \end{aligned}$$

$i = 1, 2, 3, \cdots, n$, $k = 1, 2, \cdots$.

Lemma 6.2.13. *Define an operator by* $(P\varphi)(t) = \phi(t)$ *for* $-\tau \leq t \leq 0$, *and for* $t \geq 0$,

$$\begin{aligned} (P\varphi)_i(t) &= e^{-c_i t}\varphi_i(0) + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} a_{ij}g_j(\varphi_j(s))\,ds \\ &\quad + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} b_{ij}g_j(\varphi_j(s-\tau(s)))\,ds \\ &\quad + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} l_{ij}\int_{s-r(s)}^{s} g_j(\varphi_j(u))\,du\,ds + \sum_{0<t_k<t} e^{-c_i(t-t_k)}I_{ik}(x_i(t_k)) \\ &:= I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t). \end{aligned} \tag{6.67}$$

If the conditions (i)-(iii) *in Theorem* 6.2.6 *are satisfied, then* $P : \mathcal{S}_\phi \to \mathcal{S}_\phi$ *and* P *is a contraction mapping.*

Proof. First, we prove that $P\mathcal{S}_\phi \subseteq \mathcal{S}_\phi$. In view of (6.67), it is easy to check that $(Px_i)(t)$ is continuous on fixed time $t \neq t_k$ $(k = 1, 2, \cdots)$. On the other hand, as $t = t_k$ $(k = 1, 2, \cdots)$, it is not difficult to show that $I_1(t), I_2(t), I_3(t), I_4(t)$ is continuous on fixed time $t = t_k$ $(k = 1, 2, \cdots)$. Let $r < 0$ be small enough, we obtain that

$$\begin{aligned} |J_{5i}(t_k + r) - J_{5i}(t_k)| &= \left|\sum_{0<t_m<t_k+r} e^{-c_i(t_k+r-t_m)}I_{im}(\varphi_i(t_m)) - \sum_{0<t_m<t_k} e^{-c_i(t_k-t_m)}I_{im}(\varphi_i(t_m))\right| \\ &\leq \left|\left(e^{-c_i(t_k+r)} - e^{-c_i t_k}\right)\sum_{0<t_m<t_k} e^{c_i t_m}I_{im}(\varphi_i(t_m))\right|, \end{aligned}$$

which implies that $\lim_{r\to 0^-} |J_{5i}(t_k + r) - J_{5i}(t_k)| = 0$. Let $r > 0$ be small enough, we obtain that

$$\begin{aligned}
|J_{5i}(t_k + r) - J_{5i}(t_k)| &= \left| \sum_{0<t_m<t_k+r} e^{-c_i(t_k+r-t_m)} I_{im}(\varphi_i(t_m)) - \sum_{0<t_m<t_k} e^{-c_i(t_k-t_m)} I_{im}(\varphi_i(t_m)) \right| \\
&= \left| e^{-c_i(t_k+r)} \left[\sum_{0<t_m<t_k} e^{c_i t_m} I_{im}(\varphi_i(t_m)) + e^{c_i t_k} I_{ik}(\varphi_i(t_k)) \right] \right. \\
&\qquad \left. - e^{-c_i t_k} \sum_{0<t_m<t_k} e^{c_i t_m} I_{im}(\varphi_i(t_m)) \right| \\
&= \left| \left(e^{-c_i(t_k+r)} - e^{-c_i t_k} \right) \sum_{0<t_m<t_k} e^{c_i t_m} I_{im}(\varphi_i(t_m)) + e^{-c_i r} I_{ik}(\varphi_i(t_k)) \right|,
\end{aligned}$$

which implies that $\lim_{r\to 0^+} |J_{5i}(t_k + r) - J_{5i}(t_k)| = |I_{ik}(\varphi_i(t_k))|$.

