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A fixed point approach towards stability of delay differential equations with applications to neural networks

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CHAPTER 1

Introduction

1.1 Outline

This thesis focuses on asymptotic behavior and stability of solutions of deterministic and stochastic delay differential equations.

A delay differential equation is a differential equation where the derivatives at the current time depend on the solution at previous times. Such equations are also called *differential equations with retarded argument*. Strictly speaking, a delay differential equation is a specific example of a functional differential equation, in which the functional part of the differential equation is the evaluation of a functional on the past of the process.

Suppose $r \geq 0$ is a given real number, $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^n$ is an n-dimensional linear vector space over the reals with norm $|\cdot|$, $\mathcal{C} = C([-r, 0], \mathbb{R}^n)$ is the set of continuous functions mapping $[-r, 0]$ into $\mathbb{R}^n$. Then $\mathcal{C}$ is a Banach space with respect to the supremum norm $\|\varphi\| = \sup_{-r\leq\theta\leq0} |\varphi(\theta)|$, where $\varphi \in \mathcal{C}$. If $\sigma \in \mathbb{R}$, $A \geq 0$ and $x \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$, then for any $t \in [\sigma, \sigma + A]$, we let $x_t \in C$ be defined by $x_t(\theta) = x(t + \theta)$ for $-r \leq \theta \leq 0$. If Ω is a subset of $\mathbb{R} \times \mathcal{C}$, $f : \Omega \to \mathbb{R}^n$ is a given function and "$\cdot$" represents the right-hand derivative, we say the relation

$$\dot{x}(t) = f(t, x_t), \tag{1.1}$$

is a *delay differential equation* on Ω, which is denoted by DDE (f). The number r is called the delay. The case $r = 0$ corresponds with an ordinary differential equation.

Equation (1.1) is called *linear* if $f(t, \varphi) = L(t)\varphi$, where $L(t)$ is linear for each t. Equation (1.1) is called nonhomogeneous if $f(t, \varphi) = L(t)\varphi + h(t)$, where $h(t) \not\equiv 0$. Equation (1.1) is called *autonomous* if $f(t, \varphi) = g(\varphi)$, where g does not depend on t.

Now, we show some examples of delay differential equations.

$$\dot{x}(t) = \int_{-r}^{0} x(t + \theta)\, d\theta, \tag{1.2}$$

$$\dot{x}(t) = ax(t) + bx(t - 1), \tag{1.3}$$

$$\dot{x}(t) = c(t)x(t) + d(t)x(t - \tau(t)), \tag{1.4}$$

where a, b are constants, $c(t), d(t), \tau(t)$ are continuous functions. Equation (1.2) is a linear integro-differential equation with a distributed delay, equation (1.3) is linear autonomous differential equation with a constant delay and equation (1.4) is linear nonautonomous differential equation with a time dependent delay.

Suppose that $\Omega \subseteq \mathbb{R} \times \mathcal{C}$ is open, $f : \Omega \to \mathbb{R}^n$, $D : \Omega \to \mathbb{R}^n$ are given continuous functions with D atomic at zero (See Subsection 1.3.2 on page 10 for the concept of *atomic at zero*). The relation

$$\frac{d}{dt} D(t, x_t) = f(t, x_t) \tag{1.5}$$

is called a *neutral delay differential equation*, which is denoted by NDDE (D, f), the function D is called the *difference operator* for the neutral delay differential equation. In the following, we present two examples of neutral delay differential equations.

$$\frac{d}{dt}[x(t) - Bx(t-r)] = f(t, x_t),$$

where $r > 0$, B is an $n \times n$ constant matrix, $D(\phi) = \phi(0) - B\phi(-r)$ and $f : \Omega \to \mathbb{R}^n$ is continuous.

If $D\phi = \phi(0)$ for all ϕ, then D is atomic at 0. Therefore, for any continuous $f : \Omega \to \mathbb{R}^n$, the pair (D, f) defines a neutral delay differential equation. Consequently, DDEs are NDDEs.

Delay differential equations arise from a variety of applications including control systems, electrodynamics, mixing liquids, neutron transportation and population models. In the following, we show some models to illustrate the applications of neutral delay differential equations.

Biological models

Differential equations have long been used to model various types of populations. In many cases ordinary differential equations are the starting point in the modeling process. When time delays (due to feedback, cells division time lags, etc.) become important, then delay differential equations become a natural tool for modeling in the life sciences.

Predator-prey model

The classic predator-prey model suggested by Lotka and Volterra in the 1920's has the form

$$\begin{cases} \dot{x}(t) = a_1 x(t) - b_1 x(t) y(t) \\ \dot{y}(t) = a_2 y(t) - b_2 x(t) y(t), \end{cases} \tag{1.6}$$

with initial condition

$$x(0) = x_0, \qquad y(0) = y_0, \tag{1.7}$$

where $x(t)$ represents the population of prey and $y(t)$ the population of predators at time t and a_1, a_2, b_1, b_2 are positive constants. If we consider the fact that a change in the population of the prey will not immediately affect the population of the predators and conversely, then the system (1.6) with the initial condition (1.7) becomes a delay differential equation of the form

$$\begin{cases} \dot{x}(t) = a_1 x(t) - b_1 x(t) y(t - r_1) \\ \dot{y}(t) = a_2 y(t) - b_2 x(t - r_2) y(t), \end{cases} \tag{1.8}$$

with initial conditions

$$x(0) = x_0, \quad x(s) = \phi(s), \quad y(0) = y_0, \quad y(s) = \varphi(s), \quad -\tau < s < 0, \tag{1.9}$$

where $r_1 > 0$ and $r_2 > 0$ are time delays and the functions $\phi(\cdot)$ and $\varphi(\cdot)$ are the initial past history functions, $\tau = \max\{r_1, r_2\}$, see [28, 47] for detailed information.

Australian blowfly

In the dynamic system of the blowfly population, resource limitation acts with a time delay, roughly equal to the time for a larva to grow up to an adult. Thus May [97] proposed to model the population dynamics of blowflies with a delay differential equation

$$\dot{N}(t) = rN(t)\left(1 - \frac{1}{1000K}N(t-\tau)\right), \tag{1.10}$$

where $N(t)$ is the population size of the adult blowflies, r is the rate of increase of the blowfly population, K is a resource limitation parameter set by the supply of food, and τ is the time delay, roughly equal to the time for a larva to grow up to an adult (about 11 days).

Metal cutting model

The metal cutting model (Moon and Johnson [99]) can be described by

$$\begin{aligned} m\ddot{x}(t) + \gamma_1\dot{x}(t) + k_1x(t) &= F_1(x(t) - x(t-\tau), y(t) - y(t-\tau)) \\ m\ddot{y}(t) + \gamma_2\dot{y}(t) + k_2y(t) &= F_2(x(t) - x(t-\tau), y(t) - y(t-\tau)), \end{aligned}$$

where $x(t)$ is the x component of the tool tip position, $y(t)$ is the y component of the tool tip position, γ_j, k_j $(j = 1, 2)$ are the damping and spring force constants, $\tau = \frac{C}{\omega}$ with C a constant and ω the turning speed. Normally, ω is considered constant, but during the machine startup or shut down, ω is a function of t, thus $\tau = \tau(t)$. For the other applications of delay differential equations, refer to [29, 50, 51].

Delay differential equations are studied from several different perspectives, mostly concerned with their solutions. Only the simplest equations admit solutions given by explicit formulas. However, some properties of solutions of a given equation may be determined without finding their exact form. In the case when a self-contained formula for the solution is not available, qualitative analysis, which has been proved to be a useful tool to investigate the properties of solutions, will be emphasised on. In the qualitative analysis of equations, asymptotic behavior and stability of solutions play an important role. The investigation of asymptotic behavior and stability of solutions of delay differential equations is more complicated than the case for ordinary differential equations because of the delay effects, refer to [29, 50, 51, 72] for detailed information.

Besides delay effects, impulsive effects likewise exist in a great variety of evolutionary processes in which states are changed abruptly at certain moments of time. Time-dependent impulses arise naturally in many biological and physiological systems, including ones from delayed cellular neural networks with impulsive effects.

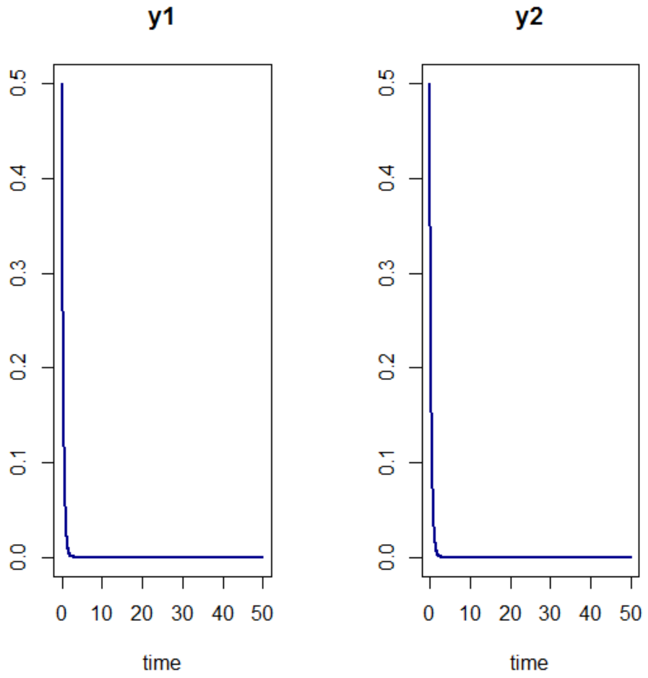

Figure 1.1: Delayed cellular neural network without impulses.

Delayed cellular neural networks with impulsive effects

Consider the following system of delayed cellular neural networks with impulsive effects

$$\begin{cases} \dot{y_1}(t) = -2y_1(t) - g(y_1(t)) + 0.5g(y_2(t)) - 0.5g(y_1(t - 0.2\sin t)) + 0.5g(y_2(t - 0.2\cos t)) \\ \\ \dot{y_2}(t) = -3.5y_2(t) + 0.5g(y_1(t)) - g(y_2(t)) \\ \qquad\qquad +0.5g(y_1(t - 0.2\sin t)) + 0.5g(y_2(t - 0.2\cos t)), \end{cases}$$

where

$$g(x) = \frac{|x+1| - |x-1|}{2}.$$

The initial condition is given by $y_1(t) = 0.5$ and $y_2(t) = 0.5$. At each impulse time $t_k = 0.2k$ an impulse is applied with $y_1(t_k)$ being replaced by $1.8y_1(t_k)$ and $y_2(t_k)$ being replaced by $1.7y_2(t_k)$.

Figure 1.1 and Figure 1.2 show that the impulses can destabilize a system.

Consider the following system of delayed cellular neural networks with impulsive effects

$$\begin{cases} \dot{y_1}(t) = -0.2y_1(t) - g(y_1(t)) + 0.5g(y_2(t)) \\ \qquad\qquad -0.5g(y_1(t - 0.2\sin t)) + 0.5g(y_2(t - 0.2\cos t)) \\ \\ \dot{y_2}(t) = -0.1y_2(t) + 0.5g(y_1(t)) - g(y_2(t)) \\ \qquad\qquad +0.5g(y_1(t - 0.2\sin t)) + 0.5g(y_2(t - 0.2\cos t)), \end{cases}$$

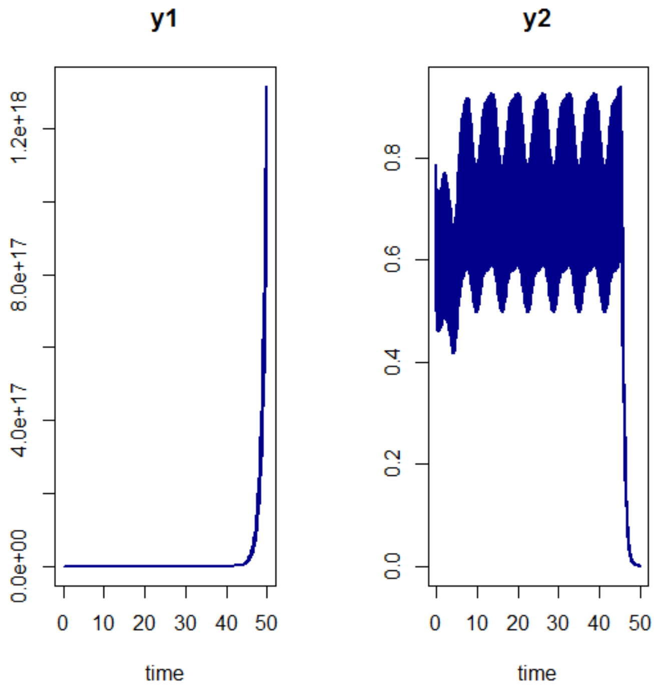

Figure 1.2: Delayed cellular neural network with impulses.

where

$$g(x) = \frac{|x+1| - |x-1|}{2}.$$

The initial condition is given by $y_1(t) = 0.5$ and $y_2(t) = 0.5$. At each impulse time $t_k = 0.2k$ an impulse is applied with $y_1(t_k)$ being replaced by $-0.8y_1(t_k)$ and $y_2(t_k)$ being replaced by $-0.7y_2(t_k)$.

Figure 1.3 and Figure 1.4 show that the impulses can stabilize a system.

When modeling systems which do not noticeably affect their environment, stochastic variables are often used to model the environmental fluctuations, which is described as *stochastic delay differential equations*. Stochastic delay differential equations can be considered as deterministic delay differential equations with random elements or stochastic differential equations with time delays. As an important mathematical model to describe real world problems more effectively, stochastic delay differential equations have been applied in many fields of science, such as automatic control, neural networks, biology, economics, chemical reaction engineering, etc. As an example, we consider an entire delayed neural network appeared in Huang et al.[56].

Stochastic neural networks

Figure 1.5 shows the scheme of the entire delayed neural network, where the nonlinear neuron transfer function S is constructed by using the voltage operational amplifiers. The time delay

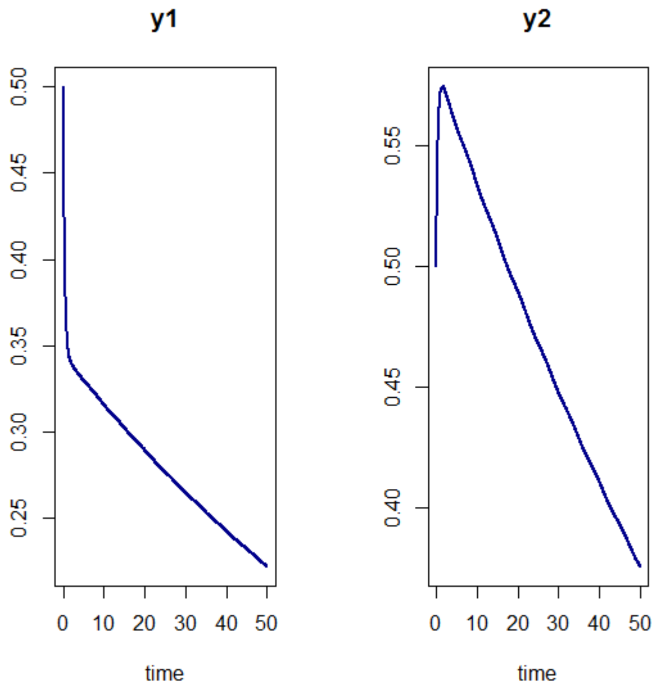

Figure 1.3: Delayed cellular neural network without impulses.

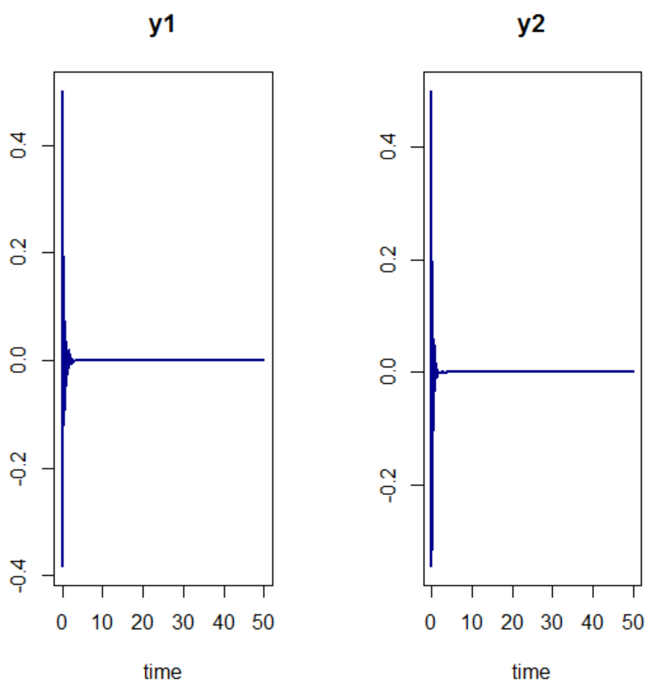

Figure 1.4: Delayed cellular neural network with impulses.

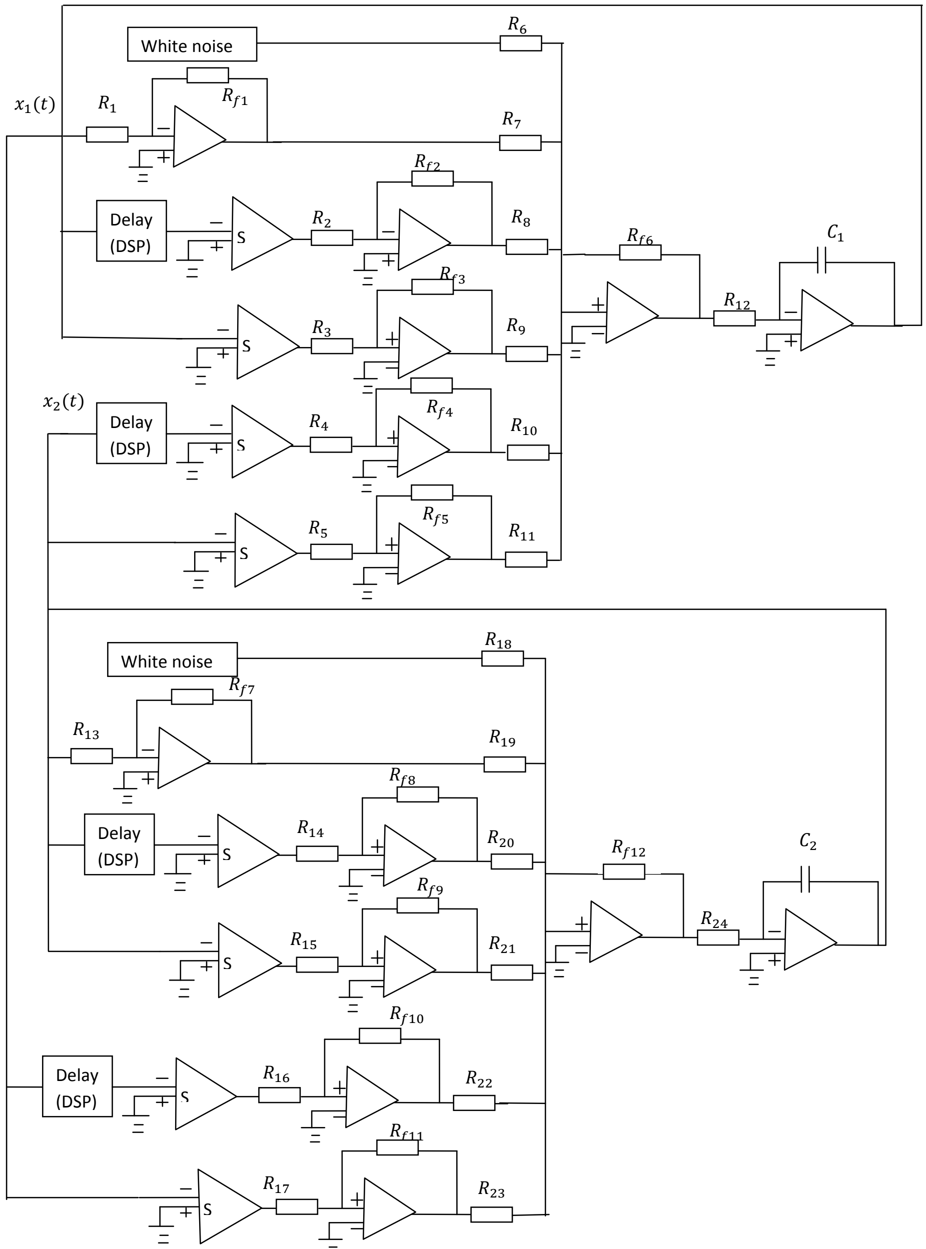

Figure 1.5: A schematic circuit diagram for system (1.11), where $R_i = 1k\Omega$ $(i = 1, \cdots, 11, 13, \cdots, 23)$, $R_{12} = R_{24} = 100k\Omega$; $R_{f_1} = 4.5k\Omega$, $R_{f_2} = 0.16k\Omega$, $R_{f_3} = R_{f_4} = 0.4k\Omega$, $R_{f_5} = 0.08k\Omega$, $R_{f_6} = 1k\Omega$, $R_{f_7} = 4.5k\Omega$, $R_{f_8} = 0.8k\Omega$, $R_{f_9} = R_{f_{10}} = 0.2k\Omega$, $R_{f_{11}} = 0.12k\Omega$, $R_{f_{12}} = 1k\Omega$; $C_1 = C_2 = 0.1\mu F$.

is achieved by using a digital signal processor (DSP) with an analog-to-digital converter (ADC) and a digital-to -analog converter (DAC). There is white noise is generated by a white noise signal generator.

