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Chapter 2

Asymptotic behavior of a class of autonomous neutral delay differential equations

In this chapter, three different methods to study the asymptotic behavior of a class of autonomous neutral delay differential equations are presented. Our approach is either based on methods from functional analysis, ordinary differential equations or fixed point theory. The relations of the method from functional analysis (called spectral method) and the method from ordinary differential equations (called ODE method) are addressed. If there are no neutral terms in the considered equations, a third method based on fixed point theory is introduced.

The organization of this chapter is as follows. In Section 2.2, the spectral approach is introduced and used to study the asymptotic behavior of the solutions of (2.1). In Section 2.3, the ODE approach is introduced to study the asymptotic behavior of solutions of (2.1). In Section 2.4, both approaches are analysed by investigating a number of examples. In Section 2.5, an approach based on fixed point theory is introduced and used to study the asymptotic behavior of (2.2). An application to the mechanical model of turning processes is presented in Section 2.6.

2.1 Introduction

In 1973, Driver, Sasser and Slater [35] studied asymptotic behavior, oscillation and stability of first order delay differential equations with small delay using an approach based on an ordinary differential equation (ODE) method. The key idea of the ODE approach is to transform the differential equation into a lower order equation by using a real root of the corresponding characteristic equation. Following this approach as presented in [35], a number of papers appeared in which the asymptotic behavior, oscillation and stability for first (or second or higher) order (neutral) delay differential equations, and integro-differential equations with unbounded delay as well as for delay difference equations were studied, see [51, 84, 101, 106, 105, 107]. A disadvantage of this ODE approach is that it does not lead to explicit formulas for the reduced lower order equations and that it only works if the characteristic equation has a real root.

In 2003, by using residue calculus and spectral theory, Frasson and Verduyn Lunel [39] presented a new approach to study the asymptotic behavior of neutral delay differential equations, the so-called spectral projection method. In this chapter, by studying asymptotic behavior of a class of second order neutral delay differential equations, we discuss the relations of the two approaches. We obtain that under the same assumptions, the ODE approach is equivalent to the spectral approach (see Section 2.4). However, the spectral approach has some advantages, since the conditions for the spectral method are weaker than those needed for the ODE method, as is illustrated by Example 2.4.2, and the asymptotic behavior of neutral delay differential equations can be presented by a general formula (see Theorem 2.2.6). Furthermore, by using the spectral approach, we can also study the asymptotic behavior of neutral delay differential equations with matrix coefficients.

In this chapter, we consider a specific class of second order neutral delay differential equations of the following form

$$\begin{cases} x''(t) + cx''(t-\tau) = p_1 x'(t) + p_2 x'(t-\tau) + q_1 x(t) + q_2 x(t-\tau), \\ x(t) = \phi(t), \quad -\tau \le t \le 0, \end{cases} \tag{2.1}$$

where $c, p_1, p_2, q_1, q_2 \in \mathbb{R}$, $\tau > 0$, the initial function ϕ is a given continuously differentiable real-valued function on the initial interval $[-\tau, 0]$.

A special case of system (2.1) is the retarded delay differential equation

$$x''(t) + ax'(t) + bx(t-r) + cx(t) = 0, \quad a, b, c \in \mathbb{R}, \quad r > 0, \tag{2.2}$$

which is often called a delayed oscillator, is well-studied in applications [59]. It appears, for example, as the basic governing equation of the regenerative model of machine tool chatter.

2.2 Asymptotic behavior by a spectral approach

Let $\mathcal{C} = C([-\tau, 0], \mathbb{C}^n)$ denote the Banach space of continuous functions endowed with the supremum norm. From the Riesz representation theorem it follows that every bounded linear mapping $L : \mathcal{C} \to \mathbb{C}^n$ can be represented by

$$L\varphi = \int_{-\tau}^{0} d\eta(\theta)\varphi(\theta),$$

where $\eta(\theta)$, $-\tau \le \theta \le 0$, is an $n \times n$-matrix whose elements are of bounded variation, normalized so that $\eta(0) = 0$ and η is continuous from the left on $(-\tau, 0)$ with values in the matrix space $\mathbb{C}^{n\times n}$. This set of functions is denoted by $\mathrm{NBV}([-\tau, 0], \mathbb{C}^{n\times n})$. For a function $x : [-\tau, \infty) \to \mathbb{C}^n$, we denote by $x_t \in \mathcal{C}$ the function $x_t(\theta) = x(t + \theta)$, $-\tau \le \theta \le 0$ and $t \ge 0$.

An initial value problem for a linear autonomous neutral delay differential equation is given by the following relation

$$\begin{cases} \frac{d}{dt} Dx_t = Lx_t, \qquad t \ge 0, \\ x_0 = \phi, \quad \phi \in \mathcal{C}, \end{cases} \tag{2.3}$$

where $D : \mathcal{C} \to \mathbb{C}^n$ is continuous, linear and atomic at zero, $L : \mathcal{C} \to \mathbb{C}^n$ is linear and continuous and, both operators are respectively, presented by

$$L\varphi = \int_{-\tau}^{0} d\eta(\theta)\varphi(\theta), \quad D\varphi = \varphi(0) - \int_{-\tau}^{0} d\mu(\theta)\varphi(\theta),$$

where $\eta, \mu \in \mathrm{NBV}([-\tau, 0], \mathbb{C}^{n\times n})$, and μ is continuous at zero. See Hale and Verduyn Lunel [51] for a detailed information.

For the second order neutral delay differential equation (2.1), let $y(t) = x'(t)$, then (2.1) can be written in the form

$$\begin{cases} x'(t) = y(t), \\ y'(t) + cy'(t-\tau) = p_1 y(t) + p_2 y(t-\tau) + q_1 x(t) + q_2 x(t-\tau). \end{cases}$$

Let $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, we have

$$X'(t) + CX'(t-\tau) = EX(t) + FX(t-\tau), \tag{2.4}$$

where

$$C = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ q_1 & p_1 \end{pmatrix} \quad and \quad F = \begin{pmatrix} 0 & 0 \\ q_2 & p_2 \end{pmatrix}.$$

By taking $\mu(\theta) = C$, for $\theta \le -\tau$, $\mu(\theta) = 0$, for $\theta > -\tau$, and $\eta(\theta) = -F$, for $\theta \le -\tau$, $\eta(\theta) = 0$, for $-\tau < \theta < 0$, $\eta(\theta) = E$, for $\theta \ge 0$, (2.1) can be written in the form (2.3).

Throughout this chapter, a continuous real-valued function x defined on the interval $[-\tau, \infty)$ is said to be a solution of the initial value problem (2.1) if x satisfies (2.1) in the mild sense, see Lemma 2.2.1. It is well known (see [35]) that for any given initial function ϕ, there exists a unique solution of the initial value problem (2.1).

Given the solution $x(\phi)$ of the initial value problem (2.3), the solution operator $T(t) : \mathcal{C} \to \mathcal{C}$ is defined by the relation

$$T(t)\phi = x_t(.;\phi), \quad t \ge 0.$$

Lemma 2.2.1. (Hale and Verduyn Lunel [51]) *The solution operator $T(t)$ is a C_0-semigroup on $\mathcal{C}$ with infinitesimal generator*

$$\begin{cases} D(A) = \left\{ \phi \in \mathcal{C} \mid \frac{d\phi}{d\theta} \in \mathcal{C},\ D\frac{d\phi}{d\theta} = L\phi \right\} \\ A\phi = \frac{d\phi}{d\theta} \end{cases} \tag{2.5}$$

Lemma 2.2.2. (Hale and Verduyn Lunel [51]) *If A is defined by equation (2.5), then $\sigma(A) = P_{\sigma(A)}$ and $\lambda \in \sigma(A)$ if and only if λ satisfies the characteristic equation $det\triangle(\lambda) = 0$, where*

$$\triangle(\lambda) = \lambda I - \int_{-\tau}^{0} \lambda e^{\lambda\theta} d\mu(\theta) - \int_{-\tau}^{0} e^{\lambda\theta} d\eta(\theta). \tag{2.6}$$

Here $P_{\sigma(A)}$ denotes the point spectrum of A.

It is well known that there is a close connection between the spectral properties of the inifinitesimal generator A and the characteristic matrix $\triangle(\lambda)$ given by (2.6). In particular, the geometric multiplicity d_λ is equal to the dimension of the null space of $\triangle(z)$ at $z = \lambda$, and the algebraic multiplicity m_λ is equal to the multiplicity of $z=\lambda$ as a zero of $\det \triangle(\lambda)=0$. Furthermore, the generalized eigenspace at λ is given by

$$\mathcal{M}_\lambda = \mathcal{N}(\lambda I - A)^{k_\lambda},$$

where k_λ denotes the order of $z=\lambda$ as a pole of $\triangle(z)^{-1}$. See Kaashoek and Verduyn Lunel [66] for more information.

Lemma 2.2.3. (Hale and Verduyn Lunel [51]) *For any λ in $\sigma(A)$, the generalized eigenspace $\mathcal{M}_\lambda(A)$ is finite dimensional and there is an integer k such that $\mathcal{M}_\lambda(A)=\mathcal{N}((\lambda I-A)^k)$ and we have a direct sum decomposition*

$$\mathcal{C} = \mathcal{N}((\lambda I - A)^k) \oplus \mathcal{R}((\lambda I - A)^k).$$

From the spectral theory [29, 51], it follows that the spectral projection onto $\mathcal{M}_\lambda(A)$ along $\mathcal{R}((\lambda I-A)^k)$ can be represented by a Dunford integral

$$P_\lambda = \frac{1}{2\pi i}\int_{\Gamma_\lambda} (zI-A)^{-1}\,dz, \tag{2.7}$$

where Γ_λ is a small circle such that λ is the only singularity of $(zI-A)^{-1}$ inside Γ_λ.