Based on the above discussion, we obtain that $(P\varphi)_i(t) : [\vartheta, \infty) \to \mathbb{R}^n$ is continuous on $t \neq t_k$ $(k = 1, 2, \cdots)$, and for $t = t_k$ $(k = 1, 2, \cdots)$, $\lim_{t\to t_k^+}(P\varphi)_i(t)$ and $\lim_{t\to t_k^-}(P\varphi)_i(t)$ exist. Furthermore, we also obtain that $\lim_{t\to t_k^-}(P\varphi)_i(t) = (P\varphi)_i(t_k) \neq \lim_{t\to t_k^+}(P\varphi)_i(t)$.

Next, we prove that $\lim_{t\to\infty}(P\varphi)_i(t) = 0$ for $\varphi_i(t) \in \mathcal{S}_{i\phi}$.

$$\begin{aligned}
|I_5(t)| = \left| \sum_{0<t_k<t} e^{-c_i(t-t_k)} I_{ik}(x_i(t_k)) \right| &\leq \left| \sum_{0<t_k<t} e^{-c_i(t-t_k)} p_i(t_k - t_{k-1}) x_i(t_k) \right| \\
&\leq p_i \int_0^t e^{-c_i(t-s)} |x_i(s)|\, ds. \qquad (6.68)
\end{aligned}$$

From the fact that $c_i > 0$ $(i = 1, 2, \cdots, n)$ and the estimate (6.33), (6.34), (6.35) and (6.68), we conclude that $\lim_{t\to\infty}(Px_i)(t) = 0$ for $x_i(t) \in \mathcal{S}_{i\phi}$. Therefore, $P : \mathcal{S}_\phi \to \mathcal{S}_\phi$.

Now, we prove that P is a contraction mapping. For any $x(t), y(t) \in \mathcal{S}_\phi$, we obtain that

$$\begin{aligned}
&\sum_{i=1}^{n} \sup_{\vartheta\leq s\leq t} |(Px)_i(t) - (Py)_i(t)| \\
&\quad \leq \sum_{i=1}^{n} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} |x_j(s) - y_j(s)|\, ds \\
&\qquad + \sum_{i=1}^{n} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} |x_j(s - \tau(s)) - y_j(s - \tau(s))|\, ds \\
&\qquad + \sum_{i=1}^{n} \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} \int_{s-r(s)}^{s} |x_j(u) - y_j(u)|\, du\, ds \\
&\qquad + \sum_{i=1}^{n} p_i \int_0^t e^{-c_i(t-s)} |x_i(s) - y_i(s)|\, ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \left[\sum_{i=1}^{n}\frac{1}{c_i}\max_{j=1,2,\cdots,n}|a_{ij}\alpha_j| + \sum_{i=1}^{n}\frac{1}{c_i}\max_{j=1,2,\cdots,n}|b_{ij}\beta_j| \right.\\
&\quad \left. + \sum_{i=1}^{n}\frac{\tau}{c_i}\max_{j=1,2,\cdots,n}|l_{ij}\gamma_j| + \max_{j=1,2,\cdots,n}\left\{\frac{p_i}{c_i}\right\}\right]\sum_{j=1}^{n}\left[\sup_{\vartheta\leq s\leq t}|x_j(s)-y_j(s)|\right]\\
&= \alpha\sum_{j=1}^{n}\left[\sup_{\vartheta\leq s\leq t}|x_j(s)-y_j(s)|\right].
\end{aligned}
$$

From (6.50), we obtain that P is a contraction mapping. □

We are now ready to prove Theorem 6.2.6.

Proof. Let P be defined as in Lemma 6.2.13, by a contraction mapping principle, P has a unique fixed point $x \in \mathcal{S}_\phi$ with $x(\theta) = \phi(\theta)$ on $-\tau \leq \theta \leq 0$ and $x(t) \to 0$ as $t \to \infty$.

To obtain asymptotically stable, we need to prove that the trivial equilibrium $x = 0$ of (6.47) is stable. From (6.50), For any $\varepsilon > 0$, choose $\sigma > 0$ and $\sigma < \varepsilon$ satisfying the condition $\sigma + \varepsilon\alpha < \varepsilon$.