The schematic circuit diagram can be described by the following stochastic recurrent neural network with time-varying delays

$$\begin{aligned} dx(t) &= -\begin{pmatrix} 4.5 & 0 \\ 0 & 4.5 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} dt + \begin{pmatrix} 2 & 0.4 \\ 0.6 & 1 \end{pmatrix}\begin{pmatrix} 0.2\tanh(x_1(t)) \\ 0.2\tanh(x_2(t)) \end{pmatrix} dt \\ &\quad + \begin{pmatrix} -0.8 & 2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} 0.2\tanh(x_1(t-\tau_1(t))) \\ 0.2\tanh(x_2(t-\tau_2(t))) \end{pmatrix} dt + \sigma(t, x(t), x(t-\tau(t)))\, dw(t), \end{aligned} \tag{1.11}$$

where $\tau(t) = (\tau_1(t), \tau_2(t))^T$, τ_i is any bounded positive function for $i = 1, 2$, and $\sigma : R_+ \times R^2 \times R^2 \to R^2 \times R^2$ satisfies trace $\left[\sigma^T(t, x, y)\sigma(t, x, y)\right] \le x_1^2 + x_2^2 + y_1^2 + y_2^2$.

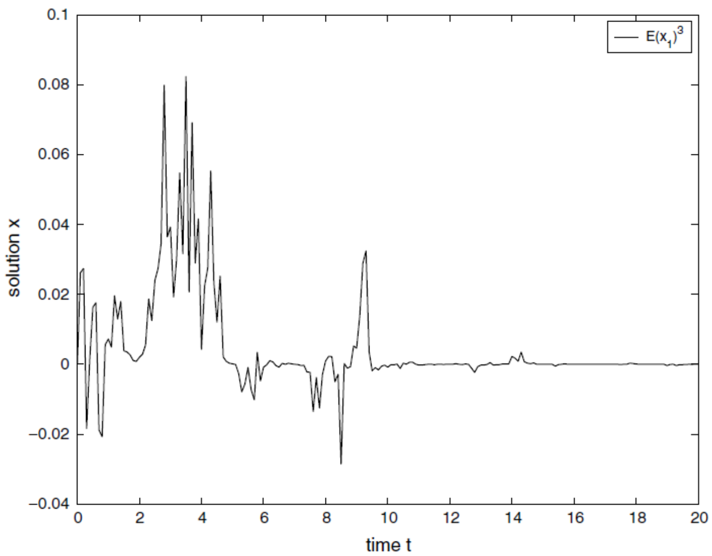

Figure 1.6: Numerical solution $E(x_1^3(t))$ of system (1.11), which comes from Huang et al.[56].

1.2 Objectives and main results of this thesis

The general aim of this thesis is to present a systematic study of different methods for stability and asymptotic stability for different types of equations. We are interested in the versatility of the methods to deal with different classes of equations and verifiability of the conditions. We also wish to understand the relations between the methods: for what equations do they eventually coincide, and what are their advantages and restrictions. In particular, we emphasize a fixed point approach to stability of delay differential equations and stability of stochastic delay differential equations.

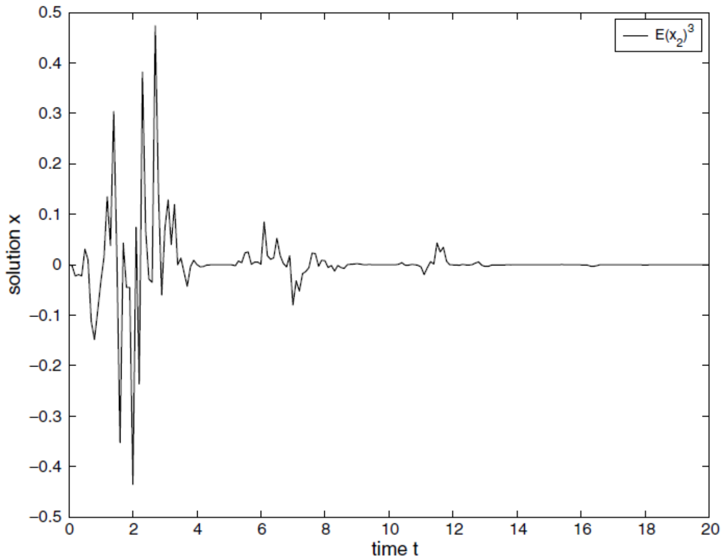

Figure 1.7: Numerical solution $E(x_2^3(t))$ of system (1.11), which comes from Huang et al.[56].

This thesis focuses on five objectives. The first objective is concerned with asymptotic behavior of autonomous delay differential equations (see Chapter 2). The ODE method and spectral method are generally viewed as effective techniques in dealing with asymptotic behavior of autonomous delay differential equations. However, there seems to be no discussion about the relations of these two methods. In Chapter 2, we will study the relations of the ODE method and spectral method by considering a class of second order neutral delay differential equations of the form

$$x''(t) + cx''(t-\tau) = p_1 x'(t) + p_2 x'(t-\tau) + q_1 x(t) + q_2 x(t-\tau), \tag{1.12}$$

where $c, p_1, p_2, q_1, q_2 \in \mathbb{R}$, $\tau > 0$. It is concluded that under the same assumptions, the results by the ODE method is equivalent to the results by the spectral method (see Section 2.4). The conditions for the spectral method are weaker than those by the ODE method, (see Example 2.4.2), and the asymptotic behavior of neutral delay differential equations can be presented by a general formula (see Theorem 2.2.6). Furthermore, the asymptotic behavior of neutral delay differential equations with matrix coefficients can be investigated by the spectral method.

The second objective focuses on asymptotic behavior of nonautonomous delay differential equations (see Chapter 3). It should be emphasized that asymptotic behavior of nonautonomous equations is much more difficult than the case of autonomous equations. Frasson and Verduyn Lunel [39] have applied a spectral method to study asymptotic behavior of a class of linear periodic delay equations of the form

$$x'(t) = a(t)x(t) + \sum_{j=1}^{k} b_j(t)x(t-\tau_j), \tag{1.13}$$

where $a(t+\omega) = a(t)$, $b_j(t+\omega) = b_j(t)$, $j = 1, 2, \cdots, k$. They considered a particular case where $\tau_j = j\omega$ (i.e. the delays are integer multiples of the period ω). Determining asymptotic behavior of general classes of nonautonomous equations seems untractable. For a special class of nonautonomous problems, we can use an approach similar to the ODE method as we discussed in Chapter 2, which is based on the application of an appropriate solution of the generalized characteristic equation. For nonautonomous equations, solving the generalized characteristic equation becomes much harder: a functional equation instead of an algebraic equation. This approach only succeeds if the generalized characteristic equation has a real solution.

The third objective concerns a fixed point approach towards stability of deterministic delay differential equations (see Chapter 4). Although there is an extensive literature on stability analysis of delay equations discussed using a fixed point approach, stability analysis of more general classes of delay equations has not been satisfactorily researched. Hence, in Chapter 4, several classes of delay equations with a combination of time-dependent delays, distributed delays and neutral terms are studied, such as, for example, a scalar neutral integro-differential equation

$$x'(t) - c(t)x'(t - r_1(t)) = -a(t)x(t - r_2(t)) + \int_{t-r_3(t)}^{t} g(t, x(s))\, d\mu(t, s). \tag{1.14}$$

The last term in (1.14) includes the following two cases:

$$(1) \quad \int_{t-r(t)}^{t} g(t, x(s))k(t, s)\, ds \qquad (2) \quad \sum_{i=1}^{n} a_i(t)g(t, x(t - r_i(t))).$$

In our result, two auxiliary continuous functions $h_1(t)$ and $h_2(t)$ are introduced and used to define an appropriate contraction mapping related to the equation. Our stability results typically say that the equation is stable if a certain expression involving the coefficients of the equation is less than one.

The fourth objective involves stability of stochastic delay differential equations with impulses (see Chapter 5). Besides delay and stochastic effects, impulsive effects are also likely to exist in mechanical, electronical or economical systems, which could stabilize or destabilize the system. Therefore, it is necessary to take delay effects, stochastic effects and impulsive effects into account when studying the dynamical behavior of the system. In Chapter 5, we consider two classes of neutral stochastic delay differential equations with impulses. The first class is an impulsive neutral stochastic delay differential equations is of the form

$$\begin{cases} d[x(t) - q(t)x(t - \tau(t))] = [a(t)x(t) + b(t)x(t - \tau(t))]dt \\ \qquad\qquad + [c(t)x(t) + e(t)x(t - \delta(t))]dw(t), \quad t \neq t_k, \\ x(t_k{}^+) - x(t_k) = b_k x(t_k), \quad t = t_k. \end{cases} \tag{1.15}$$

Equation (1.15) is a combination of a neutral term, a delay term, a stochastic term and an impulsive effect.

A fixed point method is used to study stability properties of the first class of equations. We consider two different norms:

$$\|x\|^2 := \sup_{t \geq \vartheta} \left(\mathbb{E}|x(t)|^2 \right)$$

and

$$\|x\|^2 := \sup_{t\geq 0}\left[\mathbb{E}\left(\sup_{t-\tau\leq s\leq t}|x(s)|^2\right)\right],$$

where $\vartheta = \min\{\inf_{s\geq 0}\{s-\tau(s)\}, \inf_{s\geq 0}\{s-\delta(s)\}\}$, and τ is an upper bound of $\{\tau(s), \delta(s), s\geq 0\}$. These two norms lead to different stability results. It turns out that the analysis for the second norm yields a stronger conclusion under a stronger assumption than the analysis involving the first norm.

The second class consists of equations of the form

$$\begin{cases} d[x(t)+u(t,x(t-\tau(t)))] = [Ax(t)dt + f(t,x(t-\delta(t)))]dt + g(t,x(t-\rho(t))dw(t) \\ \qquad\qquad + \int_Z h(t,x(t-\sigma(t)),y)\,\tilde{N}(dt,dy), \quad t\geq 0, \quad t\neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad t=t_k, \quad k=1,2,\cdots, \\ x_0(\theta)=\phi, \quad \theta\in[-\tau,0], \quad a.s., \end{cases} \tag{1.16}$$

which is an infinite dimensional impulsive stochastic delay differential equation. Exponential stability of this class of equations is studied by two methods, one is the method using an impulsive-integral inequality and the other one is a fixed point method. The stability criteria derived by the two methods are similar. A fixed point argument can yields existence, uniqueness and stability result in one step. However, the existence and uniqueness theorem should be provided seperately before using the method using an impulsive-integral inequality.

The fifth objective concerns an application to stochastic delayed neural networks (see Chapter 6). It is natural to consider random noise in neural networks. In real nervous, for instance, synaptic transmission is a noisy process with the noise brought on by random fluctuations from the release of neurotransmitters and other probabilistic causes. A neural network could be stabilized or destabilized by stochastic inputs. Therefore, the stochastic stability analysis problem for various neural networks has attracted considerable interest in recent years. In Chapter 6, a class of stochastic delayed neural networks is considered, which is described by

$$\begin{aligned} dx_i(t) &= \Bigg[-c_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} f_j(x_j(t-\tau(t))) \\ &\quad + \sum_{j=1}^n l_{ij}\int_{t-r(t)}^t f_j(x_j(s))\,ds\Bigg]dt + \sum_{j=1}^n \sigma_{ij}(t,x_j(t),x_j(t-\tau(t)))\,dw_j(t). \end{aligned} \tag{1.17}$$

A fixed point method is applied to study stability properties of this class of stochastic delayed neural networks. As in Chapter 5, two different types of norms are defined to study the system (1.17), that is,

$$\|x\|^p := \sup_{t\geq\vartheta}\left[\mathbb{E}\left(\sum_{i=1}^n |x_i(t)|^p\right)\right]$$

and

$$\|x\|^p = \sup_{t\geq 0}\left\{\sum_{i=1}^n \mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|x_i(s)|^p\right]\right\}.$$

Both norms lead to a complete space and a contraction mapping related to the equation. Our results neither require the boundedness, monotonicity and differentiability of the activation functions nor differentiability of the time varying delays. In addition, the case when there are impulsive effects to the system (1.17) and the case when there are no stochastic perturbations are also considered.

1.3 Preliminaries

In this section, we present basic definitions and lemmas which are frequently used in this thesis, and present some background materials on stability of deterministic and stochastic delay differential equations.

1.3.1 Delay differential equations

For $r > 0$, let $\mathcal{C} = C([-r, 0], \mathbb{R}^n)$ denote the Banach space of continuous functions from $[-r, 0]$ $(r > 0)$ with values in $\mathbb{R}^n$ endowed with the supremum norm. For $\Omega \subseteq \mathbb{R} \times \mathcal{C}$, $f : \Omega \to \mathbb{R}^n$ is a given function, consider the delay differential equation

$$\dot{x}(t) = f(t, x_t), \tag{1.18}$$

where $x_t(\theta) = x(t + \theta)$ for $-r \leq \theta \leq 0$.

It is clear that an appropriate "initial condition" at time $t = \sigma$ must at least specify the vector x for all t in $[\sigma - r, \sigma]$, i.e.,

$$x(t) = \phi(t), \qquad \sigma - r \leq t \leq \sigma. \tag{1.19}$$

Here $\phi : [\sigma - r, \sigma] \to \mathbb{R}^n$ is a known function, usually we suppose ϕ to be a continuous function. The function ϕ is called the *initial function* of the delay differential equation, σ the *initial constant* and $[\sigma - r, \sigma]$ the *initial set.*

Hence, the initial value problem of (1.18) is given by the following relation

$$\begin{cases} \dot{x}(t) = f(t, x_t) & \text{for} \quad t \geq \sigma \\ x(t) = \phi(t) & \text{for} \quad \sigma - r \leq t \leq \sigma, \end{cases} \tag{1.20}$$

where ϕ is a given function defined on $[\sigma - r, \sigma]$.

Definition 1.3.1. (Hale and Verduyn Lunel [51]) *A function x is said to be a solution of* (1.18) *on $[\sigma - r, \sigma + A]$ if there are $\sigma \in \mathbb{R}$, $A > 0$ such that $x \in C([\sigma - r, \sigma + A], \mathbb{R}^n)$, $(t, x_t) \in D$ and $x(t)$ satisfies* (1.18) *for $t \in [\sigma, \sigma + A]$. For given $\sigma \in \mathbb{R}$, $\phi \in C([-r, 0], \mathbb{R}^n)$, we say $x(t, \sigma, \phi)$ is a solution of* (1.20) *with initial value ϕ at σ or simply a solution through (σ, ϕ) if there is an $A > 0$ such that $x(t, \sigma, \phi)$ is a solution of equation* (1.20) *on $[\sigma - r, \sigma + A]$ and $x_\sigma(\sigma, \phi) = \phi$; we say $x(t, \sigma, \phi)$ is a solution of* (1.20) *on $[\sigma - r, \infty)$, if for every $A > 0$, $x(t, \sigma, \phi)$ is a solution of equation* (1.20) *on $[\sigma - r, \sigma + A]$ and $x_\sigma(\sigma, \phi) = \phi$.*

Lemma 1.3.2. (Hale and Verduyn Lunel [51]) *If $\sigma \in \mathbb{R}$, $\phi \in \mathcal{C}$ are given, and $f(t, \phi)$ is continuous, then finding a solution of equation* (1.18) *through (σ, ϕ) is equivalent to solving the integral equation*

$$\begin{cases} x(t) = \phi(\sigma) + \int_\sigma^t f(s, x_s)\, ds, & t \geq \sigma, \\ x_\sigma = \phi. \end{cases} \tag{1.21}$$

We are now consider the existence and uniqueness of the system (1.20), we assume that f is continuous. To prove the existence of the solution through a point $(\sigma, \phi) \in \mathbb{R} \times \mathcal{C}$, we consider an $\alpha > 0$ and all functions x on $[\sigma - r, \sigma + \alpha]$ that are continuous and coincide with ϕ on $[\sigma - r, \sigma]$, that is $x_\sigma = \phi$.

Theorem 1.3.3. (Existence) ([51]) *Suppose that* Ω *is an open subset in* $\mathbb{R} \times \mathcal{C}$ *and* $f \in C(\Omega, \mathbb{R}^n)$. *If* $(\sigma, \phi) \in \Omega$, *then there is a solution of the DDE* (f) *passing through* (σ, ϕ).

Definition 1.3.4. *We say* $f(t, \phi)$ *is Lipschitz in* ϕ *in a compact set* K *of* $\mathbb{R} \times \mathcal{C}$ *if there is a constant* $L > 0$ *such that, for any* $(t, \phi_i) \in K$, $i = 1, 2$,

$$\|f(t, \phi_1) - f(t, \phi_2)\| \leq L\|\phi_1 - \phi_2\|.$$

Theorem 1.3.5. (Existence and uniqueness) ([51]) *Suppose that* Ω *is an open set in* $\mathbb{R} \times \mathcal{C}$, $f : \Omega \to \mathbb{R}^n$ *is continuous and* $f(t, \phi)$ *is Lipschitz in* ϕ *in each compact set in* Ω. *If* $(\sigma, \phi) \in \Omega$, *then there is a unique solution of* (1.20) *through* (σ, ϕ).

Let x be a solution of (1.20) on $[\sigma, a)$, $a > \sigma$. We say $\hat{x}$ is a *continuation* of x if there is a $b > a$ such that $\hat{x}$ is defined on $[\sigma - r, b)$, coincides with x on $[\sigma - r, a)$, and $\hat{x}$ satisfies (1.20) on $[\sigma, b)$. A solution x is *noncontinuable* if no such continuation exists; that is, the interval $[\sigma, a)$ is the *maximal interval of existence* of the solution x.

Theorem 1.3.6. ([51]) *Suppose that* Ω *is an open set in* $\mathbb{R} \times \mathcal{C}$, $f : \Omega \to \mathbb{R}^n$ *is completely continuous* (*that is,* f *is continuous and takes closed bounded sets into compact sets*), *and* x *is a noncontinuable solution of* (1.20) *on* $[\sigma - r, b)$. *Then for any closed bounded set* U *in* $\mathbb{R} \times \mathcal{C}$, $U \subset \Omega$, *there is a* t_U *such that* $(t, x_t) \notin U$ *for* $t_U \leq t < b$.

In other words, Theorem 1.3.6 says that solution of (1.20) either exists for all $t \geq \sigma$ or becomes unbounded (with respect to Ω) at some finite time.

1.3.2 Neutral delay differential equations

Definition 1.3.7. (Hale and Verduyn Lunel [51]) *Suppose that* $\Omega \subseteq \mathbb{R} \times \mathcal{C}$ *is open with elements* (t, ϕ). *A function* $D : \Omega \to \mathbb{R}^n$ *is said to be atomic at* β *on* Ω *if* D *is continuous together with its first and second Fréchet derivatives with respect to* ϕ; *and* D_ϕ, *the derivative with respect to* ϕ, *is atomic at* β *on* Ω.

