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Chapter 1

Introduction

1.1 Outline

This thesis focuses on asymptotic behavior and stability of solutions of deterministic and stochastic delay differential equations.

A delay differential equation is a differential equation where the derivatives at the current time depend on the solution at previous times. Such equations are also called *differential equations with retarded argument.* Strictly speaking, a delay differential equation is a specific example of a functional differential equation, in which the functional part of the differential equation is the evaluation of a functional on the past of the process.

Suppose $r \geq 0$ is a given real number, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|\cdot|$, $\mathcal{C} = C([-r, 0], \mathbb{R}^n)$ is the set of continuous functions mapping $[-r, 0]$ into $\mathbb{R}^n$. Then $\mathcal{C}$ is a Banach space with respect to the supremum norm $\|\varphi\| = \sup_{-r\leq\theta\leq 0} |\varphi(\theta)|$, where $\varphi \in \mathcal{C}$. If $\sigma \in \mathbb{R}$, $A \geq 0$ and $x \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$, then for any $t \in [\sigma, \sigma + A]$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t + \theta)$ for $-r \leq \theta \leq 0$. If Ω is a subset of $\mathbb{R} \times \mathcal{C}$, $f : \Omega \to \mathbb{R}^n$ is a given function and "$\cdot$" represents the right-hand derivative, we say the relation

$$\dot{x}(t) = f(t, x_t), \tag{1.1}$$

is a *delay differential equation* on Ω, which is denoted by DDE (f). The number r is called the delay. The case $r = 0$ corresponds with an ordinary differential equation.

Equation (1.1) is called *linear* if $f(t, \varphi) = L(t)\varphi$, where $L(t)$ is linear for each t. Equation (1.1) is called nonhomogeneous if $f(t, \varphi) = L(t)\varphi + h(t)$, where $h(t) \not\equiv 0$. Equation (1.1) is called *autonomous* if $f(t, \varphi) = g(\varphi)$, where g does not depend on t.

Now, we show some examples of delay differential equations.

$$\dot{x}(t) = \int_{-r}^{0} x(t + \theta)\, d\theta, \tag{1.2}$$

$$\dot{x}(t) = ax(t) + bx(t - 1), \tag{1.3}$$

$$\dot{x}(t) = c(t)x(t) + d(t)x(t - \tau(t)), \tag{1.4}$$

where a, b are constants, $c(t), d(t), \tau(t)$ are continuous functions. Equation (1.2) is a linear integro-differential equation with a distributed delay, equation (1.3) is linear autonomous differential equation with a constant delay and equation (1.4) is linear nonautonomous differential equation with a time dependent delay.

Suppose that $\Omega \subseteq \mathbb{R} \times \mathcal{C}$ is open, $f : \Omega \to \mathbb{R}^n$, $D : \Omega \to \mathbb{R}^n$ are given continuous functions with D atomic at zero (See Subsection 1.3.2 on page 10 for the concept of *atomic at zero*). The relation

$$\frac{d}{dt} D(t, x_t) = f(t, x_t) \tag{1.5}$$

is called a *neutral delay differential equation*, which is denoted by NDDE (D, f), the function D is called the *difference operator* for the neutral delay differential equation. In the following, we present two examples of neutral delay differential equations.

$$\frac{d}{dt}[x(t) - Bx(t-r)] = f(t, x_t),$$

where $r > 0$, B is an $n \times n$ constant matrix, $D(\phi) = \phi(0) - B\phi(-r)$ and $f : \Omega \to \mathbb{R}^n$ is continuous.

If $D\phi = \phi(0)$ for all ϕ, then D is atomic at 0. Therefore, for any continuous $f : \Omega \to \mathbb{R}^n$, the pair (D, f) defines a neutral delay differential equation. Consequently, DDEs are NDDEs.

Delay differential equations arise from a variety of applications including control systems, electrodynamics, mixing liquids, neutron transportation and population models. In the following, we show some models to illustrate the applications of neutral delay differential equations.

Biological models

Differential equations have long been used to model various types of populations. In many cases ordinary differential equations are the starting point in the modeling process. When time delays (due to feedback, cells division time lags, etc.) become important, then delay differential equations become a natural tool for modeling in the life sciences.

Predator-prey model

The classic predator-prey model suggested by Lotka and Volterra in the 1920's has the form

$$\begin{cases} \dot{x}(t) = a_1 x(t) - b_1 x(t) y(t) \\ \\ \dot{y}(t) = a_2 y(t) - b_2 x(t) y(t), \end{cases} \tag{1.6}$$

with initial condition

$$x(0) = x_0, \qquad y(0) = y_0, \tag{1.7}$$

where $x(t)$ represents the population of prey and $y(t)$ the population of predators at time t and a_1, a_2, b_1, b_2 are positive constants. If we consider the fact that a change in the population of the prey will not immediately affect the population of the predators and conversely, then the system (1.6) with the initial condition (1.7) becomes a delay differential equation of the form

$$\begin{cases} \dot{x}(t) = a_1 x(t) - b_1 x(t) y(t - r_1) \\ \\ \dot{y}(t) = a_2 y(t) - b_2 x(t - r_2) y(t), \end{cases} \tag{1.8}$$

with initial conditions

$$x(0) = x_0, \quad x(s) = \phi(s), \quad y(0) = y_0, \quad y(s) = \varphi(s), \quad -\tau < s < 0, \tag{1.9}$$

where $r_1 > 0$ and $r_2 > 0$ are time delays and the functions $\phi(\cdot)$ and $\varphi(\cdot)$ are the initial past history functions, $\tau = \max\{r_1, r_2\}$, see [28, 47] for detailed information.

Australian blowfly

In the dynamic system of the blowfly population, resource limitation acts with a time delay, roughly equal to the time for a larva to grow up to an adult. Thus May [97] proposed to model the population dynamics of blowflies with a delay differential equation

$$\dot{N}(t) = rN(t)\left(1 - \frac{1}{1000K}N(t-\tau)\right), \tag{1.10}$$

where $N(t)$ is the population size of the adult blowflies, r is the rate of increase of the blowfly population, K is a resource limitation parameter set by the supply of food, and τ is the time delay, roughly equal to the time for a larva to grow up to an adult (about 11 days).

Metal cutting model

The metal cutting model (Moon and Johnson [99]) can be described by

$$\begin{aligned} m\ddot{x}(t) + \gamma_1\dot{x}(t) + k_1 x(t) &= F_1(x(t) - x(t-\tau), y(t) - y(t-\tau)) \\ m\ddot{y}(t) + \gamma_2\dot{y}(t) + k_2 y(t) &= F_2(x(t) - x(t-\tau), y(t) - y(t-\tau)), \end{aligned}$$

where $x(t)$ is the x component of the tool tip position, $y(t)$ is the y component of the tool tip position, γ_j, k_j $(j = 1, 2)$ are the damping and spring force constants, $\tau = \frac{C}{\omega}$ with C a constant and ω the turning speed. Normally, ω is considered constant, but during the machine startup or shut down, ω is a function of t, thus $\tau = \tau(t)$. For the other applications of delay differential equations, refer to [29, 50, 51].

Delay differential equations are studied from several different perspectives, mostly concerned with their solutions. Only the simplest equations admit solutions given by explicit formulas. However, some properties of solutions of a given equation may be determined without finding their exact form. In the case when a self-contained formula for the solution is not available, qualitative analysis, which has been proved to be a useful tool to investigate the properties of solutions, will be emphasised on. In the qualitative analysis of equations, asymptotic behavior and stability of solutions play an important role. The investigation of asymptotic behavior and stability of solutions of delay differential equations is more complicated than the case for ordinary differential equations because of the delay effects, refer to [29, 50, 51, 72] for detailed information.

Besides delay effects, impulsive effects likewise exist in a great variety of evolutionary processes in which states are changed abruptly at certain moments of time. Time-dependent impulses arise naturally in many biological and physiological systems, including ones from delayed cellular neural networks with impulsive effects.

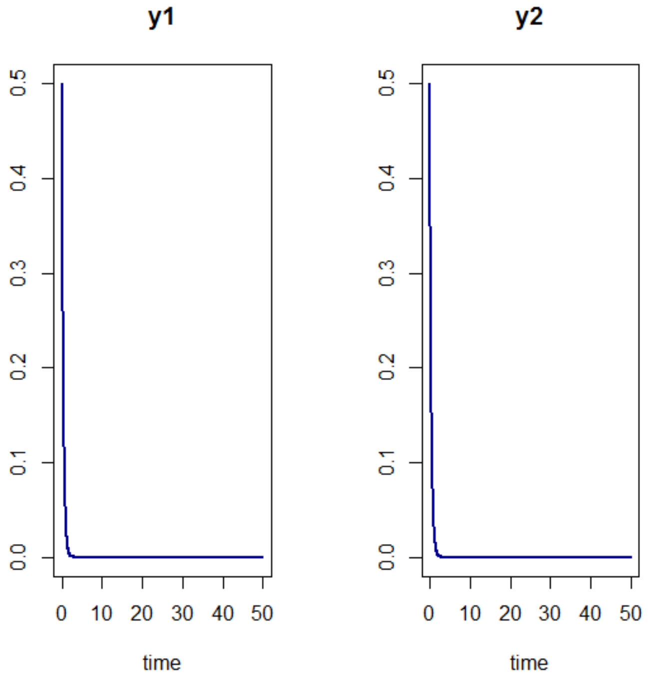

Figure 1.1: Delayed cellular neural network without impulses.

Delayed cellular neural networks with impulsive effects

Consider the following system of delayed cellular neural networks with impulsive effects

$$\begin{cases} \dot{y_1}(t) = -2y_1(t) - g(y_1(t)) + 0.5g(y_2(t)) - 0.5g(y_1(t - 0.2\sin t)) + 0.5g(y_2(t - 0.2\cos t)) \\ \\ \dot{y_2}(t) = -3.5y_2(t) + 0.5g(y_1(t)) - g(y_2(t)) \\ \qquad\qquad +0.5g(y_1(t - 0.2\sin t)) + 0.5g(y_2(t - 0.2\cos t)), \end{cases}$$

where

$$g(x) = \frac{|x+1| - |x-1|}{2}.$$

The initial condition is given by $y_1(t) = 0.5$ and $y_2(t) = 0.5$. At each impulse time $t_k = 0.2k$ an impulse is applied with $y_1(t_k)$ being replaced by $1.8y_1(t_k)$ and $y_2(t_k)$ being replaced by $1.7y_2(t_k)$.

Figure 1.1 and Figure 1.2 show that the impulses can destabilize a system.

