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Chapter 5

Stability of neutral stochastic delay differential equations with impulses

This chapter concerns the stability of two classes of neutral stochastic delay differential equations with impulses.

In Section 5.1, asymptotic stability of a class of neutral stochastic delay differential equations with linear impulses is studied by means of fixed point methods. More specifically, two theorems for the asymptotic stability of the equations are presented by using two complete metric spaces which are defined by different types of norms.

In Section 5.2, exponential stability of a class of neutral stochastic partial differential equations with delays and impulses is investigated. The equation that will be considered in this section is an infinite dimensional stochastic differential equation with variable delays. The method by using an impulsive-integral inequality and the method based on fixed point theory, are applied to study exponential stability of mild solutions of the impulsive neutral stochastic partial delay differential equations, respectively.

5.1 Asymptotic stability of a class of neutral stochastic delay differential equations with linear impulses

5.1.1 Introduction and main results

A stochastic delay differential equation is a stochastic differential equation where the increment of the process depends on values of the process (and maybe other functions) of the past. These equations can be used to model processes with a memory.

Besides delay effect on the stochastic differential equations, impulsive effect is also a common phnomenon in a wide range of physical and engineering systems including the biological stocking or harvesting, the function of the heart, the change of an economy of a state, etc. For stochastic differential equations which include delay effects and impulsive effects are described as impulsive stochastic delay differential equations.

It was recently proposed by Luo [90] and Appleby [4] to use fixed point methods to deal with the stability problems for stochastic delay differential equations. However, to the best of our knowledge, the stochastic delay differential equations which have been studied by using fixed points method are those without impulses.

Very recently, Li, Sun and Shi [75, 62, 76] obtained some interesting results for the stability of zero solution of stochastic delay differential equations with impulsive effects by using the equivalent method. This method provides a way to study the stability of impulsive equations by constructing an equivalent relation between the stability of stochastic delay differential equations under impulses and that of a corresponding stochastic delay differential equations without impulses. As a consequence of this approach, the sufficient conditions for the stability of stochastic delay differential equations with impulses are obtained by using the existing stability results of the corresponding stochastic delay differential equations without impulses.

The aim of this section is to combine the approach based on fixed point methods and the equivalent method in order to study the stability of a general class of neutral stochastic delay differential equations with linear impulses. For the class of equations, we first transform the equations into the one without impulses, and then we use fixed point argument to obtain the sufficient conditions for asymptotic stability of the considered equations. In particular, we present two different sufficient conditions for the asymptotic stability of the equations by using two appropriate contraction mappings that are defined on different complete metric spaces. It turns out that our stability results for stochastic delay differential equations with impulses do not depend on the existing stability results for the corresponding stochastic delay differential equations without impulses.

The class of neutral stochastic differential equations that we will study in this section is of the form

$$\begin{cases} d[x(t) - q(t)x(t-\tau(t))] = [a(t)x(t) + b(t)x(t-\tau(t))]\,dt \\ \qquad\qquad\qquad\qquad\qquad\qquad +[c(t)x(t) + e(t)x(t-\delta(t))]\,dw(t), \quad t \neq t_k, \\ \\ x(t_k{}^+) - x(t_k) = d_k x(t_k), \quad t = t_k, \quad k = 1, 2, 3, \cdots, \end{cases} \tag{5.1}$$

where w is a one-dimensional standard Brownian motion on some filtrated probability space $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$ which satisfies the usual conditions.

Denote by $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$, $\mathbb{Z}^+ = \{1, 2, 3, \cdots\}$. Define

$$\vartheta = \min\left\{\inf_{s\geq t_0}\{s - \tau(s)\}, \inf_{s\geq t_0}\{s - \delta(s)\}\right\}$$

and $n(t) = \max\{k \in \mathbb{Z}^+ : t_k < t\}$. For simplicity, we define $\prod_{u\leq t_k<v}(\cdot) = \prod_{k\in\{k|k\in\mathbb{Z}^+, u\leq t_k<v\}}(\cdot)$ for all $u, v \in \mathbb{R}$. Here and in the sequel, we assume that a product equals unity if the number of factors is equal to zero.

A standard fixed point argument shows that the differential equation (5.1) provided with an initial condition

$$x(t) = \phi(t), \qquad t \in [\vartheta, 0], \tag{5.2}$$

where $\phi(t) \in C([\vartheta, 0], \mathbb{R})$ defines a well-posed initial-value problem, and we denote $x(t) := x(t, \phi)$ the solution of (5.1) with initial condition (5.2).

For equation (5.1) with initial condition (5.2), we suppose that the following conditions are satisfied

(H1) $0 \le t_0 < t_1 < t_2 < t_3 < \cdots < t_k < \cdots$ are fixed impulsive points such that $t_k \to \infty$ for $k \to \infty$;

(H2) $a(t), b(t), c(t), e(t) \in C(\mathbb{R}^+, \mathbb{R})$ and $\tau(t), \delta(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$;

(H3) d_k is a real sequence such that $d_k \in (-1, \infty)$, $k = 1, 2, 3, \cdots$.

Definition 5.1.1. *For any $\phi \in C([\vartheta, 0], \mathbb{R})$, a function $x : [\vartheta, \infty) \to \mathbb{R}$ denoted by $x(t, 0, \phi)$ is said to be a solution of the system (5.1) on $[0, \infty)$ satisfying the initial value condition (5.2), if the following conditions are satisfied:*

(i) *$x : [\vartheta, \infty) \to L^2(\Omega, \mathbb{R})$ is continuous on $[\vartheta, \infty) \backslash \{t_k : k = 1, 2, 3, \cdots\}$ and adapted to $(\mathcal{F}_t)_{t \ge 0}$.*

(ii) *for any t_k, $k = 1, 2, 3, \cdots$, $x({t_k}^+) = \lim_{t \to {t_k}^+} x(t)$ and $x({t_k}^-) = \lim_{t \to {t_k}^-} x(t)$ exist in $L^2(\Omega, \mathbb{P})$ and $x({t_k}^-) = x(t)$, i.e. there exist $x({t_k}^+), x({t_k}^-) \in L^2(\Omega, \mathbb{P})$ such that $\lim_{t \to {t_k}^+} \mathbb{E}|x({t_k}^+) - x(t)|^2 = 0$, $\lim_{t \to {t_k}^-} \mathbb{E}|x({t_k}^-) - x(t)|^2 = 0$ and $x({t_k}^-) = x(t)$ $\mathbb{P}$-a.e.*

(iii) *$x(t)$ satisfies the differential equation in (5.1) in the sense that for every k and every $s_0, s_1 \in (t_k, t_{k+1}]$ with $s_0 \le s_1$, $\mathbb{P}$-almost surely,*

$$\begin{aligned}&(x(s_1) - q(s_1)x(s_1 - \tau(s_1))) - (x(s_0) - q(s_0)x(s_0 - \tau(s_0))) \\ &\quad = \int_{s_0}^{s_1} [a(s)x(s) + b(s)x(s - \tau(s))]\, ds + \int_{s_0}^{s_1} [c(s)x(s) + e(s)x(s - \delta(s))]\, dw(s).\end{aligned}$$

Remark 5.1.2. *Note that continuity of $x : (t_k, t_{k+1}) \to L^2(\Omega, \mathbb{P})$ implies that x is measurable with respect to $B_{(t_k, t_{k+1})} \otimes \mathcal{F}$, where $B_{(t_k, t_{k+1})}$ is the Borel-σ-algebra, and that $\iint |x(t, \omega)|^2\, d\mathbb{P}\, dt \le \sup_{t \in (t_k, t_{k+1})} \int |x(t, \omega)|^2\, d\mathbb{P} < \infty$.*

Denote

$$C^b_{\mathcal{F}_0}(\delta) = \left\{ \phi \mid \phi \in C^b_{\mathcal{F}_0}([\vartheta, 0], \mathbb{R}),\ \sup_{\vartheta \le s \le 0} \mathbb{E}|\phi(s)|^2 \le \delta \right\},$$

where $\mathbb{E}$ denotes expectation with respect to the probability measure $\mathbb{P}$, refer to Section 1.3 for more detailed information.

Definition 5.1.3. *The zero solution of the system (5.1) is said to be*

(i) *mean square stable if for any $\varepsilon > 0$, there is a scalar $\delta = \delta(\varepsilon) > 0$ such that for every initial function $\phi \in C^b_{\mathcal{F}_0}(\delta)$ we have that for corresponding solution $x(t, 0, \phi)$ satisfies $\mathbb{E}|x(t, 0, \phi)|^2 < \varepsilon$ for $t \ge 0$.*

(ii) *mean square asymptotically stable if it is stable and for any $\varepsilon > 0$, there exists a scalar $\delta = \delta(\varepsilon) > 0$ such that for every initial function $\phi \in C^b_{\mathcal{F}_0}(\delta)$, the corresponding solution $x(t, 0, \phi)$ satisfies $\lim_{t \to \infty} \mathbb{E}|x(t, 0, \phi)|^2 = 0$ for $t \ge 0$.*

(iii) *exponentially stable in mean square if there is a pair of positive constants λ and K such that*

$$\mathbb{E}|x(t,0,\phi)|^2 \leq K\|\phi\|_{L^2}^2 e^{-\lambda t},$$

holds for any $\phi \in C^b_{\mathcal{F}_0}([\vartheta,0],\mathbb{R})$, here λ is called the exponential convergence rate.

Denote by $\mathcal{S}_\phi$ the space of all $\mathcal{F}$-adapted processes $\varphi(t,\omega) : [\vartheta,\infty)\times\Omega \to \mathbb{R}$ such that $t \mapsto \varphi(t) : [\vartheta,\infty) \mapsto L^2(\Omega,\mathbb{R})$ is continuous in $t \neq t_k$ $(k = 1,2,\cdots)$, $\lim_{t\to t_k^-}\varphi(t)$ and $\lim_{t\to t_k^+}\varphi(t)$ exist, and $\lim_{t\to t_k^-}\varphi(t) = \varphi(t_k)$ in $L^2(\Omega;\mathbb{P})$. Moreover, $\varphi(t,\cdot) = \phi(t)$ for $t \in [\vartheta,0]$, and $\mathbb{E}|\varphi(t)|^2 \to 0$ as $t \to \infty$. If we define a norm as

$$\|\varphi\|^2 := \sup_{t\geq\vartheta}\left(\mathbb{E}|\varphi(t)|^2\right), \tag{5.3}$$

then $\mathcal{S}_\phi$ is a complete metric space with respect to the norm (5.3). Using a contraction mapping defined on the space $\mathcal{S}_\phi$, we come to our first result, which is proved in Subsection 5.1.2.

Theorem 5.1.4. *Consider the impulsive neutral stochastic differential equation (5.1) and suppose that the following conditions are satisfied*

(i) *the delay $\tau(t)$ is differentiable and $t-\tau(t) \to \infty$ as $t \to \infty$, $t-\delta(t) \to \infty$ as $t \to \infty$;*

(ii) *there exist a constant $\alpha \in (0,1)$ and a continuous function $h(t)$ such that for $t \geq 0$,*

$$\begin{aligned}
H_1(t) \quad := \quad 2&\left\{\sum_{l=1}^{n(t)}\left|\frac{d_l}{1+d_l}\prod_{t_l-\tau(t_l)\leq t_k<t_l}(1+d_k)^{-1}e^{-\int_{t_l}^t h(u)\,du}q(t_l)\right|\right.\\
&+\left|\prod_{t-\tau(t)<t_k<t}(1+d_k)^{-1}q(t)\right| + \int_{t-\tau(t)}^t |a(s)+h(s)|\,ds\\
&+\int_0^t e^{-\int_s^t h(u)\,du}|h(s)|\int_{s-\tau(s)}^s |a(u)+h(u)|\,du\,ds\\
&+\int_0^t e^{-\int_s^t h(u)\,du}\Bigg|(a(s-\tau(s))+h(s-\tau(s)))(1-\tau'(s))\\
&\left.+\prod_{s-\tau(s)\leq t_k<s}(1+d_k)^{-1}b(s) - \prod_{s-\tau(s)\leq t_k<s}(1+d_k)^{-1}h(s)q(s)\Bigg|\,ds\right\}^2\\
&+2\int_0^t e^{-2\int_s^t h(u)\,du}\left(|c(s)| + \left|\prod_{s-\delta(s)\leq t_k<s}(1+d_k)^{-1}e(s)\right|\right)^2 ds \leq \alpha,
\end{aligned}$$

(iii) *and such that $\liminf_{t\to\infty}\int_0^t h(s)\,ds > -\infty$;*

(iv) *there exists a constant $M > 0$ such that for any $t \geq 0$,*

$$\left|\prod_{0\leq t_k<t}(1+d_k)\right| \leq M.$$

Then the zero solution of (5.1) is mean square asmptotically stable if

(v)

$$\int_0^t h(s)\,ds \to \infty \quad \text{as} \quad \mathrm{t} \to \infty.$$

Remark 5.1.5. *Note that if $\varphi \in C([a,b]; L^2(\Omega,\mathbb{R}))$, then $\varphi \in L^2([a,b]; L^2(\Omega,\mathbb{R}))$ and the latter space is known to be naturally isometrically isomorphic to $L^2([a,b]\times\Omega;\mathbb{R})$ (see [30] for detail), so φ is joint measurable and $\int_a^b\int_\Omega |\varphi(t,\omega)|^2\,dt\,d\mathbb{P}(\omega) < \infty$.*

Remark 5.1.6. *Note that if $-1 < d_k \le 0$ for all k, then (iv) in Theorem 5.1.4 is always satisfied. If $d_k = 0$, Theorem 5.1.4 is Theorem 2.1 in [90] under sufficient conditions.*

Remark 5.1.7. *Impulses are small perturbation, it can stabilize or destabilize a system. As an example, consider the equation*

$$\begin{cases} x'(t) = ax(t) & a > 0, \\ x(k^+) = (1+d_k)x(k), & k \in \mathbb{N}, \end{cases} \tag{5.4}$$

From (5.4), we obtain that

(1) *if $d_k = 0$ for all $k \in \mathbb{N}$, $x(t) = e^{at}x(0)$;*

(2) *if $d_k \neq 0$ for all $k \in \mathbb{N}$,*

$$x(1) = e^a x(0) \qquad x(1^+) = (1+d_1)e^a x(0)$$

$$x(1+t) = e^{at}x(1^+) = e^{at}(1+d_1)e^a x(0), \quad 0 < t \le 1,$$

$$x(2) = (1+d_1)e^{2a}x(0) \qquad x(2^+) = (1+d_1)(1+d_2)e^{2a}x(0)$$

$$x(2+t) = e^{at}x(2^+) = e^{at}(1+d_1)(1+d_2)e^{2a}x(0), \quad 0 < t \le 1,$$

$$\vdots$$

$$x(n) = (1+d_1)(1+d_2)\cdots(1+d_{n-1})e^{na}x(0).$$

$$x(n^+) = (1+d_1)(1+d_2)\cdots(1+d_n)e^{na}x(0).$$

If $d_k = d$ $(k = 1,2,3,\cdots)$ is a constant, then

$$x(n^+) = (1+d)^n e^{na}x(0) = [(1+d)e^a]^n x(0),$$

If $(1+d)e^a < 1$, then $x(n^+) \to 0$; If $(1+d)e^a > 1$, then $x(n^+) \to \infty$.

Remark 5.1.8. *The condition (ii) in Theorem 5.1.4 can not be changed to be $H_1(t) < 1$. For example, if $H_1(t) = \int_0^t e^{-s}\,ds$ for all t, then $H_1(t) = 1 - e^{-t} < 1$. However, $H_1(t) \to 1$ as $t \to \infty$. Hence, the condition $H_1(t) < 1$ can not imply that there exsits a constant $\alpha \in (0,1)$ such that $H_1(t) \le \alpha < 1$.*

For the case when the delays $\tau(t)$ and $\delta(t)$ are bounded by a positive constant τ, we define another complete metric space as the following.

Denote by $\mathcal{C}_\phi$ the space of all $\mathcal{F}$-adapted processes $\varphi(t,\omega) : [-\tau,\infty)\times\Omega \to \mathbb{R}$, which is almost surely continuous in $t \neq t_k$ $(k = 1,2,\cdots)$ for fixed $\omega \in \Omega$, $\lim_{t\to t_k^-}\varphi(t)$ and $\lim_{t\to t_k^+}\varphi(t)$ exist, and $\lim_{t\to t_k^-}\varphi(t) = \varphi(t_k)$. Moreover, $\varphi(t,\cdot) = \phi(t)$ for $t \in [-\tau, 0]$ and for $t \to \infty$, $\mathbb{E}\left(\sup_{t-\tau\le s\le t}|\varphi(s)|^2\right) \to 0$. If we define a norm as

$$\|\varphi\|^2 := \sup_{t\ge 0}\mathbb{E}\left(\sup_{t-\tau\le s\le t}|\varphi(s)|^2\right), \tag{5.5}$$

then $\mathcal{C}_\phi$ is a complete metric space. Based on the space $\mathcal{C}_\phi$, we come to our second result, which is proved in Subsection 5.1.3.

Theorem 5.1.9. *Consider the impulsive neutral stochastic differential equation* (5.1) *and suppose that the following conditions are satisfied*

(i) *the delay $\tau(t)$ is differentiable;*

(ii) *there exist a constant $\alpha \in (0,1)$ and a continuous function $h(t)$ such that for $t \ge 0$,*

$$\begin{aligned} H_2(t) \;:=\; & 2\Bigg\{\sum_{l=1}^{n(t)}\left|\frac{d_l}{1+d_l}\prod_{t_l-\tau(t_l)\le t_k<t_l}(1+d_k)^{-1}e^{-\int_{t_l}^t h(u)\,du}q(t_l)\right| \\ & +\left|\prod_{t-\tau(t)<t_k<t}(1+d_k)^{-1}q(t)\right| + \int_{t-\tau(t)}^t |a(s)+h(s)|\,ds \\ & +\int_0^t e^{-\int_s^t h(u)\,du}|h(s)|\int_{s-\tau(s)}^s |a(u)+h(u)|\,du\,ds \\ & +\int_0^t e^{-\int_s^t h(u)\,du}\Bigg|(a(s-\tau(s))+h(s-\tau(s)))(1-\tau'(s)) \\ & +\prod_{s-\tau(s)\le t_k<s}(1+d_k)^{-1}b(s) - \prod_{s-\tau(s)\le t_k<s}(1+d_k)^{-1}h(s)q(s)\Bigg|\,ds\Bigg\}^2 \\ & +8\int_0^t\left(\sup_{t-\tau\le r\le t}e^{-2\int_s^r h(u)\,du}\right)\left(|c(s)| + \left|\prod_{s-\delta(s)\le t_k<s}(1+d_k)^{-1}e(s)\right|\right)^2 ds \\ & \le \alpha; \end{aligned}$$

where $n(t) := \sup\{k \in \mathbb{Z}^+ : t_k < t\}$.