If $x(t,\phi) = (x_1(t,\phi), x_2(t,\phi), \cdots, x_n(t,\phi))$ is the solution of (6.47) with the initial condition $\|\phi\| < \sigma$, the we claim that $\|x(t,\phi)\| < \varepsilon$ for all $t \geq 0$. Indeed, we suppose that there exists $t^* > 0$ such that

$$
\sum_{i=1}^{n}|x_i(t^*,\phi)| = \varepsilon, \qquad \text{and} \qquad \sum_{i=1}^{n}|x_i(t,\phi)| < \varepsilon \quad \text{for} \quad 0 \leq t < t^*. \tag{6.69}
$$

From (6.50), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{n}|x_i(t^*,\phi)|\\
&\leq \sum_{i=1}^{n}\left[|e^{-c_it^*}x_i(0)| + \int_0^{t^*} e^{-c_i(t^*-s)}\sum_{j=1}^{n}|a_{ij}f_j(x_j(s))|\,ds \right.\\
&\quad + \int_0^{t^*} e^{-c_i(t^*-s)}\sum_{j=1}^{n}|b_{ij}g_j(x_j(s-\tau(s)))|\,ds\\
&\quad \left. + \int_0^{t^*} e^{-c_i(t^*-s)}\sum_{j=1}^{n}|l_{ij}\int_{s-r(s)}^{s} h_j(x_j(u))|\,du\,ds + p_i\int_0^{t^*} e^{-c_i(t^*-s)}|x_i(s)|\,ds\right]\\
&< \sigma + \varepsilon\left[\sum_{i=1}^{n}\frac{1}{c_i}\max_{j=1,2,\cdots,n}|a_{ij}\alpha_j| + \sum_{i=1}^{n}\frac{1}{c_i}\max_{j=1,2,\cdots,n}|b_{ij}\beta_j| \right.\\
&\quad \left. + \sum_{i=1}^{n}\frac{\tau}{c_i}\max_{j=1,2,\cdots,n}|l_{ij}\gamma_j| + \max_{j=1,2,\cdots,n}\left\{\frac{p_i}{c_i}\right\}\right]\\
&\leq \sigma + \varepsilon\alpha < \varepsilon.
\end{aligned}
$$

which contradicts (6.69). Therefore, $\|x(t,\phi)\| < \varepsilon$ for all $t \geq 0$. This completes the proof. □

Let $l_{ij} \equiv 0$ for $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$, the system (6.47) is reduced to

$$\begin{cases} \frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau(t))), \quad t \neq t_k \\ \\ \Delta x_i(t_k) = I_{ik} x_i(t_k), \quad t = t_k, \quad k = 1, 2, 3, \cdots, \end{cases} \tag{6.70}$$

which is the description of cellular neural network with time-varying delays. Following the result of Theorem 6.2.6, we have the following corollary. Note that the delay in Corollary 6.2.14 can be unbounded.

Corollary 6.2.14. *Suppose that the conditions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *there exist constants p_i $(i = 1, 2, \cdots, n)$ such that $p_{ik} \leq p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;*

(ii) *and such that*

$$\alpha \triangleq \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \max_{i=1,2,\cdots,n} \left\{ \frac{p_i}{c_i} \right\} < 1, \tag{6.71}$$

then the trivial solution of (6.70) *is asymptotically stable.*

Remark 6.2.15. *Zhang and Guan* [143] *has studied asymptotic stability of* (6.70) *by using fixed point theory. The conditions in* [143] *are as follows*

(i) *there exists a constant μ such that $\inf_{k=1,2,\cdots}\{t_k - t_{k-1}\} \geq \mu$;*

(ii) *there exist constants p_i $(i = 1, 2, \cdots, n)$ such that $p_{ik} \leq p_i\mu$, $k = 1, 2, \cdots$;*