Consider the following system of delayed cellular neural networks with impulsive effects

$$\begin{cases} \dot{y_1}(t) = -0.2y_1(t) - g(y_1(t)) + 0.5g(y_2(t)) \\ \qquad\qquad -0.5g(y_1(t - 0.2\sin t)) + 0.5g(y_2(t - 0.2\cos t)) \\ \\ \dot{y_2}(t) = -0.1y_2(t) + 0.5g(y_1(t)) - g(y_2(t)) \\ \qquad\qquad +0.5g(y_1(t - 0.2\sin t)) + 0.5g(y_2(t - 0.2\cos t)), \end{cases}$$

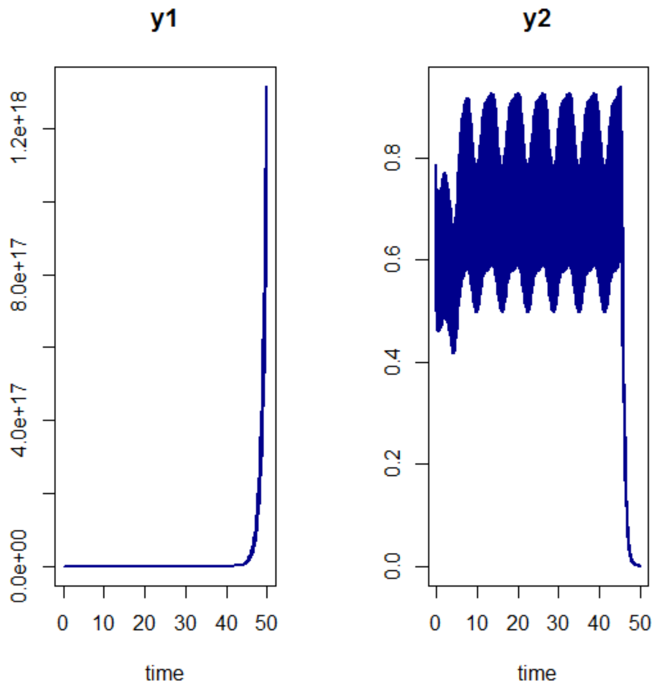

Figure 1.2: Delayed cellular neural network with impulses.

where

$$g(x) = \frac{|x+1| - |x-1|}{2}.$$

The initial condition is given by $y_1(t) = 0.5$ and $y_2(t) = 0.5$. At each impulse time $t_k = 0.2k$ an impulse is applied with $y_1(t_k)$ being replaced by $-0.8y_1(t_k)$ and $y_2(t_k)$ being replaced by $-0.7y_2(t_k)$.

Figure 1.3 and Figure 1.4 show that the impulses can stabilize a system.

When modeling systems which do not noticeably affect their environment, stochastic variables are often used to model the environmental fluctuations, which is described as *stochastic delay differential equations*. Stochastic delay differential equations can be considered as deterministic delay differential equations with random elements or stochastic differential equations with time delays. As an important mathematical model to describe real world problems more effectively, stochastic delay differential equations have been applied in many fields of science, such as automatic control, neural networks, biology, economics, chemical reaction engineering, etc. As an example, we consider an entire delayed neural network appeared in Huang et al.[56].

Stochastic neural networks

Figure 1.5 shows the scheme of the entire delayed neural network, where the nonlinear neuron transfer function S is constructed by using the voltage operational amplifiers. The time delay

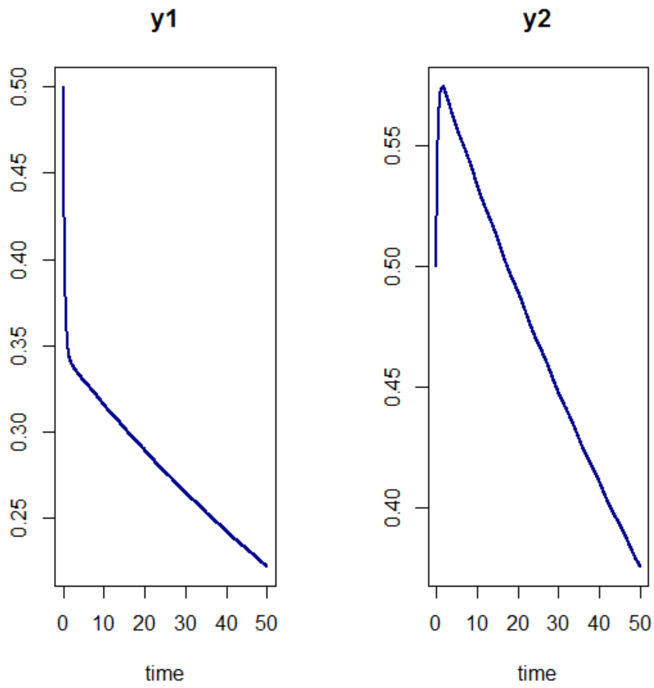

Figure 1.3: Delayed cellular neural network without impulses.

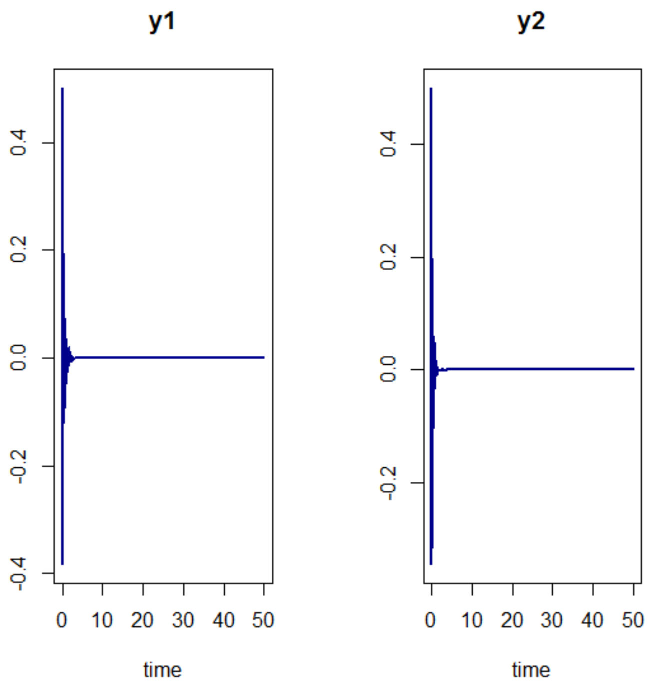

Figure 1.4: Delayed cellular neural network with impulses.

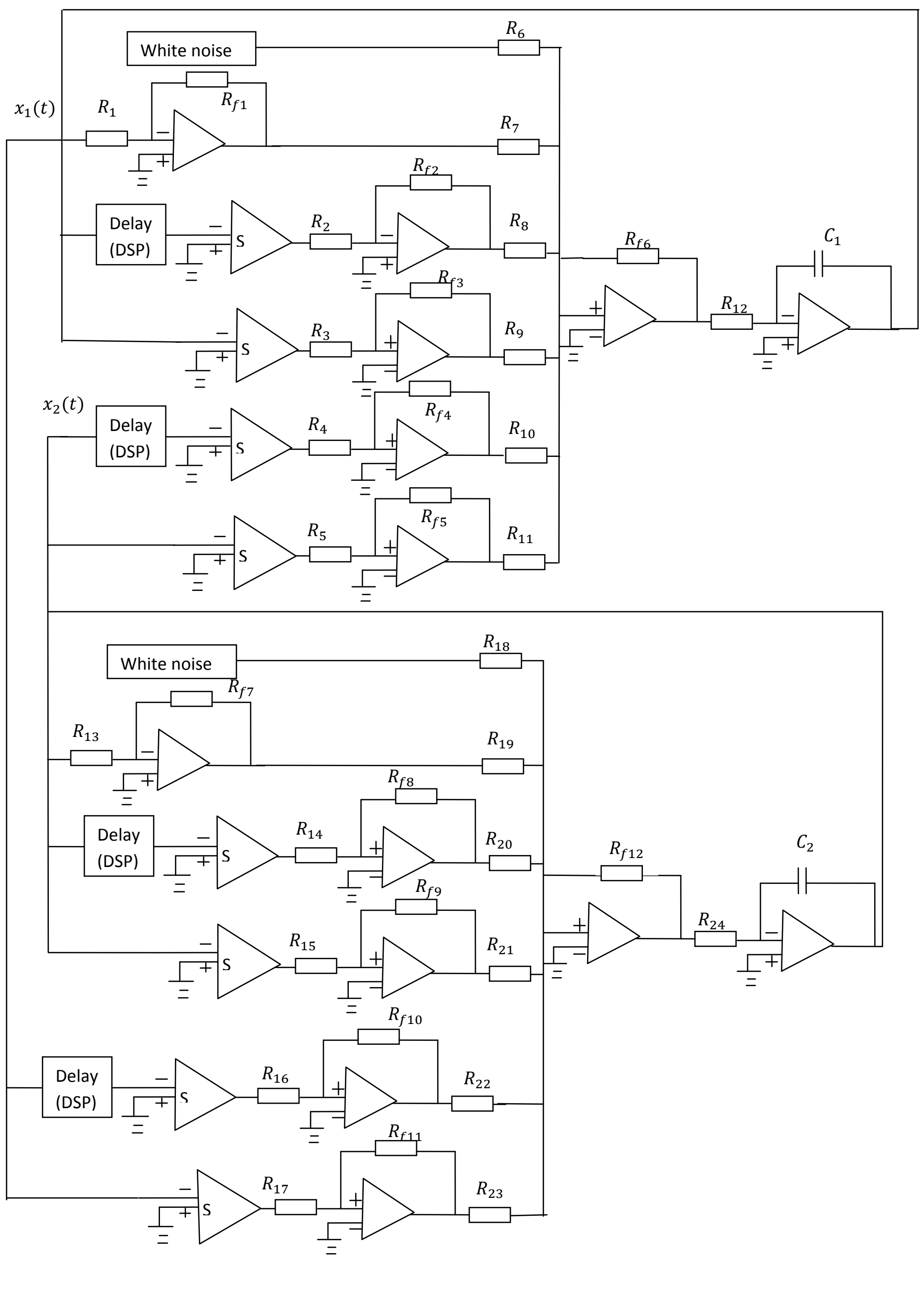

Figure 1.5: A schematic circuit diagram for system (1.11), where $R_i = 1k\Omega$ $(i = 1, \cdots, 11, 13, \cdots, 23)$, $R_{12} = R_{24} = 100k\Omega$; $R_{f_1} = 4.5k\Omega$, $R_{f_2} = 0.16k\Omega$, $R_{f_3} = R_{f_4} = 0.4k\Omega$, $R_{f_5} = 0.08k\Omega$, $R_{f_6} = 1k\Omega$, $R_{f_7} = 4.5k\Omega$, $R_{f_8} = 0.8k\Omega$, $R_{f_9} = R_{f_{10}} = 0.2k\Omega$, $R_{f_{11}} = 0.12k\Omega$, $R_{f_{12}} = 1k\Omega$; $C_1 = C_2 = 0.1\mu F$.

is achieved by using a digital signal processor (DSP) with an analog-to-digital converter (ADC) and a digital-to -analog converter (DAC). There is white noise is generated by a white noise signal generator.

The schematic circuit diagram can be described by the following stochastic recurrent neural network with time-varying delays

$$\begin{aligned} dx(t) &= -\begin{pmatrix} 4.5 & 0 \\ 0 & 4.5 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} dt + \begin{pmatrix} 2 & 0.4 \\ 0.6 & 1 \end{pmatrix}\begin{pmatrix} 0.2\tanh(x_1(t)) \\ 0.2\tanh(x_2(t)) \end{pmatrix} dt \\ &\quad + \begin{pmatrix} -0.8 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 0.2\tanh(x_1(t-\tau_1(t))) \\ 0.2\tanh(x_2(t-\tau_2(t))) \end{pmatrix} dt + \sigma(t, x(t), x(t-\tau(t)))\, dw(t), \end{aligned} \tag{1.11}$$

where $\tau(t) = (\tau_1(t), \tau_2(t))^T$, τ_i is any bounded positive function for $i = 1, 2$, and $\sigma : R_+ \times R^2 \times R^2 \to R^2 \times R^2$ satisfies trace $\left[\sigma^T(t, x, y)\sigma(t, x, y)\right] \le x_1^2 + x_2^2 + y_1^2 + y_2^2$.