(iii) *and such that $\liminf_{t\to\infty}\int_0^t h(s)\,ds > -\infty$;*

(iv) *there exists a constant $M > 0$ such that for any $t \ge 0$, $\left|\prod_{0\le t_k<t}(1+d_k)\right| \le M$.*

Then the zero solution of (5.1) *is asymptotically stable if*

(v)

$$\int_0^t h(s)\,ds \to \infty \quad \text{as} \quad \mathrm{t}\to\infty.$$

5.1.2 Proof of Theorem 5.1.4

In this subsection, we will prove Theorem 5.1.4. We start with some preparations. First we transform (5.1) into a neutral stochastic delay differential equation without impulses:

$$\begin{aligned} d\left[y(t)-\tilde{q}(t)y(t-\tau(t))\right] &= \left[a(t)y(t)+\tilde{b}(t)y(t-\tau(t))\right]dt \\ &\quad + \left[c(t)y(t)+\tilde{e}(t)y(t-\delta(t))\right]dw(t), \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} \tilde{q}(t) &= \prod_{t-\tau(t)\le t_k<t}(1+d_k)^{-1}q(t), \quad \tilde{b}(t)=\prod_{t-\tau(t)\le t_k<t}(1+d_k)^{-1}b(t), \\ \tilde{e}(t) &= \prod_{t-\delta(t)\le t_k<t}(1+d_k)^{-1}e(t). \end{aligned}$$

By a solution of the system (5.6) and (5.2) we mean a continuous function $y(t)$ on $[\vartheta,\infty)$ satisfying equation (5.6) almost everywhere for $t\ge 0$ and satisfies equation (5.2).

We start with two fundamental lemmas. The lemmas allow us to reduce the problem of stability of an impulsive neutral stochastic delay differential equation to problem of stability of a neutral stochastic delay differential equation.

Lemma 5.1.10. *Assume that* $(\mathrm{H}_1)-(\mathrm{H}_3)$ *hold, then*

(i) *if $y(t)$ is a solution of (5.6), then $x(t)=\prod_{0<t_k<t}(1+d_k)y(t)$ is a solution of (5.1);*

(ii) *if $x(t)$ is a solution of (5.1), then $y(t)=\prod_{0<t_k<t}(1+d_k)^{-1}x(t)$ is a solution of (5.6).*

Proof. First we prove (i). Let $y(t)$ be a possible solution of the problem (5.6), it is easy to see that $x(t)=\prod_{0<t_j<t}(1+d_k)y(t)$ is continuous on $(0,t_1]$ and on each interval $(t_k,t_{k+1}]$, $k\in\mathbb{N}$ and for any $t\ne t_k$, $j\in\mathbb{N}$,

$$\begin{aligned} dx(t) &= d\left[\prod_{0<t_k<t}(1+d_j)y(t)\right]=\prod_{0<t_k<t}(1+d_k)dy(t) \\ &= \prod_{0<t_k<t}(1+d_k)\Big\{d\left[\tilde{q}(t)y(t-\tau(t))\right]+\Big(a(t)y(t)+\tilde{b}(t)y(t-\tau(t))\Big)\,dt \\ &\qquad + \left(c(t)y(t)+\tilde{e}(t)y(t-\delta(t))\right)dw(t)\Big\} \\ &= d[q(t)x(t-\tau(t))]+[a(t)x(t)+b(t)x(t-\tau(t))]dt+[c(t)x(t)+e(t)x(t-\delta(t))]dw(t). \end{aligned}$$

Thus $x(t)$ satisfies the differential equation (5.1) for almost everywhere in $[0,\infty)$ when $t\ne t_k$, $k\in\mathbb{N}$, On the other hand, for every $t=t_k$, $k\in\mathbb{N}$,

$$\begin{aligned} x(t_k^+) &= \lim_{t\to t_k^+}\prod_{0<t_j<t}(1+d_j)y(t)=\prod_{0<t_j\le t_k}(1+d_j)y(t_k^+) \\ &= (1+d_k)\prod_{0<t_j<t_k}(1+d_j)y(t_k)=(1+d_k)x(t_k) \end{aligned}$$

and

$$x(t_k^-)=\lim_{t\to t_k^-}\prod_{0<t_j<t}(1+d_j)y(t)=\prod_{0<t_j<t_k}(1+d_j)y(t_k^-)=x(t_k).$$

Therefore, we have that $x(t)$ is a solution of equation (5.1) corresponding to the initial condition (5.2). In fact, if $y(t)$ is a solution of (5.6) with initial condition (5.2), then $x(t) = \prod_{0<t_k<t}(1+d_k)y(t) = y(t) = \phi(t)$ on $[\vartheta, 0]$.

Next, we prove (ii). If $x(t)$ is a solution of (5.1), then $x(t)$ is continuous on $(0, t_1]$ and on each interval $(t_k, t_{k+1}]$, $j \in \mathbb{N}$. Therefore, $y(t) = \prod_{0<t_k<t}(1+d_k)^{-1}x(t)$ is continuous on $(0, t_1]$ and on each interval $(t_k, t_{k+1}]$, $k \in \mathbb{N}$. Using the similar way as the proof for (i), we can easily check that $y(t) = \prod_{0<t_k<t}(1+d_k)^{-1}x(t)$ is the solution of (5.6) on $[\vartheta, \infty)$ corresponding to the initial condition (5.2). On the other hand, for every $t = t_k$, $k \in \mathbb{N}$,

$$\begin{aligned} y(t_k^+) &= \lim_{t \to t_k^+} \prod_{0<t_j<t}(1+d_j)^{-1}x(t) = \prod_{0<t_j\le t_k}(1+d_j)^{-1}x(t_k^+) \\ &= (1+d_k)^{-1}\prod_{0<t_j<t_k}(1+d_j)^{-1}x(t_k^+) = \prod_{0<t_j<t_k}(1+d_j)^{-1}x(t_k) = y(t_k) \end{aligned}$$

and

$$y(t_k^-) = \lim_{t \to t_k^-} \prod_{0<t_j<t}(1+d_j)^{-1}x(t) = \prod_{0<t_j<t_k}(1+d_j)^{-1}x(t_k^-) = \prod_{0<t_j<t_k}(1+d_j)^{-1}x(t_k) = y(t_k).$$

This completes the proof of the lemma. □

Lemma 5.1.11. *Assume that* $(\mathrm{H}_1) - (\mathrm{H}_3)$ *hold, and there exists a positive constant* M *such that, for any* $t > 0$,

$$\left| \prod_{0<t_k<t}(1+d_k) \right| \le M. \tag{5.7}$$

(i) *If the zero solution of equation* (5.6) *is mean square stable, then the zero solution of equation* (5.1) *is also mean square stable.*

(ii) *If the zero solution of equation* (5.6) *is mean square asymptotically stable, then the zero solution of equation* (5.1) *is also mean square asymptotically stable.*

(iii) *If the zero solution of equation* (5.6) *is mean square exponentially stable, then the zero solution of equation* (5.1) *is also mean square exponentially stable.*

Proof. First, we prove (i). Let $x(t)$ and $y(t)$ be the solution of equations (5.1) and (5.6) corresponding to initial conditions (5.2). If the zero solution of equation (5.6) is mean square stable, from the definition of mean square stable, we have that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that the initial function $\phi \in C^b_{\mathcal{F}_0}(\delta)$ implies

$$\mathbb{E}|y(t)|^2 < \frac{\varepsilon}{M^2}, \quad \text{for} \quad \mathrm{t} > 0. \tag{5.8}$$

From Lemma 5.1.10, we obatin that $x(t) = \prod_{0<t_k<t}(1+d_k)y(t)$ is a solution of (5.1) on $[\vartheta, \infty)$, and combining with (5.8), we have that

$$\mathbb{E}|x(t)|^2 = \mathbb{E}\left| \prod_{0<t_k<t}(1+d_k)y(t) \right|^2 \le \left| \prod_{0<t_k<t}(1+d_k) \right|^2 \mathbb{E}|y(t)|^2 < M^2 \frac{\varepsilon}{M^2} = \varepsilon,$$

which implies that the zero solution of (5.1) is mean square stable.

(ii)and (iii) can be proved similarly as (i). □

If we multiply both sides of (5.6) by $e^{\int_0^t h(s)\,ds}$, integrate from 0 to t, and perform an integration by parts, we obtain

$$\begin{aligned}
(Py)(t) &= \left[\phi(0) - \int_{\tau(0)}^{0} (a(s)+h(s))\phi(s)\,ds - q(0)\phi(-\tau(0)) + M(t)\right] e^{-\int_0^t h(u)\,du} \\
&\quad + \prod_{t-\tau(t)\le t_k<t} (1+d_k)^{-1} q(t) y(t-\tau(t)) + \int_{t-\tau(t)}^{t} (a(s)+h(s))y(s)\,ds \\
&\quad - \int_0^t e^{-\int_s^t h(u)\,du} h(s) \int_{s-\tau(s)}^{s} (a(u)+h(u))y(u)\,du\,ds \\
&\quad + \int_0^t e^{-\int_s^t h(u)\,du} \Bigg[(a(s-\tau(s)) + h(s-\tau(s)))(1-\tau'(s)) \\
&\quad + \prod_{s-\tau(s)\le t_k<s} (1+d_k)^{-1} b(s) - \prod_{s-\tau(s)\le t_k<s} (1+d_k)^{-1} h(s) q(s) \Bigg] y(s-\tau(s))\,ds \\
&\quad + \int_0^t e^{-\int_s^t h(u)\,du} \left[c(s)y(s) + \prod_{s-\delta(s)\le t_k<s} (1+d_k)^{-1} e(s) y(s-\delta(s)) \right] dw(s).
\end{aligned}$$

Lemma 5.1.12. *Let $\varphi \in \mathcal{S}_\phi$. Define an operator by $(P\varphi)(t) = \phi(t)$ for $t \in [\vartheta, 0]$ and for $t \geq 0$,*

$$(P\varphi)(t) = \sum_{i=1}^{5} I_i(t), \tag{5.9}$$

where

$$\begin{aligned}
M(t) &= \sum_{l=1}^{n(t)} \frac{d_l}{1+d_l} \prod_{t_l-\tau(t_l)\le t_k<t_l} (1+d_k)^{-1} e^{\int_0^{t_l} h(u)\,du} q(t_l)\varphi(t_l-\tau(t_l)), \\
I_1(t) &= \left[\phi(0) - \int_{\tau(0)}^{0} (a(s)+h(s))\phi(s)\,ds - q(0)\phi(-\tau(0))\right] e^{-\int_0^t h(u)\,du}, \\
I_2(t) &= \sum_{l=1}^{n(t)} \frac{d_l}{1+d_l} \prod_{t_l-\tau(t_l)\le t_k<t_l} (1+d_k)^{-1} e^{-\int_{t_l}^t h(u)\,du} q(t_l)\varphi(t_l-\tau(t_l)) \\
&\quad + \prod_{t-\tau(t)\le t_k<t} (1+d_k)^{-1} q(t)\varphi(t-\tau(t)), \\
I_3(t) &= \int_{t-\tau(t)}^{t} (a(s)+h(s))\varphi(s)\,ds - \int_0^t e^{-\int_s^t h(u)\,du} h(s) \int_{s-\tau(s)}^{s} (a(u)+h(u))\varphi(u)\,du\,ds,
\end{aligned}$$

$$
\begin{aligned}
I_4(t) &= \int_0^t e^{-\int_s^t h(u)\,du}\Bigg[(a(s-\tau(s))+h(s-\tau(s)))(1-\tau'(s)) \\
&\quad + \prod_{s-\tau(s)\le t_k<s}(1+d_k)^{-1}b(s) - \prod_{s-\tau(s)\le t_k<s}(1+d_k)^{-1}h(s)q(s)\Bigg]\varphi(s-\tau(s))\,ds, \\
I_5(t) &= \int_0^t e^{-\int_s^t h(u)\,du}\Bigg[c(s)y(s)+\prod_{s-\delta(s)\le t_k<s}(1+d_k)^{-1}e(s)\varphi(s-\delta(s))\Bigg]\,dw(s). \qquad (5.10)
\end{aligned}
$$

If conditions (i)-(iv) *in Theorem* 5.1.4 *are satisfied, then there exists* $\delta>0$ *such that for any* $\phi:[\vartheta,0]\to(-\delta,\delta)$, *we have that* $P:\mathcal{S}_\phi\to\mathcal{S}_\phi$ *and* P *is a contraction with respect to the norm* (5.3).

Proof. We first verify the continuity of $P\varphi$ on $[0,\infty)$ to $L^2(\Omega,\mathcal{F},\mathbb{P}).I_2(t)$ is continuous if $t\in(0,t_1)$ or $t\in(t_j,t_{j+1})$ for $j=1,2,3,\cdots$. It remains to prove that $I_2(t)$ is continuous at $t=t_j$. Let $\varphi\in S$, take the limit $r\to0^+$, we have

$$
\begin{aligned}
&\mathbb{E}|I_2(t_j+r)-I_2(t_j)|^2 \\
&= \mathbb{E}\Bigg|\left(e^{-\int_{t_j}^{t_j+r}h(u)\,du}-1\right)\sum_{l=1}^{j-1}\frac{d_l}{1+d_l}\prod_{t_l-\tau(t_l)\le t_k<t_l}(1+d_k)^{-1}e^{-\int_{t_l}^{t_j}h(u)\,du}q(t_l)\varphi(t_l-\tau(t_l)) \\
&\quad +\frac{d_j}{1+d_j}\prod_{t_j-\tau(t_j)\le t_k<t_j}(1+d_k)^{-1}e^{-\int_{t_j}^{t_j+r}h(u)\,du}q(t_j)\varphi(t_j-\tau(t_j)) \\
&\quad +\prod_{t_j+r-\tau(t_j+r)\le t_k<t_j+r}(1+d_k)^{-1}q(t_j+r)\varphi(t_j+r-\tau(t_j+r)) \\
&\quad -\prod_{t_j-\tau(t_j)\le t_k<t_j}(1+d_k)^{-1}q(t_j)\varphi(t_j-\tau(t_j))\Bigg|^2 \\
&\le 2\mathbb{E}\left|e^{-\int_{t_j}^{t_j+r}h(u)\,du}-1\right|^2\left|\sum_{l=1}^{j}\frac{d_l}{1+d_l}\prod_{t_l-\tau(t_l)\le t_k<t_l}(1+d_k)^{-1}e^{-\int_{t_l}^{t_j}h(u)\,du}q(t_l)\varphi(t_l-\tau(t_l))\right|^2 \\
&\quad +2\mathbb{E}\Bigg[\Bigg|\frac{1}{1+d_j}\prod_{t_j-\tau(t_j)\le t_k<t_j}(1+d_k)^{-1}\Bigg| \\
&\qquad \times|q(t_j+r)\varphi(t_j+r-\tau(t_j+r))-q(t_j)\varphi(t_j-\tau(t_j))| \\
&\quad +\Bigg|\prod_{t_j+r-\tau(t_j+r)\le t_k<t_j+r}(1+d_k)^{-1}-\frac{1}{1+d_j}\prod_{t_j-\tau(t_j)\le t_k<t_j}(1+d_k)^{-1}\Bigg| \\
&\qquad \times|q(t_j+r)\varphi(t_j+r-\tau(t_j+r))|\Bigg]^2 \longrightarrow 0,
\end{aligned}
$$

since

$$\lim_{r\to 0^+} \prod_{t_j+r-\tau(t_j+r)\le t_k<t_j+r} (1+d_k)^{-1} = \frac{1}{1+d_j} \prod_{t_j-\tau(t_j)\le t_k<t_j} (1+d_k)^{-1}.$$

Take the limit $r \to 0^-$, we have

$$\begin{aligned}
&\mathbb{E}|I_2(t_j+r) - I_2(t_j)|^2 \\
&= \mathbb{E}\Bigg| \left(e^{-\int_{t_j}^{t_j+r} h(u)\,du} - 1\right) \sum_{l=1}^{j-1} \frac{d_l}{1+d_l} \prod_{t_l-\tau(t_l)\le t_k<t_l} (1+d_k)^{-1} e^{-\int_{t_l}^{t_j} h(u)\,du} q(t_l)\varphi(t_l-\tau(t_l)) \\
&\quad + \prod_{t_j+r-\tau(t_j+r)\le t_k<t_j+r} (1+d_k)^{-1} q(t_j+r)\varphi(t_j+r-\tau(t_j+r)) \\
&\quad - \prod_{t_j-\tau(t_j)\le t_k<t_j} (1+d_k)^{-1} q(t_j)\varphi(t_j-\tau(t_j))\Bigg|^2 \\
&\le 2\mathbb{E}\left|e^{-\int_{t_j}^{t_j+r} h(u)\,du} - 1\right|^2 \left|\sum_{l=1}^{j-1} \frac{d_l}{1+d_l} \prod_{t_l-\tau(t_l)\le t_k<t_l} (1+d_k)^{-1} e^{-\int_{t_l}^{t_j} h(u)\,du} q(t_l) y(t_l-\tau(t_l))\right|^2 \\
&\quad + 2\mathbb{E}\Bigg| \prod_{t_j+r-\tau(t_j+r)\le t_k<t_j+r} (1+d_k)^{-1} q(t_j+r)\varphi(t_j+r-\tau(t_j+r)) \\
&\quad - \prod_{t_j-\tau(t_j)\le t_k<t_j} (1+d_k)^{-1} q(t_j)\varphi(t_j-\tau(t_j))\Bigg|^2 \to 0,
\end{aligned}$$

since

$$\lim_{r\to 0^-} \prod_{t_j+r-\tau(t_j+r)\le t_k<t_j+r} (1+d_k)^{-1} = \prod_{t_j-\tau(t_j)\le t_k<t_j} (1+d_k)^{-1}.$$

All the other terms continuous because of the following (a) and (b).