(iii) *and such that*

$$\lambda^* \triangleq \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \max_{i=1,2,\cdots,n} \left\{ \frac{p_i}{c_i} + p_i\mu \right\} < 1;$$

(iv)

$$\max_{i=1,2,\cdots,n} \{\lambda_i\} < \frac{1}{\sqrt{n}}, \quad \text{where } \lambda_i = \frac{1}{c_i} \sum_{j=1}^{n} |a_{ij}\alpha_j| + \frac{1}{c_i} \sum_{j=1}^{n} |b_{ij}\beta_j| + \left(\frac{p_i}{c_i} + p_i\mu \right).$$

It is clear that Corollary 6.2.16 is an improvement of the result in [143].

6.2.5 Proof of Theorem 6.2.7

Define the operator P as in Subsection 6.2.4. Following the proof of Theorem 6.2.6, we only need to prove that $e^{\lambda t}(P\varphi)_i(t) \to 0$ as $t \to \infty$. We estimate the right-hand terms of (6.67), we obtain that

$$\begin{aligned} e^{\lambda t}|I_2(t)| &= \left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n a_{ij}f_j(\varphi_j(s))\,ds\right| \\ &\le e^{\lambda t}\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n |a_{ij}\alpha_j||\varphi_j(s)|\,ds \\ &\le \max_{j=1,2,\cdots,n}|a_{ij}\alpha_j|\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda s}\sum_{j=1}^n|\varphi_j(s)|\,ds, \end{aligned} \tag{6.72}$$

$$\begin{aligned} e^{\lambda t}|I_3(t)| &= e^{\lambda t}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n b_{ij}g_j(\varphi_j(s-\tau(s)))\,ds\right| \\ &\le e^{\lambda t}\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n |b_{ij}\beta_j||\varphi_j(s-\tau(s))|\,ds \\ &\le e^{\lambda\tau}\max_{j=1,2,\cdots,n}|b_{ij}\beta_j|\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda(s-\tau(s))}\sum_{j=1}^n|\varphi_j(s-\tau(s))|\,ds, \end{aligned} \tag{6.73}$$

$$\begin{aligned} e^{\lambda t}|I_4(t)| &= e^{\lambda t}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n l_{ij}\int_{s-r(s)}^s h_j(\varphi_j(u))\,du\,ds\right| \\ &\le e^{\lambda t}\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n |l_{ij}\gamma_j|\int_{s-r(s)}^s|\varphi_j(u)|\,du\,ds \\ &\le e^{\lambda\tau}\max_{j=1,2,\cdots,n}|l_{ij}\gamma_j|\int_0^t e^{-(c_i-\lambda)(t-s)}\int_{s-r(s)}^s e^{\lambda u}\sum_{j=1}^n|\varphi_j(u)|\,du\,ds, \end{aligned} \tag{6.74}$$

$$\begin{aligned} e^{\lambda t}|I_5(t)| = e^{\lambda t}\left|\sum_{0<t_k<t} e^{-c_i(t-t_k)}I_{ik}(x_i(t_k))\right| &\le e^{\lambda t}\left|\sum_{0<t_k<t} e^{-c_i(t-t_k)}p_i(t_k-t_{k-1})x_i(t_k)\right| \\ &\le p_i\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda s}|x_i(s)|\,ds. \end{aligned} \tag{6.75}$$

From the fact that $\lambda < \min\{c_1, c_2, \cdots, c_n\}$, $c_i > 0$ $(i = 1, 2, \cdots, n)$ and the above estimate, we obtain that $e^{\lambda t}(P\varphi)_i(t) \to 0$ as $t \to \infty$.