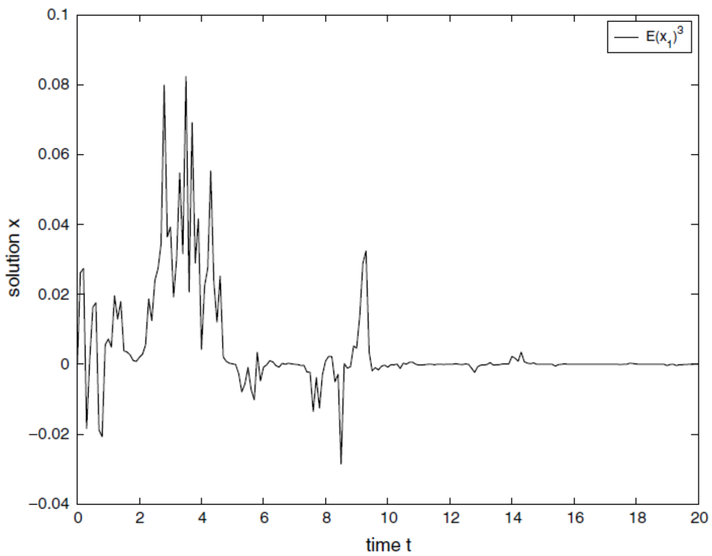

Figure 1.6: Numerical solution $E(x_1^3(t))$ of system (1.11), which comes from Huang et al.[56].

1.2 Objectives and main results of this thesis

The general aim of this thesis is to present a systematic study of different methods for stability and asymptotic stability for different types of equations. We are interested in the versatility of the methods to deal with different classes of equations and verifiability of the conditions. We also wish to understand the relations between the methods: for what equations do they eventually coincide, and what are their advantages and restrictions. In particular, we emphasize a fixed point approach to stability of delay differential equations and stability of stochastic delay differential equations.

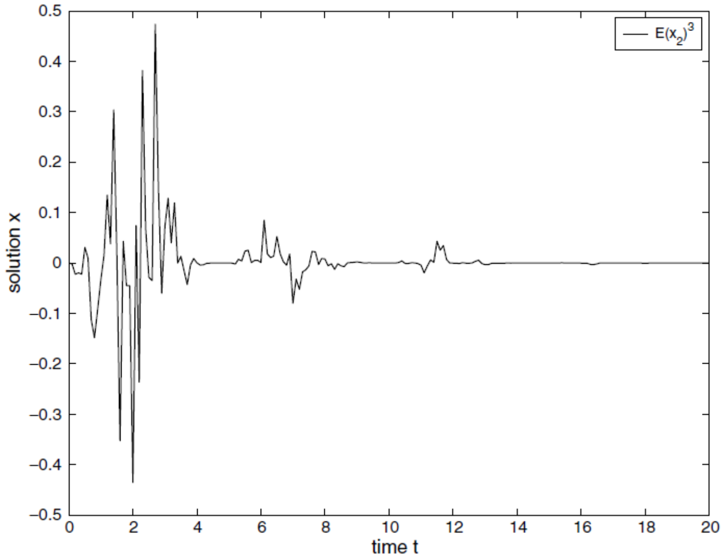

Figure 1.7: Numerical solution $E(x_2^3(t))$ of system (1.11), which comes from Huang et al.[56].

This thesis focuses on five objectives. The first objective is concerned with asymptotic behavior of autonomous delay differential equations (see Chapter 2). The ODE method and spectral method are generally viewed as effective techniques in dealing with asymptotic behavior of autonomous delay differential equations. However, there seems to be no discussion about the relations of these two methods. In Chapter 2, we will study the relations of the ODE method and spectral method by considering a class of second order neutral delay differential equations of the form

$$x''(t) + cx''(t-\tau) = p_1 x'(t) + p_2 x'(t-\tau) + q_1 x(t) + q_2 x(t-\tau), \tag{1.12}$$

where $c, p_1, p_2, q_1, q_2 \in \mathbb{R}$, $\tau > 0$. It is concluded that under the same assumptions, the results by the ODE method is equivalent to the results by the spectral method (see Section 2.4). The conditions for the spectral method are weaker than those by the ODE method, (see Example 2.4.2), and the asymptotic behavior of neutral delay differential equations can be presented by a general formula (see Theorem 2.2.6). Furthermore, the asymptotic behavior of neutral delay differential equations with matrix coefficients can be investigated by the spectral method.

The second objective focuses on asymptotic behavior of nonautonomous delay differential equations (see Chapter 3). It should be emphasized that asymptotic behavior of nonautonomous equations is much more difficult than the case of autonomous equations. Frasson and Verduyn Lunel [39] have applied a spectral method to study asymptotic behavior of a class of linear periodic delay equations of the form

$$x'(t) = a(t)x(t) + \sum_{j=1}^{k} b_j(t)x(t-\tau_j), \tag{1.13}$$

where $a(t+\omega) = a(t)$, $b_j(t+\omega) = b_j(t)$, $j = 1, 2, \cdots, k$. They considered a particular case where $\tau_j = j\omega$ (i.e. the delays are integer multiples of the period ω). Determining asymptotic behavior of general classes of nonautonomous equations seems untractable. For a special class of nonautonomous problems, we can use an approach similar to the ODE method as we discussed in Chapter 2, which is based on the application of an appropriate solution of the generalized characteristic equation. For nonautonomous equations, solving the generalized characteristic equation becomes much harder: a functional equation instead of an algebraic equation. This approach only succeeds if the generalized characteristic equation has a real solution.

The third objective concerns a fixed point approach towards stability of deterministic delay differential equations (see Chapter 4). Although there is an extensive literature on stability analysis of delay equations discussed using a fixed point approach, stability analysis of more general classes of delay equations has not been satisfactorily researched. Hence, in Chapter 4, several classes of delay equations with a combination of time-dependent delays, distributed delays and neutral terms are studied, such as, for example, a scalar neutral integro-differential equation

$$x'(t) - c(t)x'(t - r_1(t)) = -a(t)x(t - r_2(t)) + \int_{t-r_3(t)}^{t} g(t, x(s))\, d\mu(t, s). \tag{1.14}$$

The last term in (1.14) includes the following two cases:

$$(1) \quad \int_{t-r(t)}^{t} g(t, x(s))k(t, s)\, ds \qquad (2) \quad \sum_{i=1}^{n} a_i(t) g(t, x(t - r_i(t))).$$

In our result, two auxiliary continuous functions $h_1(t)$ and $h_2(t)$ are introduced and used to define an appropriate contraction mapping related to the equation. Our stability results typically say that the equation is stable if a certain expression involving the coefficients of the equation is less than one.

The fourth objective involves stability of stochastic delay differential equations with impulses (see Chapter 5). Besides delay and stochastic effects, impulsive effects are also likely to exist in mechanical, electronical or economical systems, which could stabilize or destabilize the system. Therefore, it is necessary to take delay effects, stochastic effects and impulsive effects into account when studying the dynamical behavior of the system. In Chapter 5, we consider two classes of neutral stochastic delay differential equations with impulses. The first class is an impulsive neutral stochastic delay differential equations is of the form

$$\begin{cases} d[x(t) - q(t)x(t - \tau(t))] = [a(t)x(t) + b(t)x(t - \tau(t))]dt \\ \qquad\qquad\qquad + [c(t)x(t) + e(t)x(t - \delta(t))]dw(t), \quad t \neq t_k, \\ x(t_k{}^{+}) - x(t_k) = b_k x(t_k), \quad t = t_k. \end{cases} \tag{1.15}$$

Equation (1.15) is a combination of a neutral term, a delay term, a stochastic term and an impulsive effect.

A fixed point method is used to study stability properties of the first class of equations. We consider two different norms:

$$\|x\|^2 := \sup_{t \geq \vartheta} \left(\mathbb{E}|x(t)|^2\right)$$

and

$$\|x\|^2 := \sup_{t\geq 0}\left[\mathbb{E}\left(\sup_{t-\tau\leq s\leq t}|x(s)|^2\right)\right],$$

where $\vartheta = \min\{\inf_{s\geq 0}\{s-\tau(s)\}, \inf_{s\geq 0}\{s-\delta(s)\}\}$, and τ is an upper bound of $\{\tau(s), \delta(s), s\geq 0\}$. These two norms lead to different stability results. It turns out that the analysis for the second norm yields a stronger conclusion under a stronger assumption than the analysis involving the first norm.

The second class consists of equations of the form

$$\begin{cases} d[x(t)+u(t,x(t-\tau(t)))] = [Ax(t)dt + f(t,x(t-\delta(t)))]dt + g(t,x(t-\rho(t))dw(t) \\ \qquad\qquad + \int_Z h(t,x(t-\sigma(t)),y)\,\tilde{N}(dt,dy), \quad t\geq 0, \quad t\neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad t=t_k, \quad k=1,2,\cdots, \\ x_0(\theta)=\phi, \quad \theta\in[-\tau,0], \quad a.s., \end{cases} \tag{1.16}$$

which is an infinite dimensional impulsive stochastic delay differential equation. Exponential stability of this class of equations is studied by two methods, one is the method using an impulsive-integral inequality and the other one is a fixed point method. The stability criteria derived by the two methods are similar. A fixed point argument can yields existence, uniqueness and stability result in one step. However, the existence and uniqueness theorem should be provided seperately before using the method using an impulsive-integral inequality.

The fifth objective concerns an application to stochastic delayed neural networks (see Chapter 6). It is natural to consider random noise in neural networks. In real nervous, for instance, synaptic transmission is a noisy process with the noise brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes. A neural network could be stabilized or destabilized by stochastic inputs. Therefore, the stochastic stability analysis problem for various neural networks has attracted considerable interest in recent years. In Chapter 6, a class of stochastic delayed neural networks is considered, which is described by

$$\begin{aligned} dx_i(t) &= \Bigg[-c_ix_i(t) + \sum_{j=1}^n a_{ij}f_j(x_j(t)) + \sum_{j=1}^n b_{ij}f_j(x_j(t-\tau(t))) \\ &\quad + \sum_{j=1}^n l_{ij}\int_{t-r(t)}^t f_j(x_j(s))\,ds\Bigg]dt + \sum_{j=1}^n \sigma_{ij}(t,x_j(t),x_j(t-\tau(t)))\,dw_j(t). \end{aligned} \tag{1.17}$$

A fixed point method is applied to study stability properties of this class of stochastic delayed neural networks. As in Chapter 5, two different types of norms are defined to study the system (1.17), that is,

$$\|x\|^p := \sup_{t\geq\vartheta}\left[\mathbb{E}\left(\sum_{i=1}^n |x_i(t)|^p\right)\right]$$

and

$$\|x\|^p = \sup_{t\geq 0}\left\{\sum_{i=1}^n \mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|x_i(s)|^p\right]\right\}.$$

Both norms lead to a complete space and a contraction mapping related to the equation. Our results neither require the boundedness, monotonicity and differentiability of the activation functions nor differentiability of the time varying delays. In addition, the case when there are impulsive effects to the system (1.17) and the case when there are no stochastic perturbations are also considered.

1.3 Preliminaries

In this section, we present basic definitions and lemmas which are frequently used in this thesis, and present some background materials on stability of deterministic and stochastic delay differential equations.

1.3.1 Delay differential equations

For $r > 0$, let $\mathcal{C} = C([-r, 0], \mathbb{R}^n)$ denote the Banach space of continuous functions from $[-r, 0]$ $(r > 0)$ with values in $\mathbb{R}^n$ endowed with the supremum norm. For $\Omega \subseteq \mathbb{R} \times \mathcal{C}$, $f : \Omega \to \mathbb{R}^n$ is a given function, consider the delay differential equation

$$\dot{x}(t) = f(t, x_t), \tag{1.18}$$

where $x_t(\theta) = x(t + \theta)$ for $-r \leq \theta \leq 0$.