(a) If $\sup_{s\in[0,t]} \left[\mathbb{E}g(s)^2\right] < \infty$ for each t, then $t \mapsto \int_0^t g(s)\,ds : [0,\infty) \mapsto L^2(\Omega, \mathcal{F}_0, \mathbb{P})$ is continuous.

(b) If $\sup_{s\in[0,t]} \left[\mathbb{E}g(s)^2\right] < \infty$ for each t, then $t \mapsto \int_0^t g(s)\,dw : [0,\infty) \mapsto L^2(\Omega, \mathcal{F}_0, \mathbb{P})$ is continuous.

Thus P is continuous on $[0,\infty)$.

Next, We prove that $P(\mathcal{S}_\phi) \subseteq \mathcal{S}_\phi$. For $\varphi \in \mathcal{S}_\phi$, we consider the first term of $I_3(t)$,

$$
\begin{aligned}
&\mathbb{E}\left(\int_{t-\tau(t)}^{t}(a(s)+h(s))\varphi(s)\,ds\right)^2\\
&=\mathbb{E}\left(\int_{t-\tau(t)}^{t}(a(s)+h(s))\varphi(s)\,ds\int_{t-\tau(t)}^{t}(a(v)+h(v))\varphi(v)\,dv\right)\\
&=\int_{t-\tau(t)}^{t}\int_{t-\tau(t)}^{t}(a(s)+h(s))(a(v)+h(v))\mathbb{E}\varphi(s)\varphi(v)\,ds\,dv\\
&\le\int_{t-\tau(t)}^{t}\int_{t-\tau(t)}^{t}|a(s)+h(s)||a(v)+h(v)|\,ds\,dv\sup_{s,v\in[t-\tau(t),t]}\mathbb{E}\varphi(s)\varphi(v)\\
&=\left(\int_{t-\tau(t)}^{t}|a(s)+h(s)|\,ds\right)^2\sup_{s,v\in[t-\tau(t),t]}\left(\mathbb{E}\varphi(s)^2\right)^{1/2}\left(\mathbb{E}\varphi(v)^2\right)^{1/2}.
\end{aligned}
$$

Since $\mathbb{E}\varphi(t)^2 \to 0$ and $t-\tau(t) \to \infty$ as $t \to \infty$, so for every $\varepsilon > 0$, there exists $T_0 > 0$ such that $t > T_0$ implies $\mathbb{E}\varphi(t)^2 < \varepsilon$ and $\mathbb{E}\varphi(t-\tau(t))^2 < \varepsilon$. Hence, for $t > T_0$, we have that

$$
\sup_{s,v\in[t-\tau(t),t]}\left(\mathbb{E}\varphi(s)^2\right)^{1/2}\left(\mathbb{E}\varphi(v)^2\right)^{1/2} < \varepsilon,
$$

Combining with condition (ii) in Theorem 5.1.4, we obtain

$$
\mathbb{E}\left(\int_{t-\tau(t)}^{t}(a(s)+h(s))\varphi(s)\,ds\right)^2 \to 0 \quad \text{as} \quad t\to\infty.
$$

Following the similar estimation as above, we obtain that $\mathbb{E}|I_i(s)|^2 \to 0$ as $t \to \infty$, $i = 1, 2, 3, 4$. It follows from the last term $I_5(s)$ in (5.9) that

$$
\begin{aligned}
&\mathbb{E}|I_5(t)|^2\\
&=\mathbb{E}\left|\int_0^t e^{-\int_z^t h(u)\,du}\left[c(z)\varphi(z)+\prod_{z-\delta(z)\le s_k<z}(1+d_k)^{-1}e(z)\varphi(z-\delta(z))\right]dw(z)\right|^2\\
&\le\mathbb{E}\left|\int_0^t e^{-\int_z^t h(u)\,du}\left[c(z)\varphi(z)+\prod_{z-\delta(z)\le s_k<z}(1+d_k)^{-1}e(z)\varphi(z-\delta(z))\right]dw(z)\right|^2\\
&\le\mathbb{E}\int_0^t e^{-2\int_z^t h(u)\,du}\left[c(z)\varphi(z)+\prod_{z-\delta(z)\le s_k<z}(1+d_k)^{-1}e(z)\varphi(z-\delta(z))\right]^2dz.
\end{aligned}
$$

Since $\mathbb{E}|\varphi(t)|^2 \to 0$ and $t-\delta(t) \to 0$ as $t \to \infty$, then for any $\varepsilon > 0$, there exists $T > 0$ such that $t > T$ implies $\mathbb{E}|\varphi(t)|^2 < \varepsilon$ and $\mathbb{E}|\varphi(t-\delta(t))|^2 < \varepsilon$. Hence, for $t > T$, we have that

$$
\begin{aligned}
&\mathbb{E}\int_0^t e^{-2\int_z^t h(u)\,du}\left[c(z)\varphi(z)+\prod_{z-\delta(z)\le s_k<z}(1+d_k)^{-1}e(z)\varphi(z-\delta(z))\right]^2dz\\
&\quad<\left\{\int_0^t e^{-2\int_z^t h(u)\,du}\left[|c(z)|+\left|\prod_{z-\delta(z)\le s_k<z}(1+d_k)^{-1}e(z)\right|\right]^2dz\right\}\varepsilon,
\end{aligned}
$$

which yields $\mathbb{E}|I_5(s)|^2 \to 0$ as $t \to \infty$. Hence, we obtain that $P(\mathcal{S})_\phi \subseteq \mathcal{S}_\phi$.

Finally, We show that P is contraction mapping. For $\varphi, \psi \in \mathcal{S}_\phi$, we have that

$$
\begin{aligned}
&\sup_{s\geq\vartheta}\left\{\mathbb{E}|(P\varphi)(s)-(P\psi)(s)|^2\right\}\\
&=\sup_{s\geq\vartheta}\mathbb{E}\Bigg|\sum_{l=1}^{n(t)}\frac{d_l}{1+d_l}\prod_{s_l-\tau(s_l)\leq s_k<s_l}(1+d_k)^{-1}e^{-\int_{s_l}^{s}h(u)\,du}q(s_l)\\
&\qquad\times(\varphi(s_l-\tau(s_l))-\psi(s_l-\tau(s_l)))\\
&\quad+\prod_{s-\tau(s)\leq s_k<s}(1+d_k)^{-1}q(s)(\varphi(s-\tau(s))-\psi(s-\tau(s)))\\
&\quad+\int_{s-\tau(s)}^{s}(a(z)+h(z))(\varphi(z)-\psi(z))\,dz\\
&\quad-\int_0^s e^{-\int_z^s h(u)\,du}h(z)\int_{z-\tau(z)}^{z}(a(u)+h(u))(\varphi(u)-\psi(u))\,du\,dz\\
&\quad+\int_0^s e^{-\int_z^s h(u)\,du}\Bigg[(a(z-\tau(z))+h(z-\tau(z)))(1-\tau'(z))\\
&\quad+\prod_{z-\tau(z)\leq s_k<z}(1+d_k)^{-1}b(z)-\prod_{z-\tau(z)\leq s_k<z}(1+d_k)^{-1}h(z)q(z)\Bigg]\\
&\qquad\times\left(\varphi(z-\tau(z))-\psi(z-\tau(z))\right)\,dz+\int_0^s e^{-\int_z^s h(u)\,du}\Bigg[c(z)(\varphi(z)-\psi(z))\\
&\quad+\prod_{z-\delta(z)\leq s_k<z}(1+d_k)^{-1}e(z)(\varphi(z-\delta(z))-\psi(z-\delta(z)))\Bigg]\,dw(z)\Bigg|^2\\
&\leq\sup_{s\geq\vartheta}\Bigg\{2\Bigg[\sum_{l=1}^{n(t)}\Bigg|\frac{d_l}{1+d_l}\prod_{s_l-\tau(s_l)\leq s_k<s_l}(1+d_k)^{-1}e^{-\int_{s_l}^{s}h(u)\,du}q(s_l)\Bigg|\\
&\quad+\Bigg|\prod_{s-\tau(s)<s_k<s}(1+d_k)^{-1}q(s)\Bigg|+\int_{s-\tau(s)}^{s}|a(z)+h(z)|\,dz\\
&\quad+\int_0^s e^{-\int_z^s h(u)\,du}|h(z)|\int_{z-\tau(z)}^{z}|a(u)+h(u)|\,du\,dz\\
&\quad+\int_0^s e^{-\int_z^s h(u)\,du}\Bigg|(a(z-\tau(z))+h(z-\tau(z)))(1-\tau'(z))\\
&\quad+\prod_{z-\tau(z)\leq s_k<z}(1+d_k)^{-1}b(z)-\prod_{z-\tau(z)\leq s_k<z}(1+d_k)^{-1}h(z)q(z)\Bigg|\,dz\Bigg]^2\\
&\quad+2\int_0^s e^{-2\int_z^s h(u)\,du}\Bigg(c(z)+\prod_{z-\delta(z)\leq s_k<z}(1+d_k)^{-1}e(z)\Bigg)^2\,dz\Bigg\}\\
&\qquad\times\sup_{s\geq\vartheta}\left\{\mathbb{E}|\varphi(s)-\psi(s)|^2\right\}\leq\alpha\sup_{s\geq\vartheta}\left\{\mathbb{E}|\varphi(s)-\psi(s)|^2\right\}.
\end{aligned}
$$

Therefore, $P : \mathcal{S}_\phi \to \mathcal{S}_\phi$ is a contraction mapping with respect to the norm (5.3). $\square$

We are now ready to prove Theorem 5.1.4.

Proof. Let P be defined as in Lemma 5.1.12. By a contraction mapping principle, P has a fixed point $y \in \mathcal{S}_\phi$, which is a solution of (5.6) with initial function (5.2) and $\mathbb{E}|y(s)|^2 \to 0$ as $t \to \infty$.

To obtain mean square asymptotic stability, we need to show that the zero solution of (5.6) is mean square stable. Let $\varepsilon > 0$ be given and choose $\delta > 0$ ($\delta < \varepsilon$) satisfying

$$2\delta \left(1 + \int_{\tau(0)}^{0} |a(s) + h(s)|\, ds + |q(0)|\right)^2 e^{-2\int_0^{t^*} h(u)\,du} + 2\varepsilon\alpha < \varepsilon,$$

If $y(t, 0, \phi)$ is a solution of (5.6) with $\|\phi\|^2 < \delta$, then $y(t) = (Py)(t)$ as defined in (5.9). We claim that $\mathbb{E}|y(t)|^2 < \varepsilon$ for all $t \geq 0$. Notice that $\mathbb{E}|y(t)|^2 < \varepsilon$ on $t \in [\vartheta, 0]$. Suppose there exists $t^* > 0$ such that $\mathbb{E}|y(t^*)|^2 = \varepsilon$ and $\mathbb{E}|y(s)|^2 < \varepsilon$ for all $\vartheta \leq s < t^*$, it follows from (5.9),

$$\begin{aligned}
\mathbb{E}|y(t^*)|^2 \leq\ & 2\|\phi\|^2 \left(1 + \int_{\tau(0)}^{0} |a(s) + h(s)|\, ds + |q(0)|\right)^2 e^{-2\int_0^{t^*} h(u)\,du} \\
& +2\varepsilon \Bigg\{ 2 \Bigg[\sum_{l=1}^{n(t)} \Bigg| \frac{d_l}{1+d_l} \prod_{t_l - \tau(t_l) \leq t_k < t_l} (1+d_k)^{-1} e^{\int_{t^*}^{t_l} h(u)\,du} q(t_l) \Bigg| \\
& + \Bigg| \prod_{t^* - \tau(t^*) < t_k < t^*} (1+d_k)^{-1} q(s) \Bigg| + \int_{t^*-\tau(t^*)}^{t^*} |a(z) + h(z)|\, dz \\
& + \int_0^{t^*} e^{-\int_z^{t^*} |h(u)|\,du} h(z) \int_{z-\tau(z)}^{z} |a(u) + h(u)|\, du\, dz \\
& + \int_0^{t^*} e^{-\int_z^{t^*} h(u)\,du} \Bigg| (a(z-\tau(z)) + h(z-\tau(z)))(1-\tau'(z)) \\
& + \prod_{z-\tau(z) \leq s_k < z} (1+d_k)^{-1} b(z) - \prod_{z-\tau(z) \leq s_k < z} (1+d_k)^{-1} h(z) q(z) \Bigg| dz \Bigg]^2 \\
& +2 \int_0^{t^*} e^{-2\int_z^{t^*} h(u)\,du} \left(|c(z)| + \Bigg| \prod_{z-\delta(z) \leq s_k < z} (1+d_k)^{-1} e(z) \Bigg| \right)^2 dz \Bigg\} \\
\leq\ & 2\delta \left(1 + \int_{\tau(0)}^{0} |a(s) + h(s)|\, ds + |q(0)|\right)^2 e^{-2\int_0^{t^*} h(u)\,du} + 2\varepsilon\alpha < \varepsilon,
\end{aligned}$$

which contradicts the definition of t^*. Thus, the zero solution of (5.6) is mean square stable. It follows that the zero solution of (5.6) is mean square asymptotically stable if (iii) holds.

Combining Lemma 5.1.11 and Theorem 5.1.4, we obtain that the zero solution of (5.1) is mean square asymptotically stable. $\square$

Corollary 5.1.13. *Suppose that the conditions of* Theorem 5.1.4 *hold. Moreover,*

(i) *there exists a constant* $0 < \rho \leq 1$ *such that* $1 + d_k \leq \rho$,

(ii) *there exist constants* $\zeta > 0$ *such that* $t_k \leq \zeta k$.

If there exists a constant $M > 0$ *such that* $\left|\prod_{0<t_k<t}(1+d_k)\right| \leq M$, *then the zero solution of* (5.1) *is mean square exponentially stable.*

Proof. From (i) and (ii), we obtain that

$$\begin{aligned} \mathbb{E}|x(s)|^2 = \mathbb{E}\left|\prod_{0<t_k<s}(1+d_k)y(s)\right|^2 &\leq \rho^{2n}\mathbb{E}|y(s)|^2 = \frac{1}{\rho^2}e^{-2(n+1)|\ln\rho|}\mathbb{E}|y(s)|^2 \\ &\leq \frac{1}{\rho^2}e^{-\frac{t_{n+1}}{\zeta}|2\ln\rho|}\mathbb{E}|y(s)|^2 \leq \frac{1}{\rho^2}e^{-\frac{2|\ln\rho|}{\zeta}t}\mathbb{E}|y(s)|^2 \end{aligned}$$

the proof of Theorem 5.1.4 indicates that for any $\varepsilon > 0$ and $\sigma \geq 0$, there exists a $\delta = \delta(\varepsilon, \sigma) > 0$ such that $\|\phi\|^2 < \delta$ implies $\mathbb{E}|y(t)|^2 < \varepsilon$ for $t \geq 0$. Therefore, the zero solution of (5.1) is mean square exponentially stable. ☐

If $q(t) \equiv 0$ in (5.1), then we come to the following stochastic delay differential equation

$$\begin{cases} dx(t) = [a(t)x(t) + b(t)x(t-\tau(t))]dt + [c(t)x(t) + e(t)x(t-\delta(t))]dw(t), & t \neq t_k, \\ x(t_k^+) - x(t_k) = d_k x(t_k), \quad t = t_k, \end{cases} \tag{5.11}$$

Corollary 5.1.14. *Suppose that the following conditions are satisfied*

(i) *the delay* $\tau(t)$ *is differentiable and* $t - \tau(t) \to \infty$ *as* $t \to \infty$, $t - \delta(t) \to \infty$ *as* $t \to \infty$;

(ii) *there exist a constant* $\alpha \in (0,1)$ *and a continuous function* $h(t)$ *such that for* $t \geq 0$,

$$\begin{aligned} &2\Bigg[\int_{t-\tau(t)}^{t}|a(s)+h(s)|\,ds + \int_0^t e^{-\int_s^t h(u)\,du}|h(s)|\int_{s-\tau(s)}^{s}|a(u)+h(u)|\,du\,ds \\ &\quad + \int_0^t e^{-\int_s^t h(u)\,du}\Bigg|(a(s-\tau(s)) + h(s-\tau(s)))(1-\tau'(s)) \\ &\quad + \prod_{s-\tau(s)\leq t_k<s}(1+d_k)^{-1}b(s)\Bigg|\,ds\Bigg]^2 \\ &\quad + 2\int_0^t e^{-2\int_s^t h(u)\,du}\left(|c(s)| + \left|\prod_{s-\delta(s)\leq t_k<s}(1+d_k)^{-1}e(s)\right|\right)^2 ds \leq \alpha; \end{aligned}$$

(iii) *and such that* $\liminf_{t\to\infty}\int_0^t h(s)\,ds > -\infty$;

(iv) *there exists a constant* $M > 0$ *such that*

$$\left|\prod_{0<t_k<t}(1+d_k)\right| \leq M.$$

Then the zero solution of (5.11) *is mean square asmptotically stable if*

(v)

$$\int_0^t h(s)\,ds \to \infty \quad \text{as} \quad \mathrm{t} \to \infty.$$

5.1.3 Proof of Theorem 5.1.9

Lemma 5.1.15. *Let $\varphi \in \mathcal{C}_\phi$. Define an operator by $(P\varphi)(t) = \phi(t)$ for $t \in [-\tau, 0]$ and for $t \geq 0$,*

$$(P\varphi)(t) = \sum_{i=1}^{5} I_i(t),$$

where $I_i(t)$, $I = 1, 2, 3, 4, 5$, is denoted as in (5.10).

If conditions (i)-(iv) in Theorem 5.1.9 are satisfied, then there exists $\delta > 0$ such that for any $\phi : [-\tau, 0] \to (-\delta, \delta)$, we have that $P : \mathcal{C}_\phi \to \mathcal{C}_\phi$ and P is a contraction with respect to the norm (5.5).

Proof. First, following the proof of Theorem 5.1.4, we note that P is continuous on $[0, \infty)$.