Suppose that $\Omega \subseteq \mathbb{R} \times \mathcal{C}$ is open, $f : \Omega \to \mathbb{R}^n$, $D : \Omega \to \mathbb{R}^n$ are given continuous functions with D atomic at zero. Consider the neutral delay differential equation

$$\frac{d}{dt} D(t, x_t) = f(t, x_t). \tag{1.22}$$

Definition 1.3.8. (Hale and Verduyn Lunel [51]) *A function* x *is said to be a solution of* (1.22) *on* $[\sigma - r, \sigma + A]$ *if there are* $\sigma \in \mathbb{R}$ *and* $A > 0$ *such that*

$$x \in C([\sigma - r, \sigma + A], \mathbb{R}^n), \quad (t, x_t) \in \Omega, \quad t \in [\sigma, \sigma + A],$$

$D(t, x_t)$ *is continuously differentiable and satisfies equation* (1.22) *on* $[\sigma, \sigma + A]$. *For a given* $t_0 \in \mathbb{R}$, $\phi \in \mathcal{C}$, *and* $(\sigma, \phi) \in \Omega$, *we say* $x(t, \sigma, \phi)$ *is a solution of equation* (1.22) *with initial value* ϕ *at* σ *or simply a solution through* (σ, ϕ) *if there is an* $A > 0$ *such that* $x(t, \sigma, \phi)$ *is a solution of equation* (1.22) *on* $[\sigma - r, \sigma + A]$ *and* $x_\sigma(\sigma, \phi) = \phi$; *we say* $x(t, \sigma, \phi)$ *is a solution of* (1.22) *on* $[\sigma - r, \infty)$, *if for every* $A > 0$, $x(t, \sigma, \phi)$ *is a solution of equation* (1.20) *on* $[\sigma - r, \sigma + A]$ *and* $x_\sigma(\sigma, \phi) = \phi$.

Theorem 1.3.9. (Existence) (Hale and Verduyn Lunel [51]) *If* Ω *is an open set in* $\mathbb{R} \times \mathcal{C}$ *and* $(\sigma, \phi) \in \Omega$*, then there exists a solution of the NDDE* (D, f) *through* (σ, ϕ)*.*

Theorem 1.3.10. (Existence and Uniqueness) (Hale and Verduyn Lunel [51]) *If* Ω *is an open set in* $\mathbb{R} \times \mathcal{C}$ *and* $f(t, \phi)$ *is Lipschitzian in* ϕ *on compact sets of* Ω*, then, for any* $(\sigma, \phi) \in \Omega$*, there exists a unique solution of the NDDE* (D, f) *through* (σ, ϕ)*.*

A continuation result similar to Theorem 1.3.6 also exists for neutral delay differential equations, refer to Hale and Verduyn Lunel [51] for details.

1.3.3 Stability of delay differential equations

Suppose that $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ is continuous and consider the delay differential equation

$$\dot{x}(t) = f(t, x_t). \tag{1.23}$$

The function f will be supposed to be completely continuous and to satisfy enough additional smoothness conditions to ensure the solution $x(t, \sigma, \phi)$ through (σ, ϕ) is continuous in (t, σ, ϕ) in the domain of definition of the function.

Definition 1.3.11. *Suppose that* $f(t, 0) = 0$ *for all* $t \in \mathbb{R}$*. The solution* $x = 0$ *of equation (1.23) is said to be stable if for any* $\sigma \in \mathbb{R}$*,* $\varepsilon > 0$*, there is a* $\delta = \delta(\varepsilon, \sigma) > 0$ *such that* $\phi \in \mathcal{B}(0, \delta)$ *implies* $x_t(\sigma, \phi) \in \mathcal{B}(0, \varepsilon)$ *for* $t \geq \sigma$*. The solution* $x = 0$ *of equation (1.23) is said to be uniformly stable if the number* δ *in the definition is independent of* σ*.*

Definition 1.3.12. *The solution* $x = 0$ *of equation (1.23) is said to be asymptotically stable if it is stable and there is a* $b_0 = b_0(\sigma)$ *such that* $\phi \in \mathcal{B}(0, b_0)$ *implies that* $x(t, \sigma, \phi) \to 0$ *as* $t \to \infty$*. The solution* $x = 0$ *of equation (1.23) is said to be uniformly asymptotically stable if it is uniformly stable and there is* $b_0 > 0$ *such that for every* $\eta > 0$ *there is a* $t_0(\eta)$ *such that* $\phi \in \mathcal{B}(0, b_0)$ *implies* $x_t(\sigma, \phi) \in \mathcal{B}(0, \eta)$ *for* $t \geq \sigma + t_0(\eta)$ *for every* $\sigma \in \mathbb{R}$*.*

Definition 1.3.13. *A solution* $x(t, \sigma, \phi)$ *of an DDE* (f) *is bounded if there is a* $\beta(\sigma, \phi)$ *such that* $|x(t, \sigma, \phi)| < \beta(\sigma, \phi)$ *for* $t \geq \sigma - r$*. The solutions are uniformly bounded if for any* $\alpha > 0$*, there is a* $\beta = \beta(\alpha) > 0$ *such that for all* $\sigma \in \mathbb{R}$*,* $\phi \in \mathcal{C}$ *and* $|\phi| \leq \alpha$*, we have* $|x(t, \sigma, \phi)| \leq \beta(\alpha)$ *for all* $t \geq \sigma$*.*

1.3.4 Stability by spectral theory

Consider a linear ordinary differential equation of the form

$$x'(t) = ax(t). \tag{1.24}$$

The characteristic equation of (1.24) is $\lambda = a$, the solution of (1.24) is asymptotically stable if $Re(a) < 0$ and it is unstable if $Re(a) > 0$.

What about the stability of delay differential equations? Consider the following delay differential equation

$$x'(t) = ax(t) + bx(t-1), \tag{1.25}$$

Here a, b are constants. From Figure 1.8, the solution of (1.25) is stable with $a = \frac{1}{2}$ and $b = -1$

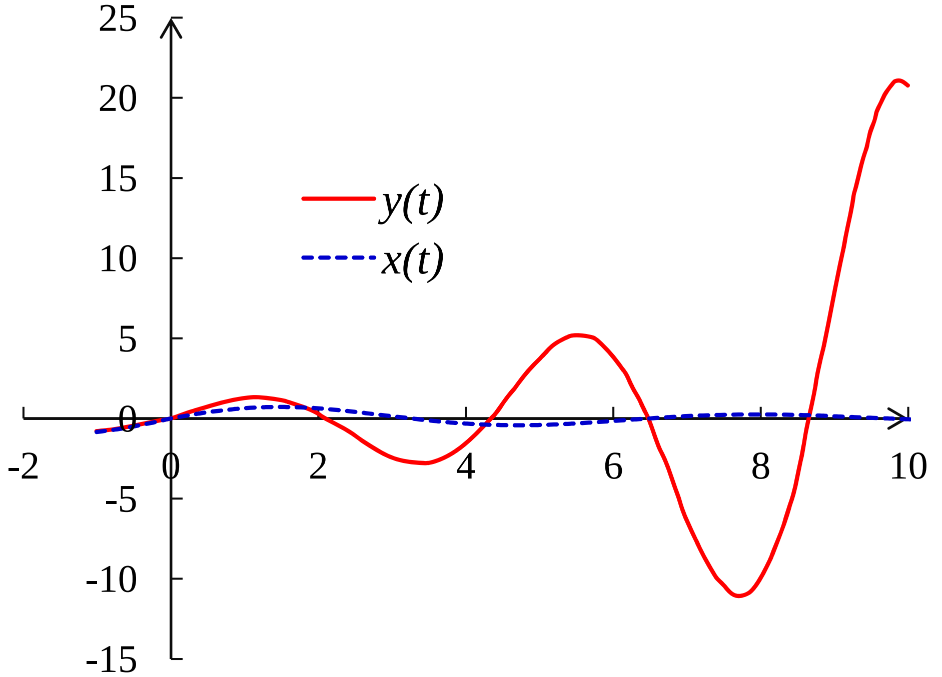

Figure 1.8: Numerical solution of (1.25) with $a = \frac{1}{2}, b = -1$ ($x(t)$) and $a = \frac{1}{2}, b = -2$ ($y(t)$).

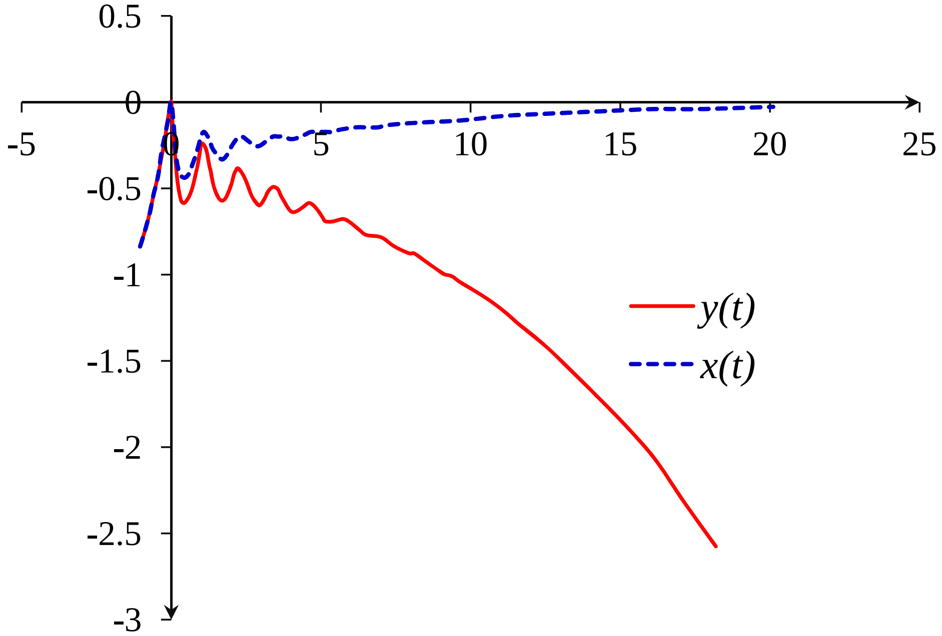

Figure 1.9: Numerical solution (1.25) with $a = -\frac{7}{2}, b = 3$ ($x(t)$) and $a = -\frac{7}{2}, b = 4$ ($y(t)$).

and unstable with $a = \frac{1}{2}$ and $b = -2$. From Figure 1.9, the solution is stable of (1.25) with $a = -\frac{7}{2}$ and $b = 3$ and unstable with $a = -\frac{7}{2}$ and $b = 4$. It is not difficult to find that the stability theory of delay differential equations is more complicated than the case for ordinary differential equations.

For such linear, autonomous delay differential equations, a simple way to study its stability is by spectral theory.

In fact, the characteristic equation of (1.25) is

$$z - a - be^{-z} = 0. \tag{1.26}$$

It is stable if all roots of the characteristic equation satisfy $Re(z) \leq \beta < 0$; It is unstable if for some root z, $Re(z) \geq 0$. Hence, to study the stability of (1.25) is to derive as much information as we can about the location of the roots of the characteristic equation (1.26) in the complex plane.

Let $z = \mu + i\nu$ in (1.26), we obtain two real equations

$$\begin{aligned} \mu - a - be^{-\mu}\cos\nu &= 0 \\ \nu + be^{-\mu}\sin\nu &= 0, \end{aligned} \tag{1.27}$$

where μ and ν are real numbers. By studying (1.27), some results towards the location of the roots in the complex plane of (1.26) are presented in Diekmann et al. [29].

Define the following strips,

$$\begin{aligned} \Sigma_k^+ &= \{\mu + i\nu \mid \nu \in I_k^+ = (2k\pi, (2k+1)\pi)\}, \\ \Sigma_k &= \{\mu + i\nu \mid \nu \in I_k = ((2k-1)\pi, (2k+1)\pi)\}, \\ \Sigma_k^- &= \{\mu + i\nu \mid \nu \in I_k^- = ((2k-1)\pi, 2k\pi)\}. \end{aligned}$$

Theorem 1.3.14. (Diekmann et al. [29]) *For $b > 0$, equation (1.26) has a unique and simple root λ_k in the strip Σ_k for $k = 0, 1, 2, \cdots$ and no other roots. For $k = 1, 2, \cdots$, the root λ_k is contained in Σ_k^-.*

Theorem 1.3.15. (Diekmann et al. [29]) *For $b < 0$, equation (1.26) has a unique and simple root λ_k in the strip Σ_k^+ for $k = 1, 2, \cdots$. There are two roots in Σ_0 (which are real and simple for $-e^{a-1} < b < 0$ and complex conjugate for $b < -e^{a-1}$). There are no other roots.*

However, in some real-world applications, the delay differential equations are nonautonomous, for example,

$$x'(t) = a(t)x(t) - b(t)x(t - r), \tag{1.28}$$

and

$$x'(t) = a(t)x(t) - b(t)x(t - r(t)). \tag{1.29}$$

What can we say about asymptotic behavior and stability of nonautonomous delay differential equations such as (1.28) and (1.29)? In the following, we present three methods, Liapunov's direct method, fixed point method and LMI method, which are extensively applied to the stability of nonautonomous equations.

1.3.5 Liapunov's direct method

Liapunov's direct method has long been viewed the main classical method of studying stability problems in many areas of differential equations. The difficulty of this method is to look for a suitable Liapunov functional or Liapunov function.

If $V : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ is continuous and $x(t, \sigma, \phi)$ is the solution of equation (1.23) through (σ, ϕ), we define

$$\dot{V}(t, \phi) = \limsup_{h \to 0^+} \frac{1}{h}[V(t + h, x_{t+h}(t, \phi)) - V(t, \phi)].$$

The function $\dot{V}(t, \phi)$ is the upper right-hand derivative of $V(t, \phi)$ along the solution of (1.23).

Theorem 1.3.16. (Hale and Verduyn Lunel [51]) *Suppose $f : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^n$ takes $\mathbb{R}\times$ (bounded sets of $\mathcal{C}$) into bounded sets of $\mathbb{R}^n$, and $u, v, w : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous nondecreasing functions, $u(s)$ and $v(s)$ are positive for $s > 0$, and $u(0) = v(0) = 0$. If there is a continuous function $V : \mathbb{R} \times \mathcal{C} \to \mathbb{R}$ such that*

$$\begin{aligned} u(|\phi(0)|) &\leq V(t, \phi) \leq v(|\phi|) \\ \dot{V}(t, \phi) &\leq -w(|\phi(0)|), \end{aligned}$$

then the solution $x = 0$ of equation (1.23) is uniformly stable. If $u(s) \to \infty$ as $s \to \infty$, the solutions of equation (1.23) are unifomly bounded. If $w(s) > 0$ for $s > 0$, then the solution $x = 0$ is uniformly asymptotically stable.

Example 1.3.17. (Burton [11]) *Consider the delay differential equation*

$$\dot{x}(t) = -b(t)x(t - r), \tag{1.30}$$

where $r > 0$ is a constant, $b : [0, \infty) \to \mathbb{R}$ is a bounded and continuous function.

The equation (1.30) can be written as the form

$$\dot{x}(t) = -b(t + r)x(t) + \frac{d}{dt}\int_{t-r}^{t} b(s + r)x(s)\, ds, \tag{1.31}$$

equation (1.31) is equivalent to

$$\left(x(t) - \int_{t-r}^{t} b(s + r)x(s)\, ds\right)' = -b(t + r)x(t). \tag{1.32}$$

By constructing the following the Liapunov functional $V(t, x_t) = V_1(t, x_t) + V_2(t, x_t)$, where

$$V_1(t, x_t) = \left(x(t) - \int_{t-r}^{t} b(s + r)x(s)\, ds\right)^2 + \int_{-r}^{0}\int_{t+s}^{t} b(u + r)x^2(u)\, du\, ds \tag{1.33}$$

and

$$V_2(t, x_t) = \gamma\left(x^2 + \int_{t-r}^{t} b(s + r)x^2(s)\, ds\right). \tag{1.34}$$

Burton [11] obtained the following theorem.

Theorem 1.3.18. (Burton [11]) *If $b(t+r) \geq 0$ for all $t \geq 0$ and $\int_0^\infty b(s)\,ds = \infty$, an $\varepsilon > 0$ with*

$$b(t+r)\int_{t-r}^{t} b(s+r)\,ds - 2 + r \leq -\varepsilon \quad \text{for} \quad \text{all} \quad t \geq 0,$$

and there is a $\gamma > 0$ with $\gamma[b(t) + b(t+r)] \leq (\varepsilon/2)b(t+r)$ for all $t \geq 0$, then the zero solution of (1.30) is asymptotically stable.

Example 1.3.19. (Burton [11]) *Let $b(t) = 1.1 + \sin t$ in (1.30), we have*

$$\dot{x}(t) = -(1.1 + \sin t)x(t-r). \tag{1.35}$$

Theorem 1.3.18 holds if there is an $\varepsilon > 0$ such that

$$2.1(1.1r + 2\sin(r/2)) - 2 + r < -\varepsilon. \tag{1.36}$$

Using a rough estimate (taking $\sin(r/2) = r/2$) on (1.36), we have that $r < 0.37$. Therefore, the zero solution of (1.35) is asymptotically stable if $r < 0.37$.

1.3.6 Fixed point method

Liapunov's direct method has been very effective in establishing stability results for a wide variety of differential equations. The success of Liapunov's direct method depends on finding good Liapunov functions or Liapunov functionals, which may be very difficult, especially for the equations with unbounded terms or unbounded delays, see the examples in Burton [13]. Therefore, it was recently proposed by Burton [13] and co-workers to use fixed point methods as an alternative. While Liapunov's direct method usually requires pointwise conditions, fixed point methods need conditions of an averaging nature.

Theorem 1.3.20. (**Banach's fixed point theorem**) *Let (X, d) be a non-empty complete metric space, let $T : X \to X$ be a contraction mapping on X, i.e. there is a nonnegative real number $q < 1$ such that $d(Tx, Ty) \leq qd(x, y)$ for all $x, y \in X$. Then the map T admits one and only one fixed point x^* in X ($Tx^* = x^*$).*

Hence, to solve a problem using a fixed point approach we have to identify:

(a) a set $\mathcal{S}$ consisting of points which would be acceptable solutions;

(b) a mapping $P : \mathcal{S} \to \mathcal{S}$ with the property that a fixed point solves the problem;

(c) a fixed point theorem stating that this mapping on the set $\mathcal{S}$ will have a fixed point.

The following steps represent the way in which we can establish stability of the zero solution of a delay differential equation by applying fixed point theory.

Step 1. An examination of the differential equation reveals that for a given initial time σ there is an initial interval we denote it to be E_σ and we require an initial function $\phi : E_\sigma \to \mathbb{R}^n$. We then must determine a set $\mathcal{S}$ of functions $\varphi : E_\sigma \cup [\sigma, \infty) \to \mathbb{R}^n$ with $\varphi(t) = \phi(t)$ on E_σ which could serve as acceptable functions. Usually, this means that we would ask some other conditions on φ, for example, the boundedness, and sometimes we require that $\varphi(t) \to 0$ as $t \to \infty$.

Step 2. Next, invert the differential equation and define a mapping from $\mathcal{S}$ to $\mathcal{S}$.

Step 3. Finally, we select a fixed point theorem which will show that the mapping P has a fixed point in $\mathcal{S}$.

Notice that the process of application of a fixed point method relies on three principles: an elementary variation of constants formula, a complete metric space and the contraction mapping principle. Moreover, in one step, a fixed point argument yields existence, uniqueness and stability. Hence, our major problem, when using fixed point theory to deal with stability analysis, is to define a suitable Banach space and a suitable mapping.

In the following, some results are presented to illustrate the application of a fixed point method. Consider the delay differential equation (1.30), by using a fixed point method, Burton [11] obtained the following result.

Theorem 1.3.21. (Burton [11]) *Suppose there exists a constant $\alpha < 1$ such that*

$$\int_{t-r}^{t} |b(s+r)|\,ds + \int_0^t |b(s+r)|e^{-\int_s^t b(u+r)\,du} \int_{s-r}^{s} |b(u+r)|\,du\,ds \le \alpha, \tag{1.37}$$

for all $t \ge 0$ and $\int_0^\infty b(s)\,ds = \infty$. Then for every continuous initial function $\phi : [-r, \infty) \to \mathbb{R}$, the solution $x(t) = x(t, 0, \phi)$ of (1.30) is bounded and tends to zero as $t \to \infty$.

Example 1.3.22. (Burton [11]) *Consider the differential equation*

$$\dot{x}(t) = -(1 + 2\sin t)x(t-r), \tag{1.38}$$

where $0 < r < 1$. The zero solution of (1.38) is asymptotically stable when $(r + 4\sin(r/2))(2 + 2e^2) < 1$, this is approximately $0 \le r < 0.02$.

Since $1 + 2\sin t$ changes sign for $t \ge 0$, Theorem 1.3.18 is not applicable to Example 1.3.22. Consider Example 1.3.19 by using Theorem 1.3.21, we obtain that the zero solution of (1.35) is asymptotically stable if $2(1.1r + 2\sin(r/2)) < 1$. This is approximated by $0 < r < 0.2$, compared to $r < 0.37$ by using Liapunov's direct method.