It is clear that an appropriate "initial condition" at time $t = \sigma$ must at least specify the vector x for all t in $[\sigma - r, \sigma]$, i.e.,

$$x(t) = \phi(t), \qquad \sigma - r \leq t \leq \sigma. \tag{1.19}$$

Here $\phi : [\sigma - r, \sigma] \to \mathbb{R}^n$ is a known function, usually we suppose ϕ to be a continuous function. The function ϕ is called the *initial function* of the delay differential equation, σ the *initial constant* and $[\sigma - r, \sigma]$ the *initial set.*

Hence, the initial value problem of (1.18) is given by the following relation

$$\begin{cases} \dot{x}(t) = f(t, x_t) & \text{for} \quad t \geq \sigma \\ x(t) = \phi(t) & \text{for} \quad \sigma - r \leq t \leq \sigma, \end{cases} \tag{1.20}$$

where ϕ is a given function defined on $[\sigma - r, \sigma]$.

Definition 1.3.1. (Hale and Verduyn Lunel [51]) *A function x is said to be a solution of* (1.18) *on $[\sigma - r, \sigma + A]$ if there are $\sigma \in \mathbb{R}$, $A > 0$ such that $x \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$, $(t, x_t) \in D$ and $x(t)$ satisfies* (1.18) *for $t \in [\sigma, \sigma + A]$. For given $\sigma \in \mathbb{R}$, $\phi \in C([-r, 0], \mathbb{R}^n)$, we say $x(t, \sigma, \phi)$ is a solution of* (1.20) *with initial value ϕ at σ or simply a solution through (σ, ϕ) if there is an $A > 0$ such that $x(t, \sigma, \phi)$ is a solution of equation* (1.20) *on $[\sigma - r, \sigma + A]$ and $x_\sigma(\sigma, \phi) = \phi$; we say $x(t, \sigma, \phi)$ is a solution of* (1.20) *on $[\sigma - r, \infty)$, if for every $A > 0$, $x(t, \sigma, \phi)$ is a solution of equation* (1.20) *on $[\sigma - r, \sigma + A]$ and $x_\sigma(\sigma, \phi) = \phi$.*

Lemma 1.3.2. (Hale and Verduyn Lunel [51]) *If $\sigma \in \mathbb{R}$, $\phi \in \mathcal{C}$ are given, and $f(t, \phi)$ is continuous, then finding a solution of equation* (1.18) *through (σ, ϕ) is equivalent to solving the integral equation*

$$\begin{cases} x(t) = \phi(\sigma) + \int_\sigma^t f(s, x_s)\, ds, & t \geq \sigma, \\ x_\sigma = \phi. \end{cases} \tag{1.21}$$

We are now consider the existence and uniqueness of the system (1.20), we assume that f is continuous. To prove the existence of the solution through a point $(\sigma, \phi) \in \mathbb{R} \times \mathcal{C}$, we consider an $\alpha > 0$ and all functions x on $[\sigma - r, \sigma + \alpha]$ that are continuous and coincide with ϕ on $[\sigma - r, \sigma]$, that is $x_\sigma = \phi$.

Theorem 1.3.3. (Existence) ([51]) *Suppose that Ω is an open subset in $\mathbb{R} \times \mathcal{C}$ and $f \in C(\Omega, \mathbb{R}^n)$. If $(\sigma, \phi) \in \Omega$, then there is a solution of the DDE (f) passing through (σ, ϕ).*

Definition 1.3.4. *We say $f(t, \phi)$ is Lipschitz in ϕ in a compact set K of $\mathbb{R} \times \mathcal{C}$ if there is a constant $L > 0$ such that, for any $(t, \phi_i) \in K$, $i = 1, 2$,*

$$\|f(t, \phi_1) - f(t, \phi_2)\| \leq L\|\phi_1 - \phi_2\|.$$

Theorem 1.3.5. (Existence and uniqueness) ([51]) *Suppose that Ω is an open set in $\mathbb{R} \times \mathcal{C}$, $f : \Omega \to \mathbb{R}^n$ is continuous and $f(t, \phi)$ is Lipschitz in ϕ in each compact set in Ω. If $(\sigma, \phi) \in \Omega$, then there is a unique solution of* (1.20) *through (σ, ϕ).*

Let x be a solution of (1.20) on $[\sigma, a)$, $a > \sigma$. We say $\hat{x}$ is a *continuation* of x if there is a $b > a$ such that $\hat{x}$ is defined on $[\sigma - r, b)$, coincides with x on $[\sigma - r, a)$, and $\hat{x}$ satisfies (1.20) on $[\sigma, b)$. A solution x is *noncontinuable* if no such continuation exists; that is, the interval $[\sigma, a)$ is the *maximal interval of existence* of the solution x.

Theorem 1.3.6. ([51]) *Suppose that Ω is an open set in $\mathbb{R} \times \mathcal{C}$, $f : \Omega \to \mathbb{R}^n$ is completely continuous (that is, f is continuous and takes closed bounded sets into compact sets), and x is a noncontinuable solution of* (1.20) *on $[\sigma - r, b)$. Then for any closed bounded set U in $\mathbb{R} \times \mathcal{C}$, $U \subset \Omega$, there is a t_U such that $(t, x_t) \notin U$ for $t_U \leq t < b$.*

In other words, Theorem 1.3.6 says that solution of (1.20) either exists for all $t \geq \sigma$ or becomes unbounded (with respect to Ω) at some finite time.

1.3.2 Neutral delay differential equations

Definition 1.3.7. (Hale and Verduyn Lunel [51]) *Suppose that $\Omega \subseteq \mathbb{R} \times \mathcal{C}$ is open with elements (t, ϕ). A function $D : \Omega \to \mathbb{R}^n$ is said to be atomic at β on Ω if D is continuous together with its first and second Fréchet derivatives with respect to ϕ; and D_ϕ, the derivative with respect to ϕ, is atomic at β on Ω.*

Suppose that $\Omega \subseteq \mathbb{R} \times \mathcal{C}$ is open, $f : \Omega \to \mathbb{R}^n$, $D : \Omega \to \mathbb{R}^n$ are given continuous functions with D atomic at zero. Consider the neutral delay differential equation

$$\frac{d}{dt} D(t, x_t) = f(t, x_t). \tag{1.22}$$

Definition 1.3.8. (Hale and Verduyn Lunel [51]) *A function x is said to be a solution of* (1.22) *on $[\sigma - r, \sigma + A]$ if there are $\sigma \in \mathbb{R}$ and $A > 0$ such that*

$$x \in C([\sigma - r, \sigma + A], \mathbb{R}^n), \quad (t, x_t) \in \Omega, \quad t \in [\sigma, \sigma + A],$$

$D(t, x_t)$ is continuously differentiable and satisfies equation (1.22) *on $[\sigma, \sigma + A]$. For a given $t_0 \in \mathbb{R}$, $\phi \in \mathcal{C}$, and $(\sigma, \phi) \in \Omega$, we say $x(t, \sigma, \phi)$ is a solution of equation* (1.22) *with initial value ϕ at σ or simply a solution through (σ, ϕ) if there is an $A > 0$ such that $x(t, \sigma, \phi)$ is a solution of equation* (1.22) *on $[\sigma - r, \sigma + A]$ and $x_\sigma(\sigma, \phi) = \phi$; we say $x(t, \sigma, \phi)$ is a solution of* (1.22) *on $[\sigma - r, \infty)$, if for every $A > 0$, $x(t, \sigma, \phi)$ is a solution of equation* (1.20) *on $[\sigma - r, \sigma + A]$ and $x_\sigma(\sigma, \phi) = \phi$.*

Theorem 1.3.9. (Existence) (Hale and Verduyn Lunel [51]) *If Ω is an open set in $\mathbb{R} \times \mathcal{C}$ and $(\sigma, \phi) \in \Omega$, then there exists a solution of the NDDE (D, f) through (σ, ϕ).*

Theorem 1.3.10. (Existence and Uniqueness) (Hale and Verduyn Lunel [51]) *If Ω is an open set in $\mathbb{R} \times \mathcal{C}$ and $f(t, \phi)$ is Lipschitzian in ϕ on compact sets of Ω, then, for any $(\sigma, \phi) \in \Omega$, there exists a unique solution of the NDDE (D, f) through (σ, ϕ).*

A continuation result similar to Theorem 1.3.6 also exists for neutral delay differential equations, refer to Hale and Verduyn Lunel [51] for details.

1.3.3 Stability of delay differential equations

Suppose that $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is continuous and consider the delay differential equation

$$\dot{x}(t) = f(t, x_t). \tag{1.23}$$

The function f will be supposed to be completely continuous and to satisfy enough additional smoothness conditions to ensure the solution $x(t, \sigma, \phi)$ through (σ, ϕ) is continuous in (t, σ, ϕ) in the domain of definition of the function.

Definition 1.3.11. *Suppose that $f(t, 0) = 0$ for all $t \in \mathbb{R}$. The solution $x = 0$ of equation (1.23) is said to be stable if for any $\sigma \in \mathbb{R}$, $\varepsilon > 0$, there is a $\delta = \delta(\varepsilon, \sigma) > 0$ such that $\phi \in \mathcal{B}(0, \delta)$ implies $x_t(\sigma, \phi) \in \mathcal{B}(0, \varepsilon)$ for $t \geq \sigma$. The solution $x = 0$ of equation (1.23) is said to be uniformly stable if the number δ in the definition is independent of σ.*

Definition 1.3.12. *The solution $x = 0$ of equation (1.23) is said to be asymptotically stable if it is stable and there is a $b_0 = b_0(\sigma)$ such that $\phi \in \mathcal{B}(0, b_0)$ implies that $x(t, \sigma, \phi) \to 0$ as $t \to \infty$. The solution $x = 0$ of equation (1.23) is said to be uniformly asymptotically stable if it is uniformly stable and there is $b_0 > 0$ such that for every $\eta > 0$ there is a $t_0(\eta)$ such that $\phi \in \mathcal{B}(0, b_0)$ implies $x_t(\sigma, \phi) \in \mathcal{B}(0, \eta)$ for $t \geq \sigma + t_0(\eta)$ for every $\sigma \in \mathbb{R}$.*

Definition 1.3.13. *A solution $x(t, \sigma, \phi)$ of an DDE (f) is bounded if there is a $\beta(\sigma, \phi)$ such that $|x(t, \sigma, \phi)| < \beta(\sigma, \phi)$ for $t \geq \sigma - r$. The solutions are uniformly bounded if for any $\alpha > 0$, there is a $\beta = \beta(\alpha) > 0$ such that for all $\sigma \in \mathbb{R}$, $\phi \in \mathcal{C}$ and $|\phi| \leq \alpha$, we have $|x(t, \sigma, \phi)| \leq \beta(\alpha)$ for all $t \geq \sigma$.*

1.3.4 Stability by spectral theory

Consider a linear ordinary differential equation of the form

$$x'(t) = ax(t). \tag{1.24}$$

The characteristic equation of (1.24) is $\lambda = a$, the solution of (1.24) is asymptotically stable if $Re(a) < 0$ and it is unstable if $Re(a) > 0$.