Next, We prove that $P(\mathcal{C}_\phi) \subseteq \mathcal{C}_\phi$. For $\varphi \in \mathcal{C}_\phi$, it is easy to check that

$$\mathbb{E}\left(\sup_{t-\tau \leq s \leq t} |I_i(s)|^2\right) \to 0 \quad \text{as} \quad t \to \infty, \quad i = 1, 2, 3, 4.$$

Note that for every r,

$$\int_0^s e^{-\int_z^r h(u)\,du}\left[c(z)\varphi(z) + \prod_{z-\delta(z) \leq s_k < z} (1 + d_k)^{-1} e(z)\varphi(z - \delta(z))\right] dw(z),$$

is a martingale, then

$$\sup_{t-\tau \leq r \leq t}\left|\int_0^s e^{-\int_z^r h(u)\,du}\left[c(z)\varphi(z) + \prod_{z-\delta(z) \leq s_k < z} (1 + d_k)^{-1} e(z)\varphi(z - \delta(z))\right] dw(z)\right|$$

is a submartingle, by Doob's inequality, we have that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t-\tau \leq s \leq t} |I_5(s)|^2\right] \\
&= \mathbb{E}\Bigg\{\sup_{t-\tau \leq s \leq t}\Bigg|\int_0^s e^{-\int_z^s h(u)\,du} \\
&\quad \times \left[c(z)y(z) + \prod_{z-\delta(z) \leq s_k < z} (1 + d_k)^{-1} e(z)\varphi(z - \delta(z))\right] dw(z)\Bigg|^2\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{E}\Bigg\{\sup_{t-\tau\leq s\leq t}\sup_{t-\tau\leq r\leq t}\Bigg|\int_0^s e^{-\int_z^r h(u)\,du}\\
&\qquad\times\Bigg[c(z)\varphi(z)+\prod_{z-\delta(z)\leq s_k<z}(1+d_k)^{-1}e(z)\varphi(z-\delta(z))\Bigg]dw(z)\Bigg|^2\Bigg\}\\
&\leq 4\mathbb{E}\Bigg\{\sup_{t-\tau\leq r\leq t}\Bigg|\int_0^t e^{-\int_z^r h(u)\,du}\\
&\qquad\times\Bigg[c(z)\varphi(z)+\prod_{z-\delta(z)\leq s_k<z}(1+d_k)^{-1}e(z)\varphi(z-\delta(z))\Bigg]dw(z)\Bigg|^2\Bigg\}\\
&\leq 4\mathbb{E}\int_0^t\Bigg[\sup_{t-\tau\leq r\leq t}e^{-2\int_z^r h(u)\,du}\Bigg]\Bigg[c(z)y(z)+\prod_{z-\delta(z)\leq s_k<z}(1+d_k)^{-1}e(z)\varphi(z-\delta(z))\Bigg]^2dz\\
&\leq 4\int_0^t\Bigg[\sup_{t-\tau\leq r\leq t}e^{-2\int_z^r h(u)\,du}\Bigg]\Bigg[c(z)+\prod_{z-\delta(z)\leq s_k<z}(1+d_k)^{-1}e(z)\Bigg]^2dz\\
&\qquad\times\sup_{t\geq 0}\mathbb{E}\Bigg[\sup_{t-\tau\leq s\leq t}|\varphi(s)|^2\Bigg], \qquad (5.12)
\end{aligned}
$$

since $\mathbb{E}\left(\sup_{t-\tau\leq s\leq t}|\varphi(s)|^2\right)\to 0$ as $t\to\infty$, then for any $\varepsilon>0$, there exists $T>0$ such that $t>T$ implies $\mathbb{E}\sup_{t-\tau\leq s\leq t}|\varphi(s)|^2<\varepsilon$. Hence, for $t>T$, we have that

$$
\begin{aligned}
&4\int_0^t\Bigg[\sup_{t-\tau\leq r\leq t}e^{-2\int_z^r h(u)\,du}\Bigg]\Bigg[c(z)+\prod_{z-\delta(z)\leq s_k<z}(1+d_k)^{-1}e(z)\Bigg]^2dz\sup_{t\geq 0}\mathbb{E}\Bigg[\sup_{t-\tau\leq s\leq t}|\varphi(s)|^2\Bigg]\\
&\quad<\Bigg\{4\int_0^t\Bigg[\sup_{t-\tau\leq r\leq t}e^{-2\int_z^r h(u)\,du}\Bigg]\Bigg[c(z)+\prod_{z-\delta(z)\leq s_k<z}(1+d_k)^{-1}e(z)\Bigg]^2dz\Bigg\}\varepsilon,
\end{aligned}
$$

which yields $\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|I_5(s)|^2\right]\to 0 \quad \text{as} \quad \mathrm{t}\to\infty$. Hence, we obtain that $P(\mathcal{C}_\phi)\subseteq\mathcal{C}_\phi$.

Finally, We show that P is contraction mapping. For $\varphi,\psi\in\mathcal{C}_\phi$, we have that

$$
\begin{aligned}
&\sup_{t\geq 0}\Bigg\{\mathbb{E}\Bigg[\sup_{t-\tau\leq s\leq t}|(P\varphi)(s)-(P\psi)(s)|^2\Bigg]\Bigg\}\\
&=\sup_{t\geq 0}\mathbb{E}\Bigg\{\sup_{t-\tau\leq s\leq t}\Bigg|\sum_{l=1}^{n(t)}\frac{d_l}{1+d_l}\prod_{s_l-\tau(s_l)\leq s_k<s_l}(1+d_k)^{-1}e^{-\int_{s_l}^s h(u)\,du}q(s_l)\\
&\qquad\times(\varphi(s_l-\tau(s_l))-\psi(s_l-\tau(s_l)))\\
&\qquad+\prod_{s-\tau(s)\leq s_k<s}(1+d_k)^{-1}q(s)(\varphi(s-\tau(s))-\psi(s-\tau(s)))\\
&\qquad+\int_{s-\tau(s)}^s(a(z)+h(z))(\varphi(z)-\psi(z))\,dz\\
&\qquad-\int_0^s e^{-\int_z^s h(u)\,du}h(z)\int_{z-\tau(z)}^z(a(u)+h(u))(\varphi(u)-\psi(u))\,du\,dz
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^s e^{-\int_z^s h(u)\,du}\Bigg[(a(z-\tau(z))+h(z-\tau(z)))(1-\tau'(z))\\
&+\prod_{z-\tau(z)\le s_k<z}(1+d_k)^{-1}b(z)-\prod_{z-\tau(z)\le s_k<z}(1+d_k)^{-1}h(z)q(z)\Bigg]\\
&\qquad\times(\varphi(z-\tau(z))-\psi(z-\tau(z)))\,dz+\int_0^s e^{-\int_z^s h(u)\,du}\Bigg[c(z)(\varphi(z)-\psi(z))\\
&+\prod_{z-\delta(z)\le s_k<z}(1+d_k)^{-1}e(z)(\varphi(z-\delta(z))-\psi(z-\delta(z)))\Bigg]\,dw(z)\Bigg|^2\Bigg\}\\
\le \sup_{s\ge 0}&\Bigg\{2\Bigg[\sum_{l=1}^{n(t)}\Bigg|\frac{d_l}{1+d_l}\prod_{s_l-\tau(s_l)\le s_k<s_l}(1+d_k)^{-1}e^{-\int_{s_l}^s h(u)\,du}q(s_l)\Bigg|\\
&+\Bigg|\prod_{s-\tau(s)<s_k<s}(1+d_k)^{-1}q(s)\Bigg|+\int_{s-\tau(s)}^s|a(z)+h(z)|\,dz\\
&+\int_0^s e^{-\int_z^s h(u)\,du}|h(z)|\int_{z-\tau(z)}^z|a(u)+h(u)|\,du\,dz\\
&+\int_0^s e^{-\int_z^s h(u)\,du}\Bigg|(a(z-\tau(z))+h(z-\tau(z)))(1-\tau'(z))\\
&+\prod_{z-\tau(z)\le s_k<z}(1+d_k)^{-1}b(z)-\prod_{z-\tau(z)\le s_k<z}(1+d_k)^{-1}h(z)q(z)\Bigg|\,dz\Bigg]^2\\
&+8\int_0^s\Bigg[\sup_{s-\tau\le r\le s}e^{-2\int_z^r h(u)\,du}\Bigg]\Bigg[c(z)+\prod_{z-\delta(z)\le s_k<z}(1+d_k)^{-1}e(z)\Bigg]^2dz\Bigg\}\\
&\qquad\times\sup_{t\ge0}\mathbb{E}\Bigg[\sup_{t-\tau\le s\le t}|\varphi(s)-\psi(s)|^2\Bigg]\le\alpha\sup_{t\ge0}\mathbb{E}\Bigg[\sup_{t-\tau\le s\le t}|\varphi(s)-\psi(s)|^2\Bigg].
\end{aligned}
$$

Therefore, P is a contraction mapping with respect to the norm (5.5).

We are now ready to prove Theorem 5.1.9.

Proof. Let P be defined as in Lemma 5.1.15. By a contraction mapping principle, P has a fixed point $y \in S$, which is a solution of (5.6) with initial function (5.2) and $\mathbb{E}\sup_{t-\tau\le s\le t}|y(s)|^2 \to 0$ as $t \to \infty$.

To obtain mean square asymptotic stability, we need to show that the zero solution of (5.6) is mean square stable. Let $\varepsilon > 0$ be given and choose $\delta > 0$ ($\delta < \varepsilon$) satisfying

$$
2\delta\left(1+\int_{\tau(0)}^0|a(s)+h(s)|\,ds+|q(0)|\right)^2e^{-2\int_0^{t^*}h(u)\,du}+2\varepsilon\alpha<\varepsilon,
$$

If $y(t,0,\phi)$ is a solution of (5.6) with $\|\phi\|^2 < \delta$, then $y(t) = (Py)(t)$ as defined in (5.9). We claim that $\mathbb{E}|y(t)|^2 < \varepsilon$ for all $t \ge 0$. Notice that $\mathbb{E}|y(t)|^2 < \varepsilon$ on $t \in [-\tau, 0]$. Suppose there exists $t^* > 0$ such that $\mathbb{E}|y(t^*)|^2 = \varepsilon$ and $\mathbb{E}|y(s)|^2 < \varepsilon$ for all $-\tau \leq s < t^*$, it follows from (5.9),

$$\begin{aligned}
\mathbb{E}|y(t^*)|^2 \quad \leq \quad & 2\|\phi\|^2 \left(1 + \int_{\tau(0)}^{0} |a(s) + h(s)|\, ds + |q(0)|\right)^2 e^{-2\int_0^{t^*} h(u)\, du} \\
& +2\varepsilon \Bigg\{ 2 \Bigg[\sum_{l=1}^{n(t)} \Bigg| \frac{d_l}{1+d_l} \prod_{t_l - \tau(t_l) \leq t_k < t_l} (1+d_k)^{-1} e^{\int_{t^*}^{t_l} h(u)\, du} q(t_l) \Bigg| \\
& + \Bigg| \prod_{t^* - \tau(t^*) < t_k < t^*} (1+d_k)^{-1} q(s) \Bigg| + \int_{t^* - \tau(t^*)}^{t^*} |a(z) + h(z)|\, dz \\
& + \int_0^{t^*} e^{-\int_z^{t^*} |h(u)|\, du} h(z) \int_{z-\tau(z)}^{z} |a(u) + h(u)|\, du\, dz \\
& + \int_0^{t^*} e^{-\int_z^{t^*} h(u)\, du} \Bigg| (a(z-\tau(z)) + h(z-\tau(z)))(1-\tau'(z)) \\
& + \prod_{z-\tau(z) \leq s_k < z} (1+d_k)^{-1} b(z) - \prod_{z-\tau(z) \leq s_k < z} (1+d_k)^{-1} h(z) q(z) \Bigg| dz \Bigg]^2 \\
& +8 \int_0^{t^*} e^{-2\int_z^{t^*} h(u)\, du} \left(|c(z)| + \Bigg| \prod_{z-\delta(z) \leq s_k < z} (1+d_k)^{-1} e(z) \Bigg| \right)^2 dz \Bigg\} \\
\leq \quad & 2\delta \left[1 + \int_{\tau(0)}^{0} |a(s) + h(s)|\, ds + |q(0)| \right]^2 e^{-2\int_0^{t^*} h(u)\, du} + 2\varepsilon\alpha < \varepsilon,
\end{aligned}$$

which contradicts the definition of t^*. Thus, the zero solution of (5.6) is mean square stable. It follows that the zero solution of (5.6) is mean square asymptotically stable if (iii) holds.

Combining Lemma 5.1.11 and Theorem 5.1.9, we have that the zero solution of (5.1) is asymptotically stable. □

Remark 5.1.16. *In Theorem 5.1.9, we obtain* $\lim_{t\to\infty} \mathbb{E} \sup_{t-\tau \leq s \leq t} |y(s, 0, \phi)|^2 = 0$*, that is, for any function* $s \mapsto y_t(s, 0, \phi)$*, we have* $\lim_{t\to\infty} \mathbb{E}|y_t(\cdot, 0, \phi)|^2_{C[-\tau,0]} = 0$*, which implies* $\lim_{t\to\infty} \mathbb{E}|y(t, 0, \phi)|^2 = 0$.

Remark 5.1.17. *In some papers, see, for example, [89, 90, 131, 132], the norm is defined by*

$$\|\psi\|_{[0,t]} = \left\{ \mathbb{E} \left(\sup_{s\in[0,t]} |\psi(s,\omega)|^2 \right) \right\}^{1/2}.$$

As in [90], to show $P(\mathcal{S}) \subseteq \mathcal{S}$*, we need to estimate* $\mathbb{E} \sup_{s\in[0,t]} |I_5(s)|^2$*, where*

$$I_5(s) = \int_0^s e^{-\int_z^s h(u)\, du} \left[c(z)x(z) + e(z)x(z-\delta(z))\right] dw(z).$$

However, $I_5(s)$ *is not a local martingale, see Lemma 1.3.40 in Section 1.3 of Chapter* **??** *for the explanation. Hence, Burkholder-Davis-Gundy Inequality can not be applied directly.*

5.1.4 Examples

Example 5.1.18. *Consider the following linear stochastic delay differential equation with linear impulsive effects*

$$\begin{cases} dx(t) = [ax(t) + b(t)x(t-\tau(t))]dt + e(t)x(t-\tau(t))dw(t), \quad t \neq t_k \\ x(t_k{}^+) - x(t_k) = d_k x(t_k), \quad t = t_k, \end{cases} \tag{5.13}$$

where $t_k = 2k\pi$, $1 + d_k = \frac{1}{2}$ for all $k = 1, 2, \cdots$, $0 \leq \tau(t) < 2\pi$ is a continuous function such that $t - \tau(t) \to \infty$ as $t \to \infty$, $a < 0$ is a constant, $c(t), e(t)$ are bounded continuous function such that $|b(t)| \leq \bar{b}$, $|e(t)| \leq \bar{e}$, where $\bar{b}, \bar{e}$ are positive constants.

Since $t_k = 2k\pi$ and $0 \leq \tau(t) < 2\pi$, we have that at most one impulse occurs at interval $[t - \tau(t), t)$, and hence $\prod_{t-\tau(t)<t_k<t}(1 + d_k)^{-1} \leq 2$. Choosing $h(t) \equiv -a$ in Corollary 5.1.14,

$$\int_{t-\tau(t)}^{t} |a(s) + h(s)|\, ds + \int_0^t e^{-\int_s^t |h(u)|\, du} h(s) \int_{s-\tau(s)}^{s} |a(u) + h(u)|\, du\, ds = 0,$$

$$\begin{aligned} &2\left[\int_0^t e^{-\int_s^t h(u)\, du}\left|(a(s-\tau(s)) + h(s-\tau(s)))(1-\tau'(s)) + \prod_{s-\tau(s)\leq t_k<s}(1+d_k)^{-1}b(s)\right| ds\right]^2 \\ &\quad \leq 2\left[\int_0^t e^{\int_s^t a\, du}\left|\prod_{s-\tau(s)\leq t_k<s}(1+d_k)^{-1}\bar{b}\right| ds\right]^2 \leq \frac{8\bar{b}^2}{a^2}, \\ &2\int_0^t e^{-2\int_s^t h(u)\, du}\left|\prod_{s-\tau(s)\leq t_k<s}(1+d_k)^{-1}e(s)\right|^2 ds \\ &\quad \leq 2\int_0^t e^{2\int_s^t a\, du}\left|\prod_{s-\tau(s)\leq t_k<s}(1+d_k)^{-1}e(s)\right|^2 ds \leq \frac{4\bar{e}^2}{-a}. \end{aligned}$$

Hence, we obtain that $H_1(t) \leq \frac{8\bar{b}^2 - 4a\bar{e}^2}{a^2}$. On the other hand, $\left|\prod_{0\leq t_k<t}(1 + d_k)\right| \leq 1$. From Corollary 5.1.14, we obtan that the zero solution of (5.13) is mean square asymptotically stable if

$$\frac{8\bar{b}^2 - 4a\bar{e}^2}{a^2} < 1. \tag{5.14}$$

Now, we check the conditions in Theorem 5.1.9,

$$8\int_0^t \left[\sup_{t-\tau\leq r\leq t} e^{-2\int_s^r h(u)\, du}\right]\left[\left|\prod_{s-\delta(s)\leq t_k<s}(1+d_k)^{-1}e(s)\right|\right]^2 ds \leq \frac{16\bar{e}^2}{-a}e^{-4a\pi}.$$

Hence, we obtain that $H_2(t) \leq \frac{8\bar{b}^2 - 16a\bar{e}^2 e^{-4a\pi}}{a^2}$. From Theorem 5.1.9, the zero solution of (5.13) is mean square asymptotically stable if

$$\frac{8\bar{b}^2 - 16a\bar{e}^2 e^{-4a\pi}}{a^2} < 1. \tag{5.15}$$

It is not difficult to find that condition (5.14) is weaker than condition (5.15).