Definition 2.2.4. *An eigenvalue λ_d is called a dominant eigenvalue of A, if there exists a $\varepsilon>0$, such that if λ is another eigenvalue of A, then $Re\lambda < Re\lambda_d - \varepsilon$.*

Consider the scalar case of initial value problem (2.3), the characteristic equation $\Delta(z)$ is given by (2.6). Define the auxiliary function $\chi:\mathbb{C}\to[0,\infty)$ by

$$\chi(z) = \int_{-\tau}^{0}(1-\theta|z|)e^{z\theta}\,dV(\mu)(\theta) + \int_{-\tau}^{0}(-\theta)e^{z\theta}\,dV(\eta)(\theta), \tag{2.8}$$

where $V(\mu)(\theta)$ denotes the total variation function of μ on $[-\tau,\theta]$ for each θ in $(-\tau,0]$.

Theorem 2.2.5. (Frasson [40]) *Suppose that $z_0\in\mathbb{C}$ is a zero of $det\Delta(z)$ in (2.6). If $\chi(z_0)<1$, then z_0 is a simple dominant zero of $\Delta(z)$.*

Next, we provide the main result of Frasson and Verduyn Lunel [39], which presents the explicit representation of asymptotic behavior of neutral delay differential equations.

Theorem 2.2.6. (Frasson and Verduyn Lunel [39]) *Let A be given by (2.5), if A has a simple and dominant eigenvalue λ_d, then there exists positive numbers ε and M such that*

$$\|e^{-\lambda_d t}T(t)\phi - P_{\lambda_d}\phi\| \le Me^{-\varepsilon t},$$

and

$$\lim_{t\to\infty} e^{-\lambda_d t}T(t)\phi = e^{\lambda_d \cdot}\left[\frac{d}{dz}\det\triangle(\lambda_d)\right]^{-1}\operatorname{adj}\triangle(\lambda_d)K(\lambda_d)\phi.$$

Furthermore, if $x(t)=x(\cdot,\phi)$ denotes the solution of (2.3) with initial data $x_0=\phi$, then

$$\lim_{t\to\infty} e^{-\lambda_d t}x(t) = \left[\frac{d}{dz}\det\triangle(\lambda_d)\right]^{-1}\operatorname{adj}\triangle(\lambda_d)K(\lambda_d)\phi,$$

where $\operatorname{adj}\triangle(\lambda_d)$ *denotes the matrix of cofactors of* $\triangle(\lambda_d)$,

$$K(\lambda_d)\phi = D\phi + \int_{-\tau}^{0}(\lambda_d d\mu(\theta) + d\eta(\theta))e^{\lambda_d\theta}\int_{\theta}^{0} e^{-\lambda_d s}\phi(s)\,ds.$$

Combining Theorem 2.2.5 with Theorem 2.2.6, we arrive at

Theorem 2.2.7. *Let $x(\cdot)$ be the solution of (2.3) subjected to the initial condition $x_0 = \phi \in C([-\tau, 0], \mathbb{R})$. If λ_d is a real zero of characteristic equation $\Delta(z)$ given by (2.6) such that $\chi(\lambda_d) < 1$, where $\chi(\cdot)$ is given by (2.8), then the asymptotic behavior of $x(\cdot)$ is given by*

$$\lim_{t\to\infty} e^{-\lambda_d t}x(t) = \frac{1}{H(\lambda_d)}K(\lambda_d)\phi,$$

where

$$\begin{aligned} H(\lambda_d) &= 1 - \int_{-\tau}^{0} e^{\lambda_d\theta}\,d\mu(\theta) - \int_{-\tau}^{0} \theta e^{\lambda_d\theta}\,(\lambda_d d\mu(\theta) + d\eta(\theta)), \\ K(\lambda_d)\phi &= M\psi + \int_{-\tau}^{0} (\lambda_d\,d\mu(\theta) + d\eta(\theta))e^{\lambda_d\theta}\int_{\theta}^{0} e^{-\lambda_d s}\psi(s)\,ds. \end{aligned}$$

Note that the result of Theorem 2.2.7 is consistent with the result in [101] which was obtained by using ODE method.

Example 2.2.8.

$$\begin{cases} x'(t) + cx'(t-\sigma) = ax(t) + bx(t-\tau), \\ \\ x(t) = \phi(t), \quad -\tau \le t \le 0. \end{cases} \tag{2.9}$$

The characteristic equation of (2.9) is

$$\Delta(\lambda) = \lambda(1 + ce^{-\lambda\sigma}) - a - be^{-\lambda\tau}. \tag{2.10}$$

Note that the characteristic equation (2.10) may have no real root. Suppose that the characteristic equation (2.10) has a real root λ_0 which satisfies

$$|c|(1 + |\lambda_0|\sigma)e^{-\lambda_0\sigma} + |b|\tau e^{-\lambda_0\tau} < 1,$$

then by Theorem 2.2.5, λ_0 is a simple dominant root of (2.10). Hence, applying Theorem 2.2.7, we obtain that

$$\lim_{t\to\infty} e^{-\lambda_0 t}x(\phi; t) = \frac{K(\lambda_0; \phi)}{H(\lambda_0)}, \tag{2.11}$$

where

$$\begin{aligned} K(\lambda_0; \phi) &= \phi(0) + c\phi(-\sigma) - c\lambda_0 e^{-\lambda_0\sigma}\int_{-\sigma}^{0} e^{\lambda_0 s}\phi(s)\,ds + be^{-\lambda_0\tau}\int_{-\tau}^{0} e^{\lambda_0 s}\phi(s)\,ds, \\ H(\lambda_0) &= 1 + c(1 - \lambda_0\sigma)e^{-\lambda_0\sigma} + b\tau e^{-\lambda_0\tau}, \end{aligned}$$

(2.11) is consistent with the result in Kordonis et al. [84] which was obtained by using ODE method .

Now, we use Theorem 2.2.7 to study the asymptotic behavior of (2.1). Let the initial condition associated with (2.4) be given by

$$X_0 = \begin{pmatrix} \phi \\ \phi' \end{pmatrix} \in C([-\tau, 0], \mathbb{R}^2),$$

The characteristic matrix corresponding to (2.4) is given by

$$\triangle(z) = zI + ze^{-\tau z}C - E - Fe^{-\tau z} = \begin{pmatrix} z & -1 \\ -q_1 - q_2e^{-\tau z} & z + cze^{-\tau z} - p_1 - p_2e^{-\tau z} \end{pmatrix}$$

so the characteristic equation is $\det \triangle(z) = z^2 + cz^2e^{-\tau z} - (p_1 + p_2e^{-\tau z})z - q_2e^{-\tau z} - q_1$. Note that the characteristic equation may have no simple dominant zero. Suppose that there exists a simple dominant zero λ_d of the characteristic equation $\det \triangle(z) = 0$, by Theorem 2.2.6, we have

$$\begin{aligned} \lim_{t\to\infty} e^{-\lambda_d t}X(t) &= \lim_{t\to\infty} \begin{pmatrix} e^{-\lambda_d t}x(t) \\ e^{-\lambda_d t}y(t) \end{pmatrix} = \left[\frac{d}{dz}\det \triangle(\lambda_d)\right]^{-1} \mathrm{adj}\triangle(\lambda_d)K(\lambda_d)\phi \\ &= \begin{pmatrix} \frac{\lambda_d + c\lambda_d e^{-\tau\lambda_d} - p_1 - p_2e^{-\tau\lambda_d}}{\beta(\lambda_d)} & \frac{1}{\beta(\lambda_d)} \\ \frac{q_1 + q_2e^{-\tau\lambda_d}}{\beta(\lambda_d)} & \frac{\lambda_d}{\beta(\lambda_d)} \end{pmatrix} \times A \end{aligned}$$

where

$$A = \begin{pmatrix} \phi(0) \\ \phi'(0) + c\phi'(-\tau) + \int_{-\tau}^0 (p_2 - c\lambda_d)e^{-\lambda_d(s+\tau)}\phi'(s)\,ds + \int_{-\tau}^0 q_2e^{-\lambda_d(s+\tau)}\phi(s)\,ds \end{pmatrix}.$$

It follows that

$$\begin{aligned} \lim_{t\to\infty} e^{-\lambda_d t}x(t) &= \frac{1}{\beta(\lambda_d)}\Big[(\lambda_d + c\lambda_d e^{-\tau\lambda_d} - p_1 - p_2e^{-\tau\lambda_d})\phi(0) + \phi'(0) + c\phi'(-\sigma) \\ &\quad + \int_{-\tau}^0 (p_2 - c\lambda_d)e^{-\lambda_d(s+\tau)}\phi'(s)\,ds + \int_{-\tau}^0 q_2e^{-\lambda_d(s+\tau)}\phi(s)\,ds\Big], \end{aligned}$$

where $\beta(\lambda_d) = 2\lambda_d + (2c\lambda_d - c\tau\lambda_d^2 - p_2 + p_2\tau\lambda_d + q_2\tau)e^{-\tau\lambda_d} - p_1 \neq 0$.

The next theorem gives a result similar to Theorem 2.2.6, in case that the real dominant eigenvalue is not simple.