What about the stability of delay differential equations? Consider the following delay differential equation

$$x'(t) = ax(t) + bx(t - 1), \tag{1.25}$$

Here a, b are constants. From Figure 1.8, the solution of (1.25) is stable with $a = \frac{1}{2}$ and $b = -1$

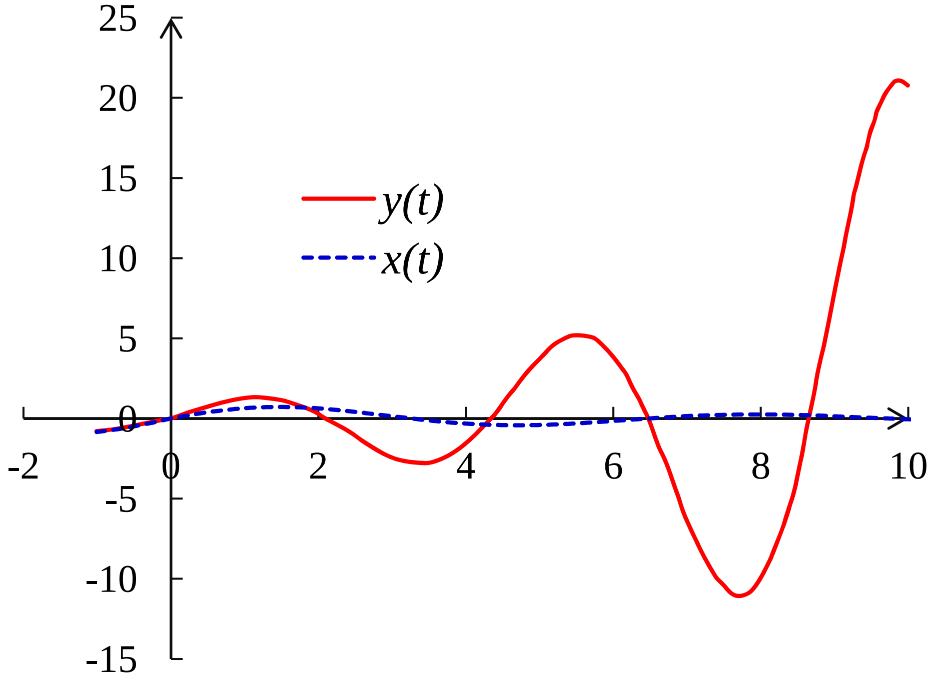

Figure 1.8: Numerical solution of (1.25) with $a = \frac{1}{2}, b = -1$ ($x(t)$) and $a = \frac{1}{2}, b = -2$ ($y(t)$).

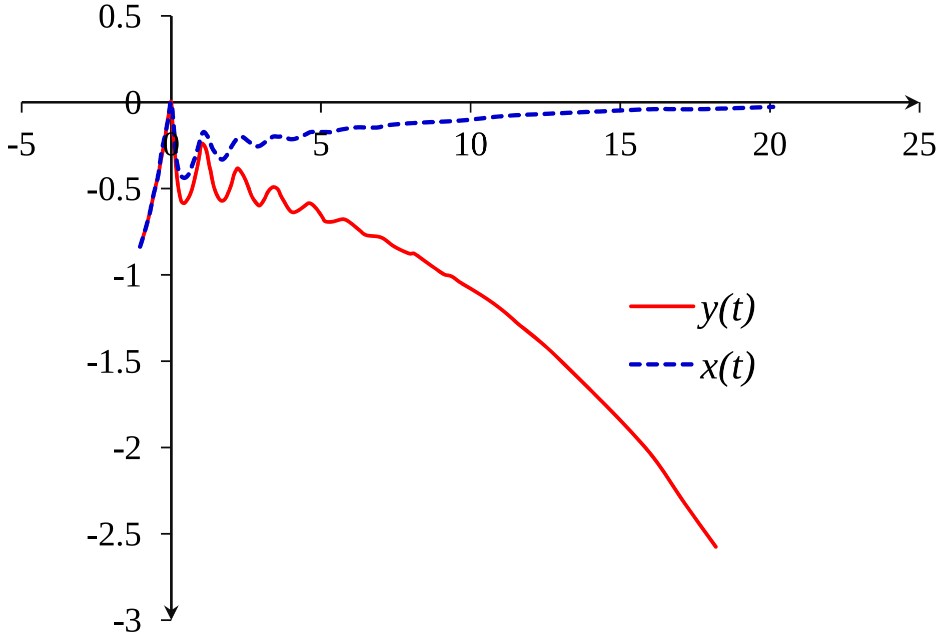

Figure 1.9: Numerical solution (1.25) with $a = -\frac{7}{2}, b = 3$ ($x(t)$) and $a = -\frac{7}{2}, b = 4$ ($y(t)$).

and unstable with $a = \frac{1}{2}$ and $b = -2$. From Figure 1.9, the solution is stable of (1.25) with $a = -\frac{7}{2}$ and $b = 3$ and unstable with $a = -\frac{7}{2}$ and $b = 4$. It is not difficult to find that the stability theory of delay differential equations is more complicated than the case for ordinary differential equations.

For such linear, autonomous delay differential equations, a simple way to study its stability is by spectral theory.

In fact, the characteristic equation of (1.25) is

$$z - a - be^{-z} = 0. \tag{1.26}$$

It is stable if all roots of the characteristic equation satisfy $Re(z) \leq \beta < 0$; It is unstable if for some root z, $Re(z) \geq 0$. Hence, to study the stability of (1.25) is to derive as much information as we can about the location of the roots of the characteristic equation (1.26) in the complex plane.

Let $z = \mu + i\nu$ in (1.26), we obtain two real equations

$$\begin{aligned} \mu - a - be^{-\mu}\cos\nu &= 0 \\ \nu + be^{-\mu}\sin\nu &= 0, \end{aligned} \tag{1.27}$$

where μ and ν are real numbers. By studying (1.27), some results towards the location of the roots in the complex plane of (1.26) are presented in Diekmann et al. [29].

Define the following strips,

$$\begin{aligned} \Sigma_k^+ &= \{\mu + i\nu \mid \nu \in I_k^+ = (2k\pi, (2k+1)\pi)\}, \\ \Sigma_k &= \{\mu + i\nu \mid \nu \in I_k = ((2k-1)\pi, (2k+1)\pi)\}, \\ \Sigma_k^- &= \{\mu + i\nu \mid \nu \in I_k^- = ((2k-1)\pi, 2k\pi)\}. \end{aligned}$$

Theorem 1.3.14. (Diekmann et al. [29]) *For $b > 0$, equation (1.26) has a unique and simple root λ_k in the strip Σ_k for $k = 0, 1, 2, \cdots$ and no other roots. For $k = 1, 2, \cdots$, the root λ_k is contained in Σ_k^-.*

Theorem 1.3.15. (Diekmann et al. [29]) *For $b < 0$, equation (1.26) has a unique and simple root λ_k in the strip Σ_k^+ for $k = 1, 2, \cdots$. There are two roots in Σ_0 (which are real and simple for $-e^{a-1} < b < 0$ and complex conjugate for $b < -e^{a-1}$). There are no other roots.*

However, in some real-world applications, the delay differential equations are nonautonomous, for example,

$$x'(t) = a(t)x(t) - b(t)x(t - r), \tag{1.28}$$

and

$$x'(t) = a(t)x(t) - b(t)x(t - r(t)). \tag{1.29}$$

What can we say about asymptotic behavior and stability of nonautonomous delay differential equations such as (1.28) and (1.29)? In the following, we present three methods, Liapunov's direct method, fixed point method and LMI method, which are extensively applied to the stability of nonautonomous equations.

1.3.5 Liapunov's direct method

Liapunov's direct method has long been viewed the main classical method of studying stability problems in many areas of differential equations. The difficulty of this method is to look for a suitable Liapunov functional or Liapunov function.

If $V : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ is continuous and $x(t, \sigma, \phi)$ is the solution of equation (1.23) through (σ, ϕ), we define

$$\dot{V}(t, \phi) = \limsup_{h \to 0^+} \frac{1}{h}[V(t + h, x_{t+h}(t, \phi)) - V(t, \phi)].$$

The function $\dot{V}(t, \phi)$ is the upper right-hand derivative of $V(t, \phi)$ along the solution of (1.23).

Theorem 1.3.16. (Hale and Verduyn Lunel [51]) *Suppose $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded sets of $\mathcal{C}$) into bounded sets of $\mathbb{R}^n$, and $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there is a continuous function $V : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ such that*

$$\begin{aligned} u(|\phi(0)|) &\leq V(t, \phi) \leq v(|\phi|) \\ \dot{V}(t, \phi) &\leq -w(|\phi(0)|), \end{aligned}$$

then the solution $x = 0$ of equation (1.23) is uniformly stable. If $u(s) \to \infty$ as $s \to \infty$, the solutions of equation (1.23) are unifomly bounded. If $w(s) > 0$ for $s > 0$, then the solution $x = 0$ is uniformly asymptotically stable.

Example 1.3.17. (Burton [11]) *Consider the delay differential equation*

$$\dot{x}(t) = -b(t)x(t - r), \tag{1.30}$$

where $r > 0$ is a constant, $b : [0, \infty) \to \mathbb{R}$ is a bounded and continuous function.

The equation (1.30) can be written as the form

$$\dot{x}(t) = -b(t + r)x(t) + \frac{d}{dt}\int_{t-r}^{t} b(s + r)x(s)\, ds, \tag{1.31}$$

equation (1.31) is equivalent to

$$\left(x(t) - \int_{t-r}^{t} b(s + r)x(s)\, ds\right)' = -b(t + r)x(t). \tag{1.32}$$

By constructing the following the Liapunov functional $V(t, x_t) = V_1(t, x_t) + V_2(t, x_t)$, where

$$V_1(t, x_t) = \left(x(t) - \int_{t-r}^{t} b(s + r)x(s)\, ds\right)^2 + \int_{-r}^{0}\int_{t+s}^{t} b(u + r)x^2(u)\, du\, ds \tag{1.33}$$

and

$$V_2(t, x_t) = \gamma\left(x^2 + \int_{t-r}^{t} b(s + r)x^2(s)\, ds\right). \tag{1.34}$$

Burton [11] obtained the following theorem.

Theorem 1.3.18. (Burton [11]) *If $b(t+r) \geq 0$ for all $t \geq 0$ and $\int_0^\infty b(s)\,ds = \infty$, an $\varepsilon > 0$ with*

$$b(t+r)\int_{t-r}^{t} b(s+r)\,ds - 2 + r \leq -\varepsilon \quad \text{for} \quad \text{all} \quad t \geq 0,$$

and there is a $\gamma > 0$ with $\gamma[b(t)+b(t+r)] \leq (\varepsilon/2)b(t+r)$ for all $t \geq 0$, then the zero solution of (1.30) is asymptotically stable.

Example 1.3.19. (Burton [11]) *Let $b(t) = 1.1 + \sin t$ in (1.30), we have*

$$\dot{x}(t) = -(1.1 + \sin t)x(t-r). \tag{1.35}$$

Theorem 1.3.18 holds if there is an $\varepsilon > 0$ such that

$$2.1(1.1r + 2\sin(r/2)) - 2 + r < -\varepsilon. \tag{1.36}$$

Using a rough estimate (taking $\sin(r/2) = r/2$) on (1.36), we have that $r < 0.37$. Therefore, the zero solution of (1.35) is asymptotically stable if $r < 0.37$.

