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Chapter 3

Asymptotic behavior of a class of nonautonomous neutral delay differential equations

In this chapter, asymptotic behavior of a class of nonautonomous neutral delay differential equations is studied. It should be emphasized that asymptotic behavior of nonautonomous equations is much more difficult than the case of autonomous equations. For instance, Frasson and Verduyn Lunel [39] studied the following linear periodic delay equation

$$x'(t) = a(t)x(t) + \sum_{j=1}^{k} b_j(t)x(t-\tau_j), \tag{3.1}$$

where $a(t+\omega) = a(t)$, $b_j(t+\omega) = b_j(t)$, $j = 1, 2, \cdots, k$, they considered a particular case where $\tau_j = j\omega$ (i.e. the delays are integer multiples of the period ω). However, it is very difficult to study general nonautonomous problems.

For a special class of nonautonomous problems, we can use an approach similar to the ODE method as we discussed in Chapter 2, which is based on the application of an appropriate solution of the generalized characteristic equation. For nonautonomous equations, solving the generalized characteristic equation becomes much harder: functional equation instead of algebraic equation. Our result can be applied in case the assumptions are satisfied, i.e., the generalized characteristic equation has a real solution.

3.1 Introduction and main result

For $r \geq 0$, let $\mathcal{C} = C([-r, 0], \mathbb{C})$ be the space of continuous functions taking $[-r, 0]$ into $\mathbb{C}$ with $\|\varphi\|$, $\varphi \in \mathcal{C}$, defined by $\|\varphi\| = \max_{-r\leq\theta\leq 0} |\varphi(\theta)|$. A delay differential equation of neutral type, or shortly, a neutral equation is a system of the form

$$\frac{d}{dt} M x_t = L(t)x_t \quad t \geqslant t_0 \in \mathbb{R}, \tag{3.2}$$

where $x_t \in \mathcal{C}$ is defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$, $M : \mathcal{C} \to \mathbb{C}$ is continuous, linear and atomic at zero, (see [51] on page 255 for the concept of atomic at zero),

$$M\varphi = \varphi(0) - \int_{-r}^{0} \varphi(\theta)\, d\mu(\theta), \tag{3.3}$$

where $\mathrm{Var}_{[s,0]}\mu \to 0$, as $s \to 0$.

For (3.2), $L(t)$ denotes a family of bounded linear functionals on $\mathcal{C}$, and by the Riesz representation theorem, for each $t \geqslant t_0$, there exists a complex valued function of bounded variation $\eta(t, \cdot)$ on $[-r, 0]$, normalized so that $\eta(t, 0) = 0$ and $\eta(t, \cdot)$ is continuous from the left in $(-r, 0)$ such that

$$L(t)\varphi = \int_{-r}^{0} \varphi(\theta)\, d_\theta \eta(t, \theta). \tag{3.4}$$

For any $\varphi \in \mathcal{C}$, $\sigma \in [t_0, \infty)$, a function $x = x(\sigma, \varphi)$ defined on $[\sigma - r, \sigma + A)$ is said to be a solution of (3.2) on $(\sigma, \sigma + A)$ with initial φ at σ if x is continuous on $[\sigma - r, \sigma + A)$, $x_\sigma = \varphi$, Mx_t is continuously differentiable on $(\sigma, \sigma + A)$ and relation (3.2) is satisfied on $(\sigma, \sigma + A)$. For more information on this type of equations, see [51].

The initial-value problem (IVP) is

$$\begin{cases} \frac{d}{dt} Mx_t = L(t)x_t & t \geq \sigma, \\ \\ x_\sigma = \varphi. \end{cases} \tag{3.5}$$

For $\mu = 0$ in (3.3), $M\varphi = \varphi(0)$ and equation (3.2) becomes a retarded functional differential equation,

$$x'(t) = L(t)x_t. \tag{3.6}$$

Consider the *generalized characteristic equation* of (3.6)

$$\lambda(t) = \int_0^r \exp\left(-\int_{t-\theta}^{t} \lambda(s)ds\right) d_\theta \eta(t, \theta) \tag{3.7}$$

which is obtained by looking for solutions to (3.6) of the form

$$x(t) = \exp\left(\int_0^t \lambda(s)\, ds\right). \tag{3.8}$$

By a solution of the generalized characteristic equation (3.7), we mean a continuous real-valued function $\lambda(\cdot)$ defined on $[t_0 - r, \infty)$ which satisfies (3.7).