Corollary 6.2.16. *Suppose that the conditions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the delay $\tau(t)$ is bounded by a constant τ $(\tau > 0)$;*

(ii) *there exist constants p_i $(i = 1, 2, \cdots, n)$ such that $p_{ik} \le p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;*

(iii) *and such that*

$$\alpha \triangleq \sum_{i=1}^n \frac{1}{c_i}\max_{j=1,2,\cdots,n}|a_{ij}\alpha_j| + \sum_{i=1}^n \frac{1}{c_i}\max_{j=1,2,\cdots,n}|b_{ij}\beta_j| + \max_{j=1,2,\cdots,n}\left\{\frac{p_i}{c_i}\right\} < 1, \tag{6.76}$$

then the trivial solution of (6.70) *is exponentially stable.*

Remark 6.2.17. *Zhang and Luo* [142] *has studied exponential stability of* (6.70) *by using fixed point theory. The conditions in* [142] *are as follows*

(i) *the delay* $\tau(t)$ *is bounded by a constant* τ $(\tau > 0)$*;*

(ii) *there exists a constant* μ *such that* $\inf_{k=1,2,\cdots}\{t_k - t_{k-1}\} \geq \mu$*;*

(iii) *there exist constants* p_i $(i = 1, 2, \cdots, n)$ *such that* $p_{ik} \leq p_i\mu$, $k = 1, 2, \cdots$*;*

(iv) *and such that*

$$\alpha \triangleq \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \max_{i=1,2,\cdots,n} \left\{\frac{p_i}{c_i} + p_i\mu\right\} < 1.$$

It is clear that Corollary 6.2.16 *is an improvement of the result in* [142].

6.2.6 Examples

Example 6.2.18. *Consider the following two-dimensional cellular neural network*

$$\begin{cases} \frac{dx_i(t)}{dt} = -c_ix_i(t) + \sum_{j=1}^{2} a_{ij}g_j(x_j(t)) + \sum_{j=1}^{2} b_{ij}g_j(x_j(t-\tau(t))) \quad i = 1, 2, \quad t \neq t_k \\ \\ \Delta x_i(t_k) = I_{ik}x_i(t_k), \quad t = t_k, \quad k = 1, 2, 3, \cdots, \end{cases} \tag{6.77}$$

with the initial conditions $x_1(s) = \cos(s)$, $x_2(s) = \sin(s)$ *on* $-\frac{1}{2} \leq s \leq 0$, *where* $c_1 = c_2 = 3$, $a_{11} = 6/7$, $a_{12} = 3/7$, $a_{21} = -1/7$, $a_{22} = -1/7$, $b_{11} = 6/7$, $b_{12} = 2/7$, $b_{21} = 3/7$, $b_{22} = 1/7$, *the activation function is described by* $g_i(x) = \frac{|x+1|-|x-1|}{2}$, $\tau(t) = 0.4t + 1$. $I_{ik}(x_i(t_k)) = \arctan(0.4x_i(t_k))$, $t_k = t_{k-1} + 0.5k$, $i = 1, 2$ *and* $k = 1, 2, \cdots$.

It is clear that $\alpha_i = \beta_i = 1$, $p_{ik} = 0.4$ for $i = 1, 2$, $k = 1, 2, \cdots$, we select $p_i = 0.8$, then

$$\begin{aligned} \sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |a_{ij}\alpha_j| + \sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |b_{ij}\beta_j| + \max_{j=1,2,\cdots,n} \left\{\frac{p_i}{c_i}\right\} &\leq \frac{1}{3} \times \left(\frac{6}{7} + \frac{1}{7} + \frac{6}{7} + \frac{3}{7}\right) + \frac{4}{35} \\ &= \frac{16}{21} + \frac{4}{35} < 0.88 < 1. \end{aligned}$$

Hence, by Corollary 6.2.14, the trivial solution of (6.77) is asymptotically stable. However,

$$\sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |a_{ij}\alpha_j| + \sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |b_{ij}\beta_j| + \max_{i=1,2,\cdots,n} \left\{\frac{p_i}{c_i} + p_i\mu\right\} > 1,$$

which implies that the result in [143] is not applicable.