1.3.6 Fixed point method

Liapunov's direct method has been very effective in establishing stability results for a wide variety of differential equations. The success of Liapunov's direct method depends on finding good Liapunov functions or Liapunov functionals, which may be very difficult, especially for the equations with unbounded terms or unbounded delays, see the examples in Burton [13]. Therefore, it was recently proposed by Burton [13] and co-workers to use fixed point methods as an alternative. While Liapunov's direct method usually requires pointwise conditions, fixed point methods need conditions of an averaging nature.

Theorem 1.3.20. **(Banach's fixed point theorem)** *Let (X, d) be a non-empty complete metric space, let $T : X \to X$ be a contraction mapping on X, i.e. there is a nonnegative real number $q < 1$ such that $d(Tx, Ty) \leq qd(x, y)$ for all $x, y \in X$. Then the map T admits one and only one fixed point x^* in X ($Tx^* = x^*$).*

Hence, to solve a problem using a fixed point approach we have to identify:

(a) a set $\mathcal{S}$ consisting of points which would be acceptable solutions;

(b) a mapping $P : \mathcal{S} \to \mathcal{S}$ with the property that a fixed point solves the problem;

(c) a fixed point theorem stating that this mapping on the set $\mathcal{S}$ will have a fixed point.

The following steps represent the way in which we can establish stability of the zero solution of a delay differential equation by applying fixed point theory.

Step 1. An examination of the differential equation reveals that for a given initial time σ there is an initial interval we denote it to be E_σ and we require an initial function $\phi : E_\sigma \to \mathbb{R}^n$. We then must determine a set $\mathcal{S}$ of functions $\varphi : E_\sigma \cup [\sigma, \infty) \to \mathbb{R}^n$ with $\varphi(t) = \phi(t)$ on E_σ which could serve as acceptable functions. Usually, this means that we would ask some other conditions on φ, for example, the boundedness, and sometimes we require that $\varphi(t) \to 0$ as $t \to \infty$.

Step 2. Next, invert the differential equation and define a mapping from $\mathcal{S}$ to $\mathcal{S}$.

Step 3. Finally, we select a fixed point theorem which will show that the mapping P has a fixed point in $\mathcal{S}$.

Notice that the process of application of a fixed point method relies on three principles: an elementary variation of constants formula, a complete metric space and the contraction mapping principle. Moreover, in one step, a fixed point argument yields existence, uniqueness and stability. Hence, our major problem, when using fixed point theory to deal with stability analysis, is to define a suitable Banach space and a suitable mapping.

In the following, some results are presented to illustrate the application of a fixed point method. Consider the delay differential equation (1.30), by using a fixed point method, Burton [11] obtained the following result.

Theorem 1.3.21. (Burton [11]) *Suppose there exists a constant $\alpha < 1$ such that*

$$\int_{t-r}^{t} |b(s+r)|\, ds + \int_{0}^{t} |b(s+r)| e^{-\int_s^t b(u+r)\, du} \int_{s-r}^{s} |b(u+r)|\, du\, ds \leq \alpha, \tag{1.37}$$

for all $t \geq 0$ and $\int_0^\infty b(s)\, ds = \infty$. Then for every continuous initial function $\phi : [-r, \infty) \to \mathbb{R}$, the solution $x(t) = x(t, 0, \phi)$ of (1.30) is bounded and tends to zero as $t \to \infty$.

Example 1.3.22. (Burton [11]) *Consider the differential equation*

$$\dot{x}(t) = -(1 + 2\sin t)x(t-r), \tag{1.38}$$

where $0 < r < 1$. The zero solution of (1.38) is asymptotically stable when $(r + 4\sin(r/2))(2 + 2e^2) < 1$, this is approximately $0 \leq r < 0.02$.

Since $1 + 2\sin t$ changes sign for $t \geq 0$, Theorem 1.3.18 is not applicable to Example 1.3.22. Consider Example 1.3.19 by using Theorem 1.3.21, we obtain that the zero solution of (1.35) is asymptotically stable if $2(1.1r + 2\sin(r/2)) < 1$. This is approximated by $0 < r < 0.2$, compared to $r < 0.37$ by using Liapunov's direct method.

From the above discussion, we find it is very difficult to find a way to interpret a relation between the fixed point method and Liapunov's direct method. Sometimes the fixed point method can provide conditions for stability when the Liapunov's direct method can not, see Example 1.3.22. Sometimes Liapunov's direct method can provide better conditions, see Example 1.3.19.

If we let $r = 0.1$ in (1.38), the condition (1.37) in Theorem 1.3.21 is not satisfied, then Theorem 1.3.21 is not applicable. Therefore, new conditions are needed to study the case of $r = 0.1$. Following the similar arguments as Burton [11], Raffoul [110] studied the following linear neutral differential equation

$$\dot{x}(t) - c(t)\dot{x}(t - r(t)) = -a(t)x(t) - b(t)x(t - r(t)), \tag{1.39}$$

and he obtained the following result.

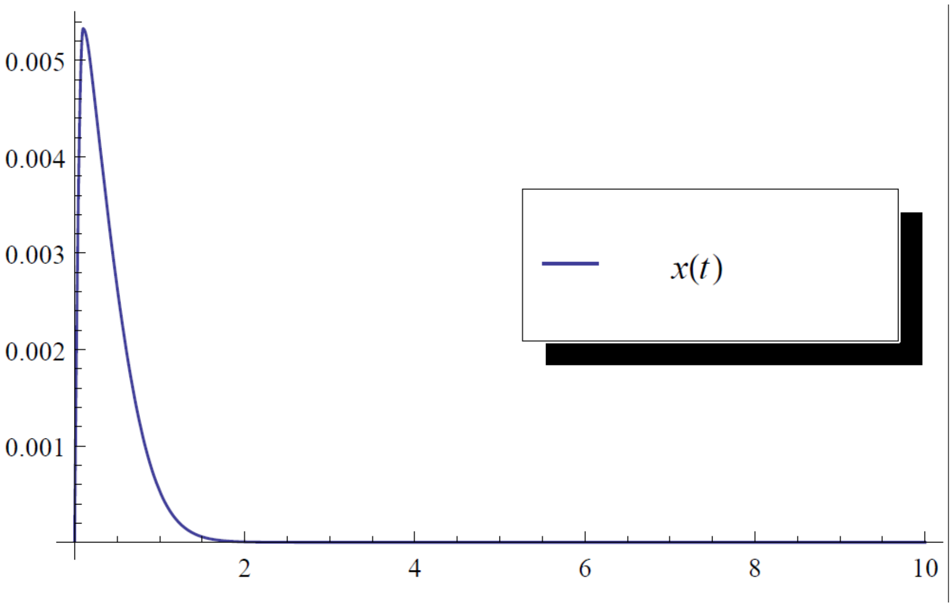

Figure 1.10: Numerical solution of (1.38).

Theorem 1.3.23. (Raffoul [110]) *Let $r(t)$ be twice differentiable and $r'(t) \neq 1$ for all $t \in R$. Suppose that there exists a constant $\alpha \in (0,1)$ such that for $t \geq 0$*

$$\int_0^t a(s)\,ds \to \infty \quad \text{as} \quad t \to \infty, \tag{1.40}$$

and such that

$$\left|\frac{c(t)}{1-r'(t)}\right| + \int_0^t e^{-\int_s^t a(u)\,du} \left| b(s) + \frac{[c(s)a(s)+c'(s)](1-r'(s)) + c(s)r''(s)}{(1-r'(s))^2} \right| ds \leq \alpha, \tag{1.41}$$

Then every solution $x(t) = x(t, 0, \phi)$ of (1.39) with a small continuous initial function ϕ is bounded and tends to zero as $t \to \infty$.

Example 1.3.24. *Consider the linear neutral delay differential equation*

$$\dot{x}(t) = -\frac{1}{t+1}x(t) + 0.48\dot{x}(t-0.05t). \tag{1.42}$$

However, the condition (1.41) in Theorem 1.3.23 is not satisfied. In fact,

$$\begin{aligned} &\left|\frac{c(t)}{1-r'(t)}\right| + \int_0^t e^{-\int_s^t a(u)\,du} \left| b(s) + \frac{[c(s)a(s)+c'(s)](1-r'(s)) + c(s)r''(s)}{(1-r'(s))^2} \right| ds \\ &= \frac{0.48(2t+1)}{0.95(t+1)}. \end{aligned} \tag{1.43}$$

Since the right-hand side of (1.43) is increasing in $t > 0$ and

$$\limsup_{t \geq 0} \frac{0.48(2t+1)}{0.95(t+1)} = 1.0105,$$

then there exists some $t_0 > 0$ such that $t \geq t_0$, we have

$$\left|\frac{c(t)}{1-r'(t)}\right| + \int_0^t e^{-\int_s^t a(u)\,du}\left|b(s) + \frac{[c(s)a(s)+c'(s)](1-r'(s))+c(s)r''(s)}{(1-r'(s))^2}\right| ds > 1.01.$$

This implies that condition (1.41) does not hold. Thus, Theorem 1.3.23 is not applicable.

Hence, weaker conditions needed to be provieded to solve such problems (Example 1.3.22 for $r = 0.1$ and Example 1.3.24). By introducing a continuous function $v(t)$ for constructing fixed point mapping, Jin and Luo [62] provided sufficient conditions for the asymptotic stability of (1.39), which can be applied to Example 1.3.22 and Example 1.3.24.

Theorem 1.3.25. (Jin and Luo [62]) *Suppose the following conditions are statisfied.*

(i) *the delay $r(t)$ is twice differentiable and $r'(t) \neq 1$ for all $t \in \mathbb{R}^+$.*

(ii) *there exists a constant $\alpha \in (0,1)$ and a continuous function $v(\mathbb{R}^+ \to \mathbb{R})$ such that $\liminf_{t\to\infty} \int_0^t v(s)\,ds > -\infty$ and*

$$\begin{aligned} P(t) \;&=\; \int_{t-r(t)}^t |v(s)-a(s)|\,ds + \left|\frac{c(t)}{1-r'(t)}\right| \\ &\quad + \int_0^t e^{-\int_s^t v(u)\,du}|-b(s) + [v(s-r(s)) - a(s-r(s))] - k(s)|\,ds \\ &\quad + \int_0^t e^{-\int_s^t v(u)\,du}|v(s)| \int_{s-r(s)}^s |v(u)-a(u)|\,du\,ds \leq \alpha, \end{aligned} \tag{1.44}$$

then the zero solution of (1.39) *is asymptotically stable if and only if $\int_0^t v(s)\,ds \to \infty$ as $t \to \infty$.*

It is clear that Theorem 1.3.25 is consistant with Theorem 1.3.23 if $v(t) = a(t)$ for $t \geq 0$ in (1.44). In addition, Theorem 1.3.25 can be applied to some equations that Theorem 1.3.23 can not. Motivated by the work as in [62], in this thesis, we will discuss some general classes of delay differential equations by using the approach as in Theorem 1.3.25.

Notice that the condition (1.44) in Theorem 1.3.25 is mainly dependent on the constraint $\left|\frac{c(t)}{1-r'(t)}\right| < 1$. However, There are some interesting examples where the constraint is not satisfied. Zhao [145] investigated (1.39) without the constraint by employing another auxiliary function $p(t)$ to construct the fixed point mapping. In this thesis, we will study the approaches used in [62] and [145] to consider some general classes of delay differential equations.

1.3.7 Linear matrix inequality (LMI) method

The linear matrix inequality (LMI) method has become one of basic approaches to study stability of delay differential equations and stochastic delay differential equations. This approach is based on constructing suitable Liapunov functionals and combining with a linear matrix inequality. To illustrate this method, we present some results from Fridman [44].