Cuevas and Frasson [26] studied the asymptotic behavior of solutions of (3.6) with initial condition $x_\sigma = \varphi$, and obtained the following result.

Theorem 3.1.1. *Assume that $\lambda(t)$ is a real solution of (3.7) such that*

$$\limsup_{t\to\infty} \int_0^r \theta |e^{-\int_{t-\theta}^t \lambda(s)ds}| d_\theta |\eta|(t, \theta) < 1.$$

Then for each solution x of (3.6), we have that the limit

$$\lim_{t\to\infty} x(t)e^{-\int_{t_0}^t \lambda(s)ds}$$

exists, and

$$\lim_{t\to\infty} \left[x(t)e^{-\int_{t_0}^t \lambda(s)ds}\right]' = 0.$$

Furthermore,

$$\lim_{t\to\infty} x'(t)e^{-\int_{t_0}^t \lambda(s)ds} = \lim_{t\to\infty} \lambda(t)x(t)e^{-\int_{t_0}^t \lambda(s)ds},$$

if $\lim_{t\to\infty} \lambda(t)x(t)e^{-\int_{t_0}^t \lambda(s)ds}$ exists.

Motivated by the work of [26], we provide a generalization of [26], as it can be applied for instance for neutral delay differential equations with distributed delays or discrete delays, as far as the delays we considered are uniformly bounded. The method for the proof of the main result is similar to [26, 33].

For equation (3.2), the generalized characteristic equation is

$$\lambda(t) = \int_{-r}^{0} d\mu(\theta)\lambda(t+\theta)\exp\left(-\int_{t+\theta}^{t}\lambda(s)ds\right) + \int_{-r}^{0} d_\theta\eta(t,\theta)\exp\left(-\int_{t+\theta}^{t}\lambda(s)ds\right), \tag{3.9}$$

which is obtained by looking for solutions of (3.2) of the form (3.8) and the solutions of (3.9) are continuous functions defined in $[\sigma - r, \infty)$ satisfying (3.9). For autonomous neutral delay differential equations, the constant solutions of (3.9) are the roots of the so called characteristic equation. The following is our main result.

Theorem 3.1.2. *Assume that a real-valued function $\lambda(t)$ is a solution of (3.9) such that*

$$\limsup_{t\to\infty} \chi_{\lambda,t} < 1, \tag{3.10}$$

where

$$\chi_{\lambda,t} = \int_{-r}^{0} e^{-\int_{t+\theta}^{t}\lambda(s)\,ds}\, d|\mu|(\theta) + \int_{-r}^{0} (-\theta)e^{-\int_{t+\theta}^{t}\lambda(s)\,ds}\left(|\lambda(t+\theta)|\, d|\mu|(\theta) + d_\theta|\eta|(t,\theta)\right).$$

Then for each solution x of (3.5), we have that the limit

$$\lim_{t\to\infty} x(t)e^{-\int_{t_0}^{t}\lambda(s)\,ds} \tag{3.11}$$

exists, and

$$\lim_{t\to\infty}\left[x(t)e^{-\int_{t_0}^{t}\lambda(s)\,ds}\right]' = 0. \tag{3.12}$$

Furthermore,

$$\lim_{t\to\infty} x'(t)e^{-\int_{t_0}^{t}\lambda(s)\,ds} = \lim_{t\to\infty}\lambda(t)x(t)e^{-\int_{t_0}^{t}\lambda(s)\,ds} \tag{3.13}$$

if the limit at the right-hand side exists.

Remark 3.1.3. *The conditions in Theorem 3.1.2 are very strong and therefore the theorem is far from providing a general theory. However, it can be applied to deal with certain examples, see Section 3.3.*

3.2 Proof of Theorem 3.1.2

In this section, we prove Theorem 3.1.2. We start with some preparations.

From (3.10), we obtain that there exists $t_1 \geq t_0$, such that $\sup_{t\geq t_1}\chi_{\lambda,t} < 1$. Without loss of generality, we assume $t_1 = 0$ and define

$$\Gamma_\lambda := \sup_{t\geq 0}\chi_{\lambda,t} < 1.$$

For solutions x of (3.5), we set

$$y(t) = x(t)e^{-\int_0^t \lambda(s)\,ds}, \quad t \geqslant -r.$$

Then (3.5) becomes

$$\begin{aligned} y'(t) + \lambda(t)y(t) - \int_{-r}^{0} d\mu(\theta)y'(t+\theta)e^{-\int_{t+\theta}^{t} \lambda(s)\,ds} \\ = \int_{-r}^{0} y(t+\theta)e^{-\int_{t+\theta}^{t} \lambda(s)\,ds} \left(\lambda(t+\theta)\,d\mu(\theta) + \,d_\theta\eta(t,\theta)\right) \end{aligned} \tag{3.14}$$

and the initial condition is equivalent to

$$y(t) = \varphi(t)e^{-\int_0^t \lambda(s)\,ds}, \quad -r \leq t \leq 0. \tag{3.15}$$