Theorem 2.2.9. (Frasson [42]) *Let λ_d be a real dominant zero of $\det \triangle(z)$ of geometric multiplicity $n \geq 1$. If $x(t) = x(t;\phi)$ denote the solution of (2.1) with initial data $x_0 = \phi$, then the large time behaviour as a function of the initial data ϕ is described as follows.*
1. If $P_{\lambda_d}\phi \neq 0$, then

$$\lim_{t\to\infty} \frac{1}{t^m}e^{-\lambda_d t}x(t) = q_m(n, \lambda_d, \phi),$$

where $m = \max\{j \in 0, 1, 2, ...n-1 : q_j(n, \lambda_d, \phi) \neq 0\}$, q_j is given by

$$q_j(n, \lambda, \phi) = \frac{1}{j!}\sum_{k=j}^{n-1} \frac{D^{n-1-k}K(\lambda)}{(n-1-k)!}\frac{D_1^{k-j}H(\lambda, \phi)}{(k-j)!}.$$

Furthermore, for integer $n \geq 1$, the $n-th$ Fréchet derivative of $H(\lambda, \phi)$ with respect to the first variable is given by

$$\begin{aligned} D_1^n H(z, \phi) &= (-1)^{n+1}n\int_0^r d\mu(\theta)\int_0^\theta \tau^{n-1}e^{-z\tau}\phi(\tau - \theta)\,d\tau \\ &\quad + (-1)^n z\int_0^r d\mu(\theta)\int_0^\theta \tau^n e^{-z\tau}\phi(\tau - \theta)\,d\tau \\ &\quad + (-1)^n\int_0^r d\eta(\theta)\int_0^\theta \tau^n e^{-z\tau}\phi(\tau - \theta)\,d\tau. \end{aligned}$$

2. If $P_{\lambda_d}\phi = 0$, then

$$\lim_{t\to\infty} e^{-\lambda_d t}x(t) = 0.$$

2.3 An ODE approach to asymptotic behavior

In this section, we study the asymptotic behavior of the solutions of a class of second order neutral delay differential equation (2.1) by employing ODE approach. An estimation of solutions of the initial value problem (2.1) is made. As a consequence of this result, the sufficient conditions for stability, the asymptotic stability and instability of the trivial solution are presented.

The characteristic equation of (2.1) is

$$\lambda^2 + c\lambda^2 e^{-\lambda\tau} = p_1\lambda + p_2\lambda e^{-\lambda\tau} + q_1 + q_2 e^{-\lambda\tau}. \tag{2.12}$$

Suppose that λ_0 is a real solution of the characteristic equation (2.12), we consider the first order neutral delay differential equation

$$\begin{aligned} &z'(t) + ce^{-\lambda_0\tau}z'(t-\tau) + (2\lambda_0 - p_1)z(t) + (2c\lambda_0 - p_2)e^{-\lambda_0\tau}z(t-\tau) \\ &= (p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0} z(s+t)\,ds. \end{aligned} \tag{2.13}$$

With (2.13), we associate the equation

$$\mu + (c\mu + 2c\lambda_0 - p_2)e^{-\tau(\lambda_0+\mu)} + 2\lambda_0 - p_1 - (p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0} e^{\mu s}\,ds = 0, \tag{2.14}$$

which is said to be the *second characteristic equation*, and it is obtained from (2.13) by seeking solutions of the form $z(t) = e^{\mu t}$.

Now, we present a proposition, which plays a crucial role in obtaining our main result presented in Theorem 2.3.2. This proposition essentially estabishes a transformation (via a solution of the characteristic equation (2.12)) of the second order neutral delay differential equation (2.1) into the first order neutral delay differential equation (2.13).

Proposition 2.3.1. *Suppose λ_0 is a real root of the characteristic equation (2.12), and let*

$$\beta(\lambda_0) = 2\lambda_d + (2c\lambda_0 - c\tau\lambda_0^2 - p_2 + p_2\tau\lambda_0 + q_2\tau)e^{-\tau\lambda_0} - p_1.$$

Suppose that $\beta(\lambda_0) \neq 0$, then a continuous real-valued function x defined on the interval $[-\tau,\infty)$ is the solution of the initial value problem (2.1) on $[0,\infty)$ if and only if z defined by

$$z(t) = e^{-\lambda_0 t}x(t) - \frac{K(\lambda_0,\phi)}{\beta(\lambda_0)} \qquad \text{for} \qquad t \geq 0, \tag{2.15}$$

is the solution of the neutral delay differential equation (2.13) with the initial condition

$$z(t) = e^{-\lambda_0 t}\phi(t) - \frac{K(\lambda_0,\phi)}{\beta(\lambda_0)} \qquad \text{for} \qquad -\tau \leq t \leq 0, \tag{2.16}$$

where $x(t) = \phi(t)$ *on* $[-\tau, 0]$ *and*

$$\begin{aligned} K(\lambda_0, \phi) &= \phi'(0) + (\lambda_0 - p_1)\phi(0) + c\phi'(-\tau) + c\lambda_0\phi(-\tau) - p_2\phi(-\tau) \\ &\quad -(p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0} e^{-\lambda_0 s}\phi(s)\,ds. \end{aligned}$$

Proof. Let x be the solution of the initial value problem (2.1) for $t \geq 0$ with $x(t) = \phi(t)$ for $-\tau \leq t \leq 0$. Define

$$y(t) = e^{-\lambda_0 t}x(t) \qquad \text{for} \qquad t \geq -\tau.$$

Using the fact that λ_0 is a real root of the characteristic equation (2.12), we have for every $t \geq 0$,

$$\begin{aligned} &[y'(t) + ce^{-\lambda_0\tau}y'(t-\tau) + (2\lambda_0 - p_1)y(t) + (2c\lambda_0 - p_2)e^{-\lambda_0\tau}y(t-\tau)]' \\ &\quad = (p_1\lambda_0 + q_1 - \lambda_0^2)y(t) + (p_2\lambda_0 + q_2 - c\lambda_0^2)e^{-\lambda_0\tau}y(t-\tau) \end{aligned} \tag{2.17}$$

with the initial condition satisfies

$$y(t) = e^{-\lambda_0 t}\phi(t) \qquad \text{for} \qquad -\tau \leq t \leq 0. \tag{2.18}$$

By integrating (2.17), and using the initial condition (2.18), we have

$$\begin{aligned} &y'(t) + ce^{-\lambda_0\tau}y'(t-\tau) + (2\lambda_0 - p_1)y(t) + (2c\lambda_0 - p_2)e^{-\lambda_0\tau}y(t-\tau) \\ &\quad = (p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0} y(s+t)\,ds + K(\lambda_0, \phi) \end{aligned} \tag{2.19}$$

for all $t \geq 0$, where $K(\lambda_0, \phi)$ is defined as in Proposition 2.3.1.

Now we suppose that $\beta(\lambda_0) \neq 0$ and define

$$z(t) = y(t) - \frac{K(\lambda_0, \phi)}{\beta(\lambda_0)} \qquad \text{for} \qquad t \geq -\tau,$$

by the definition of $\beta(\lambda_0)$, we obtain that y satisfies (2.19) if and only if z satisfies (2.15) for all $t \geq 0$. Moreover, the initial condition (2.18) is equivalent to (2.16). □

An estimate of the solution of initial value problem (2.1) will be given in the following theorem.

Theorem 2.3.2. *Suppose* λ_0 *is a real root of the characteristic equation* (2.12)*, and let* $\beta(\lambda_0)$ *and* $K(\lambda_0, \phi)$ *be defined as in Proposition* 2.3.1*. Suppose that* $\beta(\lambda_0) \neq 0$*, let* μ_0 *be a real root of the characteristic equation*(2.30)*, and set*

$$\begin{aligned} \gamma(\lambda_0, \mu_0) &= 1 + ce^{-(\lambda_0+\mu_0)\tau} - \tau(c\mu_0 + 2c\lambda_0 - p_2)e^{-(\lambda_0+\mu_0)\tau} \\ &\quad -(p_1\lambda_0 + q_1 - \lambda_0^2)\mu_0^{-2}(\mu_0\tau e^{-\mu_0\tau} + e^{-\mu_0\tau} - 1). \end{aligned}$$

Define

$$\begin{aligned} H(\lambda_0, \mu_0, \phi) &= \phi(0) + c\phi(-\tau) + (p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0} e^{\mu_0 s}\int_{s}^{0} e^{-(\lambda_0+\mu_0)u}\phi(u)\,du\,ds \\ &\quad -(c\mu_0 + 2c\lambda_0 - p_2)e^{-(\lambda_0+\mu_0)\tau}\int_{-\tau}^{0} e^{-(\lambda_0+\mu_0)s}\phi(s)\,ds \\ &\quad -\frac{K(\lambda_0, \phi)}{\beta(\lambda_0)}\Bigg[1 + ce^{-\lambda_0\tau} + (p_1\lambda_0 + q_1 - \lambda_0^2)\mu_0^{-2}(1 - e^{-\mu_0\tau} - \mu_0\tau) \\ &\quad -(c\mu_0 + 2c\lambda_0 - p_2)\mu_0^{-1}(1 - e^{-\mu_0\tau})e^{-(\lambda_0+\mu_0)\tau}\Bigg]. \end{aligned}$$

We assume that the real roots λ_0 *and* μ_0 *have the following property*

$$\chi_{\lambda_0,\mu_0} \quad := \quad |c|e^{-(\lambda_0+\mu_0)\tau} + \tau|p_1 - \mu_0 - 2\lambda_0| + \mu_0^{-2}(\mu_0\tau + e^{-\mu_0\tau} - 1)\left|p_1\lambda_0 + q_1 - \lambda_0^2\right| < 1.$$

Then for any $\phi \in C([-\tau, 0], \mathbb{R})$*, the solution* x *of* (2.1) *satisfies*

$$\left| e^{-(\mu_0+\lambda_0)t}x(t) - e^{-\mu_0 t}\frac{K(\lambda_0,\phi)}{\beta(\lambda_0)} - \frac{H(\lambda_0,\mu_0,\phi)}{\gamma(\lambda_0,\mu_0)} \right| \leq M(\lambda_0,\mu_0;\phi)\chi_{\lambda_0,\mu_0},$$

where

$$M(\lambda_0,\mu_0;\phi) = \max_{-\tau\leq t\leq 0}\left| e^{-\mu_0 t}\left(e^{-\lambda_0 t}\phi(t) - \frac{K(\lambda_0,\phi)}{\beta(\lambda_0)}\right) - \frac{H(\lambda_0,\mu_0;\phi)}{\gamma(\lambda_0,\mu_0)} \right|, \quad \text{for} \quad t \geq 0.$$

Proof. Note that $\mu_0 = 0$ is a root of (2.30) if and only if $2\lambda_0 - p_1 + (2c\lambda_0 - p_2)e^{-\tau\lambda_0} - (p_1\lambda_0 + q_1 - \lambda_0^2)\tau = 0$, from the definition of $\beta(\lambda_0)$, we obtain that if zero is a root of (2.30) if and only if $\beta(\lambda_0) = 0$. Hence, if we assume that $\beta(\lambda_0) \neq 0$, then we always have $\mu_0 \neq 0$.