Consider the following system

$$\dot{x}(t) = \sum_{i=0}^m A_i x(t-h_i), \quad x(t) = \phi(t), \quad t \in [-h, 0], \tag{1.45}$$

where $x(t) \in \mathbb{R}^n$, $h_0 = 0$, $0 < h_i \leq h$, A_i is a constant $n \times n$ matrix, ϕ is a continuously differentiable initial function. We represent (1.45) in the equivalent descriptor form

$$\dot{x}(t) = y(t), \quad y(t) = \left(\sum_{i=0}^{m} A_i\right) x(t) - \sum_{i=1}^{m} A_i \int_{t-h_i}^{t} y(s)\, ds. \tag{1.46}$$

Liapunov-Krasovskii functional for the system (1.46) has the form

$$V(t) = \begin{pmatrix} x^T(t) & y^T(t) \end{pmatrix} EP \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + V_1, \tag{1.47}$$

where

$$E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} P_1 & 0 \\ P_2 & P_3 \end{pmatrix}, \quad P_1 = P_1^T > 0,$$

$$V_1 = \sum_{i=1}^{m} \int_{-h_i}^{0} \int_{t+\theta}^{t} y^T R_i y(s)\, ds\, d\theta, \quad R_i > 0.$$

Computing $dV(t)/dt$ and using the conditions in Theorem 1.3.26, we obtain that the function V of (1.47) has a negative derivative, which implies asymptotically stable of (1.45).

Theorem 1.3.26. (Fridman [44]) *Equation* (1.45) *is asymptotically stable if there exist* $0 < P_1 = P_1^T$, P_2, P_3 *and* $R_i = R_i^T$, $i = 1, \cdots, m$ *that satisfy the following linear matrix inequality* (LMI):

$$\begin{pmatrix} (\sum_{i=0}^m A_i^T)P_2 + P_2^T(\sum_{i=0}^m A_i) & P_1 - P_2^T + (\sum_{i=0}^m A_i^T)P_3 & h_1 P_2^T A_1 & \cdots & h_m P_2^T A_m \\ P_1 - P_2 + P_3^T(\sum_{i=0}^m A_i) & -P_3 - P_3^T + \sum_{i=1}^m h_i R_i & h_1 P_3^T A_1 & \cdots & h_m P_3^T A_m \\ h_1 A_1^T P_2 & h_1 A_1^T P_3 & -h_1 R_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ h_m A_m^T P_2 & h_m A_m^T P_3 & \cdot & \cdots & -h_m R_m \end{pmatrix} < 0.$$

Example 1.3.27. (Fridman [44]) *Consider the system*

$$\dot{x} = A_0 x(t) + A_1 x(t - h_1) \tag{1.48}$$

with

$$A_0 = \begin{pmatrix} -1 & 0.5 \\ -0.5 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix}.$$

Applying LMI condition in Theorem 1.3.26, we obtain that $h_1 \leq 0.271$. Therefore, the equation (1.48) is asymptotically stable if $h_1 \leq 0.271$. For $h_1 = 0.271$ we obtain the following solution to LMI condition:

$$P_1 = \begin{pmatrix} 94.1609 & 0.1653 \\ -0.1653 & 94.0469 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 93.5589 & 0.1872 \\ 0.1872 & 94.6599 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 18.5170 & -0.0930 \\ -0.0930 & 18.4880 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 68.2748 & 0.0349 \\ 0.0349 & 68.1810 \end{pmatrix}.$$

The LMI method is also widely used to study stability of neural networks, to know more about this method, refer to [44, 45, 77, 114, 126, 127].

1.3.8 Stochastic processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is the collection of all possible outcomes, and $\mathcal{F}$ is the set of all events A to which a probability $\mathbb{P}(A)$ can be attached. $\mathcal{F}$ is $\sigma-$ algebra, and $\mathbb{P}$ a probability measure. We are often very interested in events $A \in \mathcal{F}$ which are realized for almost all $\omega \in \Omega$. Such events are called *almost sure events*, and A is an almost sure event if $\mathbb{P}[A] = 1$. We will often use the abbreviation a.s. to stand for almost sure or almost surely.

Definition 1.3.28. *Let $T \subseteq [0, \infty)$. A stochastic process is a family $(X(t))_{t\in T}$ of random variables on (Ω, F, P). For each $\omega \in \Omega$, the map $t \to X(t)(\omega)$ is called a path of X.*

Definition 1.3.29. *If $I \subset R$ is an interval, $f : I \to R$ is a function, and $x \in I$, then f is said to have a right limit at x if*

$$f(x+) := \lim_{y \downarrow x} f(y) \qquad \text{exists}$$

and a left limit if

$$f(x-) := \lim_{y \uparrow x} f(y) \qquad \text{exists}$$

f is right continuous at x if it has a right limit at x and $f(x+) = f(x)$ and left continuous if it has a left limit and $f(x-) = f(x)$. The function f is called a càdlàg function if at each $x \in I$ it is right continuous and has a left limit. f is called càglàd if it is left continuous and has a right limit at each point of I.

Definition 1.3.30. *A stochastic process is called a càdlàg process if each if its paths is a càdlàg function. A function f is of bounded variation if it equals the difference of two increasing functions. A process is said to be of bounded variation if each of its paths is of bounded variation.*

Definition 1.3.31. *A filtration in $(\Omega, \mathcal{F}, \mathbb{P})$ is a fimily of σ-algebras $(\mathcal{F}_t)_{t\in T}$ in Ω such that $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \in T$ and*

$$s \leq t \Rightarrow \mathcal{F}_s \subseteq \mathcal{F}_t.$$

A null set is a subset $A \subseteq \Omega$ for which there exists a $B \in \mathcal{F}$ such that $A \subseteq B$ and P(B) = 0. $\mathcal{F}$ is called $\mathbb{P}$-complete if each null set is a member of $\mathcal{F}$.

A filtration is said to satisfy the "usual conditions" if for every t, $\mathcal{F}_t$ is $\mathbb{P}$-complete and $\mathcal{F}_t = \cap_{u>t}\mathcal{F}_u$.

Definition 1.3.32. *A process $(X(t))_t \in T$ is called adapted to a filtration $(\mathcal{F}_t)_{t\in T}$ if $X(t)$ is $\mathcal{F}_t$-measurable for all $t \in T$.*

Definition 1.3.33. *For a random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ its expectation is defined as*

$$\mathbb{E}X = \int_\Omega X(\omega)\, d\mathbb{P}(\omega),$$

provided $X \geq 0$ a.e. on Ω or $\int_\Omega |X(\omega)|\, d\mathbb{P}(\omega) < \infty$.

For a random variable X with $\mathbb{E}|X| < \infty$ and a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$,

$$\mu(G) := \int_{\mathcal{G}} X(\omega)\, d\mathbb{P}(\omega), \qquad G \in \mathcal{G}$$

defines a measure μ on $\mathcal{G}$. Clearly, if $P(G) = 0$, then $\mu(G) = 0$. Due to the Radon-Nikodym theorem there exists a $\mathcal{G}$-measurable function $X_{\mathcal{G}} : \Omega \to \mathbb{R}$ such that

$$\mu(G) = \int_{G} X_{\mathcal{G}}(\omega)\, d\mathbb{P}(\omega), \qquad \text{for} \quad \text{all} \quad \mathrm{G} \in \mathcal{G},$$

This $\mathcal{G}$-measurable random variable $X_{\mathcal{G}}$ is called the conditional expectation of X with respect to $\mathcal{G}$ and is denoted by $E[X|\mathcal{G}]$.

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t \geq 0}$ in $\mathcal{F}$ that satifies the usual conditions.

Definition 1.3.34. *A martingale is a stochastic process $(X(t))_{t \in T}$ which is adapted, $\mathbb{E}|X(t)| < \infty$ for all t, and such that $\mathbb{E}[X(t)|\mathcal{F}_s] = X(s)$ whenever $s \leq t$, $s, t \in T$.*

Definition 1.3.35. *A random variable T with values in $[0, \infty]$ is called a stopping time if $\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t$ for every $0 \leq t \leq \infty$.*

For instance, if $(X(t))_{t \geq 0}$ is an adapted *càdlàg* process with $X(0) = 0$ and $B \subseteq \mathbb{R}$ is open, then the first time of hitting B defined by

$$T(\omega) = \inf\{t > 0 : X(t)(\omega) \in B \quad \text{or} X(t-)(\omega) \in B\}.$$

Definition 1.3.36. *If $(X(t))_{t \geq 0}$ is a stochastic process and T is a stopping time, then the stopped process X^T is given by*

$$X^T(t)(\omega) = \begin{cases} X(t)(\omega), & t < T(\omega), \\ \\ X(T(\omega))(\omega), & t \geq T(\omega), \quad \in \Omega, \quad t \geq 0 \end{cases}$$

In particular, if $(X(t))_{t \geq 0}$ is adapted and continuous, T is the first time of hitting $\mathbb{R} \setminus (-M, M)$, then X^T is uniformly bounded and $X^T(t) \leq M$ for all $t \geq 0$.

Definition 1.3.37. *A process $(X(t))_{t \geq 0}$ is called uniformly integrable if*

$$\lim_{n \to \infty} \sup_{t \geq 0} \int_{\{|X(t)| \geq n\}} |X(t)|\, d\mathbb{P} = 0.$$

If X is a random variable with $\mathbb{E}|X| < \infty$, then $(X(t))_{t \geq 0}$ given by $X(t) = \mathbb{E}[X|\mathcal{F}_t]$, $t \geq 0$, is a uniformly integrable martingale.

Definition 1.3.38. *An adapted càdlàg process $(X(t))_{t \geq 0}$ is called a local martingale, if there exists a sequence of stopping times $T_1, T_2, \cdots$ with $0 \leq T_1 \leq T_2 \cdots$ a.s. and $\lim_{n \to \infty} T_n = \infty$ a.s. such that for each n, the stopped process X^{T_n} is a uniformly integrable martingale.*

Note that any *càdlàg* martingale is a local martingale. (Taking $T_n = n$, $n \in \mathbb{N}$).

We show that the following stochastic convolution is not a martingale.

$$\int_0^t e^{-c(t-s)}\sigma(s)\,dw(s), \tag{1.49}$$

where $\sigma(t)$ is a continuous function. In fact, for $0 \le u \le t$,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t e^{-c(t-s)}\sigma(s)\,dw(s) \mid \mathcal{F}_u\right] \qquad (1.50)\\
&= \mathbb{E}\left[\int_0^u e^{-c(t-s)}\sigma(s)\,dw(s) \mid \mathcal{F}_u\right] + \mathbb{E}\left[\int_u^t e^{-c(t-s)}\sigma(s)\,dw(s) \mid \mathcal{F}_u\right]\\
&= \int_0^u e^{-c(t-s)}\sigma(s)\,dw(s) \neq \int_0^u e^{-c(u-s)}\sigma(s)\,dw(s).
\end{aligned}$$

It can also be shown that $\int_0^t e^{-c(t-s)}\sigma(s)\,d\omega(s)$ is not a local martingale. To show this, we need the following Lemma.