Combining (3.15) with (3.9), for $t \geq -r$, we have

$$\begin{aligned} y'(t) &= \int_{-r}^{0} d\mu(\theta)y'(t+\theta)e^{-\int_{t+\theta}^{t} \lambda(s)\,ds} \\ &\quad - \int_{-r}^{0} e^{-\int_{t+\theta}^{t} \lambda(s)\,ds} \int_{-r}^{0} y'(s)\,ds \left(\lambda(t+\theta)\,d\mu(\theta) + \,d_\theta\eta(t,\theta)\right). \end{aligned} \tag{3.16}$$

From the definition of the solutions to (3.5), we know that $y'(t)$ is continuous, Let

$$M_{\varphi,\lambda_1} = \max\left\{\left|\varphi'(t)e^{-\int_0^t \lambda(s)\,ds} - \lambda(t)\varphi(t)e^{-\int_0^t \lambda(s)\,ds}\right| : -r \leq t \leq 0\right\}.$$

We shall show that M_{φ,λ_1} is also a bound of y' on the whole interval $[-r, \infty)$; i.e.,

$$|y'(t)| \leq M_{\varphi,\lambda_1}, \quad t \geq -r. \tag{3.17}$$

For this purpose, take $\varepsilon > 0$, then

$$|y'(t)| < M_{\varphi,\lambda_1} + \varepsilon \quad \text{for} \quad t \geq -r. \tag{3.18}$$

In fact, we suppose that there exists a point $t^* > 0$ such that

$$\begin{aligned} |y'(t)| &< M_{\varphi,\lambda_1} + \varepsilon \quad \text{for} \quad -r \leq t < t^*, \\ |y'(t^*)| &= M_{\varphi,\lambda_1} + \varepsilon. \end{aligned} \tag{3.19}$$

Then combining (3.16) and (3.19), we obtain

$$\begin{aligned} y'(t^*) &= M_{\varphi,\lambda_1} + \varepsilon \\ &\leq \left|\int_{-r}^{0} y'(t^*+\theta)e^{-\int_{t^*+\theta}^{t^*} \lambda(s)\,ds}\,d\mu(\theta)\right| \\ &\quad + \left|\int_{-r}^{0} e^{-\int_{t^*+\theta}^{t^*} \lambda(s)\,ds} \int_{-r}^{0} y'(s)\,ds\left(\lambda(t^*+\theta)\,d\mu(\theta) + \,d_\theta\eta(t^*,\theta)\right)\right| \\ &\leq (M_{\varphi,\lambda_1} + \varepsilon)\left\{\int_{-r}^{0} |e^{-\int_{t^*+\theta}^{t^*} \lambda(s)\,ds}|\,d|\mu|(\theta) \right. \\ &\quad \left. + \int_{-r}^{0} (-\theta)|e^{-\int_{t^*+\theta}^{t^*} \lambda(s)\,ds}|\left(|\lambda(t^*+\theta)|\,d|\mu|(\theta) + d_\theta|\eta|(t^*,\theta)\right)\right\} \\ &= (M_{\varphi,\lambda_1} + \varepsilon)\Gamma_\lambda < M_{\varphi,\lambda_1} + \varepsilon, \end{aligned} \tag{3.20}$$

which is a contradiction, so (3.18) holds. Since (3.18) holds for every $\varepsilon > 0$, it follows that $|y'(t)| \le M_{\varphi,\lambda_0}$ for all $t \ge -r$.

We are now ready to prove Theorem 3.1.2.