Example 6.2.19. *Consider a two-dimensional stochastically perturbed Hopfield neural network with time-varying delays,*

$$\begin{cases} dx(t) = [-Cx(t) + Af(x(t)) + Bg(x_\tau(t))]\, dt + \sigma(t, x(t), x_\tau(t))\, dw(t), \quad t \neq t_k \\ \\ \Delta x(t_k) = I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, 3, \cdots, \end{cases} \tag{6.78}$$

where $f(x) = \frac{1}{5}\arctan x$, $g(x) = \frac{1}{5}\tanh x = \frac{1}{5}(e^x - e^{-x})/(e^x + e^{-x})$, $\tau(t) = \frac{1}{2}\sin t + \frac{1}{2}$,

$$C = \begin{pmatrix} 5 & 0 \\ 0 & 4.5 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 0.4 \\ 0.6 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -0.8 & 2 \\ 1 & 4 \end{pmatrix}.$$

In this example, let $p = 3$, *take* $\alpha_j = 0.2$, $\beta_j = 0.2$, $j = 1,2$, $\sigma : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ *satisfies*

$$\sigma_{i1}^2(t,x,y) \leq 0.01(x_1^2 + y_1^2) \quad \text{and} \quad \sigma_{i2}^2(t,x,y) \leq 0.01(x_2^2 + y_2^2), \quad i = 1,2.$$

$I_{ik}(x_i(t_k)) = 0.1x_i(t_k)$, $t_k = t_{k-1} + 0.5$, $i = 1,2$ *and* $k = 1,2,\cdots$.

It is clear that $p_{ik} = 0.1$, we choose $p_i = 0.2$, let $p = 2$, we check the condition in Corollary 6.2.10,

$$5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}$$
$$+5^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{-p/2}\left(\mu^{p/2} + \nu^{p/2}\right) + 5^{p-1}c^{-1}\max_{i=1,2}\left\{\frac{p_i^p}{c_i^{p-1}}\right\} < 0.53 < 1.$$

From Corollary 6.2.10, the trivial solution of (6.78) is asymptotically stable. On the other hand, since $|\tau(t)| = \left|\frac{1}{2}\sin t + \frac{1}{2}\right| \leq 1$, from Corollary 6.2.12, the trivial solution of (6.78) is exponentially stable.

6.3 Notes and remarks

Neural networks have received an increasing interest in various areas [34, 119]. The stability of neural networks [38, 82, 139, 140] is critical for signal processing, especially in image processing and solving some classes of optimization problems. For the stochastic effects to the dynamical behaviors of neural networks, Liao and Mao [79, 80] initiated the study of stability and instability of stochastic neural networks.

Many articles [54, 55, 56, 120, 129] have considered a special case of the stochastic equation (6.1). Hu et al.[54] and Wan and Sun [129] studied a special case of (6.1) with the delays constant and discrete. The activation functions appearing in [54] are required to be bounded. Liao and Mao [81] investigated exponential stability of stochastic delay interval systems via Razumikihin-type theorems developed in [95], several exponential stability results were provided. However, the results are not only difficult to verify but also restrict to a case of the interval matrices $\tilde{A} = \tilde{B} = \tilde{C} = 0$. Sun and Cao [120] investigated the *pth* moment exponential stability of stochastic differential equations with discrete bounded delays by using the method of variation parameter, inequality technique and stochastic analysis. This method was firstly used in [129], which does not require the boundedness, monotonicity and differentiability of the activation functions. However, the stability criteria in [120] requires that the delay functions are bounded, differentiable and their derivatives are simultaneously required to be not greater than 1, this may impose a very strict constraint on model (see [138]). Huang et al. [55, 56] investigated the exponential stability of stochastic differential equations with discrete time-varying delays with the help of the Liapunov function and Dini derivative. However, the use of their criteria depends very much on the choice of positive numbers k_{ij} etc. and a positive diagonal matrix M (see Theorem 3.3 in [55] and Theorem 3.3 in [56]).

Based on the contents of this chapter, two papers [19, 20] have been submitted for possible publication.