Example 5.1.19. *Consider the following linear neutral stochastic delay differential equations with impulsive effects*

$$\begin{cases} d\left[x(t) - qx\left(t - \frac{\sin t}{2}\right)\right] = -2x(t)dt + [cx(t) + ex(t-r)]\, dw(t), & t \neq t_k \\ x(t_k{}^+) - x(t_k) = d_k x(t_k), & t = t_k, \end{cases} \tag{5.16}$$

where $d_k = \frac{1}{2^k}$,$1+d_k = 1+\frac{1}{2^k}$ *for all* $k = 1, 2, \cdots$, $0 \le r < 1$ *is a constant,* $\inf_{k\in\mathbb{N}}\{t_{k+1} - t_k\} = 1$. *Suppose that* q, c, e *are positive constants.*

Since $\inf_{k\in\mathbb{N}}\{t_{k+1} - t_k\} = 1$ and $\tau(t) = \frac{\sin t}{2} \le \frac{1}{2}$, we have that at most one impulse occurs at interval $[t - \tau(t), t)$, and hence $\prod_{t-\tau(t)<t_k<t}(1+d_k)^{-1} \le \frac{2}{3}$. Choosing $h(t) \equiv 2$ in Theorem 5.1.4,

$$\sum_{l=1}^{n(t)} \left| \frac{d_l}{1+d_l} \prod_{t_l - \tau(t_l) \le t_k < t_l} (1+d_k)^{-1} e^{-\int_{t_l}^t h(u)\, du} q(t_l) \right| \le \frac{2q}{3} \sum_{l=1}^{n(t)} \frac{1}{1+2^l} < \frac{2qn}{9},$$

$$\left| \prod_{t-\tau(t)<t_k<t} (1+d_k)^{-1} q(t) \right| = \frac{2q}{3},$$

$$\int_{t-\tau(t)}^t |a(s) + h(s)|\, ds + \int_0^t e^{-\int_s^t |h(u)|\, du} h(s) \int_{s-\tau(s)}^s |a(u) + h(u)|\, du\, ds = 0,$$

$$\begin{aligned} &\int_0^t e^{-\int_s^t h(u)\, du} \Bigg| (a(s-\tau(s)) + h(s-\tau(s)))(1-\tau'(s)) + \prod_{s-\tau(s)\le t_k<s} (1+d_k)^{-1} b(s) \\ &\quad - \prod_{s-\tau(s)\le t_k<s} (1+d_k)^{-1} h(s) q(s) \Bigg|\, ds \le \int_0^t e^{-\int_s^t 2\, du} \frac{4q}{3}\, ds \le \frac{2q}{3}, \end{aligned}$$

$$\begin{aligned} 2\int_0^t e^{-2\int_s^t h(u)\, du} \left[|c(s)| + \left| \prod_{s-r\le t_k<s} (1+d_k)^{-1} e(s) \right| \right]^2 ds \quad &\le \quad 2\int_0^t e^{-2\int_s^t 2\, du} \left(c + \frac{2e}{3}\right)^2 ds \\ &\le \quad \frac{1}{2}\left(c + \frac{2e}{3}\right)^2. \end{aligned}$$

Hence, we obtain that $H_1(t) \le 2\left(\frac{2nq}{9} + \frac{4q}{3}\right)^2 + \frac{1}{2}\left(c + \frac{2e}{3}\right)^2$. Hence, from Theorem 5.1.4, we obtain that the zero solution of (5.13) is mean square asymptotically stable if

$$2\left(\frac{2nq}{9} + \frac{4q}{3}\right)^2 + \frac{1}{2}\left(c + \frac{2e}{3}\right)^2 < 1. \tag{5.17}$$

Now, we check the conditions in Theorem 5.1.9,

$$\begin{aligned} &8\int_0^t \left[\sup_{t-\tau\le r\le t} e^{-2\int_s^r h(u)\, du} \right] \left[|c(s)| + \left| \prod_{s-\delta(s)\le t_k<s} (1+d_k)^{-1} e(s) \right| \right]^2 ds \\ &\le 2e^2 \left(c + \frac{2d}{3}\right)^2 (1 - e^{-4t}) < 2e^2 \left(c + \frac{2e}{3}\right)^2. \end{aligned}$$

Hence, we obtain that $H_2(t) < 2\left(\frac{2nq}{9}+\frac{4q}{3}\right)^2 + 2e^2\left(c+\frac{2e}{3}\right)^2$.

On the other hand, $\left|\prod_{0\le t_k<t}(1+d_k)\right| \le 2$. From Theorem 5.1.9, the zero solution of (5.13) is mean square asymptotically stable if

$$2\left(\frac{2nq}{9}+\frac{4q}{3}\right)^2 + 2e^2\left(c+\frac{2d}{3}\right)^2 < 1. \tag{5.18}$$

It is not difficult to find that condition (5.17) is weaker than condition (5.18).

Example 5.1.20. *Consider the following linear neutral stochastic delay differential equations with impulsive effects*

$$\begin{cases} dx(t) = -a(t)x(t)dt + bx(t-r)dw(t), & t \ne t_k \\ x(t_k{}^+) - x(t_k) = d_k x(t_k), & t = t_k, \end{cases} \tag{5.19}$$

where $0 \le r < 2\pi$, $t_k = 2k\pi$, $1+d_k = c$ *for all* $k = 1, 2, \cdots$ *for some* $0 < c \le 1$, $a(t)$ *is a continuous function such that* $\int_0^t a(s)\,ds = \infty$ *and*

$$\sup_{t\ge 0}\left\{\frac{b^2}{c^2}\int_0^t e^{-2\int_s^t a(u)\,du}\,ds\right\} < 1,$$

then the zero solution of (5.19) *is mean square asymptotically stable.*

Since $t_k = 2k\pi$ and $0 \le r < 2\pi$, we have that at most one impulse occurs at interval $[t-r, t)$, and hence $\prod_{t-r<t_k<t}(1+d_k)^{-1} \le \frac{1}{c}$. If we choose $h(t) \equiv a(t)$ in Theorem 5.1.4, then by Theorem 5.1.4, we have that the zero solution of (5.19) is mean square asymptotically stable. Since $1+d_k = c \le 1$ and $t_k = 2k\pi$, from Theorem 5.1.13, we have that the zero solution of (5.19) is also mean square exponentially stable.

If $c = 1$ for all $k = 1, 2, \cdots$, the condition reduces to the condition (3.6) in [90] for equation (5.19) without impulses.

5.2 Exponential stability of a class of impulsive neutral stochastic partial differential equations with variable delays and Poisson jumps

5.2.1 Introduction and preliminaries

The classical technique applied in the study of stability of stochastic delay differential equations is based on a stochastic version of Liapunov's direct method. The success of Liapunov's direct method depending on finding good Liapunov functionals, which may be difficult, especially for equations with unbounded delays or unbounded terms. Recently, Chen [21] used an appropriate impulsive-integral inequality to establish sufficient conditions for exponential stability of impulsive stochastic partial delay differential equations, and it turns out that it is a convenient way to study exponential stability of mild solutions of impulsive stochastic delay differential equations. Sakthivel and Luo [117, 118] have discussed the asymptotic stability for mild solutions of impulsive stochastic partial delay differential equations by using fixed point methods. This powerful method is also an effective tool to deal with exponential stability for mild solutions to stochastic partial differential equations with delays, see, for example, [91] and Cui et al. [27]. However, to the best of our knowledge, there is no result about exponential stability of mild solutions of impulsive stochastic partial differential equations with variable delays and Poisson jumps.

The aim of this section is to study the impulsive effects to a class of stochastic partial differential equations with variable delays and Poisson jumps by using two methods, the method by using an appropriate impulsive-integral inequality and fixed point methods.

Let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all P-null sets). Let X, Y be two real separable Hilbert spaces which are both equipped with a norm denoted by $\|\cdot\|$. Let $\mathcal{L}(Y, X)$ denote the space of all bounded linear operators from Y into X.

Suppose $\{p(t), t \geq 0\}$ is a σ-finite stationary $\mathcal{F}_t$-adapted Possion point process taking values in measurable space $(U, \mathcal{B}(U))$. The random measure N_p defined by $N_p((0,t] \times \Lambda) := \sum_{s\in(0,t]} 1_\Lambda(p(s))$ for $\Lambda \in \mathcal{B}(U)$ is called the Poisson random measure induced by $p(\cdot)$, thus, we can define the measure $\tilde{N}$ by $\tilde{N}(dt, dy) = N_p(dt, dy) - \nu(dy)dt$, where ν is the characteristic measure of N_p, which is called the compensated Poisson random measure. Let $w = (w(t))_{t\geq 0}$, independent of the Poisson point process, be a Y-valued Wiener process defined on $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ with covariance operator Q, that is

$$E\langle w(t), x\rangle_Y \langle w(s), y\rangle_Y = (t \wedge s)\langle Qx, y\rangle_Y, \quad x, y \in Y,$$

where Q is a positive, self-adjoint, trace class operator on Y. Furthermore, $\mathcal{L}_2^0(Y, X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y to X with the norm

$$\|\xi\|^2_{\mathcal{L}_2^0} := tr(\xi Q \xi^*) < \infty, \quad \xi \in \mathcal{L}_2^0(Y, X).$$

For the construction of a stochastic integral in a Hilbert space, see Da Prato and Zabczyk [108].

For Borel set $Z \in \mathcal{B}(U \setminus \{0\})$, we consider the following impulsive neutral stochastic delay differential equation with Poisson jumps

$$\begin{cases} d[x(t) + u(t, x(t-\tau(t)))] = [Ax(t)dt + f(t, x(t-\delta(t)))]dt + g(t, x(t-\rho(t))dw(t) \\ \qquad\qquad\qquad\qquad + \int_Z h(t, x(t-\sigma(t)), y)\, \tilde{N}(dt, dy), \quad t \geq 0, \quad t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad t = t_k, \quad k = 1, 2, \cdots, \\ x_0(\theta) = \phi, \quad \theta \in [-\tau, 0], \quad a.s. \end{cases} \tag{5.20}$$

where $\phi \in PC$ and the functions $\tau(t), \delta(t), \rho(t), \sigma(t) : [0, \infty) \to [0, \tau]$ ($\tau > 0$ is a constant) are continuous functions, where $PC \equiv PC([-\tau, 0]; X)$ is the space of all almost surely bounded $\mathcal{F}_0$-measurable functions from $[-\tau, 0]$ into X that are continuous everywhere except for a finite number of points s at which the left and right limits $\phi(s-)$ and $\phi(s+)$ exist and $\phi(s^+) = \phi(s)$ as usual, equipped with the supremum norm $\|\phi\|_0 = \text{essup}_{\omega\in\Omega} \sup_{t\in[-\tau,0]} \|\phi(t)(\omega)\|$; $-A$ is a closed, densely defined linear operator generating an analytic semigroup $S(t)(t \geq 0)$ on the

Hilbert space X; then it is possible under some circumstances (we refer the readers to [100] for a detailed presentations of the definition and relevant properties of $(-A)^{\alpha}$) to define the fractional power $(-A)^{\alpha} : D((-A)^{\alpha}) \to X$ which is a closed linear operator with its domain $D((-A)^{\alpha})$, for $\alpha \in (0, 1]$. Moreover, the fixed moments of time t_k satisfy $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$; $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at time $t = t_k$, $k = 1, 2, \cdots$, respectively. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ denotes the jump in the state x at time t_k with $I_k(\cdot) : X \to X (k = 1, 2, \cdots)$ determining the size of the jump; $u, f : [0, \infty) \times X \to X$, $g : [0, \infty) \times X \to \mathcal{L}_2^0(Y, X)$, $h : [0, \infty) \times X \times U \to X$ are given functions to be specified later.

Definition 5.2.1. (Chen [21]) *Let $\sigma \in L(Y, X)$, and define*

$$\|\sigma\|^2_{\mathcal{L}_2^0} := tr(\sigma Q \sigma^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\sigma e_n\|^2$$

If $\|\sigma\|^2_{\mathcal{L}_2^0} < +\infty$, then σ is called a Q-Hilbert-Schmidt operator and $\mathcal{L}_2^0(Y, X)$ denotes the space of all Q-Hilbert-Schmidt operators $\sigma : Y \to X$.

Note that equation (5.20) could be the infinite dimensional formulation of a partial differential equation with delays, (see, e.g. Da Prato and Zabczyk [108]).

Definition 5.2.2. *An X-valued stochastic process $x(t)$, $t \in [0, +\infty)$, is called a mild solution of* (5.20) *if*

(i) *$x(t)$ is adapted to $\mathcal{F}_t$, $t \geq 0$;*

(ii) *$t \mapsto x(t)$ has* càdlàg *paths on $[0, +\infty)$ almost surely, and for $t \in [0, +\infty)$, $x(t)$ satisfies the following integral equation*

$$\begin{aligned} x(t) \quad = \quad & S(t)(\phi(0) + u(0, \phi)) - u(t, x(t - \tau(t))) \\ & - \int_0^t AS(t-s)u(s, x(s - \tau(s)))\, ds + \int_0^t S(t-s)f(s, x(s - \delta(s)))\, ds \\ & + \int_0^t \int_Z S(t-s)h(s, x(s - \theta(s)), y)\, \tilde{N}(\, ds,\, dy) \\ & + \int_0^t S(t-s)g(s, x(s - \rho(s)))\, dw(s) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)), \end{aligned} \tag{5.21}$$

where $x_0(\cdot) = \phi$, a.s.

Definition 5.2.3. *Let $p > 0$. Equation* (5.20) *is said to be exponentially stable in pth moment, if for any initial value $\phi \in PC$, there exists a pair of positive constants γ and C such that*

$$\mathbb{E}\|x(t)\|^p \leq C\|\phi\|_0^p e^{-\gamma t}, \quad t \geq 0,$$

where $\mathbb{E}$ denotes expectation with respect to the probability measure $\mathbb{P}$ and x is the mild solution of (5.20).

To obtain our main results, we impose the following assumptions:

(A1) A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in X such that $0 \in \rho(-A)$, the resolvent set of $-A$, and $S(t)$ is uniformly bounded,

$$\|S(t)\| \leq Me^{-\gamma t}, \quad t \geq 0,$$

for some constants $\gamma, M > 0$;

(A2) the mappings $f(t, \cdot), \sigma(t, \cdot)$ and $h(t, \cdot)$ satisfy Lipschitz conditions, that is, there exist $L_1, L_2, L_3 > 0$ such that for any $x, y \in H$ and $t \geq 0$,

$$\begin{aligned} \|f(t,x) - f(t,y)\| &\leq L_1\|x-y\|, \quad L_1 > 0, \\ \|g(t,x) - g(t,y)\| &\leq L_2\|x-y\|, \quad L_2 > 0, \\ \int_z \|h(t,x,z) - h(t,y,z)\|^2 \nu(dz) &\leq L_3^2\|x-y\|^2, \quad L_3 > 0, \end{aligned}$$

we further assume that $f(t, 0) = g(t, 0) = h(t, 0, z) = 0$ for all $t \geq 0$, $z \in Z$. Hence, (5.20) has a trivial solution $x = 0$ when $\phi = 0$.

(A3) The mapping $(-A)^\alpha u(t, \cdot)$ satisfies a uniformly Lipschitz condition: there exists a positive constant $K > 0$ such that for any $x, y \in X$,

$$\|(-A)^\alpha u(t,x) - (-A)^\alpha u(t,y)\| \leq K\|x-y\|, \quad u(t,0) = 0, \quad t \geq 0,$$

for $\alpha \in (1/p, 1]$ (for some $p \geq 2$) and $u(t, \cdot) \in D((-A)^\alpha)$. Moreover, for $\alpha \in (1/p, 1]$, $\kappa = \|(-A)^{-\alpha}\|K < 1$.

(A4) $I_k \in C(X, X)$ and there exists a positive constant q_k such that $\|I_k(x) - I_k(y)\| \leq q_k\|x-y\|$ and $I_k(0) = 0, k = 1, 2, 3, \cdots$, for each $x, y \in X$.

Remark 5.2.4. *Under the conditions* (A1)-(A4) *and suppose that the mappings $f(t, \cdot), \sigma(t, \cdot)$ and $h(t, \cdot)$ satisfy linear growth conditions, the existence and uniqueness of the system* (5.20) *can be shown by using Picard iterative method (see, Anguraj and Vinodkumar* [2]*).*

Lemma 5.2.5. (Theorem 6.13, [100]) *Suppose that the assumption* (A1) *holds, then for any $\beta \in (0, 1]$, we have that*

(i) *for each $x \in \mathcal{D}((-A)^\beta)$,*

$$S(t)(-A)^\beta x = (-A)^\beta S(t)x;$$

(ii) *there exist positive constant $M_\beta > 0$ such that*

$$\left\|(-A)^\beta S(t)\right\| \leq M_\beta t^{-\beta} e^{-\gamma t}, \quad t > 0.$$

Lemma 5.2.6. (Da Prato and Zabczyk [108]) *For any $p \geq 2$ and for an arbitrary $\mathcal{L}_2^0$-valued adapted caglad process $\Phi(\cdot)$,*

$$\sup_{s\in[0,t]} \mathbb{E}\left\|\int_0^s \Phi(u)dw(u)\right\|^p \leq c_p \left(\int_0^t \left(\mathbb{E}\|\Phi(s)\|_{\mathcal{L}_2^0}^p\right)^{2/p} ds\right)^{p/2}.$$

where $c_p = (p(p-1)/2)^{p/2}$.

Lemma 5.2.7. (Mao [96]) *Let $p \in [1, \infty)$ and $\nu \in (0, 1)$. For any two real positive numbers $a, b > 0$,*

$$(a+b)^p \leq \nu^{1-p}a^p + (1-\nu)^{1-p}b^p.$$

5.2.2 Exponential stability by an impulsive-integral inequality

In this subsection, we study exponential stability in *pth* moment of mild solution of the system (5.20) by using an impulsive-integral inequality.

Theorem 5.2.8. *Consider the neutral stochastic partial differential equations* (5.20)*, let* $p \geq 2$ *and suppose that the conditions* (A1)-(A4) *are satisfied. Then* (5.20) *is exponentially stable in pth moment, if the inequality*

$$\begin{aligned} & 9^{p-1}(1-\kappa)^{-p}\left[M_{1-\alpha}^p K^p \gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} + M^p L_3^p \gamma^{1-p}\right] \\ & +3^{p-1}(1-\kappa)^{-p}\left[M^p L_1^p \gamma^{1-p} + c_p M^p L_2^p \left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\right] \\ & +9^{p-1}\gamma(1-\kappa)^{-p} M^p \left(\sum_{t_k<t} q_k\right)^p < \gamma \end{aligned} \tag{5.22}$$

and $\sum_{t_k<t} q_k < +\infty$ *hold, where* $c_p = (p(p-1)/2)^{p/2}$*,* $\kappa = \|(-A)^{-\alpha}\|K < 1$.