Proof. By using (3.16) and (3.17), for $t \ge 0$, we have

$$\begin{aligned}
|y'(t)| &\le \left| \int_{-r}^{0} y'(t+\theta) e^{-\int_{t+\theta}^{t} \lambda(s) ds} \, d\mu(\theta) \right| \\
&\quad + \left| \int_{-r}^{0} e^{-\int_{t+\theta}^{t} \lambda(s)\, ds} \int_{-r}^{0} y'(s)\, ds\, (\lambda(t+\theta)\, d\mu(\theta) + \, d_\theta \eta(t,\theta)) \right| \\
&\le M_{\varphi,\lambda_1} \left\{ \int_{-r}^{0} |e^{-\int_{t+\theta}^{t} \lambda(s)\, ds}| \, d|\mu|(\theta) \right. \\
&\quad \left. + \int_{-r}^{0} (-\theta) |e^{-\int_{t+\theta}^{t} \lambda(s)\, ds}| \, (|\lambda(t+\theta)| \, d|\mu|(\theta) + \, d_\theta |\eta|(t,\theta)) \right\} \\
&= M_{\varphi,\lambda_1} \Gamma_\lambda
\end{aligned} \tag{3.21}$$

which means $|y'(t)| \le M_{\varphi,\lambda_1} \Gamma_{\lambda_1}$ for $t \ge 0$.

One can show by induction, that $y'(t)$ satisfies

$$|y'(t)| \le M_{\varphi,\lambda_1} (\Gamma_\lambda)^n \quad \text{for } t \ge nr - r, \quad (n = 0, 1, 2, 3, \dots). \tag{3.22}$$

Since $0 \le \chi_{\lambda,t} < 1$, it follows that $y'(t)$ tends to zero as $t \to \infty$. So we proved (3.12) and hence (3.13) holds. In the following, we will show (3.11) holds.

To prove that $\lim_{t\to\infty} y(t)$ exists, we consider (3.22). For an arbitrary $t \ge 0$, we set $n = [t/r]+1$ (the greatest integer less than or equal to $t/r+1$), then from $n = [t/r]+1 \le t/r+1 \le [t/r]+2 = n+1$, we have $t/r \le n$. From (3.22),

$$|y'(t)| \le M_{\varphi,\lambda_1} (\Gamma_\lambda)^n \le M_{\varphi,\lambda_1} (\Gamma_\lambda)^{t/r} \quad \text{for } t \ge nr - r. \tag{3.23}$$

Now we use the Cauchy convergence criterion. For $t > T \ge 0$, from (3.23), we have

$$\begin{aligned}
|y(t) - y(T)| \le \int_T^t |y'(s)|\, ds \le \int_T^t M_{\varphi,\lambda_1} (\Gamma_\lambda)^{s/r}\, ds &= M_{\varphi,\lambda_1} \frac{r}{\ln \Gamma_\lambda} \left[(\Gamma_\lambda)^{s/r} \right]_{s=T}^{s=t} \qquad (3.24) \\
&= M_{\varphi,\lambda_1} \frac{r}{\ln \Gamma_\lambda} \left[(\Gamma_\lambda)^{t/r} - (\Gamma_\lambda)^{T/r} \right].
\end{aligned}$$

Let $T \to \infty$, we have $t \to \infty$, and by (3.25), we have

$$M_{\varphi,\lambda} \frac{r}{\ln \Gamma_\lambda} \left[(\Gamma_\lambda)^{t/r} - (\Gamma_\lambda)^{T/r} \right] \to 0;$$

and $\lim_{T\to\infty} |y(t) - y(T)| = 0$. The Cauchy convergence criterion implies the existence of $\lim_{t\to\infty} y(t)$. $\square$

Remark 3.2.1. *Under the conditions of Theorem 3.1.2, a solution of (3.5) can not grow faster than exponential; i.e., there exists a constant $M > 0$, such that*

$$|x(t)| \le M e^{\int_0^t \lambda(s)\, ds} \quad \textit{for } t \ge 0. \tag{3.25}$$

From (3.25), it is not difficult to show that:

(i) *Every solution of* (3.5) *is bounded if and only if* $\limsup_{t\to\infty}\int_0^t \lambda(s)\,ds < \infty$*;*

(ii) *Every solution of* (3.5) *tends to zero if and only if* $\limsup_{t\to\infty}\int_0^t \lambda(s)\,ds \to -\infty$*.*

Remark 3.2.2. *If the generalized characteristic equation* (3.9) *has a constant solution* $\lambda(t) = \lambda_0$*, then from Theorem* 3.1.2*,* $\lim_{t\to\infty} x(t)e^{-\lambda_0 t}$ *exists.*

3.3 Examples

Example 3.3.1. *Consider the linear differential equation with distributed delay*

$$x'(t) - \frac{1}{2}x'(t-1) = \int_{-1}^{0} \frac{x(t+\theta)}{2(t+\theta)}\,d\theta, \quad t > 1. \tag{3.26}$$

This equation can be written in the form (3.2) by setting $\mu(\theta) = -\frac{1}{2}$ for $\theta \le -1$, $\mu(\theta) = 0$ for $\theta > -1$, $\eta(t,\theta) = \ln t + \frac{1}{2}\ln(t+\theta)$ for $t > 1$ and $\theta \in [-1, 0]$. Since both $\theta \mapsto \eta(t,\theta)$ and $\theta \mapsto \mu(\theta)$ are increasing functions, $|\mu| = \mu, |\eta| = \eta$.