From the above discussion, we find it is very difficult to find a way to interpret a relation between the fixed point method and Liapunov's direct method. Sometimes the fixed point method can provide conditions for stability when the Liapunov's direct method can not, see Example 1.3.22. Sometimes Liapunov's direct method can provide better conditions, see Example 1.3.19.

If we let $r = 0.1$ in (1.38), the condition (1.37) in Theorem 1.3.21 is not satisfied, then Theorem 1.3.21 is not applicable. Therefore, new conditions are needed to study the case of $r = 0.1$. Following the similar arguments as Burton [11], Raffoul [110] studied the following linear neutral differential equation

$$\dot{x}(t) - c(t)\dot{x}(t - r(t)) = -a(t)x(t) - b(t)x(t - r(t)), \tag{1.39}$$

and he obtained the following result.

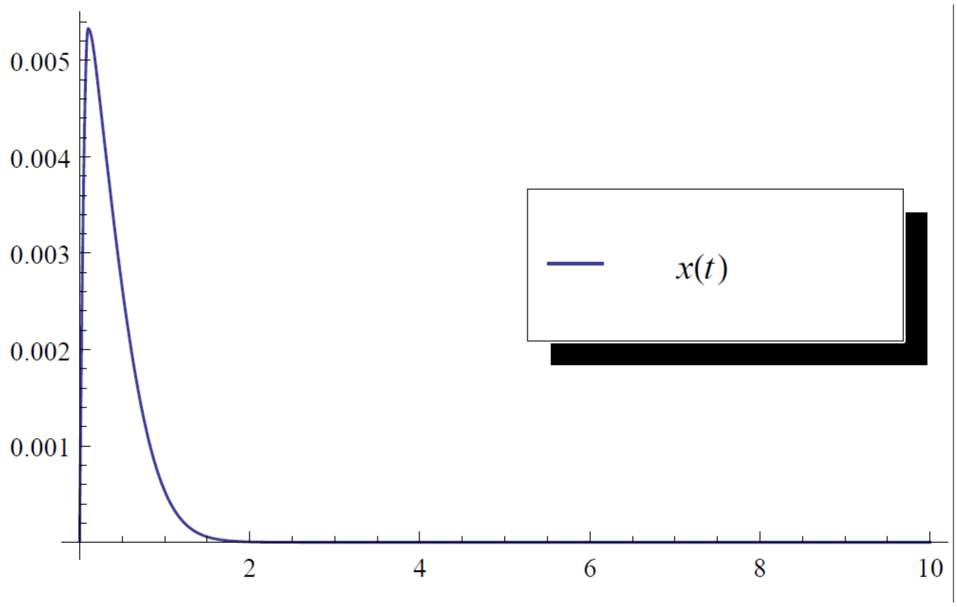

Figure 1.10: Numerical solution of (1.38).

Theorem 1.3.23. (Raffoul [110]) *Let $r(t)$ be twice differentiable and $r'(t) \neq 1$ for all $t \in R$. Suppose that there exists a constant $\alpha \in (0,1)$ such that for $t \geq 0$*

$$\int_0^t a(s)\,ds \to \infty \quad \text{as} \quad t \to \infty, \tag{1.40}$$

and such that

$$\left|\frac{c(t)}{1-r'(t)}\right| + \int_0^t e^{-\int_s^t a(u)\,du}\left|b(s) + \frac{[c(s)a(s)+c'(s)](1-r'(s)) + c(s)r''(s)}{(1-r'(s))^2}\right| ds \leq \alpha, \tag{1.41}$$

Then every solution $x(t) = x(t,0,\phi)$ of (1.39) with a small continuous initial function ϕ is bounded and tends to zero as $t \to \infty$.

Example 1.3.24. *Consider the linear neutral delay differential equation*

$$\dot{x}(t) = -\frac{1}{t+1}x(t) + 0.48\dot{x}(t-0.05t). \tag{1.42}$$

However, the condition (1.41) in Theorem 1.3.23 is not satisfied. In fact,

$$\begin{aligned}&\left|\frac{c(t)}{1-r'(t)}\right| + \int_0^t e^{-\int_s^t a(u)\,du}\left|b(s) + \frac{[c(s)a(s)+c'(s)](1-r'(s)) + c(s)r''(s)}{(1-r'(s))^2}\right| ds \\ &= \frac{0.48(2t+1)}{0.95(t+1)}.\end{aligned} \tag{1.43}$$

Since the right-hand side of (1.43) is increasing in $t > 0$ and

$$\limsup_{t\geq 0} \frac{0.48(2t+1)}{0.95(t+1)} = 1.0105,$$

then there exists some $t_0 > 0$ such that $t \geq t_0$, we have

$$\left|\frac{c(t)}{1-r'(t)}\right| + \int_0^t e^{-\int_s^t a(u)\,du}\left|b(s) + \frac{[c(s)a(s)+c'(s)](1-r'(s))+c(s)r''(s)}{(1-r'(s))^2}\right| ds > 1.01.$$

This implies that condition (1.41) does not hold. Thus, Theorem 1.3.23 is not applicable.

Hence, weaker conditions needed to be provieded to solve such problems (Example 1.3.22 for $r = 0.1$ and Example 1.3.24). By introducing a continuous function $v(t)$ for constructing fixed point mapping, Jin and Luo [62] provided sufficient conditions for the asymptotic stability of (1.39), which can be applied to Example 1.3.22 and Example 1.3.24.

Theorem 1.3.25. (Jin and Luo [62]) *Suppose the following conditions are statisfied.*

(i) *the delay $r(t)$ is twice differentiable and $r'(t) \neq 1$ for all $t \in \mathbb{R}^+$.*

(ii) *there exists a constant $\alpha \in (0,1)$ and a continuous function $v(\mathbb{R}^+ \to \mathbb{R})$ such that $\liminf_{t\to\infty} \int_0^t v(s)\,ds > -\infty$ and*

$$\begin{aligned} P(t) &= \int_{t-r(t)}^t |v(s)-a(s)|\,ds + \left|\frac{c(t)}{1-r'(t)}\right| \\ &\quad + \int_0^t e^{-\int_s^t v(u)\,du}|-b(s)+[v(s-r(s))-a(s-r(s))]-k(s)|\,ds \\ &\quad + \int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r(s)}^s |v(u)-a(u)|\,du\,ds \leq \alpha, \end{aligned} \tag{1.44}$$

then the zero solution of (1.39) is asymptotically stable if and only if $\int_0^t v(s)\,ds \to \infty$ as $t \to \infty$.

It is clear that Theorem 1.3.25 is consistant with Theorem 1.3.23 if $v(t) = a(t)$ for $t \geq 0$ in (1.44). In addition, Theorem 1.3.25 can be applied to some equations that Theorem 1.3.23 can not. Motivated by the work as in [62], in this thesis, we will discuss some general classes of delay differential equations by using the approach as in Theorem 1.3.25.

Notice that the condition (1.44) in Theorem 1.3.25 is mainly dependent on the constraint $\left|\frac{c(t)}{1-r'(t)}\right| < 1$. However, There are some interesting examples where the constraint is not satisfied. Zhao [145] investigated (1.39) without the constraint by employing another auxiliary function $p(t)$ to construct the fixed point mapping. In this thesis, we will study the approaches used in [62] and [145] to consider some general classes of delay differential equations.

1.3.7 Linear matrix inequality (LMI) method

The linear matrix inequality (LMI) method has become one of basic approaches to study stability of delay differential equations and stochastic delay differential equations. This approach is based on constructing suitable Liapunov functionals and combining with a linear matrix inequality. To illustrate this method, we present some results from Fridman [44].

Consider the following system

$$\dot{x}(t) = \sum_{i=0}^m A_i x(t-h_i), \quad x(t) = \phi(t), \quad t \in [-h, 0], \tag{1.45}$$

where $x(t) \in \mathbb{R}^n$, $h_0 = 0$, $0 < h_i \leq h$, A_i is a constant $n \times n$ matrix, ϕ is a continuously differentiable initial function. We represent (1.45) in the equivalent descriptor form

$$\dot{x}(t) = y(t), \quad y(t) = \left(\sum_{i=0}^{m} A_i\right) x(t) - \sum_{i=1}^{m} A_i \int_{t-h_i}^{t} y(s)\, ds. \tag{1.46}$$

Liapunov-Krasovskii functional for the system (1.46) has the form

$$V(t) = \begin{pmatrix} x^T(t) & y^T(t) \end{pmatrix} EP \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} + V_1, \tag{1.47}$$

where

$$E = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad P = \begin{pmatrix} P_1 & 0 \\ P_2 & P_3 \end{pmatrix}, \quad P_1 = P_1^T > 0,$$

$$V_1 = \sum_{i=1}^{m} \int_{-h_i}^{0} \int_{t+\theta}^{t} y^T R_i y(s)\, ds\, d\theta, \quad R_i > 0.$$

Computing $dV(t)/dt$ and using the conditions in Theorem 1.3.26, we obtain that the function V of (1.47) has a negative derivative, which implies asymptotically stable of (1.45).

Theorem 1.3.26. (Fridman [44]) *Equation (1.45) is asymptotically stable if there exist* $0 < P_1 = P_1^T$, P_2, P_3 *and* $R_i = R_i^T$, $i = 1, \cdots, m$ *that satisfy the following linear matrix inequality* (LMI):

$$\begin{pmatrix} (\sum_{i=0}^{m} A_i^T)P_2 + P_2^T(\sum_{i=0}^{m} A_i) & P_1 - P_2^T + (\sum_{i=0}^{m} A_i^T)P_3 & h_1 P_2^T A_1 & \cdots & h_m P_2^T A_m \\ P_1 - P_2 + P_3^T(\sum_{i=0}^{m} A_i) & -P_3 - P_3^T + \sum_{i=1}^{m} h_i R_i & h_1 P_3^T A_1 & \cdots & h_m P_3^T A_m \\ h_1 A_1^T P_2 & h_1 A_1^T P_3 & -h_1 R_1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ h_m A_m^T P_2 & h_m A_m^T P_3 & \cdot & \cdots & -h_m R_m \end{pmatrix} < 0.$$

Example 1.3.27. (Fridman [44]) *Consider the system*

$$\dot{x} = A_0 x(t) + A_1 x(t - h_1) \tag{1.48}$$

with

$$A_0 = \begin{pmatrix} -1 & 0.5 \\ -0.5 & -1 \end{pmatrix}, \quad A_1 = \begin{pmatrix} -2 & 2 \\ -2 & -2 \end{pmatrix}.$$

Applying LMI condition in Theorem 1.3.26, we obtain that $h_1 \leq 0.271$. Therefore, the equation (1.48) is asymptotically stable if $h_1 \leq 0.271$. For $h_1 = 0.271$ we obtain the following solution to LMI condition:

$$P_1 = \begin{pmatrix} 94.1609 & 0.1653 \\ -0.1653 & 94.0469 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 93.5589 & 0.1872 \\ 0.1872 & 94.6599 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 18.5170 & -0.0930 \\ -0.0930 & 18.4880 \end{pmatrix}, \quad R_1 = \begin{pmatrix} 68.2748 & 0.0349 \\ 0.0349 & 68.1810 \end{pmatrix}.$$

The LMI method is also widely used to study stability of neural networks, to know more about this method, refer to [44, 45, 77, 114, 126, 127].

1.3.8 Stochastic processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is the collection of all possible outcomes, and $\mathcal{F}$ is the set of all events A to which a probability $\mathbb{P}(A)$ can be attached. $\mathcal{F}$ is $\sigma-$ algebra, and $\mathbb{P}$ a probability measure. We are often very interested in events $A \in \mathcal{F}$ which are realized for almost all $\omega \in \Omega$. Such events are called *almost sure events*, and A is an almost sure event if $\mathbb{P}[A] = 1$. We will often use the abbreviation a.s. to stand for almost sure or almost surely.

Definition 1.3.28. *Let $T \subseteq [0, \infty)$. A stochastic process is a family $(X(t))_{t \in T}$ of random variables on (Ω, F, P). For each $\omega \in \Omega$, the map $t \to X(t)(\omega)$ is called a path of X.*

Definition 1.3.29. *If $I \subset R$ is an interval, $f : I \to R$ is a function, and $x \in I$, then f is said to have a right limit at x if*

$$f(x+) := \lim_{y \downarrow x} f(y) \qquad \text{exists}$$

and a left limit if

$$f(x-) := \lim_{y \uparrow x} f(y) \qquad \text{exists}$$

f is right continuous at x if it has a right limit at x and $f(x+) = f(x)$ and left continuous if it has a left limit and $f(x-) = f(x)$. The function f is called a càdlàg function if at each $x \in I$ it is right continuous and has a left limit. f is called càglàd if it is left continuous and has a right limit at each point of I.

Definition 1.3.30. *A stochastic process is called a càdlàg process if each if its paths is a càdlàg function. A function f is of bounded variation if it equals the difference of two increasing functions. A process is said to be of bounded variation if each of its paths is of bounded variation.*

Definition 1.3.31. *A filtration in $(\Omega, \mathcal{F}, \mathbb{P})$ is a fimily of σ-algebras $(\mathcal{F}_t)_{t \in T}$ in Ω such that $\mathcal{F}_t \subseteq \mathcal{F}$ for all $t \in T$ and*

$$s \leq t \Rightarrow \mathcal{F}_s \subseteq \mathcal{F}_t.$$

A null set is a subset $A \subseteq \Omega$ for which there exists a $B \in \mathcal{F}$ such that $A \subseteq B$ and P(B) = 0. $\mathcal{F}$ is called $\mathbb{P}$-complete if each null set is a member of $\mathcal{F}$.

A filtration is said to satisfy the "usual conditions" if for every t, $\mathcal{F}_t$ is $\mathbb{P}$-complete and $\mathcal{F}_t = \cap_{u>t} \mathcal{F}_u$.

Definition 1.3.32. *A process $(X(t))_t \in T$ is called adapted to a filtration $(\mathcal{F}_t)_{t \in T}$ if $X(t)$ is $\mathcal{F}_t$-measurable for all $t \in T$.*

Definition 1.3.33. *For a random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ its expectation is defined as*

$$\mathbb{E}X = \int_\Omega X(\omega)\, d\mathbb{P}(\omega),$$

provided $X \geq 0$ a.e. on Ω or $\int_\Omega |X(\omega)|\, d\mathbb{P}(\omega) < \infty$.

For a random variable X with $\mathbb{E}|X| < \infty$ and a σ-algebra $\mathcal{G} \subseteq \mathcal{F}$,

$$\mu(G) := \int_{\mathcal{G}} X(\omega)\, d\mathbb{P}(\omega), \qquad G \in \mathcal{G}$$

defines a measure μ on $\mathcal{G}$. Clearly, if $P(G) = 0$, then $\mu(G) = 0$. Due to the Radon-Nikodym theorem there exists a $\mathcal{G}$-measurable function $X_{\mathcal{G}} : \Omega \to \mathbb{R}$ such that

$$\mu(G) = \int_G X_{\mathcal{G}}(\omega)\, d\mathbb{P}(\omega), \qquad \text{for} \quad \text{all} \quad \mathrm{G} \in \mathcal{G},$$

This $\mathcal{G}$-measurable random variable $X_{\mathcal{G}}$ is called the conditional expectation of X with respect to $\mathcal{G}$ and is denoted by $E[X|\mathcal{G}]$.

Fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $(\mathcal{F}_t)_{t\geq 0}$ in $\mathcal{F}$ that satifies the usual conditions.

Definition 1.3.34. *A martingale is a stochastic process $(X(t))_{t\in T}$ which is adapted, $\mathbb{E}|X(t)| < \infty$ for all t, and such that $\mathbb{E}[X(t)|\mathcal{F}_s] = X(s)$ whenever $s \leq t$, $s, t \in T$.*

Definition 1.3.35. *A random variable T with values in $[0, \infty]$ is called a stopping time if $\{\omega \in \Omega : T(\omega) \leq t\} \in \mathcal{F}_t$ for every $0 \leq t \leq \infty$.*

For instance, if $(X(t))_{t\geq 0}$ is an adapted *càdlàg* process with $X(0) = 0$ and $B \subseteq \mathbb{R}$ is open, then the first time of hitting B defined by

$$T(\omega) = \inf\{t > 0 : X(t)(\omega) \in B \quad \text{or} X(t-)(\omega) \in B\}.$$

Definition 1.3.36. *If $(X(t))_{t\geq 0}$ is a stochastic process and T is a stopping time, then the stopped process X^T is given by*

$$X^T(t)(\omega) = \begin{cases} X(t)(\omega), & t < T(\omega), \\ X(T(\omega))(\omega), & t \geq T(\omega), \quad \in \Omega, \quad t \geq 0 \end{cases}$$

In particular, if $(X(t))_{t\geq 0}$ is adapted and continuous, T is the first time of hitting $\mathbb{R} \setminus (-M, M)$, then X^T is uniformly bounded and $X^T(t) \leq M$ for all $t \geq 0$.

Definition 1.3.37. *A process $(X(t))_{t\geq 0}$ is called uniformly integrable if*

$$\lim_{n\to\infty} \sup_{t\geq 0} \int_{\{|X(t)|\geq n\}} |X(t)|\, d\mathbb{P} = 0.$$

If X is a random variable with $\mathbb{E}|X| < \infty$, then $(X(t))_{t\geq 0}$ given by $X(t) = \mathbb{E}[X|\mathcal{F}_t]$, $t \geq 0$, is a uniformly integrable martingale.

Definition 1.3.38. *An adapted càdlàg process $(X(t))_{t\geq 0}$ is called a local martingale, if there exists a sequence of stopping times $T_1, T_2, \cdots$ with $0 \leq T_1 \leq T_2 \cdots$ a.s. and $\lim_{n\to\infty} T_n = \infty$ a.s. such that for each n, the stopped process X^{T_n} is a uniformly integrable martingale.*

Note that any *càdlàg* martingale is a local martingale. (Taking $T_n = n$, $n \in \mathbb{N}$).

We show that the following stochastic convolution is not a martingale.

$$\int_0^t e^{-c(t-s)}\sigma(s)\,dw(s), \tag{1.49}$$

where $\sigma(t)$ is a continuous function. In fact, for $0 \le u \le t$,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t e^{-c(t-s)}\sigma(s)\,dw(s) \mid \mathcal{F}_u\right] \\
&= \mathbb{E}\left[\int_0^u e^{-c(t-s)}\sigma(s)\,dw(s) \mid \mathcal{F}_u\right] + \mathbb{E}\left[\int_u^t e^{-c(t-s)}\sigma(s)\,dw(s) \mid \mathcal{F}_u\right] \\
&= \int_0^u e^{-c(t-s)}\sigma(s)\,dw(s) \neq \int_0^u e^{-c(u-s)}\sigma(s)\,dw(s).
\end{aligned} \tag{1.50}$$

It can also be shown that $\int_0^t e^{-c(t-s)}\sigma(s)\,d\omega(s)$ is not a local martingale. To show this, we need the following Lemma.

Lemma 1.3.39. ([109]) *If $M(t)$ is a local martingale and for every t, $\mathbb{E}\sup_{s\in[0,t]}|M(s)| < \infty$, then $M(t)$ is a martingale.*

Lemma 1.3.40. *For continuous function $\sigma(t)$, $\int_0^t e^{-c(t-s)}\sigma(s)\,dw(s)$ is not a local martingale.*

Proof. We suppose that $\int_0^t e^{-c(t-s)}\sigma(s)\,dw(s)$ is a local martingale. For every t, we have that

$$\begin{aligned}
\mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s e^{-c(s-u)}\sigma(u)\,dw(u)\right| &= \mathbb{E}\sup_{s\in[0,t]} e^{-cs}\left|\int_0^s e^{cu}\sigma(u)\,dw(u)\right| \\
&\le \mathbb{E}\sup_{s\in[0,t]}\left|\int_0^s e^{cu}\sigma(u)\,dw(u)\right| \\
&\le K_1\mathbb{E}\left(\int_0^t e^{2cu}\sigma^2(u)\,du\right)^{1/2} \\
&\le K_1\left(\int_0^t e^{2cu}\mathbb{E}\sigma^2(u)\,du\right)^{1/2} < \infty.
\end{aligned}$$

From Lemma 1.3.39, we obtain that M is a martingale. However, from (1.50), we know that $\int_0^t e^{-c(t-s)}\sigma(s)\,dw(s)$ is not a martingale, which is a contradiction. □

Lemma 1.3.41. (Mao [96] Burkholder-Davis-Gundy Inequality) *There exists a universal constant K_p for any $0 < p < \infty$ such that for every continuous local martingale M vanishing at zero and any stopping time τ,*

$$\mathbb{E}\left(\sup_{0\le s\le\tau}|M_s|^p\right) \le K_p\mathbb{E}((M,M)_\tau)^{p/2},$$

where $(M,M)_\tau$ is the cross-variation of M and in particular, one can take

$$\begin{aligned}
K_p &= \left(\frac{32}{p}\right)^{p/2} \quad \text{if} \quad 0 < p < 2, \\
K_p &= 4 \quad \text{if} \quad p = 2, \\
K_p &= \left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{p/2} \quad \text{if} \quad p > 2.
\end{aligned}$$

Lemma 1.3.42. ([109] Doob's inequality, on Page 11) *Let X be a positive submartingale. For all $p > 1$, with q conjugate to p (i.e. $\frac{1}{p} + \frac{1}{q} = 1$), we have*

$$\| \sup_t X_t \|_{L_p} \leq q \sup_t \| X_t \|_{L_p}.$$

For a real valued process, we let X^* denote $\sup_s |X_s|$. Note that if M is a martingale with $M_\infty \in L^p$, then $|M|$ is a positive submartingale, and we have

$$\mathbb{E}\{(M^*)^p\} \leq q^p \mathbb{E}\{M_\infty^p\}.$$

For $p = 2$, we have $\mathbb{E}\{(M^*)^2\} \leq 4\mathbb{E}\{M_\infty^2\}$. The last inequality is called Doob's maximal quadratic inequality.