We start with an impulsive-integral inequality lemma, which is essential to the proof of Theorem 5.2.8.

Lemma 5.2.9. *If* $\gamma > 0$*, and* $\lambda_0, \lambda, \lambda^*, \lambda_k (k = 1, 2, \cdots)$ *are positive constants such that* $\frac{\lambda^*}{\gamma} + \lambda + \sum_{k=1}^{\infty} \lambda_k < 1$*, and a function* $y : [-\tau, \infty) \to [0, \infty)$ *such that the inequality*

$$y(t) \leq \begin{cases} \lambda_0 e^{-p\gamma t} + \lambda^* \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} y(s+\theta)\, ds \\ \quad +\lambda \sup_{\theta\in[-\tau,0]} y(t+\theta) + \sum_{t_k<t} \lambda_k e^{-p\gamma(t-t_k)} y(t_k^-), \quad t \geq 0, \\ \lambda_0 e^{-p\gamma t}, \quad t \in [-\tau, 0], \end{cases} \tag{5.23}$$

holds, then we have that $y(t) \leq M_2 e^{-p\mu t} (t \geq -\tau)$*, where* μ *is a positive root of the algebraic equation* $\left(\lambda + \frac{\lambda^*}{\gamma - p\mu}\right) e^{p\mu\tau} + \sum_{k=1}^{\infty} \lambda_k = 1$ *and* $M_2 = \max\left\{\frac{\lambda_0(\gamma-p\mu)}{\lambda^* e^{p\mu\tau}}, \lambda_0\right\} > 0$.

Proof. Let $F(\mu) = \left(\lambda + \frac{\lambda^*}{\gamma - p\mu}\right) e^{p\mu\tau} + \sum_{k=1}^{\infty} \lambda_k - 1$. We have $F(0)F(\gamma^-/p) < 0$, hence, there exists constant $\mu \in (0, \gamma)$ such that $F(\mu) = 0$. For any $\varepsilon > 0$, let

$$C_\varepsilon = \max\left\{\frac{(\varepsilon + \lambda_0)(\gamma - p\mu)}{\lambda^* e^{p\mu\tau}}, \varepsilon + \lambda_0\right\}.$$

We claim that (5.23) implies

$$y(t) \leq C_\varepsilon e^{-p\mu t}, \quad t \geq -\tau. \tag{5.24}$$

It is easily shown that (5.24) holds for $t \in [-\tau, 0]$. Assume that there exists $t_1^* > 0$ such that

$$y(t) < C_\varepsilon e^{-p\mu t}, \quad t \in [-\tau, t_1^*), \qquad y(t_1^*) = C_\varepsilon e^{-p\mu t_1^*}. \tag{5.25}$$

Combining with (5.23), we have

$$\begin{aligned}
y(t_1^*) &\leq \lambda_0 e^{-p\gamma t_1^*} + \lambda C_\varepsilon \sup_{\theta\in[-\tau,0]} e^{-p\mu(t_1^*+\theta)} \\
&\quad +\lambda^* C_\varepsilon \int_0^{t_1^*} e^{-\gamma(t_1^*-s)} \sup_{\theta\in[-\tau,0]} e^{-p\mu(s+\theta)}\, ds + C_\varepsilon \sum_{t_k<t_1^*} \lambda_k e^{-p\gamma(t_1^*-t_k)} e^{-p\mu t_k} \\
&\leq \lambda_0 e^{-p\gamma t_1^*} + \lambda C_\varepsilon e^{-p\mu t_1^*} e^{p\mu\tau} - \frac{\lambda^* C_\varepsilon e^{p\mu\tau}}{\gamma - p\mu} e^{-\gamma t_1^*} + \frac{\lambda^* e^{p\mu\tau}}{\gamma - p\mu} C_\varepsilon e^{-p\mu t_1^*} + \left(\sum_{k=1}^{\infty} \lambda_k\right) C_\varepsilon e^{-p\mu t_1^*} \\
&= \lambda_0 e^{-p\gamma t_1^*} - \frac{\lambda^* C_\varepsilon e^{p\mu\tau}}{\gamma - p\mu} e^{-\gamma t_1^*} + \left[\left(\lambda + \frac{\lambda^*}{\gamma - p\mu}\right) e^{p\mu\tau} + \sum_{k=1}^{\infty} \lambda_k\right] C_\varepsilon e^{-p\mu t_1^*}.
\end{aligned}$$

From the definition of C_ε, we have

$$\lambda_0 e^{-p\gamma t_1^*} - \frac{\lambda^* C_\varepsilon e^{p\mu\tau}}{\gamma - p\mu} e^{-\gamma t_1^*} \leq \lambda_0 e^{-p\gamma t_1^*} - \frac{\lambda^* e^{p\mu\tau}}{\gamma - p\mu} e^{-\gamma t_1^*} \cdot \frac{(\varepsilon + \lambda_0)(\gamma - p\mu)}{\lambda^* e^{p\mu\tau}} < 0.$$

Then, together with the definition of μ, we obtain that $y(t_1^*) < C_\varepsilon e^{-p\mu t_1^*}$, which contradicts (5.25), so (5.24) holds. As $\varepsilon > 0$ is arbitrarily small, in view of (5.24), it follows that $y(t) \leq M_2 e^{-p\mu t}$, for $t \geq -\tau$, where $M_2 = \max\left\{\frac{\lambda_0(\gamma-p\mu)}{\lambda^* e^{p\mu\tau}}, \lambda_0\right\}$. □

We are now ready to prove Theorem 5.2.8.

Proof. Based on an elementary inequality and Lemma 5.2.7, for any real numbers a, b, c, d, e and f, we have

$$\begin{aligned}
&(a+b+c+d+e+f)^p \\
&\quad \leq 3^{p-1}(a+b+c+d)^p + 3^{p-1}e^p + 3^{p-1}f^p \\
&\quad \leq 3^{p-1}\left[\left(1+\frac{1}{\varepsilon}\right)^{p-1} a^p + (1+\varepsilon)^{p-1}(b+c+d)^p\right] + 3^{p-1}e^p + 3^{p-1}f^p \\
&\quad \leq 3^{p-1}\left(1+\frac{1}{\varepsilon}\right)^{p-1} a^p + 9^{p-1}(1+\varepsilon)^{p-1}(b^p+c^p+d^p) + 3^{p-1}e^p + 3^{p-1}f^p. \quad (5.26)
\end{aligned}$$

From Lemma 5.2.7 and (5.21), we obtain

$$\begin{aligned}
&\mathbb{E}\|x(t)\|^p \\
&\quad \leq \kappa^{1-p}\mathbb{E}\left\|u(t, x(t-\tau(t)))\right\|^p \\
&\qquad +(1-\kappa)^{1-p}\mathbb{E}\Bigg\| S(t)(\phi(0)+u(0,\phi)) - \int_0^t AS(t-s)u(s, x(s-\tau(s)))\, ds \\
&\qquad + \int_0^t S(t-s)f(s, x(s-\delta(s)))\, ds + \int_0^t S(t-s)g(s, x(s-\rho(s)))\, dw(s) \\
&\qquad + \int_0^t \int_Z S(t-s)h(s, x(s-\sigma(s)), y)\, \tilde{N}(ds, dy) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))\Bigg\|^p
\end{aligned}$$

$$
\begin{aligned}
&\leq \kappa^{1-p}\mathbb{E}\left\|u(t,x(t-\tau(t)))\right\|^p + 3^{p-1}(1-\kappa)^{1-p}\left(1+\frac{1}{\varepsilon}\right)^{p-1}\mathbb{E}\left\|S(t)(\phi(0)+u(0,\phi))\right\|^p \\
&\quad +9^{p-1}(1-\kappa)^{1-p}(1+\varepsilon)^{p-1}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,x(s-\tau(s)))\,ds\right\|^p \\
&\quad +9^{p-1}(1-\kappa)^{1-p}(1+\varepsilon)^{p-1}\mathbb{E}\left\|\int_0^t\int_Z S(t-s)h(s,x(s-\sigma(s)),y)\,\tilde{N}(ds,dy)\right\|^p \\
&\quad +9^{p-1}(1-\kappa)^{1-p}(1+\varepsilon)^{p-1}\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))\right\|^p \\
&\quad +3^{p-1}(1-\kappa)^{1-p}\mathbb{E}\left\|\int_0^t S(t-s)f(s,x(s-\delta(s)))\,ds\right\|^p \\
&\quad +3^{p-1}(1-\kappa)^{1-p}\mathbb{E}\left\|\int_0^t S(t-s)g(s,x(s-\rho(s)))\,dw(s)\right\|^p. \qquad (5.27)
\end{aligned}
$$

Now, we estimate the right-hand side of (5.27). By assumption (A3), we obtain

$$
\begin{aligned}
\mathbb{E}\left\|u(t,x(t-\tau(t)))\right\|^p &\leq \|(-A)^{-\alpha}\|^p\mathbb{E}\|(-A)^{\alpha}u(t,x(t-\tau(t)))\|^p \\
&\leq K^p\|(-A)^{-\alpha}\|^p E\|x(t-\tau(t))\|^p. \qquad (5.28)
\end{aligned}
$$

Applying Lemma 5.2.5, Hölder inequality and assumption (A3), we obtain

$$
\begin{aligned}
&\mathbb{E}\left\|\int_0^t AS(t-s)u(s,x(s-\tau(s)))\,ds\right\|^p \qquad (5.29) \\
&\leq M_{1-\alpha}^p K^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\int_0^t e^{-\gamma(t-s)}E\|x(s-\tau(s))\|^p\,ds.
\end{aligned}
$$

Similarly, we obtain that

$$
\begin{aligned}
&\mathbb{E}\left\|\int_0^t S(t-s)f(s,x(s-\delta(s)))\,ds\right\|^p \\
&\quad\leq \mathbb{E}\left[\int_0^t \|S(t-s)f(s,x(s-\delta(s)))\|\,ds\right]^p \\
&\quad\leq \mathbb{E}\left[\int_0^t Me^{-\gamma(t-s)}\|f(s,x(s-\delta(s)))\|\,ds\right]^p \\
&\quad\leq M^pL_1^p\mathbb{E}\left[\int_0^t e^{-\gamma(t-s)}\|x(s-\delta(s))\|\,ds\right]^p \\
&\quad= M^pL_1^p\mathbb{E}\left[\int_0^t e^{-(\gamma(p-1)/p)(t-s)}e^{-(\gamma/p)(t-s)}\|x(s-\delta(s))\|\,ds\right]^p \\
&\quad\leq M^pL_1^p\left[\int_0^t e^{-\gamma(t-s)}\,ds\right]^{p-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\delta(s))\|^p\,ds \\
&\quad\leq M^pL_1^p\gamma^{1-p}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\delta(s))\|^p\,ds; \qquad (5.30)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left\|\int_0^t S(t-s)g(s,x(s-\rho(s)))\,dw(s)\right\|^p \\
&\quad\le c_pM^p\left[\int_0^t \left(e^{-\gamma p(t-s)}\mathbb{E}\|g(s,x(s-\rho(s)))\|^p\right)^{2/p}\,ds\right]^{p/2} \\
&\quad\le c_pM^pL_2^p\left[\int_0^t \left(e^{-\gamma p(t-s)}\mathbb{E}\|x(s-\rho(s))\|^p\right)^{2/p}\,ds\right]^{p/2} \\
&\quad= c_pM^pL_2^p\left[\int_0^t \left(e^{-\gamma(p-1)(t-s)}e^{-\gamma(t-s)}\mathbb{E}\|x(s-\rho(s))\|^p\right)^{2/p}\,ds\right]^{p/2} \\
&\quad\le c_pM^pL_2^p\left[\int_0^t e^{-\left(\frac{2(p-1)}{p-2}\right)\gamma(t-s)}\,ds\right]^{p-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\rho(s))\|^p\,ds \\
&\quad\le c_pM^pL_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\rho(s))\|^p\,ds
\end{aligned}
\tag{5.31}
$$

$$
\begin{aligned}
&\mathbb{E}\left\|\int_0^t\int_Z S(t-s)h(s,x(s-\sigma(s)),z)\,\tilde{N}(ds,dz)\right\|^p \\
&\quad\le c_p\mathbb{E}\left[\int_0^t\int_Z \|S(t-s)h(s,x(s-\sigma(s)),z)\|^2\,ds\nu(dz)\right]^{p/2} \\
&\quad\le c_pM^p\mathbb{E}\left[\int_0^t\int_Z e^{-2\gamma(t-s)}\|h(s,x(s-\sigma(s)),z)\|^2\,ds\nu(dz)\right]^{p/2} \\
&\quad\le c_pM^p\mathbb{E}\left[\int_0^t e^{-2\gamma(t-s)}\int_Z \|h(s,x(s-\sigma(s)),z)\|^2\nu(dz)\,ds\right]^{p/2} \\
&\quad\le c_pM^pL_3^p\left[\int_0^t e^{-2\gamma(t-s)}\mathbb{E}\|x(s-\sigma(s))\|^2\,ds\right]^{p/2} \\
&\quad= c_pM^pL_3^p\left[\int_0^t e^{-\frac{2(p-1)}{p}\gamma(t-s)}e^{-\frac{2}{p}\gamma(t-s)}\mathbb{E}\|x(s-\sigma(s))\|^2\,ds\right]^{p/2} \\
&\le c_pM^pL_3^p\left(\int_0^t e^{-\frac{2(p-1)}{p}\cdot\frac{p}{p-2}\gamma(t-s)}\,ds\right)^{(p-2)/2} e^{-\gamma(t-s)}\mathbb{E}\|x(s-\sigma(s))\|^p\,ds \\
&\le c_pM^pL_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\sigma(s))\|^p\,ds.
\end{aligned}
\tag{5.32}
$$

Furthermore, we obtain that

$$
\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))\right\|^p \le M^p\left(\sum_{t_k<t} q_k\right)^{p-1}\sum_{t_k<t} q_k e^{-\gamma p(t-t_k)}\mathbb{E}\|x(t_k^-)\|^p. \tag{5.33}
$$

Hence, from (5.27)-(5.33), we obtain

$$
\begin{aligned}
&\mathbb{E}\|x(t)\|^p \\
&\quad \le \kappa \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(t+\theta)\|^p + 3^{p-1}(1-\kappa)^{1-p}\left(1+\frac{1}{\varepsilon}\right)^{p-1} \\
&\qquad\quad \times M^p e^{-\gamma p t}(1+K\|(-A)^{-\alpha}\|)^p \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(\theta)\|^p \\
&\qquad +9^{p-1}(1-\kappa)^{1-p}(1+\varepsilon)^{p-1}M_{1-\alpha}^p K^p \gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} \\
&\qquad\quad \times \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(s+\theta(s))\|^p\,ds \\
&\qquad +9^{p-1}(1-\kappa)^{1-p}(1+\varepsilon)^{p-1}c_p M^p L_3^p \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2} \\
&\qquad\quad \times \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(s+\theta(s))\|^p\,ds \\
&\qquad +9^{p-1}(1-\kappa)^{1-p}(1+\varepsilon)^{p-1}M^p\left(\sum_{t_k<t} q_k\right)^{p-1} \sum_{t_k<t} q_k e^{-\gamma p(t-t_k)}\mathbb{E}\|x(t_k^-)\|^p \\
&\qquad +3^{p-1}(1-\kappa)^{1-p}M^p L_1^p \gamma^{1-p}\int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(s+\theta(s))\|^p\,ds \qquad (5.34)\\
&\qquad +3^{p-1}(1-\kappa)^{1-p}c_p M^p L_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(s+\theta(s))\|^p\,ds.
\end{aligned}
$$

From inequality (5.22), (5.34) and the condition (A3), we can always find a number $\varepsilon>0$ small enough such that

$$
\begin{aligned}
&\kappa + 9^{p-1}\gamma^{-1}(1-\kappa)^{1-p}(1+\varepsilon)^{p-1} \\
&\quad \times\left[M_{1-\alpha}^p K^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} + c_p M^p L_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\right] \\
&\quad +3^{p-1}\gamma^{-1}(1-\kappa)^{1-p}\left[M^p L_1^p\gamma^{1-p} + c_p M^p L_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\right] \\
&\quad +9^{p-1}(1-\kappa)^{1-p}(1+\varepsilon)^{p-1}M^p\left(\sum_{t_k<t} q_k\right)^p < 1.
\end{aligned}
$$

Hence, from Lemma 5.2.9, we obtain that $E\|x(t)\|^p \le \tilde{M}e^{-p\mu t}(\tilde{M}>0, \mu\in(0,\gamma))$. Therefore, the mild solution of the system (5.20) is exponentially stable in pth moment. □

If $h\equiv 0$ or $\nu\equiv 0$, system (5.20) becomes

$$
\begin{cases}
d[x(t)+u(t,x(t-\tau(t)))] = [Ax(t)dt + f(t,x(t-\delta(t)))]dt \\
\qquad\qquad\qquad\qquad\qquad +g(t,x(t-\rho(t))dw(t), \quad t\ge 0, \quad t\ne t_k, \\
\Delta x(t_k) = I_k(x(t_k^-)), \quad t=t_k, \quad k=1,2,\cdots, \\
x_0(\cdot) = \phi \in PC,
\end{cases}
\qquad (5.35)
$$

Corollary 5.2.10. *Consider the impulsive stochastic partial differential equation* (5.35) *and suppose that the conditions* (A1)-(A4) *are satisfied. Then the mild solution of* (5.35) *is exponential stability in pth moment, if the following inequality*

$$\begin{aligned}&6^{p-1}(1-\kappa)^{-p}M_{1-\alpha}^{p}K^{p}\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\\&\quad+3^{p-1}(1-\kappa)^{-p}\left[M^{p}L_{1}^{p}\gamma^{1-p}+c_{p}M^{p}L_{2}^{p}\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\right]\\&\quad+6^{p-1}\gamma(1-\kappa)^{-p}M^{p}\left(\sum_{t_k<t}q_k\right)^{p}<\gamma\end{aligned}$$

holds, where $c_p=(p(p-1)/2)^{p/2}$.