From the assumption that $|\chi_{\lambda_0,\mu_0}| < 1$, we conclude $\gamma(\lambda_0,\mu_0) > 0$. Suppose that x is the solution of the initial value problem (2.1) with $x(\theta) = \phi(\theta)$ for $-\tau \leq \theta \leq 0$, by Proposition 2.3.1, the fact that x is the solution of the initial value problem (2.1) is equivalent to the fact that z is the solution of the delay differential equation (2.15) which satisfies the initial condition (2.16). Set

$$w(t) = e^{-\mu_0 t}z(t) \qquad \text{for} \qquad t \geq -\tau,$$

then by using the fact that μ_0 is a real root of the characteristic equation (2.30), we obtain, for every $t \geq 0$,

$$\begin{aligned}[w(t) + ce^{-(\lambda_0+\mu_0)\tau}w(t-\tau)]' \quad &= \quad (p_1 - \mu_0 - 2\lambda_0)w(t) - (c\mu_0 + 2c\lambda_0 - p_2)e^{-(\lambda_0+\mu_0)\tau}w(t-\tau) \\ &\quad +(p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0} e^{\mu_0 s}w(s+t)\,ds, \end{aligned} \tag{2.20}$$

and the initial condition satisfies

$$w(t) = e^{-(\mu_0+\lambda_0)t}\phi(t) - e^{-\mu_0 t}\frac{K(\lambda_0,\phi)}{\beta(\lambda_0)} \qquad \text{for} \qquad -\tau \leq t \leq 0. \tag{2.21}$$

By integrating (2.20) and using the initial condition of (2.21), we obtain

$$\begin{aligned} w(t) + ce^{-(\lambda_0+\mu_0)\tau}w(t-\tau) \quad &= \quad (p_1 - \mu_0 - 2\lambda_0)\int_{t-\tau}^{t} w(s)\,ds \\ &\quad +(p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0} e^{\mu_0 s}\int_{t-\tau}^{s+t} w(u)\,du\,ds + H(\lambda_0,\mu_0;\phi), \end{aligned} \tag{2.22}$$

for all $t \geq 0$, where $H(\lambda_0,\mu_0;\phi)$ is defined in Theorem 2.3.2.

Define

$$v(t) = w(t) - \frac{H(\lambda_0,\mu_0;\phi)}{\gamma(\lambda_0,\mu_0)} \qquad \text{for} \qquad t \geq -\tau,$$

then by the definition of $\gamma(\lambda_0,\mu_0)$ in Theorem 2.3.2, we obtain that the fact that w satisfies (2.22) is equivalent to the fact that v satisfies the following equation

$$\begin{aligned} v(t)+ce^{-(\lambda_0+\mu_0)\tau}v(t-\tau) &= (p_1-\mu_0-2\lambda_0)\int_{t-\tau}^{t} v(s)\,ds \\ &\quad +(p_1\lambda_0+q_1-\lambda_0^2)\int_{-\tau}^{0} e^{\mu_0 s}\int_{t-\tau}^{s+t} v(u)\,du\,ds, \end{aligned} \tag{2.23}$$

for all $t \geq 0$. Moreover, the initial condition is equivalent to

$$v(t) = e^{-(\mu_0+\lambda_0)t}\phi(t) - e^{-\mu_0 t}\frac{K(\lambda_0,\phi)}{\beta(\lambda_0)} - \frac{H(\lambda_0,\mu_0;\phi)}{\gamma(\lambda_0,\mu_0)} \qquad \text{for} \qquad -\tau \leq t \leq 0. \tag{2.24}$$

Define

$$M(\lambda_0,\mu_0;\phi) := \max_{-\tau\leq t\leq 0}\left| e^{-(\mu_0+\lambda_0)t}\phi(t) - e^{-\mu_0 t}\frac{K(\lambda_0,\phi)}{\beta(\lambda_0)} - \frac{H(\lambda_0,\mu_0;\phi)}{\gamma(\lambda_0,\mu_0)}\right|.$$

In view of (2.24), we have

$$|v(t)| \leq M(\lambda_0,\mu_0;\phi) \qquad \text{for} \qquad -\tau \leq t \leq 0.$$

We will next show that $M(\lambda_0,\mu_0;\phi)$ is also a bound of v on the whole positive half line. For this purpose, we take an arbitrary $\varepsilon > 0$ and claim that $|v(t)| < M(\lambda_0,\mu_0;\phi)+\varepsilon$ for $t \geq -\tau$. Indeed, suppose that there exists a point $t_0 > 0$ such that

$$\begin{aligned} |v(t)| &< M(\lambda_0,\mu_0;\phi)+\varepsilon \qquad \text{for} \qquad -\tau \leq t < t_0, \\ |v(t_0)| &= M(\lambda_0,\mu_0;\phi)+\varepsilon. \end{aligned} \tag{2.25}$$

Then by (2.23) and the definition of χ_{λ_0,μ_0}, we have

$$\begin{aligned} M(\lambda_0,\mu_0;\phi)+\varepsilon &= |v(t_0)| \\ &\leq |c|e^{-(\lambda_0+\mu_0)\tau}|v(t_0-\tau)| + |p_1-\mu_0-2\lambda_0|\int_{t_0-\tau}^{t_0}|v(s)|\,ds \\ &\quad +|p_1\lambda_0+q_1-\lambda_0^2|\int_{-\tau}^{0}e^{\mu_0 s}\int_{t_0-\tau}^{s+t}|v(u)|\,du\,ds \\ &\leq (M(\lambda_0,\mu_0;\phi)+\varepsilon)\Big(|c|e^{-(\lambda_0+\mu_0)\tau}+\tau|p_1-\mu_0-2\lambda_0| \\ &\quad +\mu_0^{-2}(\mu_0\tau+e^{-\mu_0\tau}-1)\left|p_1\lambda_0+q_1-\lambda_0^2\right|\Big) \\ &= (M(\lambda_0,\mu_0;\phi)+\varepsilon)\,\chi_{\lambda_0,\mu_0} < M(\lambda_0,\mu_0;\phi)+\varepsilon, \end{aligned}$$

and we arrive at a contradiction. This implies that our claim is true and since ε is arbitrary, it follows that $|v(t)| \leq M(\lambda_0,\mu_0;\phi)$ for $t \geq -\tau$. Together with (2.23), we arrive at

$$\begin{aligned} |v(t)| &\leq |c|e^{-(\lambda_0+\mu_0)\tau}|v(t-\tau)| + |p_1-\mu_0-2\lambda_0|\int_{t-\tau}^{t}|v(s)|\,ds \\ &\quad +|p_1\lambda_0+q_1-\lambda_0^2|\int_{-\tau}^{0}e^{\mu_0 s}\int_{t-\tau}^{s+t}|v(u)|\,du\,ds, \\ &\leq M(\lambda_0,\mu_0;\phi)\Big(|c|e^{-(\lambda_0+\mu_0)\tau}+\tau|p_1-\mu_0-2\lambda_0| \\ &\quad +\mu_0^{-2}(\mu_0\tau+e^{-\mu_0\tau}-1)\left|p_1\lambda_0+q_1-\lambda_0^2\right|\Big) \\ &= M(\lambda_0,\mu_0;\phi)\chi_{\lambda_0,\mu_0} < M(\lambda_0,\mu_0;\phi), \end{aligned}$$

for all $t \geq 0$. This implies

$$|v(t)| = \left| e^{-(\mu_0+\lambda_0)t} x(t) - e^{-\mu_0 t} \frac{K(\lambda_0, \phi)}{\beta(\lambda_0)} - \frac{H(\lambda_0, \mu_0; \phi)}{\gamma(\lambda_0, \mu_0)} \right| \leq M(\lambda_0, \mu_0; \phi) \chi_{\lambda_0, \mu_0},$$

for all $t \geq 0$. This completes the proof of Theorem 2.3.2. □

Using the result of Theorem 2.3.2, we can next discuss the asymptotic behavior of the solution of initial value problem (2.1).

Theorem 2.3.3. *Suppose λ_0 and μ_0 are real roots of the characteristic equations (2.12) and (2.30), respectively. Let $\beta(\lambda_0), \chi_{\lambda_0,\mu_0}, \gamma(\lambda_0, \mu_0)$ be defined as in Proposition 2.3.1 and Theorem 2.3.2. Then for any $\phi \in C([-\tau, 0], \mathbb{R})$, the solution x of initial value problem (2.1) with $x(\theta) = \phi(\theta)$ for $-\tau \leq \theta \leq 0$ satisfies*

$$\lim_{t\to\infty} e^{-(\mu_0+\lambda_0)t} x(t) - e^{-\mu_0 t} \frac{K(\lambda_0, \phi)}{\beta(\lambda_0)} = \frac{H(\lambda_0, \mu_0; \phi)}{\gamma(\lambda_0, \mu_0)},$$

where $K(\lambda_0, \phi), \beta(\lambda_0), H(\lambda_0, \mu_0; \phi), \gamma(\lambda_0, \mu_0)$ are given in Proposition 2.3.1 and Theorem 2.3.2 respectively.