Lemma 1.3.43. (Hölder inequality) *Assume that there exists two continuous functions $f(x)$, $g(x)$ and a set Ω, p and q satisfying $\frac{1}{p} + \frac{1}{q} = 1$, for any $p > 0$, $q > 0$, if $p > 1$, then the following inequality holds.*

$$\int_\Omega |f(x)g(x)|\, dx \leq \left(\int_\Omega |f(x)|^p\, dx \right)^{1/p} \left(\int_\Omega |g(x)|^q\, dx \right)^{1/q}.$$

Lemma 1.3.44. ([120]) *For any real numbers $a_k \geq 0$, $k = 1, 2, 3, \cdots n$, and $p > 1$, the following inequality holds,*

$$\left(\sum_{k=1}^n a_k \right)^p \leq n^{p-1} \sum_{k=1}^n a_k^p.$$

1.3.9 Stochastic delay differential equations

The existence, uniqueness and stability of stochastic delay differential equations have been extensively investigated by many authors, see, for example, Friedman [43], Ikeda and Watanabe [60], Mao [96].

The techniques dealing the existence and uniqueness of stochastic delay differential equations have been developed mainly by using two different methods, the iterative method [2, 22, 96] and the fixed point method [1, 3, 7, 46].

One of the powerful techniques employed in the study of the stability problems of stochastic delay differential equations is the method of the Liapunov function or functional, see, for example, Kolmanovskii [71], Mao [93, 94]. Further, a great number classes of stochastic neural networks with delays are studied by using LMI method, see the work [73, 77, 113, 115, 133].

For the stochastic differential equations with infinite delays, it was recently proposed by Luo [90] and Appleby [4] to use fixed point methods to deal with the stability problems for stochastic delay differential equations. Many authors, e.g., Luo [90, 91], Luo and Taniguchi [92], Sakthivel and Luo [117, 118], Cui et al. [27] have applied fixed point methods to study stability properties of many classes of stochastic delay differential equations. It turns out the fixed point method is a powerful technique to deal with asymptotic stability and exponential stability of stochastic delay differential equations.

1.3.10 Some examples of Banach spaces

A normed linear space is a metric space with respect to the metric d derived from its norm, where $d(x,y)=\|x-y\|$.

Definition 1.3.45. *A Banach space is a normed linear space that is complete metric space with respect to the metric derived from its norm.*

Here are some examples.

Example 1.3.46. *The space $C([a,b])$ of continuous, real-valued (or complex-valued) functions on $[a,b]$ with the sup-normed is a Banach space. More generally, we have the following examples.*

(i) *If X is a Banach space, the space $C([a,b];X)$ of continuous, X-valued functions on $[a,b]$ equipped with the sup-norm is a Banach space.*

(ii) *If X is a Banach space, the space $BC([a,b];X) := \{\varphi\in C([a,b];X), \|\varphi\|<\infty\}$ of bounded continuous, X-valued functions on $[a,b]$ equipped with the sup-norm is a Banach space.*

(iii) *If X is a Banach space, the space $\{\varphi \mid \varphi\in C([a,b];X),\ \lim_{t\to\infty}\varphi(t)=0\}$ with the sup-norm is a Banach space. Further, the space*

$$\left\{\varphi \mid \varphi\in C([a,b];X),\ \varphi(t)\to 0 \text{ as } t\to\infty\right\}$$

and the sapce

$$\left\{\varphi \mid \varphi\in C([a,b];X),\ \|\varphi\|=\sup_{s\in[a,b]}|\varphi(s)| \text{ is bounded and } \varphi(t)\to 0 \text{ as } t\to\infty\right\}$$

are Banach spaces with respect to the sup-norm. Clearly, the space

$$\left\{\varphi \mid \varphi\in C([a,b];L^p(\Omega,\mathbb{R}^n)),\ \lim_{t\to\infty}\mathbb{E}|\varphi(t)|^p=0\right\}$$

is a Banach spaces with respect to the norm defined as

$$\|\varphi\| := \left(\sup_s \mathbb{E}|\varphi(s)|^p\right)^{1/p}.$$

Lemma 1.3.47. *Suppose that $\mathcal{F}_t$ is complete, (that is, contains all null sets). Denote by*

$$C_0([0,\infty);L^p(\Omega,\mathbb{R}^n)) := \left\{\varphi \mid \varphi\in C([0,\infty);L^p(\Omega,\mathbb{R}^n)),\ \lim_{t\to\infty}\mathbb{E}|\varphi(t)|^p=0\right\},$$

then the space

$$D := \left\{\varphi \mid \varphi\in C_0([0,\infty);L^p(\Omega,\mathbb{R}^n)),\ \varphi(t) \text{ is } \mathcal{F}_t-\text{measurable for all } t\right\}$$

is a closed subspace of $C_0([0,\infty);L^p(\Omega,\mathbb{R}^n))$.

Proof. If $\varphi(t),\psi(t)\in D$, then $\varphi(t)$ and ψ are $\mathcal{F}_t$-measurable, so $\varphi(t)+\psi(t)$ and $\alpha\varphi(t)$ $(\alpha\in\mathbb{C})$ are $\mathcal{F}_t$-measurable.

Suppose that the sequences $\varphi_1(t),\varphi_2(t),\cdots\varphi_n(t)\cdots\in D$, $\varphi\in C_0([0,\infty);L^p(\Omega,\mathbb{R}^n))$, and $\varphi_n(t)\to\varphi(t)$, we claim that $\varphi(t)$ is $\mathcal{F}_t$-measurable. In fact, since $\varphi_n(t)\to\varphi(t)$, then

$$\sup_{s\in\Omega}\left(\mathbb{E}|\varphi_n(s)-\varphi(s)|^p\right)\to 0 \quad \text{as} \quad n\to\infty.$$

So, for every t, we have that $\mathbb{E}|\varphi_n(s)-\varphi(s)|^p\to 0$ as $n\to\infty$, then there exists a squence $(\varphi_{n_k}(t))_k$ such that $\varphi_{n_k}(t)\to\varphi(t)$ a.e. on Ω. On the other hand, $\mathcal{F}_t$ is complete. Hence, we obtain that $\varphi(t)$ is $\mathcal{F}_t$-measurable, which implies that D is a closed subspace of $C_0([0,\infty);L^p(\Omega,\mathbb{R}^n))$.

1.4 Structure of this thesis

This thesis is divided into two parts. The first part deals with asymptotic behavior and stability of deterministic delay differential equations. The second part is concerned with the stability properties of stochastic delay differential equations. Each chapter starts with an introduction, in which we summarize the main results. A brief overview of the contents of the thesis is given below.

Chapter 2 presents three methods concerning asymptotic behavior of autonomous neutral delay differential equations. One method based on spectral theory, another method that treats the equation as an ordinary differential equation (ODE) with the other state-dependent terms considered as perturbations, and a third method using Banach's fixed point theorem. We also address the relations of the spectral method and the ODE method. To a retarded form of the autonomous neutral delay differential equation, we illustrate a third method, fixed point method.

Chapter 3 focuses on asymptotic behavior of a class of nonautonomous neutral delay differential equations in which the coefficient for neutral term is constant. Such equations can not be treated by spectral theory, but in some special cases, a generalized characteristic equation can be used. This is a functional equation. If it can be solved, the precise asymptotic behavior of solution of the neutral equation and their derivative can be determined. Examples are given in which the generalized characteristic equation can be solved.

Chapter 4 addresses a fixed point approach to a series of differential and difference equations. In Section 4.1, four general classes of equations are considered by unifying recent results in the literature. For each of these classes of equations, different techniques are combined to prove new stability theorems. In addition, various examples are presented to illustrate our results. In Section 4.2, the stability of two classes of nonlinear neutral differential equations is studied by introducing two auxiliary functions. In Section 4.3, the stability of one class of nonlinear delay difference equations is investigated. The obtained theorems show the general applicability of the fixed point method.

Chapter 5 discusses the stability of two classes of neutral stochastic delay differential equations with impulses. In Section 5.1, asymptotic stability of a class of neutral stochastic delay differential equations with linear impulses is studied by means of the fixed point method. More specifically, two theorems for the asymptotic stability of the equations are presented by using two contraction mapping which are defined on different complete metric spaces. In Section 5.2, exponential stability of a class of neutral stochastic partial differential equations with variable delays and impulses is investigated. The equation is considered as an infinite dimensional stochastic differential equation with delays. The method by using an impulsive-integral inequality and a fixed point method are applied to study exponential stability of mild solutions of the impulsive neutral stochastic partial delay differential equations, respectively.

Chapter 6 studies stability properties of stochastic delayed neural networks without impulses and stochastic delayed neural networks with impulses. Our approaches are based on a fixed point method and the method by using an approporiate integral inequality. In Section 6.1, asymptotic stability and exponential stability of a class of stochastic delayed neural networks with discrete and distributed delays are studied. In particular, a class of delayed neural networks without stochastic perturbations is considered. In Section 6.2, impulsive effects to the class of stochastic delayed neural networks are studied.

Chapter 2

Asymptotic behavior of a class of autonomous neutral delay differential equations

In this chapter, three different methods to study the asymptotic behavior of a class of autonomous neutral delay differential equations are presented. Our approach is either based on methods from functional analysis, ordinary differential equations or fixed point theory. The relations of the method from functional analysis (called spectral method) and the method from ordinary differential equations (called ODE method) are addressed. If there are no neutral terms in the considered equations, a third method based on fixed point theory is introduced.

The organization of this chapter is as follows. In Section 2.2, the spectral approach is introduced and used to study the asymptotic behavior of the solutions of (2.1). In Section 2.3, the ODE approach is introduced to study the asymptotic behavior of solutions of (2.1). In Section 2.4, both approaches are analysed by investigating a number of examples. In Section 2.5, an approach based on fixed point theory is introduced and used to study the asymptotic behavior of (2.2). An application to the mechanical model of turning processes is presented in Section 2.6.

2.1 Introduction

In 1973, Driver, Sasser and Slater [35] studied asymptotic behavior, oscillation and stability of first order delay differential equations with small delay using an approach based on an ordinary differential equation (ODE) method. The key idea of the ODE approach is to transform the differential equation into a lower order equation by using a real root of the corresponding characteristic equation. Following this approach as presented in [35], a number of papers appeared in which the asymptotic behavior, oscillation and stability for first (or second or higher) order (neutral) delay differential equations, and integro-differential equations with unbounded delay as well as for delay difference equations were studied, see [51, 84, 101, 106, 105, 107]. A disadvantage of this ODE approach is that it does not lead to explicit formulas for the reduced lower order equations and that it only works if the characteristic equation has a real root.

In 2003, by using residue calculus and spectral theory, Frasson and Verduyn Lunel [39] presented a new approach to study the asymptotic behavior of neutral delay differential equations, the so-called spectral projection method. In this chapter, by studying asymptotic behavior of a class of second order neutral delay differential equations, we discuss the relations of the two approaches. We obtain that under the same assumptions, the ODE approach is equivalent to the spectral approach (see Section 2.4). However, the spectral approach has some advantages, since the conditions for the spectral method are weaker than those needed for the ODE method, as is illustrated by Example 2.4.2, and the asymptotic behavior of neutral delay differential equations can be presented by a general formula (see Theorem 2.2.6). Furthermore, by using the spectral approach, we can also study the asymptotic behavior of neutral delay differential equations with matrix coefficients.

In this chapter, we consider a specific class of second order neutral delay differential equations of the following form

$$\begin{cases} x''(t) + cx''(t-\tau) = p_1 x'(t) + p_2 x'(t-\tau) + q_1 x(t) + q_2 x(t-\tau), \\ \\ x(t) = \phi(t), \quad -\tau \le t \le 0, \end{cases} \tag{2.1}$$

where $c, p_1, p_2, q_1, q_2 \in \mathbb{R}$, $\tau > 0$, the initial function ϕ is a given continuously differentiable real-valued function on the initial interval $[-\tau, 0]$.

A special case of system (2.1) is the retarded delay differential equation

$$x''(t) + ax'(t) + bx(t-r) + cx(t) = 0, \quad a, b, c \in \mathbb{R}, \quad r > 0, \tag{2.2}$$

which is often called a delayed oscillator, is well-studied in applications [59]. It appears, for example, as the basic governing equation of the regenerative model of machine tool chatter.

2.2 Asymptotic behavior by a spectral approach

Let $\mathcal{C} = C([-\tau, 0], \mathbb{C}^n)$ denote the Banach space of continuous functions endowed with the supremum norm. From the Riesz representation theorem it follows that every bounded linear mapping $L : \mathcal{C} \to \mathbb{C}^n$ can be represented by

$$L\varphi = \int_{-\tau}^{0} d\eta(\theta)\varphi(\theta),$$

where $\eta(\theta)$, $-\tau \le \theta \le 0$, is an $n \times n$-matrix whose elements are of bounded variation, normalized so that $\eta(0) = 0$ and η is continuous from the left on $(-\tau, 0)$ with values in the matrix space $\mathbb{C}^{n\times n}$. This set of functions is denoted by $\mathrm{NBV}([-\tau, 0], \mathbb{C}^{n\times n})$. For a function $x : [-\tau, \infty) \to \mathbb{C}^n$, we denote by $x_t \in \mathcal{C}$ the function $x_t(\theta) = x(t + \theta)$, $-\tau \le \theta \le 0$ and $t \ge 0$.

An initial value problem for a linear autonomous neutral delay differential equation is given by the following relation

$$\begin{cases} \frac{d}{dt} Dx_t = Lx_t, \qquad t \ge 0, \\ \\ x_0 = \phi, \quad \phi \in \mathcal{C}, \end{cases} \tag{2.3}$$

where $D : \mathcal{C} \to \mathbb{C}^n$ is continuous, linear and atomic at zero, $L : \mathcal{C} \to \mathbb{C}^n$ is linear and continuous and, both operators are respectively, presented by

$$L\varphi = \int_{-\tau}^{0} d\eta(\theta)\varphi(\theta), \quad D\varphi = \varphi(0) - \int_{-\tau}^{0} d\mu(\theta)\varphi(\theta),$$

where $\eta, \mu \in \mathrm{NBV}([-\tau, 0], \mathbb{C}^{n\times n})$, and μ is continuous at zero. See Hale and Verduyn Lunel [51] for a detailed information.

For the second order neutral delay differential equation (2.1), let $y(t) = x'(t)$, then (2.1) can be written in the form

$$\begin{cases} x'(t) = y(t), \\ y'(t) + cy'(t-\tau) = p_1 y(t) + p_2 y(t-\tau) + q_1 x(t) + q_2 x(t-\tau). \end{cases}$$

Let $X(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, we have

$$X'(t) + CX'(t-\tau) = EX(t) + FX(t-\tau), \tag{2.4}$$

where

$$C = \begin{pmatrix} 0 & 0 \\ 0 & c \end{pmatrix}, E = \begin{pmatrix} 0 & 1 \\ q_1 & p_1 \end{pmatrix} \quad \textit{and} \quad F = \begin{pmatrix} 0 & 0 \\ q_2 & p_2 \end{pmatrix}.$$

By taking $\mu(\theta) = C$, for $\theta \leq -\tau$, $\mu(\theta) = 0$, for $\theta > -\tau$, and $\eta(\theta) = -F$, for $\theta \leq -\tau$, $\eta(\theta) = 0$, for $-\tau < \theta < 0$, $\eta(\theta) = E$, for $\theta \geq 0$, (2.1) can be written in the form (2.3).

Throughout this chapter, a continuous real-valued function x defined on the interval $[-\tau, \infty)$ is said to be a solution of the initial value problem (2.1) if x satisfies (2.1) in the mild sense, see Lemma 2.2.1. It is well known (see [35]) that for any given initial function ϕ, there exists a unique solution of the initial value problem (2.1).

Given the solution $x(\phi)$ of the initial value problem (2.3), the solution operator $T(t) : \mathcal{C} \to \mathcal{C}$ is defined by the relation

$$T(t)\phi = x_t(.;\phi), \quad t \geq 0.$$

Lemma 2.2.1. (Hale and Verduyn Lunel [51]) *The solution operator $T(t)$ is a C_0-semigroup on $\mathcal{C}$ with infinitesimal generator*

$$\begin{cases} D(A) = \left\{\phi \in \mathcal{C} \mid \frac{d\phi}{d\theta} \in \mathcal{C},\ D\frac{d\phi}{d\theta} = L\phi\right\} \\ A\phi = \frac{d\phi}{d\theta} \end{cases} \tag{2.5}$$

Lemma 2.2.2. (Hale and Verduyn Lunel [51]) *If A is defined by equation (2.5), then $\sigma(A) = P_{\sigma(A)}$ and $\lambda \in \sigma(A)$ if and only if λ satisfies the characteristic equation $det\triangle(\lambda) = 0$, where*

$$\triangle(\lambda) = \lambda I - \int_{-\tau}^{0} \lambda e^{\lambda\theta} d\mu(\theta) - \int_{-\tau}^{0} e^{\lambda\theta} d\eta(\theta). \tag{2.6}$$

Here $P_{\sigma(A)}$ denotes the point spectrum of A.

It is well known that there is a close connection between the spectral properties of the inifinitesimal generator A and the characteristic matrix $\triangle(\lambda)$ given by (2.6). In particular, the geometric multiplicity d_λ is equal to the dimension of the null space of $\triangle(z)$ at $z = \lambda$, and the algebraic multiplicity m_λ is equal to the multiplicity of $z = \lambda$ as a zero of $\det \triangle(\lambda) = 0$. Furthermore, the generalized eigenspace at λ is given by

$$\mathcal{M}_\lambda = \mathcal{N}(\lambda I - A)^{k_\lambda},$$

where k_λ denotes the order of $z = \lambda$ as a pole of $\triangle(z)^{-1}$. See Kaashoek and Verduyn Lunel [66] for more information.

Lemma 2.2.3. (Hale and Verduyn Lunel [51]) *For any λ in $\sigma(A)$, the generalized eigenspace $\mathcal{M}_\lambda(A)$ is finite dimensional and there is an integer k such that $\mathcal{M}_\lambda(A) = \mathcal{N}((\lambda I - A)^k)$ and we have a direct sum decomposition*

$$\mathcal{C} = \mathcal{N}((\lambda I - A)^k) \oplus \mathcal{R}((\lambda I - A)^k).$$

From the spectral theory [29, 51], it follows that the spectral projection onto $\mathcal{M}_\lambda(A)$ along $\mathcal{R}((\lambda I - A)^k)$ can be represented by a Dunford integral

$$P_\lambda = \frac{1}{2\pi i} \int_{\Gamma_\lambda} (zI - A)^{-1}\, dz, \tag{2.7}$$

where Γ_λ is a small circle such that λ is the only singularity of $(zI - A)^{-1}$ inside Γ_λ.

Definition 2.2.4. *An eigenvalue λ_d is called a dominant eigenvalue of A, if there exists a $\varepsilon > 0$, such that if λ is another eigenvalue of A, then $Re\lambda < Re\lambda_d - \varepsilon$.*

Consider the scalar case of initial value problem (2.3), the characteristic equation $\Delta(z)$ is given by (2.6). Define the auxiliary function $\chi : \mathbb{C} \to [0, \infty)$ by

$$\chi(z) = \int_{-\tau}^{0} (1 - \theta|z|) e^{z\theta}\, dV(\mu)(\theta) + \int_{-\tau}^{0} (-\theta) e^{z\theta}\, dV(\eta)(\theta), \tag{2.8}$$

where $V(\mu)(\theta)$ denotes the total variation function of μ on $[-\tau, \theta]$ for each θ in $(-\tau, 0]$.