Remark 5.2.11. *If* $I_k(\cdot)\equiv 0$ $(k=1,2,\cdots)$ *in* (5.35), Theorem 3.2 *in* [21] *can be obtained from* Corollary 5.2.10.

If $h\equiv 0$ and the impulsive effects $I_k(\cdot)\equiv 0$ $(k=1,2,\cdots)$, system (5.20) becomes

$$\begin{cases} dx(t)=[Ax(t)dt+f(t,x(t-\delta(t)))]dt+g(t,x(t-\rho(t))dw(t) & t\geq 0,\\ x_0(\cdot)=\phi\in PC. \end{cases}\tag{5.36}$$

Corollary 5.2.12. *Consider the stochastic partial differential equation* (5.36) *and suppose that the conditions* (A1)-(A4) *are satisfied. Then the mild solution of* (5.36) *is exponential stability in pth moment, if the following inequality*

$$3^{p-1}M^{p}\left[L_{1}^{p}\gamma^{1-p}+c_{p}L_{2}^{p}\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\right]<\gamma\tag{5.37}$$

holds, where $c_p=(p(p-1)/2)^{p/2}$.

Remark 5.2.13. *Corollary* 5.2.12 *is consistent with the result in Luo* [90] *which was studied by using fixed point methods.*

Example 5.2.14. *Consider the following neutral stochastic partial differential equation with delays and poisson jumps of the form*

$$\begin{aligned}&d\left[x(t,\xi)+\frac{\alpha_3}{M_{1-\alpha}\|(-A)^{\alpha}\|}x(t-\tau(t)),\xi)\right]\\&\quad=\left[\frac{\partial^2}{\partial\xi^2}x(t,\xi)dt+\alpha_1x(t-\delta(t),\xi)\right]dt+\alpha_2x(t-\rho(t),\xi)dw(t)\\&\qquad+\int_Z\alpha_4yx(t-\theta(t),\xi)\,\tilde{N}(dt,dy),\quad t\geq 0,\quad t\neq t_k,\end{aligned}\tag{5.38}$$

and for $t=t_k,\quad k=1,2,\cdots m$,

$$\Delta x(t_k,\xi)=d_kx(t_k,\xi),\tag{5.39}$$

where $d_k\geq 0,\sum_{k=1}^{m}d_k<\infty$, $x(t,0)=x(t,\pi)=0$, $\alpha_i>0,i=1,2,3,\quad 0<\tau(t),\delta(t),\rho(t),\theta(t)<\tau$, $x(s,\xi)=\phi(s,\xi)$, $\phi(\cdot,\xi)\in C$, $\phi(s,\cdot)\in\mathcal{L}^2[0,\pi]$, $-\tau\leq s\leq 0$, $0\leq\xi\leq\pi$, $\tau\geq 0$, $t\geq 0$, $w(t)$ *is a standard one-dimensional Wiener process and* $\|\phi\|_C<+\infty$ *a.s., and* $M_{1-\alpha}\geq 1$ $(\alpha\in(1/2,1])$.

Take $X = L^2[0,\pi]$,$Y = R^1$, define $A : X \to X$ by $-A = \frac{\partial^2}{\partial \xi^2}$ with domain

$$D(-A) = \left\{ \omega \in X : \quad \omega, \quad \frac{\partial^2 \omega}{\partial \xi} \quad \text{are} \quad \text{continuous}, \frac{\partial^2 \omega}{\partial \xi^2} \in X, \quad \omega(0) = \omega(\pi) = 0 \right\}.$$

Then

$$(-A)\omega = \sum_{n=1}^{\infty} n^2(\omega, \omega_n)\omega_n, \quad \omega \in D(-A),$$

where $\omega_n(\xi) = \sqrt{2/\pi}\sin n\xi$, $n = 1, 2, 3, \cdots$, is orthonormal set of eigenvector of $-A$. It is well known that A is the infinitesimal generator of an analytic semigroup $S(t)(t \geq 0)$ in X and is given (see Pazy[100], Page 70) by

$$S(t)\omega = \sum_{n=1}^{\infty} \exp(-n^2 t)(\omega, \omega_n)\omega_n, \quad \omega \in X,$$

that satisfies $\|S(t)\| \leq \exp\left(-\pi^2 t\right)$, $t \geq 0$, and hence is a contraction semigroup. Let

$$\begin{aligned}
u(t, x(t-\tau(t), \xi)) &= \frac{\alpha_3}{M_{1-\alpha}\|(-A)^{\alpha}\|} x(t-\tau(t)), \xi), \\
f(t, x(t-\delta(t), \xi)) &= \alpha_1 x(t-\delta(t), \xi), \quad g(t, x(t-\rho(t), \xi) = \alpha_2 x(t-\rho(t), \xi), \\
h(t, x(t-\theta(t), \xi), z) &= \alpha_4 z x(t-\theta(t), \xi).
\end{aligned}$$

We obtain that

$$\begin{aligned}
&\|f(t, x(t-\delta(t), \xi)) - f(t, y(t-\delta(t), \xi))\| \\
&\quad \leq \alpha_1 \|x(t-\delta(t), \xi) - y(t-\delta(t), \xi)\|, \quad f(t, 0) = 0.
\end{aligned}$$

$$\begin{aligned}
&\|g(t, x(t-\rho(t), \xi) - g(t, y(t-\rho(t), \xi)\| \\
&\quad \leq \alpha_2 \|x(t-\delta(t), \xi) - y(t-\delta(t), \xi)\|, \quad g(t, 0) = 0.
\end{aligned}$$

$$\begin{aligned}
&\int_Z \|h(t, x(t-\theta(t), \xi), z) - h(t, y(t-\theta(t), \xi), z)\| \, \nu(dz) \\
&\quad \leq \alpha_4 \int_Z z \, \nu(dz) \|x(t-\theta(t), \xi) - y(t-\theta(t), \xi)\|, \quad h(t, 0, z) = 0.
\end{aligned}$$

$$\begin{aligned}
&\|(-A)^{\alpha} u(t, x(t-\tau(t), \xi)) - (-A)^{\alpha} u(t, y(t-\tau(t), \xi))\| \\
&\quad \leq \frac{\alpha_3}{M_{1-\alpha}} \|x(t-\tau(t)), \xi) - y(t-\tau(t)), \xi)\|, \quad (-A)^{\alpha} u(t, 0, \xi)) = 0.
\end{aligned}$$

From the definition of $(-A)^{-\alpha}$, we obtain

$$\|(-A)^{-\alpha}\| \leq \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} \|S(t)\| \, dt \leq \frac{1}{\pi^{2\alpha}}.$$

Thus, when $\alpha_3 < M_{1-\alpha}\pi^{2\alpha}(\alpha \in (1/2, 1])$. By Theorem 5.2.8, the mild solution of (5.20) is exponential stability in mean square provide that

$$9\alpha_3^2\pi^{2-4\alpha}\Gamma(2\alpha-1) + 9\alpha_4^2\pi^{-2}\left(\int_Z z\,\nu(dz)\right)^2 + 3\left(\alpha_1^2\pi^{-2}+\alpha_2^2\right)$$
$$+9\pi^2\left(\sum_{k=1}^m q_k\right)^2 < \left(\pi - \frac{\alpha_3}{M_{1-\alpha}\pi^{2\alpha-1}}\right)^2, \quad \alpha \in \left(\frac{1}{2}, 1\right].$$

5.2.3 Exponential stability by using fixed point methods

In this subsection, we study existence and exponential stability in *pth* moment of mild solution of the system (5.20) by using fixed point methods.

Denote by $\mathcal{S}_\phi$ the space of all $\mathcal{F}$-adapted càdlàg processes: $\varphi(t,\omega) : [-\tau,\infty) \times \Omega \to X$ such that $\phi(s,\cdot) = \phi(s)$ for $s \in [-\tau, 0]$ and $e^{\eta t}\mathbb{E}\|\varphi(t)\|^p \to 0$ as $t \to \infty$, where $0 < \eta < \gamma$. If we define the metric as

$$\|\varphi\|_{\mathcal{S}_\phi} := \sup_{s \geq -\tau} \mathbb{E}\|\varphi(s)\|^p, \tag{5.40}$$

then $\mathcal{S}_\phi$ is a complete metric space with respect to (5.40). Using a contraction mapping defined on the space $\mathcal{S}_\phi$ and applying a contraction mapping principle, we obtain the following result.

Theorem 5.2.15. *Suppose that the assumptions* (A1)-(A4) *hold for some* $\alpha \in (1/p, 1]$, $p \geq 2$, *and the following conditions also hold,*

(i) *there exists a constant* $\tilde{q}$ *such that* $q_k \leq \tilde{q}(t_k - t_{k-1})$, $k = 1, 2, \cdots$,

(ii) *and such that*

$$6^{p-1}\|(-A)^{-\alpha}\|^p K^p + 6^{p-1}M_{1-\alpha}^p K^p \gamma^{-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}$$
$$+6^{p-1}M^p L_1^p \gamma^{-p} + 6^{p-1}c_p M^p L_2^p \left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\gamma^{-1}$$
$$+6^{p-1}c_p M^p L_3^p \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\gamma^{-1} + 6^{p-1}M^p\tilde{q}^p\gamma^{-p} < 1,$$

where $\Gamma(\cdot)$ *is the Gamma function,* $M_{1-\alpha}$ *is the corresponding constant as in Lemma* 5.2.5,

then the mild solution of the system (5.20) *is exponentially stable.*

Remark 5.2.16. *In our results, we do not require the monotone decreasing behavior of the delays, i.e.* $\tau'(t) \leq 0$, $\delta'(t) \leq 0$, $\rho'(t) \leq 0$, $\theta'(t) \leq 0$ *for* $t \geq 0$, *which is needed in* [14].

Remark 5.2.17. *Sakthivel and Luo* [117, 118] *and Jiang and Shen* [61] *studied asymptotic stability of special cases of the system* (5.20) *by using fixed point theory. In* [61, 117, 118], *the estimate of the impulsive term is*

$$\sup_{t\in[0,T]} \mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)(I_k(x(t_k^-)) - I_k(y(t_k^-)))\right\|^p \leq M^p e^{-\gamma p T}\mathbb{E}\left(\sum_{k=1}^m q_k^p\right)\sup_{t\in[0,T]}\mathbb{E}\|x(t)-y(t)\|^p$$

which seems to be a mistake. It should be

$$
\begin{aligned}
&\sup_{t\geq 0}\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)(I_k(x(t_k^-))-I_k(y(t_k^-)))\right\|^p\\
&\leq M^p\sup_{t\geq 0}\mathbb{E}\left[\sum_{0<t_k<t} e^{-\gamma(t-t_k)}q_k\|x(t_k^-)-y(t_k^-)\|\right]^p.
\end{aligned}
$$

To estimate the impulsive effects to the system (5.20)*, we consider the case which satisfy condition* (i) *in Theorem* 5.2.15*. Note that k in condition* (i) *can be equal to infinity.*

To prove Theorem 5.2.15, we start with a lemma.

Lemma 5.2.18. *Define an operator by $(\pi\varphi)(t)=\phi(t)$ for $t\in[-\tau,0]$, and for $t\geq 0$,*

$$
\begin{aligned}
(\pi\varphi)(t) &= S(t)(\phi(0)+u(0,\phi))-u(t,\varphi(t-\tau(t)))-\int_0^t AS(t-s)u(s,\varphi(s-\tau(s)))\,ds\\
&\quad+\int_0^t S(t-s)f(s,\varphi(s-\delta(s)))\,ds+\int_0^t S(t-s)g(s,\varphi(s-\rho(s)))\,dw(s)\\
&\quad+\int_0^t\int_Z S(t-s)h(s,\varphi(s-\sigma(s)),y)\,\tilde{N}(ds,dy)+\sum_{0<t_k<t}S(t-t_k)I_k(\varphi(t_k^-))\\
&:= \sum_{i=1}^{7}J_i(t).
\end{aligned}
\tag{5.41}
$$

Suppose that the assumptions (A1) – (A4) *hold. If the conditions (i) and (ii) in Theorem* 5.2.15 *are satisfied, then $\pi:\mathcal{S}_\phi\to\mathcal{S}_\phi$ and π is a contraction mapping.*

Proof. First, we prove the continuity in pth moment of π on $[0,\infty)$. Let $\varphi\in\mathcal{S}_\phi$, $t_1\geq 0$, and $|r|$ be sufficiently small, from (5.41), we have

$$
\mathbb{E}\|(\pi\varphi)(t_1+r)-(\pi\varphi)(t_1)\|^p\leq 7^{p-1}\sum_{i=1}^{7}\mathbb{E}\|J_i(t_1+r)-J_i(t_1)\|^p. \tag{5.42}
$$

It is easily to check that

$$
\mathbb{E}\|J_i(t_1+r)-J_i(t_1)\|^p\to 0 \quad\text{as}\quad r\to 0, i=1,2,3,4,7.
$$

Further, by using Hölder inequality and Lemma 5.2.6, we obtain

$$
\begin{aligned}
&\mathbb{E}\|J_5(t_1+r)-J_5(t_1)\|^p\\
&\quad\leq 2^{p-1}\mathbb{E}\left\|\int_0^{t_1}(S(t_1+r-s)-S(t_1))g(s,\varphi(s-\rho(s)))\,dw(s)\right\|^p\\
&\qquad+2^{p-1}\mathbb{E}\left\|\int_{t_1}^{t_1+r}S(t_1+r-s)g(s,\varphi(s-\rho(s)))\,dw(s)\right\|^p\\
&\quad\leq 2^{p-1}c_p\left(\int_0^{t_1}(\mathbb{E}\|(S(t_1+r-s)-S(t_1))g(s,\varphi(s-\rho(s)))\|^p)^{2/p}\,ds\right)^{p/2}\\
&\qquad+2^{p-1}c_p\left(\int_{t_1}^{t_1+r}(\mathbb{E}\|S(t_1+r-s)g(s,\varphi(s-\rho(s)))\|^p)^{2/p}\,ds\right)^{p/2}\to 0\quad\text{as}\quad r\to 0.
\end{aligned}
$$

Similarly, we can verify that $\mathbb{E}\|J_6(t_1+r)-J_6(t_1)\|^p \to 0 \quad \text{as} \quad r \to 0$. Thus, π is indeed continuous in pth moment on $[0,\infty)$.

Next, We show that $\pi(\mathcal{S}_\phi) \subseteq \mathcal{S}_\phi$. It follows form (5.41) that

$$\begin{aligned}
e^{\eta t}\mathbb{E}\|(\pi\varphi)(t)\|^p \leq\ & 7^{p-1}e^{\eta t}\mathbb{E}\|S(t)(\phi(0)+u(0,\phi))\|^p + 7^{p-1}e^{\eta t}\mathbb{E}\|u(t,\varphi(t-\tau(t)))\|^p \\
& +7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,\varphi(s-\tau(s)))\,ds\right\|^p \\
& +7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t S(t-s)f(s,\varphi(s-\delta(s)))\,ds\right\|^p \\
& +7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t S(t-s)g(s,\varphi(s-\rho(s)))\,dw(s)\right\|^p \\
& +7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t\int_Z S(t-s)h(s,\varphi(s-\sigma(s)),y)\,\tilde{N}(ds,dy)\right\|^p \\
& +7^{p-1}e^{\eta t}\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)I_k\varphi(t_k^-)\right\|^p. \qquad (5.43)
\end{aligned}$$

Now, we estimate the terms on the right-hand side of (5.43). By the condition (A1) and (A3), we obtain

$$7^{p-1}e^{\eta t}\mathbb{E}\|S(t)(\phi(0)+u(0,\phi))\|^p \leq M^p e^{(\eta-p\gamma)t}\mathbb{E}\|\phi(0)+u(0,\phi)\|^p \to 0 \quad \text{as} \quad t\to\infty \qquad (5.44)$$

and

$$\begin{aligned}
7^{p-1}e^{\eta t}\mathbb{E}\|u(t,\varphi(t-\tau(t)))\|^p \leq\ & 7^{p-1}e^{\eta t}\|(-A)^{-\alpha}\|^p E\|(-A)^{\alpha}u(t,\varphi(t-\tau(t)))\|_H^p \\
\leq\ & 7^{p-1}e^{\eta\tau}K^p\|(-A)^{-\alpha}\|^p e^{\eta(t-\tau(t))}E\|\varphi(t-\tau(t))\|^p \\
& \to 0 \quad \text{as} \quad t\to\infty. \qquad (5.45)
\end{aligned}$$

Using Hölder inequality, we obtain

$$\begin{aligned}
& 7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,\varphi(s-\tau(s)))\,ds\right\|^p \\
& \leq 7^{p-1}e^{\eta t}M_{1-\alpha}^p K^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\int_0^t e^{-\gamma(t-s)}E\|\varphi(s-\tau(s))\|^p\,ds \\
& \leq 7^{p-1}M_{1-\alpha}^p K^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} \\
& \quad \times e^{\eta\tau}\int_0^t e^{-(\gamma-\eta)(t-s)}e^{\eta(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p\,ds. \qquad (5.46)
\end{aligned}$$

For any $\varphi\in\mathcal{S}_\phi$ and $\varepsilon>0$, there exists $t_1>0$ such that $e^{\eta(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p<\varepsilon$ for $t\geq t_1$.