Proof. By the definition of x, y, z, w and v, we have to prove that

$$\lim_{t\to\infty} v(t) = 0.$$

From Theorem 2.3.2, one can show by induction that v satisfies

$$|v(t)| \leq M(\lambda_0, \mu_0; \phi)(\chi_{\lambda_0,\mu_0})^n \quad \text{for} \quad \text{all} \quad t \geq n\tau - \tau. \tag{2.26}$$

Since $0 \leq \chi_{\lambda_0,\mu_0} < 1$, thus from (2.26), we obtain that v tends to zero as $t \to \infty$. □

Definition 2.3.4. *The trivial solution of (2.1) is said to be stable if for any $t_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|x_{t_0}\| < \delta$ implies $|x(t)| < \varepsilon$ for $t \geq t_0$. The solution is said to be asymptotically stable if it is stable and for any $t_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there exists a $\delta_a = \delta_a(t_0, \varepsilon) > 0$ such that $\|x_{t_0}\| < \delta_a$ implies $\lim_{t\to\infty} x(t) = 0$.*

As a consequence of Theorem 2.3.2 and Theorem 2.3.3, we have the following stability criterion.

Theorem 2.3.5. *Let λ_0 and μ_0 be real roots of the characteristic equations (2.12) and (2.30), and let $\beta(\lambda_0), \chi_{\lambda_0,\mu_0}, \gamma(\lambda_0, \mu_0)$ be defined as in Proposition 2.3.1 and Theorem 2.3.2 respectively, and satisfy the conditions in Theorem 2.3.2. Then for any $\phi \in C([-\tau, 0], \mathbb{R})$, the solution x of (2.1) with $x(\theta) = \phi(\theta)$ for $-\tau \leq \theta \leq 0$ satisfies*

$$\begin{aligned} |x(t)| \leq{} & \frac{k_{\lambda_0}}{|\beta(\lambda_0)|} N(\lambda_0, \mu_0; \phi) e^{\lambda_0 t} + \left[\frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0, \mu_0)|} \right. \\ & \left. + \left(1 + \frac{K_{\lambda_0} e_{\mu_0}}{|\beta(\lambda_0)|} + \frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0, \mu_0)|}\right) \chi_{\lambda_0,\mu_0} \right] N(\lambda_0, \mu_0; \phi) e^{(\lambda_0+\mu_0)t}, \end{aligned}$$

where

$$\begin{aligned}
k_{\lambda_0} &= 1+|c|+|\lambda_0-p_1|+|c||\lambda_0|+|p_2|+|p_1\lambda_0+q_1-\lambda_0^2|\tau,\\
e_{\mu_0} &= \max_{-\tau\le t\le 0}\{e^{-\mu_0 t}\},\\
h_{\lambda_0,\mu_0} &= 1+|c|+|p_1\lambda_0+q_1-\lambda_0^2|\mu_0^{-2}(1-e^{-\mu_0\tau}-\mu_0\tau e^{-\mu_0\tau})\\
&\quad +|c\mu_0+2c\lambda_0-p_2|\tau e^{-(\lambda_0+\mu_0)\tau}\\
&\quad +\frac{k_{\lambda_0}}{|\beta(\lambda_0)|}\Big[1+|c|e^{-\lambda_0\tau}+|p_1\lambda_0+q_1-\lambda_0^2|\mu_0^{-2}(\mu_0\tau+e^{-\mu_0\tau}-1)\\
&\quad +|c\mu_0+2c\lambda_0-p_2||\mu_0^{-1}(1-e^{-\mu_0\tau})|e^{-(\lambda_0+\mu_0)\tau}\Big],\\
N(\lambda_0,\mu_0;\phi) &= \max\left\{\max_{-\tau\le t\le 0}|e^{-\lambda_0 t}\phi(t)|,\ \max_{-\tau\le t\le 0}|e^{-(\lambda_0+\mu_0)t}\phi(t)|,\ \max_{-\tau\le t\le 0}|\phi'(t)|,\ \max_{-\tau\le t\le 0}|\phi(t)|\right\}.
\end{aligned}$$

Furthermore, the trivial solution of (2.1) is stable if $\lambda_0\le 0,\lambda_0+\mu_0\le 0$; it is asymptotically stable if $\lambda_0<0,\lambda_0+\mu_0<0$; and it is unstable if $\mu_0>0,\lambda_0+\mu_0>0$.

Proof. From Theorem 2.3.2, it follows that

$$e^{-(\mu_0+\lambda_0)t}|x(t)|\le\frac{|K(\lambda_0,\phi)|}{|\beta(\lambda_0)|}e^{-\mu_0 t}+\frac{|H(\lambda_0,\mu_0;\phi)|}{|\gamma(\lambda_0,\mu_0)|}+|M(\lambda_0,\mu_0;\phi)|\chi_{\lambda_0,\mu_0},$$

where $K(\lambda_0,\phi),H(\lambda_0,\mu_0,\phi),M(\lambda_0,\mu_0;\phi),\beta(\lambda_0),\gamma(\lambda_0,\mu_0),\chi_{\lambda_0,\mu_0}$ are defined as in Theorem 2.3.2 respectively. From the representation of $K(\lambda_0,\phi)$, $H(\lambda_0,\mu_0,\phi)$ and $M(\lambda_0,\mu_0;\phi)$ we have

$$\begin{aligned}
|K(\lambda_0,\phi)| &\le k_{\lambda_0}N(\lambda_0,\mu_0;\phi),\quad |H(\lambda_0,\mu_0,\phi)|\le h_{\lambda_0,\mu_0}N(\lambda_0,\mu_0;\phi),\\
|M(\lambda_0,\mu_0;\phi)| &\le \left(1+\frac{K_{\lambda_0}e_{\mu_0}}{|\beta(\lambda_0)|}+\frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0,\mu_0)|}\right)N(\lambda_0,\mu_0;\phi).
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
e^{-(\mu_0+\lambda_0)t}|x(t)| &\le \frac{k_{\lambda_0}}{|\beta(\lambda_0)|}e^{-\mu_0 t}N(\lambda_0,\mu_0;\phi)+\frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0,\mu_0)|}N(\lambda_0,\mu_0;\phi)\\
&\quad +\left(1+\frac{K_{\lambda_0}e_{\mu_0}}{|\beta(\lambda_0)|}+\frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0,\mu_0)|}\right)N(\lambda_0,\mu_0;\phi)\chi_{\lambda_0,\mu_0},
\end{aligned}$$

which yields

$$\begin{aligned}
|x(t)| &\le \left[\frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0,\mu_0)|}+\left(1+\frac{K_{\lambda_0}e_{\mu_0}}{|\beta(\lambda_0)|}+\frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0,\mu_0)|}\right)\chi_{\lambda_0,\mu_0}\right]N(\lambda_0,\mu_0;\phi)e^{(\lambda_0+\mu_0)t}\\
&\quad +\frac{k_{\lambda_0}}{|\beta(\lambda_0)|}N(\lambda_0,\mu_0;\phi)e^{\lambda_0 t} \qquad (2.27)
\end{aligned}$$

for $t\ge 0$. Next, we consider three cases to discuss the stability of the trivial solution.

Case 1. Suppose that $\lambda_0\le 0,\lambda_0+\mu_0\le 0$, then $e^{\lambda_0 t}\le 1$, $e^{(\lambda_0+\mu_0)t}\le 1$. Define $\|\phi\|=\max_{-\tau\le t\le 0}|\phi(t)|$, it is not difficult to obtain that $\|\phi\|\le N(\lambda_0,\mu_0;\phi)$. From (2.27), we have

$$|x(t)|\le\left[\frac{k_{\lambda_0}}{|\beta(\lambda_0)|}+\left(1+\frac{K_{\lambda_0}e_{\mu_0}}{|\beta(\lambda_0)|}\right)\chi_{\lambda_0,\mu_0}+(1+\chi_{\lambda_0,\mu_0})\frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0,\mu_0)|}\right]N(\lambda_0,\mu_0;\phi) \qquad (2.28)$$

for every $t \geq 0$. Define

$$\rho := \frac{k_{\lambda_0}}{|\beta(\lambda_0)|} + \left(1 + \frac{K_{\lambda_0} e_{\mu_0}}{|\beta(\lambda_0)|}\right) \chi_{\lambda_0,\mu_0} + (1 + \chi_{\lambda_0,\mu_0}) \frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0,\mu_0)|}.$$

For any $\varepsilon > 0$, we choose $\delta = \varepsilon\rho^{-1}$ such that $N(\lambda_0, \mu_0; \phi) < \delta$, since $\|\phi\| \leq N(\lambda_0, \mu_0; \phi)$, we obtain that $\|\phi\| \leq \delta$. From estimate (2.28), we obtain $|x(t)| \leq \rho N(\lambda_0, \mu_0; \phi) < \rho\delta = \varepsilon$. This implies the trivial solution of (2.1) is stable.

Case 2. Suppose that $\lambda_0 < 0, \lambda_0 + \mu_0 < 0$. From estimate (2.27), it follows that $\lim_{t\to\infty} x(t) = 0$. Hence, the trivial solution of (2.1) is asymptotically stable.