Theorem 2.2.5. (Frasson [40]) *Suppose that $z_0 \in \mathbb{C}$ is a zero of $\det\Delta(z)$ in (2.6). If $\chi(z_0) < 1$, then z_0 is a simple dominant zero of $\Delta(z)$.*

Next, we provide the main result of Frasson and Verduyn Lunel [39], which presents the explicit representation of asymptotic behavior of neutral delay differential equations.

Theorem 2.2.6. (Frasson and Verduyn Lunel [39]) *Let A be given by (2.5), if A has a simple and dominant eigenvalue λ_d, then there exists positive numbers ε and M such that*

$$\|e^{-\lambda_d t} T(t)\phi - P_{\lambda_d}\phi\| \le M e^{-\varepsilon t},$$

and

$$\lim_{t\to\infty} e^{-\lambda_d t} T(t)\phi = e^{\lambda_d \cdot} \left[\frac{d}{dz} \det \triangle(\lambda_d) \right]^{-1} \mathrm{adj}\triangle(\lambda_d) K(\lambda_d)\phi.$$

Furthermore, if $x(t) = x(\cdot, \phi)$ denotes the solution of (2.3) with initial data $x_0 = \phi$, then

$$\lim_{t\to\infty} e^{-\lambda_d t} x(t) = \left[\frac{d}{dz} \det \triangle(\lambda_d) \right]^{-1} \mathrm{adj}\triangle(\lambda_d) K(\lambda_d)\phi,$$

where $\mathrm{adj}\triangle(\lambda_d)$ denotes the matrix of cofactors of $\triangle(\lambda_d)$,

$$K(\lambda_d)\phi = D\phi + \int_{-\tau}^{0} (\lambda_d d\mu(\theta) + d\eta(\theta)) e^{\lambda_d \theta} \int_{\theta}^{0} e^{-\lambda_d s} \phi(s)\, ds.$$

Combining Theorem 2.2.5 with Theorem 2.2.6, we arrive at

Theorem 2.2.7. *Let* $x(\cdot)$ *be the solution of* (2.3) *subjected to the initial condition* $x_0 = \phi \in C([-\tau, 0], \mathbb{R})$*. If* λ_d *is a real zero of characteristic equation* $\Delta(z)$ *given by* (2.6) *such that* $\chi(\lambda_d) < 1$*, where* $\chi(\cdot)$ *is given by* (2.8)*, then the asymptotic behavior of* $x(\cdot)$ *is given by*

$$\lim_{t\to\infty} e^{-\lambda_d t}x(t) = \frac{1}{H(\lambda_d)}K(\lambda_d)\phi,$$

where

$$\begin{aligned} H(\lambda_d) &= 1 - \int_{-\tau}^{0} e^{\lambda_d\theta}\, d\mu(\theta) - \int_{-\tau}^{0} \theta e^{\lambda_d\theta}\, (\lambda_d d\mu(\theta) + d\eta(\theta)), \\ K(\lambda_d)\phi &= M\psi + \int_{-\tau}^{0} (\lambda_d\, d\mu(\theta) + d\eta(\theta))e^{\lambda_d\theta} \int_{\theta}^{0} e^{-\lambda_d s}\psi(s)\, ds. \end{aligned}$$

Note that the result of Theorem 2.2.7 is consistent with the result in [101] which was obtained by using ODE method.

Example 2.2.8.

$$\begin{cases} x'(t) + cx'(t-\sigma) = ax(t) + bx(t-\tau), \\ \\ x(t) = \phi(t), \quad -\tau \le t \le 0. \end{cases} \tag{2.9}$$

The characteristic equation of (2.9) is

$$\Delta(\lambda) = \lambda(1 + ce^{-\lambda\sigma}) - a - be^{-\lambda\tau}. \tag{2.10}$$

Note that the characteristic equation (2.10) may have no real root. Suppose that the characteristic equation (2.10) has a real root λ_0 which satisfies

$$|c|(1 + |\lambda_0|\sigma)e^{-\lambda_0\sigma} + |b|\tau e^{-\lambda_0\tau} < 1,$$

then by Theorem 2.2.5, λ_0 is a simple dominant root of (2.10). Hence, applying Theorem 2.2.7, we obtain that

$$\lim_{t\to\infty} e^{-\lambda_0 t}x(\phi; t) = \frac{K(\lambda_0; \phi)}{H(\lambda_0)}, \tag{2.11}$$

where

$$\begin{aligned} K(\lambda_0; \phi) &= \phi(0) + c\phi(-\sigma) - c\lambda_0 e^{-\lambda_0\sigma}\int_{-\sigma}^{0} e^{\lambda_0 s}\phi(s)\, ds + be^{-\lambda_0\tau}\int_{-\tau}^{0} e^{\lambda_0 s}\phi(s)\, ds, \\ H(\lambda_0) &= 1 + c(1 - \lambda_0\sigma)e^{-\lambda_0\sigma} + b\tau e^{-\lambda_0\tau}, \end{aligned}$$

(2.11) is consistent with the result in Kordonis et al. [84] which was obtained by using ODE method .

Now, we use Theorem 2.2.7 to study the asymptotic behavior of (2.1). Let the initial condition associated with (2.4) be given by

$$X_0 = \begin{pmatrix} \phi \\ \phi' \end{pmatrix} \in C([-\tau, 0], \mathbb{R}^2),$$

The characteristic matrix corresponding to (2.4) is given by

$$\triangle(z) = zI + ze^{-\tau z}C - E - Fe^{-\tau z} = \begin{pmatrix} z & -1 \\ -q_1 - q_2 e^{-\tau z} & z + cze^{-\tau z} - p_1 - p_2 e^{-\tau z} \end{pmatrix}$$

so the characteristic equation is $\det\triangle(z) = z^2 + cz^2e^{-\tau z} - (p_1 + p_2 e^{-\tau z})z - q_2 e^{-\tau z} - q_1$. Note that the characteristic equation may have no simple dominant zero. Suppose that there exists a simple dominant zero λ_d of the characteristic equation $\det\triangle(z) = 0$, by Theorem 2.2.6, we have

$$\begin{aligned}\lim_{t\to\infty} e^{-\lambda_d t}X(t) &= \lim_{t\to\infty}\begin{pmatrix} e^{-\lambda_d t}x(t) \\ e^{-\lambda_d t}y(t)\end{pmatrix} = \left[\frac{d}{dz}\det\triangle(\lambda_d)\right]^{-1} \mathrm{adj}\triangle(\lambda_d)K(\lambda_d)\phi \\ &= \begin{pmatrix} \frac{\lambda_d + c\lambda_d e^{-\tau\lambda_d} - p_1 - p_2 e^{-\tau\lambda_d}}{\beta(\lambda_d)} & \frac{1}{\beta(\lambda_d)} \\ \frac{q_1 + q_2 e^{-\tau\lambda_d}}{\beta(\lambda_d)} & \frac{\lambda_d}{\beta(\lambda_d)}\end{pmatrix} \times A\end{aligned}$$

where

$$A = \begin{pmatrix} \phi(0) \\ \phi'(0) + c\phi'(-\tau) + \int_{-\tau}^0 (p_2 - c\lambda_d)e^{-\lambda_d(s+\tau)}\phi'(s)\,ds + \int_{-\tau}^0 q_2 e^{-\lambda_d(s+\tau)}\phi(s)\,ds\end{pmatrix}.$$

It follows that

$$\begin{aligned}\lim_{t\to\infty} e^{-\lambda_d t}x(t) &= \frac{1}{\beta(\lambda_d)}\Big[(\lambda_d + c\lambda_d e^{-\tau\lambda_d} - p_1 - p_2 e^{-\tau\lambda_d})\phi(0) + \phi'(0) + c\phi'(-\sigma) \\ &\quad + \int_{-\tau}^0 (p_2 - c\lambda_d)e^{-\lambda_d(s+\tau)}\phi'(s)\,ds + \int_{-\tau}^0 q_2 e^{-\lambda_d(s+\tau)}\phi(s)\,ds\Big],\end{aligned}$$

where $\beta(\lambda_d) = 2\lambda_d + (2c\lambda_d - c\tau\lambda_d^2 - p_2 + p_2\tau\lambda_d + q_2\tau)e^{-\tau\lambda_d} - p_1 \neq 0$.

The next theorem gives a result similar to Theorem 2.2.6, in case that the real dominant eigenvalue is not simple.

Theorem 2.2.9. (Frasson [42]) *Let λ_d be a real dominant zero of $\det\triangle(z)$ of geometric multiplicity $n \geq 1$. If $x(t) = x(t;\phi)$ denote the solution of (2.1) with initial data $x_0 = \phi$, then the large time behaviour as a function of the initial data ϕ is described as follows.*
1. If $P_{\lambda_d}\phi \neq 0$, then

$$\lim_{t\to\infty}\frac{1}{t^m}e^{-\lambda_d t}x(t) = q_m(n, \lambda_d, \phi),$$

where $m = \max\{j \in 0, 1, 2, ...n-1 : q_j(n, \lambda_d, \phi) \neq 0\}$, q_j is given by

$$q_j(n, \lambda, \phi) = \frac{1}{j!}\sum_{k=j}^{n-1}\frac{D^{n-1-k}K(\lambda)}{(n-1-k)!}\frac{D_1^{k-j}H(\lambda, \phi)}{(k-j)!}.$$

Furthermore, for integer $n \geq 1$, the $n-th$ Fréchet derivative of $H(\lambda, \phi)$ with respect to the first variable is given by

$$\begin{aligned}D_1^n H(z, \phi) &= (-1)^{n+1}n\int_0^r d\mu(\theta)\int_0^\theta \tau^{n-1}e^{-z\tau}\phi(\tau - \theta)\,d\tau \\ &\quad + (-1)^n z\int_0^r d\mu(\theta)\int_0^\theta \tau^n e^{-z\tau}\phi(\tau - \theta)\,d\tau \\ &\quad + (-1)^n\int_0^r d\eta(\theta)\int_0^\theta \tau^n e^{-z\tau}\phi(\tau - \theta)\,d\tau.\end{aligned}$$

2. If $P_{\lambda_d}\phi = 0$, then

$$\lim_{t\to\infty} e^{-\lambda_d t}x(t) = 0.$$

2.3 An ODE approach to asymptotic behavior

In this section, we study the asymptotic behavior of the solutions of a class of second order neutral delay differential equation (2.1) by employing ODE approach. An estimation of solutions of the initial value problem (2.1) is made. As a consequence of this result, the sufficient conditions for stability, the asymptotic stability and instability of the trivial solution are presented.

The characteristic equation of (2.1) is

$$\lambda^2 + c\lambda^2 e^{-\lambda\tau} = p_1\lambda + p_2\lambda e^{-\lambda\tau} + q_1 + q_2 e^{-\lambda\tau}. \tag{2.12}$$

Suppose that λ_0 is a real solution of the characteristic equation (2.12), we consider the first order neutral delay differential equation

$$\begin{aligned} &z'(t) + ce^{-\lambda_0\tau}z'(t-\tau) + (2\lambda_0 - p_1)z(t) + (2c\lambda_0 - p_2)e^{-\lambda_0\tau}z(t-\tau) \\ &= (p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0} z(s+t)\,ds. \end{aligned} \tag{2.13}$$

With (2.13), we associate the equation

$$\mu + (c\mu + 2c\lambda_0 - p_2)e^{-\tau(\lambda_0+\mu)} + 2\lambda_0 - p_1 - (p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0} e^{\mu s}\,ds = 0, \tag{2.14}$$

which is said to be the *second characteristic equation*, and it is obtained from (2.13) by seeking solutions of the form $z(t) = e^{\mu t}$.

Now, we present a proposition, which plays a crucial role in obtaining our main result presented in Theorem 2.3.2. This proposition essentially estabishes a transformation (via a solution of the characteristic equation (2.12)) of the second order neutral delay differential equation (2.1) into the first order neutral delay differential equation (2.13).

Proposition 2.3.1. *Suppose λ_0 is a real root of the characteristic equation (2.12), and let*

$$\beta(\lambda_0) = 2\lambda_d + (2c\lambda_0 - c\tau\lambda_0^2 - p_2 + p_2\tau\lambda_0 + q_2\tau)e^{-\tau\lambda_0} - p_1.$$

Suppose that $\beta(\lambda_0) \neq 0$, then a continuous real-valued function x defined on the interval $[-\tau, \infty)$ is the solution of the initial value problem (2.1) on $[0, \infty)$ if and only if z defined by

$$z(t) = e^{-\lambda_0 t}x(t) - \frac{K(\lambda_0, \phi)}{\beta(\lambda_0)} \qquad \text{for} \qquad t \geq 0, \tag{2.15}$$

is the solution of the neutral delay differential equation (2.13) with the initial condition

$$z(t) = e^{-\lambda_0 t}\phi(t) - \frac{K(\lambda_0, \phi)}{\beta(\lambda_0)} \qquad \text{for} \qquad -\tau \leq t \leq 0, \tag{2.16}$$

where $x(t) = \phi(t)$ *on* $[-\tau, 0]$ *and*

$$\begin{aligned} K(\lambda_0, \phi) &= \phi'(0) + (\lambda_0 - p_1)\phi(0) + c\phi'(-\tau) + c\lambda_0\phi(-\tau) - p_2\phi(-\tau) \\ &\quad -(p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0} e^{-\lambda_0 s}\phi(s)\, ds. \end{aligned}$$

Proof. Let x be the solution of the initial value problem (2.1) for $t \geq 0$ with $x(t) = \phi(t)$ for $-\tau \leq t \leq 0$. Define

$$y(t) = e^{-\lambda_0 t}x(t) \qquad \text{for} \qquad t \geq -\tau.$$

Using the fact that λ_0 is a real root of the characteristic equation (2.12), we have for every $t \geq 0$,

$$\begin{aligned} &[y'(t) + ce^{-\lambda_0\tau}y'(t-\tau) + (2\lambda_0 - p_1)y(t) + (2c\lambda_0 - p_2)e^{-\lambda_0\tau}y(t-\tau)]' \\ &= (p_1\lambda_0 + q_1 - \lambda_0^2)y(t) + (p_2\lambda_0 + q_2 - c\lambda_0^2)e^{-\lambda_0\tau}y(t-\tau) \end{aligned} \tag{2.17}$$

with the initial condition satisfies

$$y(t) = e^{-\lambda_0 t}\phi(t) \qquad \text{for} \qquad -\tau \leq t \leq 0. \tag{2.18}$$

By integrating (2.17), and using the initial condition (2.18), we have

$$\begin{aligned} &y'(t) + ce^{-\lambda_0\tau}y'(t-\tau) + (2\lambda_0 - p_1)y(t) + (2c\lambda_0 - p_2)e^{-\lambda_0\tau}y(t-\tau) \\ &= (p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0} y(s+t)\, ds + K(\lambda_0, \phi) \end{aligned} \tag{2.19}$$

for all $t \geq 0$, where $K(\lambda_0, \phi)$ is defined as in Proposition 2.3.1.

Now we suppose that $\beta(\lambda_0) \neq 0$ and define

$$z(t) = y(t) - \frac{K(\lambda_0, \phi)}{\beta(\lambda_0)} \qquad \text{for} \qquad t \geq -\tau,$$

by the definition of $\beta(\lambda_0)$, we obtain that y satisfies (2.19) if and only if z satisfies (2.15) for all $t \geq 0$. Moreover, the initial condition (2.18) is equivalent to (2.16). ☐

An estimate of the solution of initial value problem (2.1) will be given in the following theorem.

Theorem 2.3.2. *Suppose* λ_0 *is a real root of the characteristic equation* (2.12)*, and let* $\beta(\lambda_0)$ *and* $K(\lambda_0, \phi)$ *be defined as in Proposition* 2.3.1*. Suppose that* $\beta(\lambda_0) \neq 0$*, let* μ_0 *be a real root of the characteristic equation*(2.30)*, and set*

$$\begin{aligned} \gamma(\lambda_0, \mu_0) &= 1 + ce^{-(\lambda_0+\mu_0)\tau} - \tau(c\mu_0 + 2c\lambda_0 - p_2)e^{-(\lambda_0+\mu_0)\tau} \\ &\quad -(p_1\lambda_0 + q_1 - \lambda_0^2)\mu_0^{-2}(\mu_0\tau e^{-\mu_0\tau} + e^{-\mu_0\tau} - 1). \end{aligned}$$

Define

$$\begin{aligned} H(\lambda_0, \mu_0, \phi) &= \phi(0) + c\phi(-\tau) + (p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0} e^{\mu_0 s}\int_{s}^{0} e^{-(\lambda_0+\mu_0)u}\phi(u)\, du\, ds \\ &\quad -(c\mu_0 + 2c\lambda_0 - p_2)e^{-(\lambda_0+\mu_0)\tau}\int_{-\tau}^{0} e^{-(\lambda_0+\mu_0)s}\phi(s)\, ds \\ &\quad -\frac{K(\lambda_0, \phi)}{\beta(\lambda_0)}\Bigg[1 + ce^{-\lambda_0\tau} + (p_1\lambda_0 + q_1 - \lambda_0^2)\mu_0^{-2}(1 - e^{-\mu_0\tau} - \mu_0\tau) \\ &\quad -(c\mu_0 + 2c\lambda_0 - p_2)\mu_0^{-1}(1 - e^{-\mu_0\tau})e^{-(\lambda_0+\mu_0)\tau}\Bigg]. \end{aligned}$$

We assume that the real roots λ_0 and μ_0 have the following property

$$\chi_{\lambda_0,\mu_0} := |c|e^{-(\lambda_0+\mu_0)\tau} + \tau|p_1 - \mu_0 - 2\lambda_0| + \mu_0^{-2}(\mu_0\tau + e^{-\mu_0\tau} - 1)\left|p_1\lambda_0 + q_1 - \lambda_0^2\right| < 1.$$

Then for any $\phi \in C([-\tau, 0], \mathbb{R})$, the solution x of (2.1) satisfies

$$\left|e^{-(\mu_0+\lambda_0)t}x(t) - e^{-\mu_0 t}\frac{K(\lambda_0,\phi)}{\beta(\lambda_0)} - \frac{H(\lambda_0,\mu_0,\phi)}{\gamma(\lambda_0,\mu_0)}\right| \leq M(\lambda_0,\mu_0;\phi)\chi_{\lambda_0,\mu_0},$$

where

$$M(\lambda_0,\mu_0;\phi) = \max_{-\tau\leq t\leq 0}\left|e^{-\mu_0 t}\left(e^{-\lambda_0 t}\phi(t) - \frac{K(\lambda_0,\phi)}{\beta(\lambda_0)}\right) - \frac{H(\lambda_0,\mu_0;\phi)}{\gamma(\lambda_0,\mu_0)}\right|, \quad \text{for} \quad t \geq 0.$$

Proof. Note that $\mu_0 = 0$ is a root of (2.30) if and only if $2\lambda_0 - p_1 + (2c\lambda_0 - p_2)e^{-\tau\lambda_0} - (p_1\lambda_0 + q_1 - \lambda_0^2)\tau = 0$, from the definition of $\beta(\lambda_0)$, we obtain that if zero is a root of (2.30) if and only if $\beta(\lambda_0) = 0$. Hence, if we assume that $\beta(\lambda_0) \neq 0$, then we always have $\mu_0 \neq 0$.

From the assumption that $|\chi_{\lambda_0,\mu_0}| < 1$, we conclude $\gamma(\lambda_0,\mu_0) > 0$. Suppose that x is the solution of the initial value problem (2.1) with $x(\theta) = \phi(\theta)$ for $-\tau \leq \theta \leq 0$, by Proposition 2.3.1, the fact that x is the solution of the initial value problem (2.1) is equivalent to the fact that z is the solution of the delay differential equation (2.15) which satisfies the initial condition (2.16). Set

$$w(t) = e^{-\mu_0 t}z(t) \qquad \text{for} \qquad t \geq -\tau,$$

then by using the fact that μ_0 is a real root of the characteristic equation (2.30), we obtain, for every $t \geq 0$,

$$\begin{aligned}[w(t) + ce^{-(\lambda_0+\mu_0)\tau}w(t-\tau)]' &= (p_1 - \mu_0 - 2\lambda_0)w(t) - (c\mu_0 + 2c\lambda_0 - p_2)e^{-(\lambda_0+\mu_0)\tau}w(t-\tau)\\ &\quad +(p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0} e^{\mu_0 s}w(s+t)\,ds,\end{aligned} \tag{2.20}$$

and the initial condition satisfies

$$w(t) = e^{-(\mu_0+\lambda_0)t}\phi(t) - e^{-\mu_0 t}\frac{K(\lambda_0,\phi)}{\beta(\lambda_0)} \qquad \text{for} \qquad -\tau \leq t \leq 0. \tag{2.21}$$

By integrating (2.20) and using the initial condition of (2.21), we obtain

$$\begin{aligned}w(t) + ce^{-(\lambda_0+\mu_0)\tau}w(t-\tau) &= (p_1 - \mu_0 - 2\lambda_0)\int_{t-\tau}^{t} w(s)\,ds \\ &\quad +(p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0} e^{\mu_0 s}\int_{t-\tau}^{s+t} w(u)\,du\,ds + H(\lambda_0,\mu_0;\phi),\end{aligned} \tag{2.22}$$

for all $t \geq 0$, where $H(\lambda_0,\mu_0;\phi)$ is defined in Theorem 2.3.2.