Thus, from (5.46), we obtain

$$
\begin{aligned}
&7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,\varphi(s-\tau(s)))\,ds\right\|^p\\
&\le 7^{p-1}M_{1-\alpha}^pK^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\\
&\qquad\times e^{\eta\tau}\int_0^{t_1} e^{-(\gamma-\alpha)(t-s)}e^{\eta(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p\,ds\\
&\quad+7^{p-1}M_{1-\alpha}^pK^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\\
&\qquad\times e^{\eta\tau}\int_{t_1}^{t} e^{-(\gamma-\alpha)(t-s)}e^{\eta(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p\,ds\\
&\le 7^{p-1}M_{1-\alpha}^pK^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\\
&\qquad\times e^{\eta\tau}\int_0^{t_1} e^{-(\gamma-\eta)(t-s)}e^{\eta(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p\,ds\\
&\quad+7^{p-1}M_{1-\alpha}^pK^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\,(e^{\eta\tau}/(\gamma-\eta))\,\varepsilon.
\end{aligned}
\tag{5.47}
$$

Since $e^{-(\gamma-\eta)t}\to 0$ as $t\to\infty$, then there exists $t_2\ge t_1$ such that for $t\ge t_2$, we have

$$
\begin{aligned}
&7^{p-1}M_{1-\alpha}^pK^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\\
&e^{\eta\tau}\int_0^{t_1} e^{-(\gamma-\eta)(t-s)}e^{\eta(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p\,ds\\
&\quad\le\varepsilon-7^{p-1}M_{1-\alpha}^pK^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\,(e^{\eta\tau}/(\gamma-\eta))\,\varepsilon.
\end{aligned}
\tag{5.48}
$$

So, from (5.47)and (5.48), we obtain that for any $t\ge t_2$,

$$
7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,\varphi(s-\tau(s)))\,ds\right\|^p<\varepsilon. \tag{5.49}
$$

Hence

$$
7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,\varphi(s-\tau(s)))\,ds\right\|^p\to 0\quad\text{as}\quad t\to\infty.
$$

As for the fourth term on the right-hand side of (5.43), for any $\varphi\in\mathcal{S}_\phi$, we have that for $p>2$,

$$
\begin{aligned}
&7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t S(t-s)f(s,\varphi(s-\delta(s)))\,ds\right\|^p\\
&\le 7^{p-1}e^{\eta t}E\left[\int_0^t Me^{-\gamma(t-s)}\|f(s,\varphi(s-\delta(s)))\|\,ds\right]^p\\
&\le 7^{p-1}e^{\eta t}M^pL_1^pE\left[\int_0^t e^{-\gamma(t-s)}\|\varphi(s-\delta(s))\|\,ds\right]^p\\
&\le 7^{p-1}e^{\eta t}M^pL_1^p\left[\int_0^t e^{-\gamma(t-s)}\,ds\right]^{p-1}\int_0^t e^{-\gamma(t-s)}E\|\varphi(s-\delta(s))\|^p\,ds\\
&\le 7^{p-1}e^{\eta\tau}M^pL_1^p\gamma^{1-p}\int_0^t e^{-(\gamma-\eta)(t-s)}e^{\eta(s-\delta(s))}E\|\varphi(s-\delta(s))\|^p\,ds.
\end{aligned}
\tag{5.50}
$$

From Lemma 5.2.6 and Hölder inequality, we obtain

$$
\begin{aligned}
&7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t S(t-s)g(s,\varphi(s-\rho(s)))\,dw(s)\right\|^p \\
&\le 7^{p-1}e^{\eta t}c_pM^p\left[\int_0^t\left(e^{-\gamma p(t-s)}\mathbb{E}\|g(s,\varphi(s-\rho(s)))\|^p\right)^{2/p}ds\right]^{p/2} \\
&\le 7^{p-1}e^{\eta t}c_pM^pL_2^p\left[\int_0^t\left(e^{-\gamma p(t-s)}\mathbb{E}\|\varphi(s-\rho(s))\|^p\right)^{2/p}ds\right]^{p/2} \\
&\le 7^{p-1}e^{\eta t}c_pM^pL_2^p\left[\int_0^t e^{-\left(\frac{2(p-1)}{p-2}\right)\gamma(t-s)}\,ds\right]^{p-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|\varphi(s-\rho(s))\|^p\,ds \\
&\le 7^{p-1}e^{\eta t}c_pM^pL_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|\varphi(s-\rho(s))\|^p\,ds \qquad (5.51)\\
&\le 7^{p-1}c_pM^pL_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}e^{\eta\tau}\int_0^t e^{-(\gamma-\eta)(t-s)}e^{\eta(s-\rho(s))}\mathbb{E}\|\varphi(s-\rho(s))\|^p\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
&7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t\int_Z S(t-s)h(s,\varphi(s-\sigma(s)),z)\,\tilde N(ds,dz)\right\|^p \\
&\le 7^{p-1}e^{\eta t}c_p\mathbb{E}\left[\int_0^t\int_Z\|S(t-s)h(s,\varphi(s-\sigma(s)),z)\|^2\,ds\nu(dz)\right]^{p/2} \\
&\le 7^{p-1}e^{\eta t}c_pM^p\mathbb{E}\left[\int_0^t\int_Z e^{-2\gamma(t-s)}\|h(s,\varphi(s-\sigma(s)),z)\|^2\,ds\nu(dz)\right]^{p/2} \\
&\le 7^{p-1}e^{\eta t}c_pM^p\mathbb{E}\left[\int_0^t e^{-2\gamma(t-s)}\int_Z\|h(s,\varphi(s-\sigma(s)),z)\|^2\nu(dz)\,ds\right]^{p/2} \\
&\le 7^{p-1}e^{\eta t}c_pM^pL_3^p\left[\int_0^t e^{-2\gamma(t-s)}\mathbb{E}\|\varphi(s-\sigma(s))\|^2\,ds\right]^{p/2} \\
&\le 7^{p-1}e^{\eta t}c_pM^pL_3^p\left(\int_0^t e^{-\frac{2(p-1)}{p}\cdot\frac{p}{p-2}\gamma(t-s)}\,ds\right)^{(p-2)/2}e^{-\gamma(t-s)}\mathbb{E}\|\varphi(s-\sigma(s))\|^p\,ds \\
&\le 7^{p-1}e^{\eta t}c_pM^pL_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|\varphi(s-\sigma(s))\|^p\,ds \qquad (5.52)\\
&\le 7^{p-1}c_pM^pL_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}e^{\eta\tau}\int_0^t e^{-(\gamma-\eta)(t-s)}e^{\eta(s-\sigma(s))}\mathbb{E}\|\varphi(s-\sigma(s))\|^p\,ds,
\end{aligned}
$$

where $c_p=(p(p-1)/2)^{p/2}$. We remark that if $p=2$, the inequality (5.51) also holds with $0^0:=1$. Similar to the proof of (5.46), from estimate (5.50), (5.51) and (5.52), we obtain that

$$
\begin{aligned}
&7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t S(t-s)f(s,\varphi(s-\delta(s)))\,ds\right\|^p\to 0\quad\text{as}\quad t\to\infty,\\
&7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t S(t-s)g(s,\varphi(s-\rho(s)))\,dw(s)\right\|^p\to 0\quad\text{as}\quad t\to\infty,\\
&7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t\int_Z S(t-s)h(s,\varphi(s-\sigma(s)),z)\,\tilde N(ds,dz)\right\|^p\to 0\quad\text{as}\quad t\to\infty. \qquad (5.53)
\end{aligned}
$$

Now, we estimate the impulsive term, from the condition (i), we obtain

$$
\begin{aligned}
&7^{p-1}e^{\eta t}\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)I_k\varphi(t_k^-)\right\|^p \\
&\le 7^{p-1}e^{\eta t}\mathbb{E}\left(\sum_{0<t_k<t} Me^{-\gamma(t-t_k)}q_k\|\varphi(t_k^-)\|\right)^p \\
&\le 7^{p-1}e^{\eta t}\mathbb{E}\left(\sum_{0<t_k<t} Me^{-\gamma(t-t_k)}\tilde{q}\|\varphi(t_k^-)(t_k-t_{k-1})\|\right)^p \\
&\le 7^{p-1}e^{\eta t}\mathbb{E}\left(\int_0^t Me^{-\gamma(t-s)}\tilde{q}\|\varphi(s)\|\,ds\right)^p \\
&\le 7^{p-1}M^p\tilde{q}^p\left(\int_0^t e^{-\gamma(t-s)}\,ds\right)^{p-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|\varphi(s)\|^p\,ds \\
&\le 7^{p-1}M^p\tilde{q}^p\gamma^{1-p}\int_0^t e^{-(\gamma-\eta)(t-s)}e^{\eta s}\mathbb{E}\|\varphi(s)\|^p\,ds.
\end{aligned} \tag{5.54}
$$

From (5.54), we obtain

$$
7^{p-1}e^{\eta t}\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)I_k\varphi(t_k^-)\right\|^p \to 0 \quad \text{as} \quad t\to\infty. \tag{5.55}
$$

Hence, from (5.43), (5.53) and (5.55), we obtain that

$$
e^{\eta t}\mathbb{E}\,\|(\pi\varphi)(t)\|^p \to 0 \quad \text{as} \quad t\to\infty.
$$

Therefore, we conclude that $\pi(\mathcal{S}_\phi)\subseteq\mathcal{S}_\phi$.

Finally, we show that π is a contraction mapping. For any $\varphi,\psi\in\mathcal{S}_\phi$, using the estimate (5.27)-(5.33), we obtain

$$
\begin{aligned}
&\sup_{t\ge-\tau}\mathbb{E}\|(\pi\varphi)(t)-(\pi\psi)(t)\|^p \\
&\quad\le 6^{p-1}\sup_{t\ge-\tau}\mathbb{E}\|u(t,\varphi(t-\tau(t)))-u(t,\psi(t-\tau(t)))\|^p \\
&\qquad+6^{p-1}\sup_{t\ge-\tau}\mathbb{E}\left\|\int_0^t AS(t-s)(u(s,\varphi(s-\tau(s)))-u(s,\psi(s-\tau(s))))\,ds\right\|^p \\
&\qquad+6^{p-1}\sup_{t\ge-\tau}\mathbb{E}\left\|\int_0^t S(t-s)(f(s,\varphi(s-\delta(s)))-f(s,\psi(s-\delta(s))))\,ds\right\|^p \\
&\qquad+6^{p-1}\sup_{t\ge-\tau}\mathbb{E}\left\|\int_0^t S(t-s)(g(s,\varphi(s-\rho(s)))-g(s,\psi(s-\rho(s)))))\,dw(s)\right\|^p \\
&\qquad+6^{p-1}\sup_{t\ge-\tau}\mathbb{E}\left\|\int_0^t\int_Z S(t-s)(h(s,\varphi(s-\sigma(s)),y)-h(s,\psi(s-\sigma(s)),y))\,\tilde{N}(ds,dy)\right\|^p \\
&\qquad+6^{p-1}\sup_{t\ge-\tau}\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)(I_k(\varphi(t_k^-))-I_k(\psi(t_k^-)))\right\|^p
\end{aligned}
$$

$$\leq \Bigg[6^{p-1}\|(-A)^{-\alpha}\|^p K^p + 6^{p-1}M_{1-\alpha}^p K^p \gamma^{-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}$$
$$+6^{p-1}M^p L_1^p \gamma^{-p} + 6^{p-1}c_p M^p L_2^p \left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\gamma^{-1}$$
$$+6^{p-1}c_p M^p L_3^p \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\gamma^{-1} + 6^{p-1}M^p\tilde{q}^p\gamma^{-p}\Bigg]\sup_{t\geq 0}\mathbb{E}\|\varphi(t)-\psi(t)\|^p.$$

Thus, by the condition (ii), we know that π is a contraction mapping.

Hence, using a contraction mapping principle, π has a unique fixed point $x(t)$ in $\mathcal{S}_\phi$, which is a solution of the system (5.20) with $x(s)=\phi(s)$ on $[-\tau,0]$ and $e^{\eta t}\mathbb{E}\|x(t)\|^p \to 0$ as $t\to\infty$. □

If the impulsive effects $I_k(\cdot)\equiv 0$ $(k=1,2,\cdots)$, system (5.20) reduces to the following neutral stochastic partial differential equations with delays

$$\begin{cases} d[x(t)+u(t,x(t-\tau(t)))] = [Ax(t)dt + f(t,x(t-\delta(t)))]dt + g(t,x(t-\rho(t))dw(t) \\ \qquad\qquad + \int_Z h(t,x(t-\theta(t)),y)\,\tilde{N}(dt,dy), \quad t\geq 0 \\ x_0(\cdot)=\phi\in PC. \end{cases} \tag{5.56}$$

Corollary 5.2.19. *Consider the stochastic partial differential equation* (5.56) *and suppose that the conditions* (A1)-(A4) *are satisfied. Then the mild solution of* (5.56) *is exponential stability in pth moment, if the following inequality*

$$5^{p-1}\|(-A)^{-\alpha}\|^p K^p + 5^{p-1}M_{1-\alpha}^p K^p \gamma^{-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}$$
$$+5^{p-1}M^p L_1^p \gamma^{-p} + 5^{p-1}c_p M^p L_2^p \left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\gamma^{-1}$$
$$+5^{p-1}c_p M^p L_3^p \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\gamma^{-1} < 1$$

holds, where $c_p=(p(p-1)/2)^{p/2}$.

Example 5.2.20. Consider a neutral stochastic partial differential equation with delays and Poisson jumps of the form

$$\begin{cases} d\left[x(t,\xi)+\alpha_0 x(t-\tau(t),\xi)\right] = \left[\frac{\partial^2}{\partial\xi^2}x(t,\xi)dt + \alpha_1 x(t-\delta(t),\xi)\right]dt \\ \qquad\qquad +\alpha_2 x(t-\rho(t),\xi)dw(t), \quad t\neq t_k, \\ \Delta x(t_k,\xi) = b_k x(t_k,\xi), \quad t=t_k, \quad k=1,2,\cdots, \\ x(s,\xi)=\phi(s,\xi), \quad \phi(\cdot,\xi)\in C, \quad \phi(s,\cdot)\in\mathcal{L}^2[0,\pi], \quad -\tau\leq s\leq 0, \quad 0\leq\xi\leq\pi, \end{cases} \tag{5.57}$$

for $t\geq 0$, where $\tau\geq 0$ is a constant, $w(t)$ is a standard one-dimensional Wiener process and $\|\phi\|_C < +\infty$ a.s., and $\alpha\in(1/p,1]$, $p\geq 2$. We suppose that $x(t,0)=x(t,\pi)=0$ and $0\leq\tau(t),\delta(t),\rho(t)\leq\tau$, and we suppose that α_0, α_1 and α_2 are positive constants. Further, we suppose that there exists a constant b such that $b_k\leq b(t_k-t_{k-1})$.

Take $X = L^2[0,\pi]$,$Y = R^1$, define $A : X \to X$ by $A = \frac{\partial^2}{\partial \xi^2}$ with domain

$$D(A) = \left\{ x \in X \mid x, \ \ \partial x/\partial \xi \ \text{ are } \ \text{continuous}, \ \ \partial^2 x/\partial \xi^2 \in X, \ \ x(0) = x(\pi) = 0 \right\}.$$

Then $Ax = \sum_{n=1}^{\infty} n^2 \langle x, e_n \rangle e_n, \quad x \in D(A)$, where $e_n(\xi) = \sqrt{2/\pi} \sin n\xi$, $n = 1, 2, 3, \cdots$, is an orthonormal set of eigenvectors of $-A$. It is well known that A is the infinitesimal generator of an analytic semigroup $S(t)$ in X and is given (see [100], Page 70) by

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t} \langle x, e_n \rangle e_n, \quad x \in X.$$

Hence, $\|S(t)\| \le e^{-\pi^2 t}$, $t \ge 0$. We define A^α (actually $|A|^\alpha$) for the self-adjoint operator A by the classical spectral theorem,

$$|A|^\alpha S(t)x = \sum_{n=1}^{\infty} \left(n^2\right)^\alpha e^{-n^2 t} \langle x, e_n \rangle e_n.$$

From the arguments in [125], we have that for $0 < m < 1$ and $M_\alpha = \left(\frac{\alpha}{1-m}\right)^\alpha$,

$$\|(-A)^\alpha S(t)x\| \le M_\alpha e^{-mt} t^{-\alpha} \|x\|, \quad t > 0$$

holds for all $x \in X$. Let $u(t, x(t-\tau(t), \xi)) = \alpha_0 x(t-\tau(t), \xi)$, $f(t, x(t-\delta(t), \xi)) = \alpha_1 x(t-\delta(t), \xi)$, $g(t, x(t-\rho(t), \xi)) = \alpha_2 x(t-\rho(t), \xi)$. It is clear that $u(t, \cdot)$, $f(t, \cdot)$ and $g(t, \cdot)$ satisfy conditions (A2) and (A3). Further, from the definition of $(-A)^{-\alpha}$, we have that

$$\|(-A)^{-\alpha}\| \le \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} \|S(t)\| \, dt \le \frac{1}{\pi^{2\alpha}}.$$

Thus, by Theorem 5.2.15, the mild solution of (5.57) is exponentially stable in *pth* moment provided that

$$\left(\frac{\alpha_0}{\pi^{2\alpha}}\right)^p + \left(\frac{1-\alpha}{1-m}\right)^{1-\alpha} \alpha_0^p \pi^{-2p\alpha} \left(\Gamma\left(1 + \frac{p(\alpha-1)}{p-1}\right)\right)^{p-1} + \alpha_1^p \pi^{-2p}$$
$$+ c_p \alpha_2^p \pi^{-2} \left(\frac{2\pi^2(p-1)}{p-2}\right)^{1-p/2} + b^p \pi^{-2p} < 6^{1-p},$$

where $\Gamma(\cdot)$ is the Gamma function, $\alpha \in (1/p, 1]$, $p \ge 2$, $0 < m < 1$, $c_p = (p(p-1)/2)^{p/2}$.

5.3 Notes and remarks

Luo [90] and Appleby [4] have applied fixed point method to deal with the stability problems for stochastic differential equations with stochastic effects. Following the ideas of [4, 90], by employing a contraction mapping principle and stochastic integral technique, many other investigators [89, 131, 132, 147] considered its generalization.

Many methods which are used frequently to investigate the stability problems for stochastic partial differential equations are ineffective to study the exponential stability of mild solutions for impulsive stochastic delay differential equations with delays, see, for example, the comparison theorem in Govindan [48, 49], the Gronwall inequality in Caraballo and Liu [14], the analytic technique in Liu and Truman [87], Taniguchi [121] and the semigroup method in Taniguchi et al.[123], the methods proposed in Caraballo et al.[122], Ichikawa [57, 58], Liu and Mao [86], Wan and Duan [130], Liu [85, 88] and Taniguchi [124].

Based on the contents of Section 5.2, a paper [18] has been submitted to a journal for possible publication.