Case 3. Let $\mu_0 > 0, \lambda_0 + \mu_0 > 0$. If the trivial solution of (2.1) is stable, then there exists a number $l = l(1) > 0$ such that, for any $\phi \in C([-\tau, 0], \mathbb{R})$ with $\|\phi\| < l$ the solution x of (2.1) with $x(\theta) = \phi(\theta)$ for $-\tau \leq \theta \leq 0$ satisfies $|x(t)| < 1$ for $t \geq 0$. Define

$$\phi_0(t) = e^{(\lambda_0+\mu_0)t} - e^{\lambda_0 t} \quad \text{for} \quad t \in [-\tau, 0].$$

By definition of $K(\lambda_0, \phi)$and $H(\lambda_0, \mu_0, \phi)$, and using the relation of (2.12) , we have $K(\lambda_0, \phi_0) = -\beta(\lambda_0)$ and $H(\lambda_0, \mu_0, \phi_0) = \gamma(\lambda_0, \mu_0)$. Let $\phi \in C([-\tau, 0], \mathbb{R})$ be defined by $\phi = \frac{l_1}{\|\phi_0\|}\phi_0$ with $0 < l_1 < l$. From Theorem 2.3.3, we have

$$\lim_{t\to\infty} e^{-(\mu_0+\lambda_0)t} x(t) - e^{-\mu_0 t} \frac{K(\lambda_0, \phi)}{\beta(\lambda_0)} = \frac{H(\lambda_0, \mu_0; \phi)}{\gamma(\lambda_0, \mu_0)}. \tag{2.29}$$

On the other hand,

$$\lim_{t\to\infty} e^{-(\mu_0+\lambda_0)t} x(t) - e^{-\mu_0 t} \frac{K(\lambda_0, \phi)}{\beta(\lambda_0)} = \lim_{t\to\infty} e^{-(\mu_0+\lambda_0)t} x(t) + \frac{l_1}{\|\phi_0\|} e^{-\mu_0 t} = 0,$$

but

$$\frac{H(\lambda_0, \mu_0; \phi)}{\gamma(\lambda_0, \mu_0)} = \frac{(l_1/\|\phi_0\|) H(\lambda_0, \mu_0; \phi)}{\gamma(\lambda_0, \mu_0)} = \frac{l_1}{\|\phi_0\|} > 0.$$

This is a contradiction to (2.29) and this shows that the trivial solution of (2.1) is unstable. □

2.4 Discussion of the two approaches

In this section, we discuss the relations of spectral method and ODE method. First, we consider the conditions of Theorem 2.3.3 in more detail. Suppose μ_0 is a real root of second characteristic equation (2.30). If μ_0 satisfies $\chi_{\lambda_0,\mu_0} < 1$, we claim that μ_0 is a simple dominant zero. Let

$$G(\mu) := \mu + (c\mu + 2c\lambda_0 - p_2) e^{-\tau(\lambda_0+\mu)} + 2\lambda_0 - p_1 - (p_1\lambda_0 + q_1 - \lambda_0^2) \int_{-\tau}^{0} e^{\mu s}\, ds. \tag{2.30}$$

By the condition $\chi_{\lambda_0,\mu_0} < 1$ in Theorem 2.3.3, we have $G'(\mu_0) \neq 0$.

Indeed, since $\chi_{\lambda_0,\mu_0} < 1$,

$$|G'(\mu_0)| \geq 1 - \left[|c| e^{-(\lambda_0+\mu_0)\tau} + \tau|p_1 - \mu_0 - 2\lambda_0| + \mu_0^{-2}(\mu_0\tau + e^{-\mu_0\tau} - 1)\,|p_1\lambda_0 + q_1 - \lambda_0^2|\right] > 0.$$

Since $\chi_{\lambda_0,\mu_0} < 1$, let $0 < \delta < 1$ such that $\chi_{\lambda_0,\mu_0} < \delta$. From the representation of χ_{λ_0,μ_0}, we can estimate

$$1 - \frac{1}{\delta}|c|e^{-(\lambda_0+\mu_0)\tau} > \frac{1}{\delta}\left[\tau|p_1 - \mu_0 - 2\lambda_0| + \mu_0^{-2}(\mu_0\tau + e^{-\mu_0\tau} - 1)\left|p_1\lambda_0 + q_1 - \lambda_0^2\right|\right].$$

Let $\varepsilon > 0$ such that $1 < e^{\varepsilon r} \le \frac{1}{\delta}$, and we let Ω denote the right half plane given by

$$\Omega = \{\mu \in \mathbb{C} : \mathrm{Re}\mu > \mu_0 - \varepsilon\}.$$

For $\mu \in \Omega$ and $0 \le s \le r$, we have $|e^{-s\mu}| = e^{-sRe\mu} < e^{-s\mu_0}e^{\varepsilon s} \le \frac{e^{-s\mu_0}}{\delta}$. If $\mu \in \Omega$, let γ denote the line segment between μ_0 and μ, such that the segment is in Ω, then for $0 \le s \le r$,

$$|e^{-s\mu} - e^{-s\mu_0}| = \left|\int_{\mu_0}^{\mu} se^{-ts}\,dt\right| = s\left|\int_{\gamma} e^{-ts}\,dt\right| \le \frac{e^{-s\mu_0}}{\delta}|\mu - \mu_0|s, \tag{2.31}$$

since $G(\mu_0) = 0$, we have

$$\begin{aligned} G(\mu) &= (\mu - \mu_0)(1 + ce^{-\tau(\lambda_0+\mu)}) + c\mu_0 e^{-\tau\lambda_0}(e^{-\tau\mu} - e^{-\tau\mu_0}) \\ &\quad + (2c\lambda_0 - p_2)e^{-\tau\lambda_0}(e^{-\tau\mu} - e^{-\tau\mu_0}) - (p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0}(e^{-\tau\mu} - e^{-\tau\mu_0})\,ds. \end{aligned}$$

Now, we estimate $|G(\mu)|$ by using (2.31),

$$\begin{aligned} |G(\mu)| &\ge |\mu - \mu_0|\left(1 - \left|\frac{1}{\delta}ce^{-\tau(\lambda_0+\mu_0)}\right|\right) \\ &\quad - \frac{|\mu - \mu_0|}{\delta}\left\{\tau|p_1 - \mu_0 - 2\lambda_0| + \mu_0^{-2}(\mu_0\tau + e^{-\mu_0\tau} - 1)\left|p_1\lambda_0 + q_1 - \lambda_0^2\right|\right\} > 0, \end{aligned}$$

which means μ_0 is the only zero in the right half plane Ω, so μ_0 is a simple dominant zero.

For the case when the space $\mathcal{C} = C([-\tau, 0], \mathbb{R})$. The main result in [101] (using ODE method) and Theorem 2.2.7 implies that spectral approach is equivalent to the ODE appraoch in this case.

For the case when the space $\mathcal{C} = C([-\tau, 0], \mathbb{C}^n)$, as we discussed in Section 2.2, the spectral approach can be applied to study the asymptotic behavior of the functional differential equations with the solutions in this space. However, for this case, the ODE approach is not applicable.

In the following, we present two examples to illustrate the relations of the two approaches.

Example 2.4.1. *We suppose $a = 1, b = 1, c = 1$ and $\sigma = \tau = 1$ in (2.1). We have*

$$\begin{cases} x''(t) + x''(t-1) = x(t) + x(t-1), \\ x(t) = \phi(t), \quad -1 \le t \le 0. \end{cases} \tag{2.32}$$

The characteristic equation of (2.32) is $\lambda^2 + \lambda^2 e^{-\lambda} = 1 + e^{-\lambda}$. We denote $F_1(\lambda) = \lambda^2 + \lambda^2 e^{-\lambda} - 1 - e^{-\lambda} = (\lambda^2 - 1)(1 + e^{-\lambda})$, Since $F_1(1) = 0$, $F_1'(1) = 2 + \frac{2}{e} \ne 0$, we have that $\lambda_0 = 1.$ is a simple zero of $F_1(\lambda)$. Hence, (2.32) becomes

$$z'(t) + e^{-1}z'(t-1) + 2z(t) + 2e^{-1}z(t-1) = 0, \tag{2.33}$$

and the characteristic equation of (2.33) is

$$\mu + (\mu+2)e^{-(\mu+1)} + 2 = (\mu+2)(1+e^{-\mu-1}) = 0.$$

We denote $G_1(\mu) = (\mu+2)(1+e^{-\mu-1})$. Since $\mu_0 = -2$ is a real zero of $G_1(\mu)$, the condition of Theorem 2.3.3 is $\chi_{1,-2} = e > 1$, so Theorem 2.3.3 is not applicable.

But $\lambda = -1$ is another root of $F_1(\lambda)$ and satisfies $F_1'(-1) = -2-2e \neq 0$, so (2.32) becomes

$$z'(t) + ez'(t-1) - 2z(t) - 2ez(t-1) = 0, \tag{2.34}$$

and the characteristic equation of (2.34) is

$$\mu + (\mu-2)e^{-(\mu-1)} - 2 = (\mu-2)(1+e^{-(\mu-1)}) = 0. \tag{2.35}$$

It is easy to check that $\mu = \mu_0 = 2$ is a real root of (2.35). Corresponding to the roots $\lambda_0 = -1$ and $\mu_0 = 2$, the condition of Theorem 2.3.3 becomes $\chi_{-1,2} = e^{-1} < 1$. Therefore by using the result of Theorem 2.3.3, the asymptotic behavior of initial value problem (2.32) is

$$\lim_{t\to\infty} e^{-t}x(t) = \frac{H(-1,2;\phi)}{\gamma(-1,2)} \quad = \quad \frac{\phi(0)+\phi(-1)+\phi'(0)+\phi'(-1)}{2+2e^{-1}}.$$

Next, we apply Theorem 2.2.6 to study the asymptotic behavior of initial value problem (2.32). The characteristic matrix of (2.32) is

$$\triangle(z) \quad = \quad \begin{pmatrix} z & -1 \\ -e^{-z}-1 & z+ze^{-z} \end{pmatrix}.$$

Since $z = z_0 = 1$ is a dominant zero of $\det\triangle(z)$, and $\frac{d}{dz}(\det\triangle(z))|_{z=z_0} = 2+2e^{-1} \neq 0$, we obtain that $z_0 = 1$ is a simple dominant zero of $\det\triangle(z)$, which satisfies the condition of Theorem 2.2.6. Therefore, we have

$$\lim_{t\to\infty} e^{-t}x(t) = \frac{\phi(0)+\phi(-1)+\phi'(0)+\phi'(-1)}{2+2e^{-1}}.$$

From this example, we see that the result by the spectral approach is the same as the one by the ODE approach.