Define

$$v(t) = w(t) - \frac{H(\lambda_0,\mu_0;\phi)}{\gamma(\lambda_0,\mu_0)} \qquad \text{for} \qquad t \geq -\tau,$$

then by the definition of $\gamma(\lambda_0, \mu_0)$ in Theorem 2.3.2, we obtain that the fact that w satisfies (2.22) is equivalent to the fact that v satisfies the following equation

$$\begin{aligned} v(t) + ce^{-(\lambda_0+\mu_0)\tau}v(t-\tau) &= (p_1 - \mu_0 - 2\lambda_0)\int_{t-\tau}^{t} v(s)\,ds \\ &\quad +(p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0} e^{\mu_0 s}\int_{t-\tau}^{s+t} v(u)\,du\,ds, \end{aligned} \tag{2.23}$$

for all $t \geq 0$. Moreover, the initial condition is equivalent to

$$v(t) = e^{-(\mu_0+\lambda_0)t}\phi(t) - e^{-\mu_0 t}\frac{K(\lambda_0,\phi)}{\beta(\lambda_0)} - \frac{H(\lambda_0,\mu_0;\phi)}{\gamma(\lambda_0,\mu_0)} \qquad \text{for} \qquad -\tau \leq t \leq 0. \tag{2.24}$$

Define

$$M(\lambda_0,\mu_0;\phi) := \max_{-\tau\leq t\leq 0}\left| e^{-(\mu_0+\lambda_0)t}\phi(t) - e^{-\mu_0 t}\frac{K(\lambda_0,\phi)}{\beta(\lambda_0)} - \frac{H(\lambda_0,\mu_0;\phi)}{\gamma(\lambda_0,\mu_0)}\right|.$$

In view of (2.24), we have

$$|v(t)| \leq M(\lambda_0,\mu_0;\phi) \qquad \text{for} \qquad -\tau \leq t \leq 0.$$

We will next show that $M(\lambda_0,\mu_0;\phi)$ is also a bound of v on the whole positive half line. For this purpose, we take an arbitrary $\varepsilon > 0$ and claim that $|v(t)| < M(\lambda_0,\mu_0;\phi) + \varepsilon$ for $t \geq -\tau$. Indeed, suppose that there exists a point $t_0 > 0$ such that

$$\begin{aligned} |v(t)| &< M(\lambda_0,\mu_0;\phi) + \varepsilon \qquad \text{for} \qquad -\tau \leq t < t_0, \\ |v(t_0)| &= M(\lambda_0,\mu_0;\phi) + \varepsilon. \end{aligned} \tag{2.25}$$

Then by (2.23) and the definition of χ_{λ_0,μ_0}, we have

$$\begin{aligned} M(\lambda_0,\mu_0;\phi) + \varepsilon &= |v(t_0)| \\ &\leq |c|e^{-(\lambda_0+\mu_0)\tau}|v(t_0-\tau)| + |p_1 - \mu_0 - 2\lambda_0|\int_{t_0-\tau}^{t_0}|v(s)|\,ds \\ &\quad +|p_1\lambda_0 + q_1 - \lambda_0^2|\int_{-\tau}^{0} e^{\mu_0 s}\int_{t_0-\tau}^{s+t}|v(u)|\,du\,ds \\ &\leq (M(\lambda_0,\mu_0;\phi) + \varepsilon)\Big(|c|e^{-(\lambda_0+\mu_0)\tau} + \tau|p_1 - \mu_0 - 2\lambda_0| \\ &\quad +\mu_0^{-2}(\mu_0\tau + e^{-\mu_0\tau} - 1)\left|p_1\lambda_0 + q_1 - \lambda_0^2\right|\Big) \\ &= (M(\lambda_0,\mu_0;\phi) + \varepsilon)\,\chi_{\lambda_0,\mu_0} < M(\lambda_0,\mu_0;\phi) + \varepsilon, \end{aligned}$$

and we arrive at a contradiction. This implies that our claim is true and since ε is arbitrary, it follows that $|v(t)| \leq M(\lambda_0,\mu_0;\phi)$ for $t \geq -\tau$. Together with (2.23), we arrive at

$$\begin{aligned} |v(t)| &\leq |c|e^{-(\lambda_0+\mu_0)\tau}|v(t-\tau)| + |p_1 - \mu_0 - 2\lambda_0|\int_{t-\tau}^{t}|v(s)|\,ds \\ &\quad +|p_1\lambda_0 + q_1 - \lambda_0^2|\int_{-\tau}^{0} e^{\mu_0 s}\int_{t-\tau}^{s+t}|v(u)|\,du\,ds, \\ &\leq M(\lambda_0,\mu_0;\phi)\Big(|c|e^{-(\lambda_0+\mu_0)\tau} + \tau|p_1 - \mu_0 - 2\lambda_0| \\ &\quad +\mu_0^{-2}(\mu_0\tau + e^{-\mu_0\tau} - 1)\left|p_1\lambda_0 + q_1 - \lambda_0^2\right|\Big) \\ &= M(\lambda_0,\mu_0;\phi)\chi_{\lambda_0,\mu_0} < M(\lambda_0,\mu_0;\phi), \end{aligned}$$

for all $t \geq 0$. This implies

$$|v(t)| = \left| e^{-(\mu_0+\lambda_0)t} x(t) - e^{-\mu_0 t} \frac{K(\lambda_0, \phi)}{\beta(\lambda_0)} - \frac{H(\lambda_0, \mu_0; \phi)}{\gamma(\lambda_0, \mu_0)} \right| \leq M(\lambda_0, \mu_0; \phi) \chi_{\lambda_0, \mu_0},$$

for all $t \geq 0$. This completes the proof of Theorem 2.3.2. □

Using the result of Theorem 2.3.2, we can next discuss the asymptotic behavior of the solution of initial value problem (2.1).

Theorem 2.3.3. *Suppose λ_0 and μ_0 are real roots of the characteristic equations* (2.12) *and* (2.30)*, respectively. Let $\beta(\lambda_0), \chi_{\lambda_0,\mu_0}$, $\gamma(\lambda_0, \mu_0)$ be defined as in Proposition 2.3.1 and Theorem 2.3.2. Then for any $\phi \in C([-\tau, 0], \mathbb{R})$, the solution x of initial value problem* (2.1) *with $x(\theta) = \phi(\theta)$ for $-\tau \leq \theta \leq 0$ satisfies*

$$\lim_{t\to\infty} e^{-(\mu_0+\lambda_0)t} x(t) - e^{-\mu_0 t} \frac{K(\lambda_0, \phi)}{\beta(\lambda_0)} = \frac{H(\lambda_0, \mu_0; \phi)}{\gamma(\lambda_0, \mu_0)},$$

where $K(\lambda_0, \phi), \beta(\lambda_0), H(\lambda_0, \mu_0; \phi), \gamma(\lambda_0, \mu_0)$ are given in Proposition 2.3.1 *and Theorem* 2.3.2 *respectively.*

Proof. By the definition of x, y, z, w and v, we have to prove that

$$\lim_{t\to\infty} v(t) = 0.$$

From Theorem 2.3.2, one can show by induction that v satisfies

$$|v(t)| \leq M(\lambda_0, \mu_0; \phi)(\chi_{\lambda_0,\mu_0})^n \quad \text{for} \quad \text{all} \quad t \geq n\tau - \tau. \tag{2.26}$$

Since $0 \leq \chi_{\lambda_0,\mu_0} < 1$, thus from (2.26), we obtain that v tends to zero as $t \to \infty$. □

Definition 2.3.4. *The trivial solution of* (2.1) *is said to be stable if for any $t_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there exists $\delta = \delta(t_0, \varepsilon) > 0$ such that $\|x_{t_0}\| < \delta$ implies $|x(t)| < \varepsilon$ for $t \geq t_0$. The solution is said to be asymptotically stable if it is stable and for any $t_0 \in \mathbb{R}$ and any $\varepsilon > 0$, there exists a $\delta_a = \delta_a(t_0, \varepsilon) > 0$ such that $\|x_{t_0}\| < \delta_a$ implies* $\lim_{t\to\infty} x(t) = 0$.

As a consequence of Theorem 2.3.2 and Theorem 2.3.3, we have the following stability criterion.

Theorem 2.3.5. *Let λ_0 and μ_0 be real roots of the characteristic equations* (2.12) *and* (2.30)*, and let $\beta(\lambda_0), \chi_{\lambda_0,\mu_0}$, $\gamma(\lambda_0, \mu_0)$ be defined as in Proposition* 2.3.1 *and Theorem* 2.3.2 *respectively, and satisfy the conditions in Theorem* 2.3.2*. Then for any $\phi \in C([-\tau, 0], \mathbb{R})$, the solution x of* (2.1) *with $x(\theta) = \phi(\theta)$ for $-\tau \leq \theta \leq 0$ satisfies*

$$\begin{aligned} |x(t)| \quad \leq \quad & \frac{k_{\lambda_0}}{|\beta(\lambda_0)|} N(\lambda_0, \mu_0; \phi) e^{\lambda_0 t} + \left[\frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0, \mu_0)|} \right. \\ & \left. + \left(1 + \frac{K_{\lambda_0} e_{\mu_0}}{|\beta(\lambda_0)|} + \frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0, \mu_0)|} \right) \chi_{\lambda_0,\mu_0} \right] N(\lambda_0, \mu_0; \phi) e^{(\lambda_0+\mu_0)t}, \end{aligned}$$

where

$$\begin{aligned}
k_{\lambda_0} &= 1+|c|+|\lambda_0-p_1|+|c||\lambda_0|+|p_2|+|p_1\lambda_0+q_1-\lambda_0^2|\tau,\\
e_{\mu_0} &= \max_{-\tau\le t\le 0}\{e^{-\mu_0 t}\},\\
h_{\lambda_0,\mu_0} &= 1+|c|+|p_1\lambda_0+q_1-\lambda_0^2|\mu_0^{-2}(1-e^{-\mu_0\tau}-\mu_0\tau e^{-\mu_0\tau})\\
&\quad +|c\mu_0+2c\lambda_0-p_2|\tau e^{-(\lambda_0+\mu_0)\tau}\\
&\quad +\frac{k_{\lambda_0}}{|\beta(\lambda_0)|}\Big[1+|c|e^{-\lambda_0\tau}+|p_1\lambda_0+q_1-\lambda_0^2|\mu_0^{-2}(\mu_0\tau+e^{-\mu_0\tau}-1)\\
&\quad +|c\mu_0+2c\lambda_0-p_2||\mu_0^{-1}(1-e^{-\mu_0\tau})|e^{-(\lambda_0+\mu_0)\tau}\Big],\\
N(\lambda_0,\mu_0;\phi) &= \max\left\{\max_{-\tau\le t\le 0}|e^{-\lambda_0 t}\phi(t)|,\ \max_{-\tau\le t\le 0}|e^{-(\lambda_0+\mu_0)t}\phi(t)|,\ \max_{-\tau\le t\le 0}|\phi'(t)|,\ \max_{-\tau\le t\le 0}|\phi(t)|\right\}.
\end{aligned}$$

Furthermore, the trivial solution of (2.1) is stable if $\lambda_0\le 0, \lambda_0+\mu_0\le 0$; it is asymptotically stable if $\lambda_0<0, \lambda_0+\mu_0<0$; and it is unstable if $\mu_0>0, \lambda_0+\mu_0>0$.

Proof. From Theorem 2.3.2, it follows that

$$e^{-(\mu_0+\lambda_0)t}|x(t)|\le \frac{|K(\lambda_0,\phi)|}{|\beta(\lambda_0)|}e^{-\mu_0 t}+\frac{|H(\lambda_0,\mu_0;\phi)|}{|\gamma(\lambda_0,\mu_0)|}+|M(\lambda_0,\mu_0;\phi)|\chi_{\lambda_0,\mu_0},$$

where $K(\lambda_0,\phi), H(\lambda_0,\mu_0,\phi), M(\lambda_0,\mu_0;\phi), \beta(\lambda_0), \gamma(\lambda_0,\mu_0), \chi_{\lambda_0,\mu_0}$ are defined as in Theorem 2.3.2 respectively. From the representation of $K(\lambda_0,\phi)$, $H(\lambda_0,\mu_0,\phi)$ and $M(\lambda_0,\mu_0;\phi)$ we have

$$\begin{aligned}
|K(\lambda_0,\phi)| &\le k_{\lambda_0}N(\lambda_0,\mu_0;\phi),\quad |H(\lambda_0,\mu_0,\phi)|\le h_{\lambda_0,\mu_0}N(\lambda_0,\mu_0;\phi),\\
|M(\lambda_0,\mu_0;\phi)| &\le \left(1+\frac{K_{\lambda_0}e_{\mu_0}}{|\beta(\lambda_0)|}+\frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0,\mu_0)|}\right)N(\lambda_0,\mu_0;\phi).
\end{aligned}$$

Hence, it follows that

$$\begin{aligned}
e^{-(\mu_0+\lambda_0)t}|x(t)| &\le \frac{k_{\lambda_0}}{|\beta(\lambda_0)|}e^{-\mu_0 t}N(\lambda_0,\mu_0;\phi)+\frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0,\mu_0)|}N(\lambda_0,\mu_0;\phi)\\
&\quad +\left(1+\frac{K_{\lambda_0}e_{\mu_0}}{|\beta(\lambda_0)|}+\frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0,\mu_0)|}\right)N(\lambda_0,\mu_0;\phi)\chi_{\lambda_0,\mu_0},
\end{aligned}$$

which yields

$$\begin{aligned}
|x(t)| &\le \left[\frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0,\mu_0)|}+\left(1+\frac{K_{\lambda_0}e_{\mu_0}}{|\beta(\lambda_0)|}+\frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0,\mu_0)|}\right)\chi_{\lambda_0,\mu_0}\right]N(\lambda_0,\mu_0;\phi)e^{(\lambda_0+\mu_0)t}\\
&\quad +\frac{k_{\lambda_0}}{|\beta(\lambda_0)|}N(\lambda_0,\mu_0;\phi)e^{\lambda_0 t} \qquad (2.27)
\end{aligned}$$

for $t\ge 0$. Next, we consider three cases to discuss the stability of the trivial solution.

Case 1. Suppose that $\lambda_0\le 0, \lambda_0+\mu_0\le 0$, then $e^{\lambda_0 t}\le 1$, $e^{(\lambda_0+\mu_0)t}\le 1$. Define $\|\phi\|=\max_{-\tau\le t\le 0}|\phi(t)|$, it is not difficult to obtain that $\|\phi\|\le N(\lambda_0,\mu_0;\phi)$. From (2.27), we have

$$|x(t)|\le\left[\frac{k_{\lambda_0}}{|\beta(\lambda_0)|}+\left(1+\frac{K_{\lambda_0}e_{\mu_0}}{|\beta(\lambda_0)|}\right)\chi_{\lambda_0,\mu_0}+(1+\chi_{\lambda_0,\mu_0})\frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0,\mu_0)|}\right]N(\lambda_0,\mu_0;\phi) \qquad (2.28)$$

for every $t \geq 0$. Define

$$\rho := \frac{k_{\lambda_0}}{|\beta(\lambda_0)|} + \left(1 + \frac{K_{\lambda_0} e_{\mu_0}}{|\beta(\lambda_0)|}\right) \chi_{\lambda_0,\mu_0} + (1 + \chi_{\lambda_0,\mu_0}) \frac{h_{\lambda_0,\mu_0}}{|\gamma(\lambda_0,\mu_0)|}.$$

For any $\varepsilon > 0$, we choose $\delta = \varepsilon\rho^{-1}$ such that $N(\lambda_0, \mu_0; \phi) < \delta$, since $\|\phi\| \leq N(\lambda_0, \mu_0; \phi)$, we obtain that $\|\phi\| \leq \delta$. From estimate (2.28), we obtain $|x(t)| \leq \rho N(\lambda_0, \mu_0; \phi) < \rho\delta = \varepsilon$. This implies the trivial solution of (2.1) is stable.

Case 2. Suppose that $\lambda_0 < 0, \lambda_0 + \mu_0 < 0$. From estimate (2.27), it follows that $\lim_{t\to\infty} x(t) = 0$. Hence, the trivial solution of (2.1) is asymptotically stable.

Case 3. Let $\mu_0 > 0, \lambda_0 + \mu_0 > 0$. If the trivial solution of (2.1) is stable, then there exists a number $l = l(1) > 0$ such that, for any $\phi \in C([-\tau, 0], \mathbb{R})$ with $\|\phi\| < l$ the solution x of (2.1) with $x(\theta) = \phi(\theta)$ for $-\tau \leq \theta \leq 0$ satisfies $|x(t)| < 1$ for $t \geq 0$. Define

$$\phi_0(t) = e^{(\lambda_0+\mu_0)t} - e^{\lambda_0 t} \quad \text{for} \quad t \in [-\tau, 0].$$

By definition of $K(\lambda_0, \phi)$and $H(\lambda_0, \mu_0, \phi)$, and using the relation of (2.12) , we have $K(\lambda_0, \phi_0) = -\beta(\lambda_0)$ and $H(\lambda_0, \mu_0, \phi_0) = \gamma(\lambda_0, \mu_0)$. Let $\phi \in C([-\tau, 0], \mathbb{R})$ be defined by $\phi = \frac{l_1}{\|\phi_0\|}\phi_0$ with $0 < l_1 < l$. From Theorem 2.3.3, we have

$$\lim_{t\to\infty} e^{-(\mu_0+\lambda_0)t} x(t) - e^{-\mu_0 t} \frac{K(\lambda_0, \phi)}{\beta(\lambda_0)} = \frac{H(\lambda_0, \mu_0; \phi)}{\gamma(\lambda_0, \mu_0)}. \tag{2.29}$$

On the other hand,

$$\lim_{t\to\infty} e^{-(\mu_0+\lambda_0)t} x(t) - e^{-\mu_0 t} \frac{K(\lambda_0, \phi)}{\beta(\lambda_0)} = \lim_{t\to\infty} e^{-(\mu_0+\lambda_0)t} x(t) + \frac{l_1}{\|\phi_0\|} e^{-\mu_0 t} = 0,$$

but

$$\frac{H(\lambda_0, \mu_0; \phi)}{\gamma(\lambda_0, \mu_0)} = \frac{(l_1/\|\phi_0\|) H(\lambda_0, \mu_0; \phi)}{\gamma(\lambda_0, \mu_0)} = \frac{l_1}{\|\phi_0\|} > 0.$$

This is a contradiction to (2.29) and this shows that the trivial solution of (2.1) is unstable. □

2.4 Discussion of the two approaches

In this section, we discuss the relations of spectral method and ODE method. First, we consider the conditions of Theorem 2.3.3 in more detail. Suppose μ_0 is a real root of second characteristic equation (2.30). If μ_0 satisfies $\chi_{\lambda_0,\mu_0} < 1$, we claim that μ_0 is a simple dominant zero. Let

$$G(\mu) := \mu + (c\mu + 2c\lambda_0 - p_2) e^{-\tau(\lambda_0+\mu)} + 2\lambda_0 - p_1 - (p_1\lambda_0 + q_1 - \lambda_0^2) \int_{-\tau}^{0} e^{\mu s}\, ds. \tag{2.30}$$

By the condition $\chi_{\lambda_0,\mu_0} < 1$ in Theorem 2.3.3, we have $G'(\mu_0) \neq 0$.