Lemma 1.3.39. ([109]) *If $M(t)$ is a local martingale and for every t, $\mathbb{E}\sup_{s\in[0,t]}|M(s)| < \infty$, then $M(t)$ is a martingale.*

Lemma 1.3.40. *For continuous function $\sigma(t)$, $\int_0^t e^{-c(t-s)}\sigma(s)\,dw(s)$ is not a local martingale.*

Proof. We suppose that $\int_0^t e^{-c(t-s)}\sigma(s)\,dw(s)$ is a local martingale. For every t, we have that

$$\begin{aligned}
\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s e^{-c(s-u)}\sigma(u)\,dw(u)\right| &= \mathbb{E}\sup_{s\in[0,t]} e^{-cs}\left|\int_0^s e^{cu}\sigma(u)\,dw(u)\right|\\
&\le \mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s e^{cu}\sigma(u)\,dw(u)\right|\\
&\le K_1\mathbb{E}\left(\int_0^t e^{2cu}\sigma^2(u)\,du\right)^{1/2}\\
&\le K_1\left(\int_0^t e^{2cu}\mathbb{E}\sigma^2(u)\,du\right)^{1/2} < \infty.
\end{aligned}$$

From Lemma 1.3.39, we obtain that M is a martingale. However, from (1.50), we know that $\int_0^t e^{-c(t-s)}\sigma(s)\,dw(s)$ is not a martingale, which is a contradiction. □

Lemma 1.3.41. (Mao [96] Burkholder-Davis-Gundy Inequality) *There exists a universal constant K_p for any $0 < p < \infty$ such that for every continuous local martingale M vanishing at zero and any stopping time τ,*

$$\mathbb{E}\left(\sup_{0\le s\le\tau}|M_s|^p\right) \le K_p\mathbb{E}((M,M)_\tau)^{p/2},$$

where $(M,M)_\tau$ is the cross-variation of M and in particular, one can take

$$\begin{aligned}
K_p &= \left(\frac{32}{p}\right)^{p/2} \quad \text{if} \quad 0 < p < 2,\\
K_p &= 4 \quad \text{if} \quad p = 2,\\
K_p &= \left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{p/2} \quad \text{if} \quad p > 2.
\end{aligned}$$

Lemma 1.3.42. ([109] Doob's inequality, on Page 11) *Let X be a positive submartingale. For all $p > 1$, with q conjugate to p (i.e. $\frac{1}{p} + \frac{1}{q} = 1$), we have*

$$\| \sup_t X_t \|_{L_p} \leq q \sup_t \|X_t\|_{L_p}.$$

For a real valued process, we let X^* denote $\sup_s |X_s|$. Note that if M is a martingale with $M_\infty \in L^p$, then $|M|$ is a positive submartingale, and we have

$$\mathbb{E}\{(M^*)^p\} \leq q^p \mathbb{E}\{M_\infty^p\}.$$

For $p = 2$, we have $\mathbb{E}\{(M^*)^2\} \leq 4\mathbb{E}\{M_\infty^2\}$. The last inequality is called Doob's maximal quadratic inequality.

Lemma 1.3.43. (Hölder inequality) *Assume that there exists two continuous functions $f(x)$, $g(x)$ and a set Ω, p and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$, for any $p > 0$, $q > 0$, if $p > 1$, then the following inequality holds.*

$$\int_\Omega |f(x)g(x)|\, dx \leq \left(\int_\Omega |f(x)|^p \, dx \right)^{1/p} \left(\int_\Omega |g(x)|^q \, dx \right)^{1/q}.$$

Lemma 1.3.44. ([120]) *For any real numbers $a_k \geq 0$, $k = 1, 2, 3, \cdots n$, and $p > 1$, the following inequality holds,*

$$\left(\sum_{k=1}^n a_k \right)^p \leq n^{p-1} \sum_{k=1}^n a_k^p.$$

1.3.9 Stochastic delay differential equations

The existence, uniqueness and stability of stochastic delay differential equations have been extensively investigated by many authors, see, for example, Friedman [43], Ikeda and Watanabe [60], Mao [96].

The techniques dealing the existence and uniqueness of stochastic delay differential equations have been developed mainly by using two different methods, the iterative method [2, 22, 96] and the fixed point method [1, 3, 7, 46].

One of the powerful techniques employed in the study of the stability problems of stochastic delay differential equations is the method of the Liapunov function or functional, see, for example, Kolmanovskii [71], Mao [93, 94]. Further, a great number classes of stochastic neural networks with delays are studied by using LMI method, see the work [73, 77, 113, 115, 133].

For the stochastic differential equations with infinite delays, it was recently proposed by Luo [90] and Appleby [4] to use fixed point methods to deal with the stability problems for stochastic delay differential equations. Many authors, e.g., Luo [90, 91], Luo and Taniguchi [92], Sakthivel and Luo [117, 118], Cui et al. [27] have applied fixed point methods to study stability properties of many classes of stochastic delay differential equations. It turns out the fixed point method is a powerful technique to deal with asymptotic stability and exponential stability of stochastic delay differential equations.

1.3.10 Some examples of Banach spaces

A normed linear space is a metric space with respect to the metric d derived from its norm, where $d(x,y) = \|x-y\|$.

Definition 1.3.45. *A Banach space is a normed linear space that is complete metric space with respect to the metric derived from its norm.*

Here are some examples.

Example 1.3.46. *The space $C([a,b])$ of continuous, real-valued (or complex-valued) functions on $[a,b]$ with the sup-normed is a Banach space. More generally, we have the following examples.*

(i) *If X is a Banach space, the space $C([a,b];X)$ of continuous, X-valued functions on $[a,b]$ equipped with the sup-norm is a Banach space.*

(ii) *If X is a Banach space, the space $BC([a,b];X) := \{\varphi \in C([a,b];X), \|\varphi\| < \infty\}$ of bounded continuous, X-valued functions on $[a,b]$ equipped with the sup-norm is a Banach space.*

(iii) *If X is a Banach space, the space $\{\varphi \mid \varphi \in C([a,b];X), \lim_{t\to\infty}\varphi(t) = 0\}$ with the sup-norm is a Banach space. Further, the space*

$$\{\varphi \mid \varphi \in C([a,b];X),\ \varphi(t) \to 0 \text{ as } t \to \infty\}$$

and the sapce

$$\left\{\varphi \mid \varphi \in C([a,b];X),\ \|\varphi\| = \sup_{s\in[a,b]} |\varphi(s)| \text{ is bounded and } \varphi(t) \to 0 \text{ as } t \to \infty\right\}$$

are Banach spaces with respect to the sup-norm. Clearly, the space

$$\left\{\varphi \mid \varphi \in C([a,b];L^p(\Omega,\mathbb{R}^n)),\ \lim_{t\to\infty} \mathbb{E}|\varphi(t)|^p = 0\right\}$$

is a Banach spaces with respect to the norm defined as

$$\|\varphi\| := \left(\sup_s \mathbb{E}|\varphi(s)|^p\right)^{1/p}.$$

Lemma 1.3.47. *Suppose that $\mathcal{F}_t$ is complete, (that is, contains all null sets). Denote by*

$$C_0([0,\infty);L^p(\Omega,\mathbb{R}^n)) := \left\{\varphi \mid \varphi \in C([0,\infty);L^p(\Omega,\mathbb{R}^n)),\ \lim_{t\to\infty} \mathbb{E}|\varphi(t)|^p = 0\right\},$$

then the space

$$D := \{\varphi \mid \varphi \in C_0([0,\infty);L^p(\Omega,\mathbb{R}^n)),\ \varphi(t) \text{ is } \mathcal{F}_t - \text{measurable for all } t\}$$

is a closed subspace of $C_0([0,\infty);L^p(\Omega,\mathbb{R}^n))$.

Proof. If $\varphi(t),\psi(t) \in D$, then $\varphi(t)$ and ψ are $\mathcal{F}_t$-measurable, so $\varphi(t)+\psi(t)$ and $\alpha\varphi(t)$ $(\alpha \in \mathbb{C})$ are $\mathcal{F}_t$-measurable.

Suppose that the sequences $\varphi_1(t), \varphi_2(t), \cdots \varphi_n(t) \cdots \in D$, $\varphi \in C_0([0,\infty);L^p(\Omega,\mathbb{R}^n))$, and $\varphi_n(t) \to \varphi(t)$, we claim that $\varphi(t)$ is $\mathcal{F}_t$-measurable. In fact, since $\varphi_n(t) \to \varphi(t)$, then

$$\sup_{s\in\Omega} (\mathbb{E}|\varphi_n(s) - \varphi(s)|^p) \to 0 \quad \text{as} \quad n \to \infty.$$

So, for every t, we have that $\mathbb{E}|\varphi_n(s) - \varphi(s)|^p \to 0$ as $n \to \infty$, then there exists a squence $(\varphi_{n_k}(t))_k$ such that $\varphi_{n_k}(t) \to \varphi(t)$ a.e. on Ω. On the other hand, $\mathcal{F}_t$ is complete. Hence, we obtain that $\varphi(t)$ is $\mathcal{F}_t$-measurable, which implies that D is a closed subspace of $C_0([0,\infty);L^p(\Omega,\mathbb{R}^n))$.

1.4 Structure of this thesis

This thesis is divided into two parts. The first part deals with asymptotic behavior and stability of deterministic delay differential equations. The second part is concerned with the stability properties of stochastic delay differential equations. Each chapter starts with an introduction, in which we summarize the main results. A brief overview of the contents of the thesis is given below.

Chapter 2 presents three methods concerning asymptotic behavior of autonomous neutral delay differential equations. One method based on spectral theory, another method that treats the equation as an ordinary differential equation (ODE) with the other state-dependent terms considered as perturbations, and a third method using Banach's fixed point theorem. We also address the relations of the spectral method and the ODE method. To a retarded form of the autonomous neutral delay differential equation, we illustrate a third method, fixed point method.

Chapter 3 focuses on asymptotic behavior of a class of nonautonomous neutral delay differential equations in which the coefficient for neutral term is constant. Such equations can not be treated by spectral theory, but in some special cases, a generalized characteristic equation can be used. This is a functional equation. If it can be solved, the precise asymptotic behavior of solution of the neutral equation and their derivative can be determined. Examples are given in which the generalized characteristic equation can be solved.

Chapter 4 addresses a fixed point approach to a series of differential and difference equations. In Section 4.1, four general classes of equations are considered by unifying recent results in the literature. For each of these classes of equations, different techniques are combined to prove new stability theorems. In addition, various examples are presented to illustrate our results. In Section 4.2, the stability of two classes of nonlinear neutral differential equations is studied by introducing two auxiliary functions. In Section 4.3, the stability of one class of nonlinear delay difference equations is investigated. The obtained theorems show the general applicability of the fixed point method.

Chapter 5 discusses the stability of two classes of neutral stochastic delay differential equations with impulses. In Section 5.1, asymptotic stability of a class of neutral stochastic delay differential equations with linear impulses is studied by means of the fixed point method. More specifically, two theorems for the asymptotic stability of the equations are presented by using two contraction mapping which are defined on different complete metric spaces. In Section 5.2, exponential stability of a class of neutral stochastic partial differential equations with variable delays and impulses is investigated. The equation is considered as an infinite dimensional stochastic differential equation with delays. The method by using an impulsive-integral inequality and a fixed point method are applied to study exponential stability of mild solutions of the impulsive neutral stochastic partial delay differential equations, respectively.

Chapter 6 studies stability properties of stochastic delayed neural networks without impulses and stochastic delayed neural networks with impulses. Our approaches are based on a fixed point method and the method by using an approporiate integral inequality. In Section 6.1, asymptotic stability and exponential stability of a class of stochastic delayed neural networks with discrete and distributed delays are studied. In particular, a class of delayed neural networks without stochastic perturbations is considered. In Section 6.2, impulsive effects to the class of stochastic delayed neural networks are studied.