Example 2.4.2. *We suppose $a = 1$, $\sigma = \tau = 1$, $b = c$ in (2.1), we have*

$$\begin{cases} x''(t) + cx''(t-1) = x(t) + cx(t-1), \\ x(t) = \phi(t), \quad -1 \le t \le 0. \end{cases} \tag{2.36}$$

The characteristic equation of (2.36) is

$$\lambda^2 + c\lambda^2 e^{-\lambda} = 1 + ce^{-\lambda},$$

we denotes $F_2(\lambda) = \lambda^2 + c\lambda^2 e^{-\lambda} - 1 - ce^{-\lambda} = (\lambda^2-1)(1+ce^{-\lambda})$. Since $F_2(-1) = 0$, $F_2'(-1) = -2-2ce \neq 0$, So $\lambda_0 = -1$ is a simple zero of $F_2(\lambda)$, (2.36) becomes

$$z'(t) + cez'(t-1) - 2z(t) - 2cez(t-1) = 0. \tag{2.37}$$

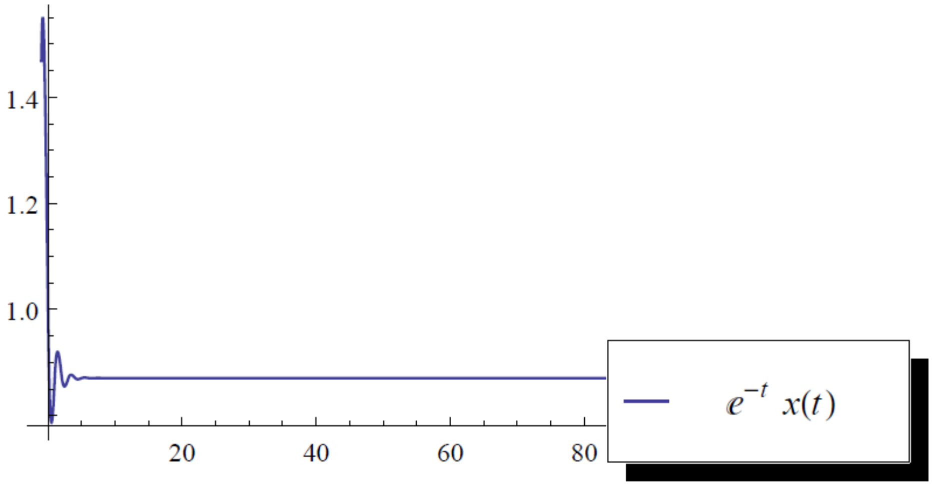

Figure 2.1: Numerical solution of equation (2.32).

The characteristic equation of (2.37) is

$$\mu + (\mu - 2)ce^{-(\mu-1)} - 2 = (\mu - 2)(1 + ce^{-(\mu-1)}) = 0.$$

We denote $G_2(\mu) = (\mu - 2)(1 + ce^{-(\mu-1)})$. Since $\mu_0 = 2$ is a real zero of $G_2(\mu)$, corresponding to the roots $\lambda_0 = -1$ and $\mu_0 = 2$, the condition of Theorem 2.3.3 is $\chi_{-1,2} = |c|e^{-1}$. If $|c| < e$, we have $\chi_{-1,2} < 1$. Therefore by using the result of Theorem 2.3.3, the asymptotic behavior of initial value problem (2.36) is

$$\lim_{t\to\infty} e^{-t}x(t) = \frac{H(-1, 2; \phi)}{\gamma(-1, 2)} = \frac{\phi(0) + \phi'(0) + c(\phi(-1) + \phi'(-1))}{2 + 2ce^{-1}}.$$

Next, we consider (2.36) by applying spectral approach. For (2.36), the characteristic matrix is given by

$$\triangle(z) = \begin{pmatrix} z & -1 \\ -ce^{-z} - 1 & z + cze^{-z} \end{pmatrix}.$$

(1) Case $-e < c$. It is not difficult to check $z_0 = 1$ is a dominant zero of $\det \triangle(z)$, since $\frac{d}{dz} \det \triangle(z)|_{z=z_0} = 2 + 2ce^{-1} \neq 0$, so $z_0 = 1$ is a simple dominant zero of $\det \triangle(z)$. Therefore, by applying the result of Theorem 2.2.6,

$$\lim_{t\to\infty} e^{-t}x(t) = \frac{\phi(0) + \phi'(0) + c(\phi(-1) + \phi'(-1))}{2 + 2ce^{-1}}.$$

(2) Case $c < -e$. After checking the roots of $\det \triangle(z)$, we find $z_0 = \ln(-c)$ is a dominant zero of $\det \triangle(z)$ and we can also use Theorem 2.2.6 to obtain the asymptotic behavior of the equation initial value problem (2.36).

(3) Case $c = -e$. We learned that $z_0 = 1$ is a dominant zero with order 2, so by the spectral approach in [6], we can have the asymptotic behavior of the equation initial value problem (2.36).

From this example, we derive that for the ODE approach, the coefficient c should satisfy $|c| < e$. However, for every $c \in \mathbb{R}$, the asymptotic behavior of the equation initial value problem (2.36) can be obtained by the spectral projection approach, and the result is the same as the one by the ODE approach when c satisfies $|c| < e$.

2.5 A fixed point method towards asymptotic behavior

In this section, we study the special case of the system (2.1) with $c = 0$ and $p_2 = 0$. Since it is not easy to apply the ODE approach or the spectral approach to discuss the asymptotic behavior because of the difficulty in computing the roots of the characteristic eequation, we introduce a third approach, based on a fixed point method, to study the asymptotic behavior of such equations.

This approach is based on fixed point theory and relies on three principles: a complete metric space, the contraction mapping principle, and an elementary variation of parameters formula. Together this yields existence, uniqueness and stability.

By using a fixed point approach, Burton and Furumochi [10] have considered asymptotic stability of the following linear equation

$$x''(t) + ax'(t) + bx(t-r) = 0 \tag{2.38}$$

and obtained the following.

Theorem 2.5.1. (Burton and Furumochi [10]) *Let $a > 0$ and $b > 0$. If*

$$br\left(1 + \int_0^t |Ae^{A(t-s)}|\, ds\right) < 1$$

holds, where $A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$, then every solution of equation (2.38) and its derivative tend to 0 as $t \to \infty$.

By using a similar technique as Burton and Furumochi [10], we consider the retarded delay differential equation

$$x''(t) + ax'(t) + bx(t-r) + cx(t) = 0. \tag{2.39}$$

Let $x' = y$, (2.39) can be written in the following form

$$y' = -ay - (b+c)x + (d/dt)\int_{t-r}^{t} bx(s)\, ds,$$

which is then expressed as the vector system

$$z' = Az + (d/dt)\int_{t-r}^{t} Bz(s)\, ds,$$

where A and B are

$$A = \begin{pmatrix} 0 & 1 \\ -(b+c) & -a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}. \tag{2.40}$$

By the variation of parameters formula

$$z(t) = e^{At}z_0 + \int_0^t e^{A(t-s)}(d/ds)\int_{s-r}^{s} Bz(u)\,du\,ds,$$

employing an integration by parts, we have

$$z(t) = e^{At}z_0 + \int_{t-r}^{t} Bz(u)\,du - e^{At}\int_{-r}^{0} Bz(u)\,du + A\int_0^t e^{A(t-s)}\int_{s-r}^{s} Bz(u)\,du\,ds.$$

In order to have $e^{At} \to 0$ as $t \to \infty$, we need

$$b + c > 0, \quad a > 0.$$

Let $C([-r,0],\mathbb{R}^2)$ be the space of continuous functions, let $\phi \in \mathcal{C}$ be an initial function and define

$$S_\phi := \left\{\varphi : \varphi \in C([-r,0],\mathbb{R}^2), \varphi(t) = \phi(t) \quad \text{on} \quad [-r,0], \quad \varphi(t) \to 0 \quad \text{as} \quad t \to \infty\right\}.$$

Define a mapping $P : S_\phi \to S_\phi$

$$(P\varphi)(t) = e^{At}\phi(0) + \int_{t-r}^{t} B\varphi(u)\,du - e^{At}\int_{-r}^{0} B\varphi(u)\,du + A\int_0^t e^{A(t-s)}\int_{s-r}^{s} B\varphi(u)\,du\,ds.$$

We choose a suitable norm for a vector or matrix. For

$$z = \begin{pmatrix} x \\ y \end{pmatrix}$$

let $|z|_0 := |x| + |y|$. Let Q be a fixed 2×2 nonsingular matrix such that $|q|_0 \le 1$, where q is the second column of Q, and let $|z| := |Qz|_0$. For a 2×2 matrix M, let

$$|M| := \sup\left\{|QMQ^{-1}z|_0 : |z|_0 = 1\right\}, \tag{2.41}$$

then $|M|$ is the norm of M. We have the following theorem.

Theorem 2.5.2. *Let $b + c > 0$, $b > 0$ and $a > 0$, if the following condition is satisfied*

$$br\left(1 + \int_0^t |Ae^{A(t-s)}|\,ds\right) < 1, \tag{2.42}$$

where A is given by (2.40), then every solution of equation (2.39) and its derivative tend to 0 as $t \to \infty$.