The generalized characteristic equation associated with (3.26) is

$$\lambda(t) = \frac{\lambda(t-1)}{2}\exp\left(-\int_{t-1}^{t}\lambda(s)\,ds\right) + \int_{-1}^{0}\frac{1}{2(t+\theta)}\exp\left(-\int_{t+\theta}^{t}\lambda(s)\,ds\right)d\theta,$$

which has a solution

$$\lambda(t) = 1/t. \tag{3.27}$$

For this $\lambda(t)$ and for $t > 1$, using the expression of $\chi_{\lambda,t}$, we obtain that

$$\begin{aligned}\chi_{\lambda,t} &= \frac{1}{2}\left(1 - \frac{1}{2t}\right) + \frac{1}{4t} + \int_{-1}^{0}\frac{-\theta}{2(t+\theta)}\exp\left[-\int_{t+\theta}^{t}\frac{ds}{s}\right]d\theta \\ &= \frac{1}{2} + \frac{1}{4(t)} \to \frac{1}{2} < 1 \quad \text{as } t \to \infty.\end{aligned}$$

Hence the hypothesis (3.10) of Theorem 3.1.2 is fulfilled. So we obtain that for each solution of (3.3.1)

$$\lim_{t\to\infty}\frac{x(t)}{t} \text{ exists}, \quad \lim_{t\to\infty}\left[\frac{x(t)}{t}\right]' = 0 \quad \text{and} \quad \lim_{t\to\infty}\frac{x'(t)}{t} = 0. \tag{3.28}$$

Example 3.3.2. *Consider the linear differential equation with distributed delay*

$$x'(t) - \frac{1}{p}x'(t-1) = \int_{-1}^{0}\frac{x(t+\theta)}{q(t+\varepsilon+\theta)}\,d\theta, \quad t > 1, \tag{3.29}$$

ε is any constant, p and q are positive constants such that $1/p + 1/q = 1$. This equation can be written in the form (3.2) by setting $\mu(\theta) = -\frac{1}{p}$ for $\theta \le -1$, $\mu(\theta) = 0$ for $\theta > -1$, $\eta(t,\theta) = \ln t + \frac{1}{q}\ln(t+\varepsilon+\theta)$ for $t > 1$ and $\theta \in [-1, 0]$. Since both $\theta \mapsto \eta(t,\theta)$ and $\theta \mapsto \mu(\theta)$ are increasing functions, $|\mu| = \mu, |\eta| = \eta$.

The generalized characteristic equation associated with (3.29) is

$$\lambda(t) = \frac{\lambda(t-1)}{p}\exp\left(-\int_{t-1}^{t}\lambda(s)\,ds\right) + \int_{-1}^{0}\frac{1}{q(t+\varepsilon+\theta)}\exp\left(-\int_{t+\theta}^{t}\lambda(s)\,ds\right)d\theta,$$

which has a solution

$$\lambda(t) = \frac{1}{t+\varepsilon}.$$

For this $\lambda(t)$ and for $t > 1$, using the expression of $\chi_{\lambda,t}$, we obtain that

$$\begin{aligned}\chi_{\lambda,t} &= \frac{1}{p}\left(1 - \frac{1}{2(t+\varepsilon)}\right) + \frac{1}{2p(t+\varepsilon)} + \int_{-1}^{0} \frac{-\theta}{2q(t+\varepsilon+\theta)} \exp\left[-\int_{t+\theta}^{t} \frac{ds}{s+\varepsilon}\right] d\theta \\ &= \frac{1}{p} + \frac{1}{2q(t+\varepsilon)} \to \frac{1}{p} < 1 \quad \text{as } t \to \infty.\end{aligned}$$

Hence the hypothesis (3.10) of Theorem 3.1.2 is fulfilled. So we obtain that for each solution of (3.3.1)

$$\lim_{t\to\infty} \frac{x(t)}{t} \text{ exists,} \quad \lim_{t\to\infty} \left[\frac{x(t)}{t}\right]' = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{x'(t)}{t} = 0. \tag{3.30}$$

Remark 3.3.3. *Note that if the generalized characteristic equation* (3.9) *has a solution is difficult to verify. Example* 3.3.2 *is an extension of Example* 3.3.1*, we added an* ε*, and the coefficients* $1/2$ *and* $1/2$ *changed to be* $1/p$ *and* $1/q$*, which has to be satisfied* $1/p + 1/q = 1$.