Proof. Since e^{At} is a L^1-function on $\mathbb{R}^+$, if $\varphi \in S_\phi$, then $(P\varphi)(t) \to 0$ as $t \to \infty$. Thus, $P : S_\phi \to S_\phi$. Furthermore, from condition (2.42), we have that P is a contraction with respect to the norm (2.41) on S_ϕ. The proof proceeds similarly to the proof as presented by Burton and Furumochi [10]. □

Example 2.5.3. *Consider the equation*

$$x''(t) + \frac{7}{12}x'(t) + \frac{1}{6}x(t-2) - \frac{1}{12}x(t) = 0, \quad t \in \mathbb{R}^+. \tag{2.43}$$

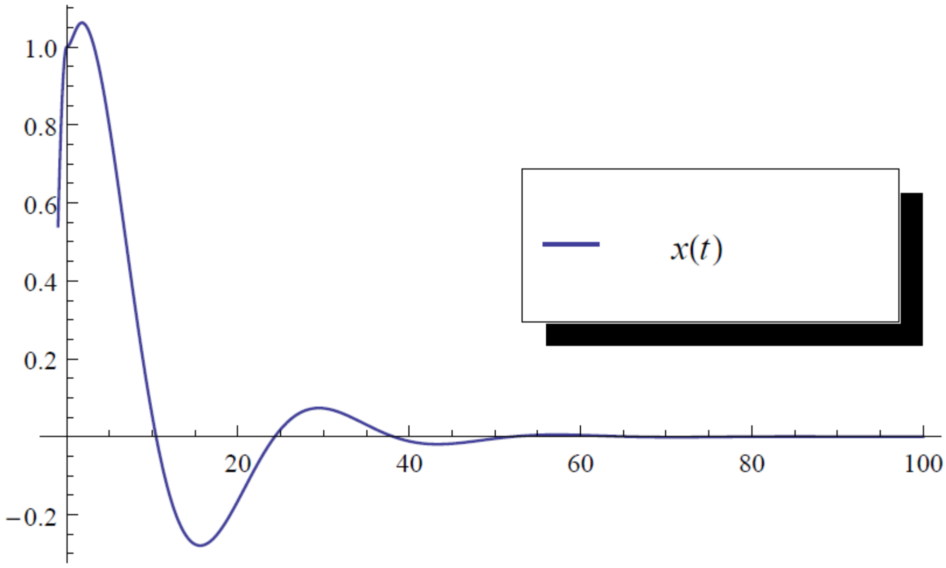

Figure 2.2: Numerical solution of equation (2.43).

We have

$$A = \begin{pmatrix} 0 & 1 \\ -1/12 & -7/12 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 0 & 0 \\ 1/6 & 0 \end{pmatrix}.$$

The eigenvalues of A are $-\frac{1}{3}$ and $-\frac{1}{4}$, let Q be a 2×2 nonsingular matrix such that

$$QAQ^{-1} = \begin{pmatrix} -1/3 & 0 \\ 0 & -1/4 \end{pmatrix}.$$

Then we have $Ae^{A(t-s)} = QEQ^{-1}$, where

$$E = \begin{pmatrix} -(1/3)e^{-(t-s)/3} & 0 \\ 0 & -(1/4)e^{-(t-s)/4} \end{pmatrix},$$

and

$$\begin{aligned} |Ae^{A(t-s)}| &= \sup\{|Ez|_0 : |z|_0 = 1\} \\ &= \sup\{(|x|/3)e^{-(t-s)/3} + (|y|/4)e^{-(t-s)/4} : |x| + |y| = 1\} \\ &\leq (1/3)e^{-(t-s)/3} + (1/4)e^{-(t-s)/4}. \end{aligned}$$

Hence,

$$\int_0^t |Ae^{A(t-s)}|\, ds \leq \int_0^t \left[(1/3)e^{-(t-s)/3} + (1/4)e^{-(t-s)/4}\right] ds = 2 - e^{-t/3} - e^{-t/4} < 2$$

for $t \geq 0$, which together with $br = \frac{1}{3}$ implies that (2.42) holds. Thus, by Theorem 2.5.2, we have that the zero solution of (2.43) tends to 0 as $t \to \infty$.

2.6 An application to a mechanical model of turning processes

Systems governed by (neutral) delay differential equations (DDEs) often come up in different fields of science and engineering. One of the most important mechanical application is the turning processes. For the simplest model of turning, the governing equation of motion is an autonomous DDE with a corresponding infinite dimensional state space. This fact results in an infinite number of characteristic roots, most of them having negative real parts referring to damped components of the vibration signals. There may be some finite number of characteristic roots that have positive real parts.

From the detailed introduction of mechanical models of turning processes in [59], we focus on a linear autonomous delay differential equation

$$m\xi''(t) + c\xi'(t) + k\xi(t) = -wh(\xi(t) - \xi(t-\tau)), \tag{2.44}$$

where m, c, k, w, h, τ are constants. For the meaning of every parameter, refer to [59].

Using the model parameters, equation (2.44) reads

$$\xi''(t) + 2\zeta w_n \xi'(t) + w_n^2 \xi(t) = -\frac{wh}{m}(\xi(t) - \xi(t-\tau)), \tag{2.45}$$

where $w_n = \sqrt{k/m}$, $\zeta = c/(2mw_n)$. Generally, $\zeta \approx 0.005 \sim 0.02$. Equation (2.45) is the standard linear delay differential equation model of the turning process.

Equation (2.45) can be even further simplified. Introduce the dimensionless time $\tilde{t}$ by $\tilde{t} = tw_n$, and by abuse of notation, drop the tilde immediately. This gives the dimensionless equation of motion

$$\xi''(t) + 2\zeta\xi'(t) + \xi(t) = -\tilde{w}(\xi(t) - \xi(t - w_n\tau)), \tag{2.46}$$

where $\tilde{w} = \frac{wh}{mw_n^2}$. In the following, we study the asymptotic stability of equation (2.46) by Theorem 2.5.2.

Equation (2.46) can be written as the following.

$$\xi''(t) + 2\zeta\xi'(t) + (1+\tilde{w})\xi(t) - \tilde{w}\xi(t - w_n\tau) = 0, \tag{2.47}$$

We denote $w_n\tau = r$. From (2.47), we have

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2\zeta \end{pmatrix} \quad and \quad B = \begin{pmatrix} 0 & 0 \\ -\tilde{w} & 0 \end{pmatrix}.$$

The characteristic equation of A is

$$\lambda^2 + 2\zeta\lambda + 1 = 0.$$

The eigenvalues are

$$\lambda_1 = -\zeta + i\sqrt{1-\zeta^2}, \quad \lambda_2 = -\zeta - i\sqrt{1-\zeta^2},$$

so $|\lambda_1| = |\lambda_2| = 1$, $|\lambda_1 e^{\lambda_1(t-s)}| = |\lambda_1|e^{(Re\lambda_1)(t-s)}$ and $|\lambda_2 e^{\lambda_2(t-s)}| = |\lambda_2|e^{(Re\lambda_2)(t-s)}$.

If $\zeta \notin \{-1, 1\}$, the two different eigenvalues λ_1, λ_2 have eigenvectors V_1 and V_2, which are linearly independent. Suppose that $Q = (V_1, V_2)^{-1}$, then

$$QAQ^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \triangleq \Lambda.$$

Hence,

$$A = Q^{-1}\Lambda Q = (V_1, V_2)A(V_1, V_2)^{-1}, \quad e^{A(t-s)} = Q^{-1}e^{\Lambda(t-s)}Q.$$

Then we have

$$Ae^{A(t-s)} = Q^{-1}\Lambda QQ^{-1}e^{\Lambda(t-s)}Q = Q^{-1}\Lambda e^{\Lambda(t-s)}Q = Q^{-1}EQ,$$

where

$$E = \begin{pmatrix} \lambda_1 e^{\lambda_1(t-s)} & 0 \\ 0 & \lambda_2 e^{\lambda_2(t-s)} \end{pmatrix}.$$

Using the norm in Theorem 2.5.2, we have

$$\begin{aligned} |Ae^{A(t-s)}| &= \sup\{|Ez|_0 : |z|_0 = 1\} \\ &= \sup\left\{\left|x\lambda_1 e^{\lambda_1(t-s)}\right| + \left|y\lambda_2 e^{\lambda_2(t-s)}\right| : |x| + |y| = 1\right\} \\ &= \sup\left\{|x||\lambda_1|e^{(Re\lambda_1)(t-s)} + |y||\lambda_2|e^{(Re\lambda_2)(t-s)} : |x| + |y| = 1\right\} \\ &\le 2e^{-\zeta(t-s)}. \end{aligned}$$

Hence,

$$\int_0^t |Ae^{A(t-s)}|\, ds \le \int_0^t 2e^{-\zeta(t-s)}\, ds \le 2\zeta^{-1}(1 - e^{-\zeta t}), \quad t \ge 0,$$

$$(-\tilde{w})r\left(1 + \int_0^t |Ae^{A(t-s)}|\, ds\right) \le (-\tilde{w})r(1 + 2\zeta^{-1}(1 - e^{-\zeta t})) \le (-\tilde{w})r(1 + 2\zeta^{-1}).$$

If $(-\tilde{w})r/\zeta < 1/3$, the conditions of Theorem 2.5.2 are satisfied, that is to say, if $\tilde{w} < 0$ very large or r very small, then every solution of (2.47) and its derivative tends to 0 as $t \to \infty$.

2.7 Notes and remarks

For more results on asymptotic behavior of autonomous delay differential equations, see the overview books by Hale and Verduyn Lunel [51], Diekmann, van Gils, Verduyn Lunel and Walther [29], Driver [35], papers by Driver [36], Philos and Purnaras [101, 102, 105, 106], Dix, Philos and Purnaras [33], Frasson [41, 40, 42], Frasson and Verduyn Lunel [39].

In this chapter, we used three methods to study asymptotic behavior of the solutions of functional differential equations, that is, ordinary differential equation (ODE) method, spectral method and fixed point method. The basic idea for the ODE method in Section 2.3 essentially originated in a very interesting asymptotic result due to Driver [37] concerning the solutions of linear differential systems with small delays. Motivated by the treatment in [37], Philos, Purnaras and many other authors [84, 101, 102, 103, 104, 105, 106, 137] have obtained some interesting results on the asymptotic behavior of the solutions to autonomous differential and difference equations with delays. For example, equations with neutral terms, equations with variable delays, periodic differential and difference equations with delays. Continuing the study for asymptotic behavior of a wide class of functional differential equations, Frasson and Verduyn Lunel [39] explored a new approach, the so-called spectral approach. Frasson [40, 41, 42] established some interesting results based on the results in [39]. Towards asymptotic behavior, a fixed point method (see Burton [13]) is introduced. This method is one of the main methods in this thesis. For more detailed information about the fixed point method, refer to Chapter 4, Chapter 5 and Chapter 6.

A paper based on the contents of this chapter has been submitted for publication ([17]).