Example 3.3.4. *Consider the equation with variable delay*

$$x'(t) - \frac{2}{3}x'(t-1) = \frac{x(t-\tau(t))}{3(t+c-\tau(t))}, \quad t \geqslant t_0. \tag{3.31}$$

where $c \in \mathbb{R}$ *and* $\tau : [0,\infty) \to [-1,0]$ *is a continuous function such that* $t + c - \tau(t) > 0$ *for* $t \geqslant t_0$.

Equation (3.31) can be written in the form (3.2) by letting $\mu(\theta) = -\frac{2}{3}$ for $\theta \leq -1$, $\mu(\theta) = 0$ for $\theta > -1$, $\eta(t,\theta) = 0$ for $\theta < \tau(t)$, $\eta(t,\theta) = 1/3(t+c-\tau(t))$ for $\theta \geqslant \tau(t)$. Since both $\theta \mapsto \eta(t,\theta)$ and $\theta \mapsto \mu(\theta)$ are increasing functions, we have that $|\mu| = \mu$, $|\eta| = \eta$.

The generalzied characteristic equation associated with (3.31) is

$$\lambda(t) = \frac{2\lambda(t-1)}{3} \exp\left(-\int_{t-1}^{t} \lambda(s)ds\right) + \frac{1}{3(t+c-\tau(t))} \exp\left(-\int_{t-\tau(t)}^{t} \lambda(s)ds\right) \tag{3.32}$$

and we have that a solution of (3.32) is

$$\lambda(t) = \frac{1}{t+c}. \tag{3.33}$$

For (3.33), the left hand side of (3.10) reads

$$\begin{aligned}&\limsup_{t\to\infty}\left[\frac{2}{3}\left(1 - \frac{1}{t+c}\right) + \frac{1}{6(t+c)} + \int_{-1}^{0} (-\theta)|e^{-\int_{t-\theta}^{t}\lambda(s)ds}|d_\theta|\eta|(t,\theta)\right] \\ &= \limsup_{t\to\infty}\left[\frac{2}{3} - \frac{\tau(t)}{3(t+c)}\right] = \frac{2}{3} < 1.\end{aligned}$$

and hence hypothesis (3.10) of Theorem 3.1.2 is fulfilled and therefore, for each solution $x(t)$ of (3.31), we have that

$$\lim_{t\to\infty} \frac{x(t)}{t+c} \text{ exists,} \quad \text{and} \quad \lim_{t\to\infty} \left(\frac{x(t)}{t+c}\right)' = 0.$$

Manipulating further the limits in (3.31), we are able to establish that $x(t) = O(t)$ and $x'(t) = o(t)$ as $t \to \infty$.

3.4 Notes and remarks

A paper based on the contents of this chapter has been published in [15].

Dix et al. [32] studied the asymptotic behavior of solutions to a class of nonautonomous d-ifferential equation with discrete delays of the form

$$x'(t) = a(t)x(t) + \sum_{j=1}^{k} b_j(t)x(t - \tau_j), \quad t \geq 0$$

where the coefficients $a(t)$ and $b_j(t)$ are continuous real-valued functions on $[0, \infty)$, $\tau_j > 0$ for $j = 1, 2, \cdots, k$, by introducing the concept of the generalized characteristic equation and using an appropriate solution of this generalized characteristic equation. Existence of such a solution, however, is quite a restrictive condition. The basic idea in [32] is essentially originated in the work in Driver [37]. The extended results for asymptotic behavior of neutral delay differential equations can be found in Dix et al [33]. An asymptotic property of the solutions to second order linear nonautonomous delay differential equations is discussed in [107]. Cuevas and Frasson [26] provide a generalization of [32], as it can be applied for instance for retarded delay differential equations with distributed delays or discrete variable delays, as far as the delays are uniformly bounded. Our results in this chapter was motivated by the work in Cuevas and Frasson [26], we generalized the class of delay differential equations studied in Cuevas and Frasson [26] by adding a neutral term, the coefficient for the neutral term is restricted to be constant.