Indeed, since $\chi_{\lambda_0,\mu_0} < 1$,

$$|G'(\mu_0)| \geq 1 - \left[|c| e^{-(\lambda_0+\mu_0)\tau} + \tau |p_1 - \mu_0 - 2\lambda_0| + \mu_0^{-2}(\mu_0\tau + e^{-\mu_0\tau} - 1) \left|p_1\lambda_0 + q_1 - \lambda_0^2\right| \right] > 0.$$

Since $\chi_{\lambda_0,\mu_0} < 1$, let $0 < \delta < 1$ such that $\chi_{\lambda_0,\mu_0} < \delta$. From the representation of χ_{λ_0,μ_0}, we can estimate

$$1 - \frac{1}{\delta}|c|e^{-(\lambda_0+\mu_0)\tau} > \frac{1}{\delta}\left[\tau|p_1 - \mu_0 - 2\lambda_0| + \mu_0^{-2}(\mu_0\tau + e^{-\mu_0\tau} - 1)\left|p_1\lambda_0 + q_1 - \lambda_0^2\right|\right].$$

Let $\varepsilon > 0$ such that $1 < e^{\varepsilon r} \leq \frac{1}{\delta}$, and we let Ω denote the right half plane given by

$$\Omega = \{\mu \in \mathbb{C} : \mathrm{Re}\mu > \mu_0 - \varepsilon\}.$$

For $\mu \in \Omega$ and $0 \leq s \leq r$, we have $|e^{-s\mu}| = e^{-sRe\mu} < e^{-s\mu_0}e^{\varepsilon s} \leq \frac{e^{-s\mu_0}}{\delta}$. If $\mu \in \Omega$, let γ denote the line segment between μ_0 and μ, such that the segment is in Ω, then for $0 \leq s \leq r$,

$$|e^{-s\mu} - e^{-s\mu_0}| = \left|\int_{\mu_0}^{\mu} se^{-ts}\,dt\right| = s\left|\int_{\gamma} e^{-ts}\,dt\right| \leq \frac{e^{-s\mu_0}}{\delta}|\mu - \mu_0|s, \tag{2.31}$$

since $G(\mu_0) = 0$, we have

$$\begin{aligned} G(\mu) &= (\mu - \mu_0)(1 + ce^{-\tau(\lambda_0+\mu)}) + c\mu_0 e^{-\tau\lambda_0}(e^{-\tau\mu} - e^{-\tau\mu_0}) \\ &\quad + (2c\lambda_0 - p_2)e^{-\tau\lambda_0}(e^{-\tau\mu} - e^{-\tau\mu_0}) - (p_1\lambda_0 + q_1 - \lambda_0^2)\int_{-\tau}^{0}(e^{-\tau\mu} - e^{-\tau\mu_0})\,ds. \end{aligned}$$

Now, we estimate $|G(\mu)|$ by using (2.31),

$$\begin{aligned} |G(\mu)| &\geq |\mu - \mu_0|\left(1 - \left|\frac{1}{\delta}ce^{-\tau(\lambda_0+\mu_0)}\right|\right) \\ &\quad - \frac{|\mu - \mu_0|}{\delta}\left\{\tau|p_1 - \mu_0 - 2\lambda_0| + \mu_0^{-2}(\mu_0\tau + e^{-\mu_0\tau} - 1)\left|p_1\lambda_0 + q_1 - \lambda_0^2\right|\right\} > 0, \end{aligned}$$

which means μ_0 is the only zero in the right half plane Ω, so μ_0 is a simple dominant zero.

For the case when the space $\mathcal{C} = C([-\tau, 0], \mathbb{R})$. The main result in [101] (using ODE method) and Theorem 2.2.7 implies that spectral approach is equivalent to the ODE appraoch in this case.

For the case when the space $\mathcal{C} = C([-\tau, 0], \mathbb{C}^n)$, as we discussed in Section 2.2, the spectral approach can be applied to study the asymptotic behavior of the functional differential equations with the solutions in this space. However, for this case, the ODE approach is not applicable.

In the following, we present two examples to illustrate the relations of the two approaches.

Example 2.4.1. *We suppose $a = 1, b = 1, c = 1$ and $\sigma = \tau = 1$ in (2.1). We have*

$$\begin{cases} x''(t) + x''(t-1) = x(t) + x(t-1), \\ x(t) = \phi(t), \quad -1 \leq t \leq 0. \end{cases} \tag{2.32}$$

The characteristic equation of (2.32) is $\lambda^2 + \lambda^2 e^{-\lambda} = 1 + e^{-\lambda}$. We denote $F_1(\lambda) = \lambda^2 + \lambda^2 e^{-\lambda} - 1 - e^{-\lambda} = (\lambda^2 - 1)(1 + e^{-\lambda})$, Since $F_1(1) = 0$, $F_1'(1) = 2 + \frac{2}{e} \neq 0$, we have that $\lambda_0 = 1.$ is a simple zero of $F_1(\lambda)$. Hence, (2.32) becomes

$$z'(t) + e^{-1}z'(t-1) + 2z(t) + 2e^{-1}z(t-1) = 0, \tag{2.33}$$

and the characteristic equation of (2.33) is

$$\mu + (\mu+2)e^{-(\mu+1)} + 2 = (\mu+2)(1+e^{-\mu-1}) = 0.$$

We denote $G_1(\mu) = (\mu+2)(1+e^{-\mu-1})$. Since $\mu_0 = -2$ is a real zero of $G_1(\mu)$, the condition of Theorem 2.3.3 is $\chi_{1,-2} = e > 1$, so Theorem 2.3.3 is not applicable.

But $\lambda = -1$ is another root of $F_1(\lambda)$ and satisfies $F_1'(-1) = -2 - 2e \neq 0$, so (2.32) becomes

$$z'(t) + ez'(t-1) - 2z(t) - 2ez(t-1) = 0, \tag{2.34}$$

and the characteristic equation of (2.34) is

$$\mu + (\mu-2)e^{-(\mu-1)} - 2 = (\mu-2)(1+e^{-(\mu-1)}) = 0. \tag{2.35}$$

It is easy to check that $\mu = \mu_0 = 2$ is a real root of (2.35). Corresponding to the roots $\lambda_0 = -1$ and $\mu_0 = 2$, the condition of Theorem 2.3.3 becomes $\chi_{-1,2} = e^{-1} < 1$. Therefore by using the result of Theorem 2.3.3, the asymptotic behavior of initial value problem (2.32) is

$$\lim_{t\to\infty} e^{-t}x(t) = \frac{H(-1,2;\phi)}{\gamma(-1,2)} \quad = \quad \frac{\phi(0)+\phi(-1)+\phi'(0)+\phi'(-1)}{2+2e^{-1}}.$$

Next, we apply Theorem 2.2.6 to study the asymptotic behavior of initial value problem (2.32). The characteristic matrix of (2.32) is

$$\triangle(z) \quad = \quad \begin{pmatrix} z & -1 \\ -e^{-z}-1 & z+ze^{-z} \end{pmatrix}.$$

Since $z = z_0 = 1$ is a dominant zero of $\det\triangle(z)$, and $\frac{d}{dz}(\det\triangle(z))|_{z=z_0} = 2+2e^{-1} \neq 0$, we obtain that $z_0 = 1$ is a simple dominant zero of $\det\triangle(z)$, which satisfies the condition of Theorem 2.2.6. Therefore, we have

$$\lim_{t\to\infty} e^{-t}x(t) = \frac{\phi(0)+\phi(-1)+\phi'(0)+\phi'(-1)}{2+2e^{-1}}.$$

From this example, we see that the result by the spectral approach is the same as the one by the ODE approach.

Example 2.4.2. *We suppose $a = 1$, $\sigma = \tau = 1$, $b = c$ in (2.1), we have*

$$\begin{cases} x''(t) + cx''(t-1) = x(t) + cx(t-1), \\ x(t) = \phi(t), \quad -1 \le t \le 0. \end{cases} \tag{2.36}$$

The characteristic equation of (2.36) is

$$\lambda^2 + c\lambda^2 e^{-\lambda} = 1 + ce^{-\lambda},$$

we denotes $F_2(\lambda) = \lambda^2 + c\lambda^2 e^{-\lambda} - 1 - ce^{-\lambda} = (\lambda^2-1)(1+ce^{-\lambda})$. Since $F_2(-1) = 0$, $F_2'(-1) = -2 - 2ce \neq 0$, So $\lambda_0 = -1$ is a simple zero of $F_2(\lambda)$, (2.36) becomes

$$z'(t) + cez'(t-1) - 2z(t) - 2cez(t-1) = 0. \tag{2.37}$$

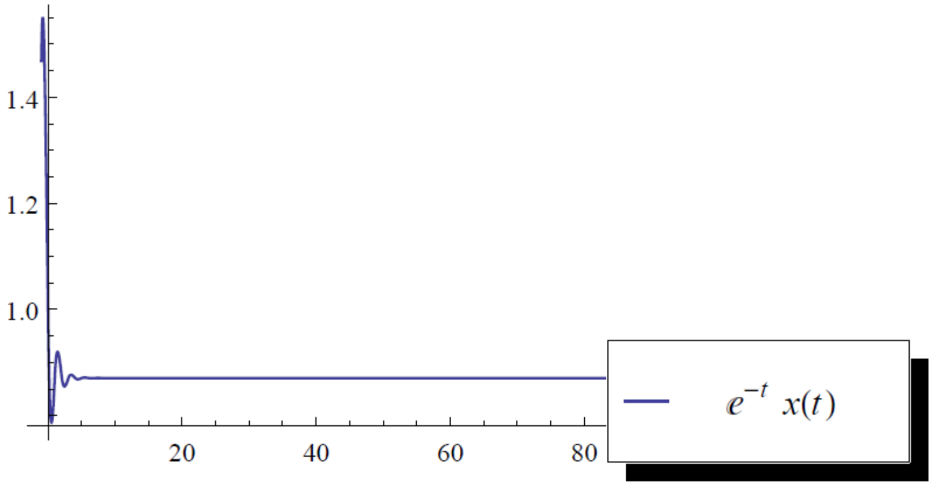

Figure 2.1: Numerical solution of equation (2.32).

The characteristic equation of (2.37) is

$$\mu + (\mu - 2)ce^{-(\mu-1)} - 2 = (\mu - 2)(1 + ce^{-(\mu-1)}) = 0.$$

We denote $G_2(\mu) = (\mu - 2)(1 + ce^{-(\mu-1)})$. Since $\mu_0 = 2$ is a real zero of $G_2(\mu)$, corresponding to the roots $\lambda_0 = -1$ and $\mu_0 = 2$, the condition of Theorem 2.3.3 is $\chi_{-1,2} = |c|e^{-1}$. If $|c| < e$, we have $\chi_{-1,2} < 1$. Therefore by using the result of Theorem 2.3.3, the asymptotic behavior of initial value problem (2.36) is

$$\lim_{t\to\infty} e^{-t}x(t) = \frac{H(-1, 2; \phi)}{\gamma(-1, 2)} = \frac{\phi(0) + \phi'(0) + c(\phi(-1) + \phi'(-1))}{2 + 2ce^{-1}}.$$

Next, we consider (2.36) by applying spectral approach. For (2.36), the characteristic matrix is given by

$$\triangle(z) = \begin{pmatrix} z & -1 \\ -ce^{-z} - 1 & z + cze^{-z} \end{pmatrix}.$$

(1) Case $-e < c$. It is not difficult to check $z_0 = 1$ is a dominant zero of $\det \triangle(z)$, since $\frac{d}{dz} \det \triangle(z)|_{z=z_0} = 2 + 2ce^{-1} \neq 0$, so $z_0 = 1$ is a simple dominant zero of $\det \triangle(z)$. Therefore, by applying the result of Theorem 2.2.6,

$$\lim_{t\to\infty} e^{-t}x(t) = \frac{\phi(0) + \phi'(0) + c(\phi(-1) + \phi'(-1))}{2 + 2ce^{-1}}.$$

(2) Case $c < -e$. After checking the roots of $\det \triangle(z)$, we find $z_0 = \ln(-c)$ is a dominant zero of $\det \triangle(z)$ and we can also use Theorem 2.2.6 to obtain the asymptotic behavior of the equation initial value problem (2.36).

(3) Case $c = -e$. We learned that $z_0 = 1$ is a dominant zero with order 2, so by the spectral approach in [6], we can have the asymptotic behavior of the equation initial value problem (2.36).

From this example, we derive that for the ODE approach, the coefficient c should satisfy $|c| < e$. However, for every $c \in \mathbb{R}$, the asymptotic behavior of the equation initial value problem (2.36) can be obtained by the spectral projection approach, and the result is the same as the one by the ODE approach when c satisfies $|c| < e$.

2.5 A fixed point method towards asymptotic behavior

In this section, we study the special case of the system (2.1) with $c = 0$ and $p_2 = 0$. Since it is not easy to apply the ODE approach or the spectral approach to discuss the asymptotic behavior because of the difficulty in computing the roots of the characteristic eequation, we introduce a third approach, based on a fixed point method, to study the asymptotic behavior of such equations.

This approach is based on fixed point theory and relies on three principles: a complete metric space, the contraction mapping principle, and an elementary variation of parameters formula. Together this yields existence, uniqueness and stability.

By using a fixed point approach, Burton and Furumochi [10] have considered asymptotic stability of the following linear equation

$$x''(t) + ax'(t) + bx(t-r) = 0 \tag{2.38}$$

and obtained the following.

Theorem 2.5.1. (Burton and Furumochi [10]) *Let $a > 0$ and $b > 0$. If*

$$br\left(1 + \int_0^t |Ae^{A(t-s)}|\, ds\right) < 1$$

holds, where $A = \begin{pmatrix} 0 & 1 \\ -b & -a \end{pmatrix}$, then every solution of equation (2.38) and its derivative tend to 0 as $t \to \infty$.

By using a similar technique as Burton and Furumochi [10], we consider the retarded delay differential equation

$$x''(t) + ax'(t) + bx(t-r) + cx(t) = 0. \tag{2.39}$$

Let $x' = y$, (2.39) can be written in the following form

$$y' = -ay - (b+c)x + (d/dt)\int_{t-r}^{t} bx(s)\, ds,$$

which is then expressed as the vector system

$$z' = Az + (d/dt)\int_{t-r}^{t} Bz(s)\, ds,$$

where A and B are

$$A = \begin{pmatrix} 0 & 1 \\ -(b+c) & -a \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ b & 0 \end{pmatrix}. \tag{2.40}$$

By the variation of parameters formula

$$z(t) = e^{At}z_0 + \int_0^t e^{A(t-s)}(d/ds)\int_{s-r}^{s} Bz(u)\,du\,ds,$$

employing an integration by parts, we have

$$z(t) = e^{At}z_0 + \int_{t-r}^{t} Bz(u)\,du - e^{At}\int_{-r}^{0} Bz(u)\,du + A\int_0^t e^{A(t-s)}\int_{s-r}^{s} Bz(u)\,du\,ds.$$

In order to have $e^{At} \to 0$ as $t \to \infty$, we need

$$b + c > 0, \quad a > 0.$$

Let $C([-r,0],\mathbb{R}^2)$ be the space of continuous functions, let $\phi \in \mathcal{C}$ be an initial function and define

$$S_\phi := \left\{\varphi : \varphi \in C([-r,0],\mathbb{R}^2), \varphi(t) = \phi(t) \quad \text{on} \quad [-r,0], \quad \varphi(t) \to 0 \quad \text{as} \quad t \to \infty\right\}.$$

Define a mapping $P : S_\phi \to S_\phi$

$$(P\varphi)(t) = e^{At}\phi(0) + \int_{t-r}^{t} B\varphi(u)\,du - e^{At}\int_{-r}^{0} B\varphi(u)\,du + A\int_0^t e^{A(t-s)}\int_{s-r}^{s} B\varphi(u)\,du\,ds.$$

We choose a suitable norm for a vector or matrix. For

$$z = \begin{pmatrix} x \\ y \end{pmatrix}$$

let $|z|_0 := |x| + |y|$. Let Q be a fixed 2×2 nonsingular matrix such that $|q|_0 \le 1$, where q is the second column of Q, and let $|z| := |Qz|_0$. For a 2×2 matrix M, let

$$|M| := \sup\left\{|QMQ^{-1}z|_0 : |z|_0 = 1\right\}, \tag{2.41}$$

then $|M|$ is the norm of M. We have the following theorem.

Theorem 2.5.2. *Let $b + c > 0$, $b > 0$ and $a > 0$, if the following condition is satisfied*

$$br\left(1 + \int_0^t |Ae^{A(t-s)}|\,ds\right) < 1, \tag{2.42}$$

where A is given by (2.40), then every solution of equation (2.39) and its derivative tend to 0 as $t \to \infty$.

Proof. Since e^{At} is a L^1-function on $\mathbb{R}^+$, if $\varphi \in S_\phi$, then $(P\varphi)(t) \to 0$ as $t \to \infty$. Thus, $P : S_\phi \to S_\phi$. Furthermore, from condition (2.42), we have that P is a contraction with respect to the norm (2.41) on S_ϕ. The proof proceeds similarly to the proof as presented by Burton and Furumochi [10]. □

Example 2.5.3. *Consider the equation*

$$x''(t) + \frac{7}{12}x'(t) + \frac{1}{6}x(t-2) - \frac{1}{12}x(t) = 0, \quad t \in \mathbb{R}^+. \tag{2.43}$$

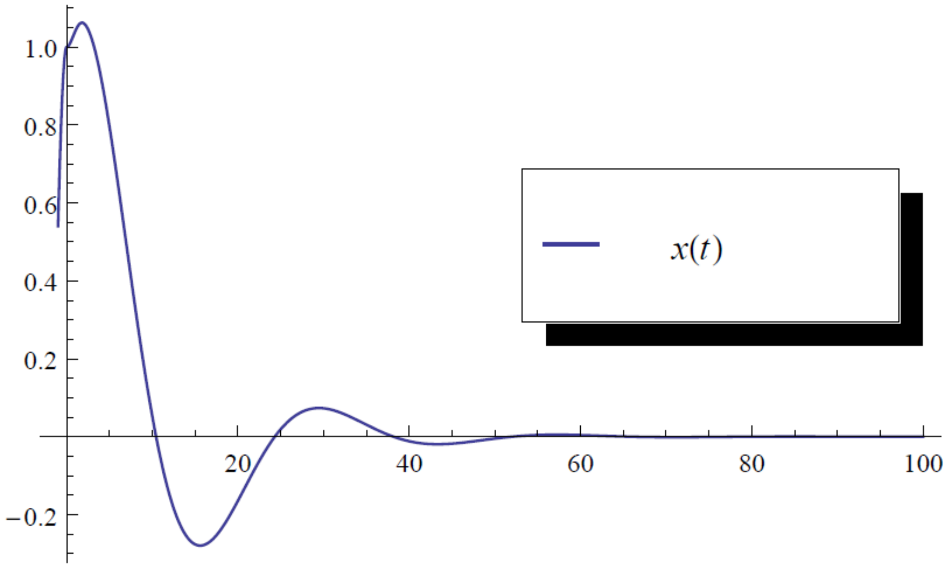

Figure 2.2: Numerical solution of equation (2.43).

We have

$$A = \begin{pmatrix} 0 & 1 \\ -1/12 & -7/12 \end{pmatrix} \quad \textit{and} \quad B = \begin{pmatrix} 0 & 0 \\ 1/6 & 0 \end{pmatrix}.$$

The eigenvalues of A are $-\frac{1}{3}$ and $-\frac{1}{4}$, let Q be a 2×2 nonsingular matrix such that

$$QAQ^{-1} = \begin{pmatrix} -1/3 & 0 \\ 0 & -1/4 \end{pmatrix}.$$

Then we have $Ae^{A(t-s)} = QEQ^{-1}$, where

$$E = \begin{pmatrix} -(1/3)e^{-(t-s)/3} & 0 \\ 0 & -(1/4)e^{-(t-s)/4} \end{pmatrix},$$

and

$$\begin{aligned} |Ae^{A(t-s)}| &= \sup\{|Ez|_0 : |z|_0 = 1\} \\ &= \sup\{(|x|/3)e^{-(t-s)/3} + (|y|/4)e^{-(t-s)/4} : |x| + |y| = 1\} \\ &\le (1/3)e^{-(t-s)/3} + (1/4)e^{-(t-s)/4}. \end{aligned}$$

Hence,

$$\int_0^t |Ae^{A(t-s)}|\, ds \le \int_0^t \left[(1/3)e^{-(t-s)/3} + (1/4)e^{-(t-s)/4}\right] ds = 2 - e^{-t/3} - e^{-t/4} < 2$$

for $t \ge 0$, which together with $br = \frac{1}{3}$ implies that (2.42) holds. Thus, by Theorem 2.5.2, we have that the zero solution of (2.43) tends to 0 as $t \to \infty$.

2.6 An application to a mechanical model of turning processes

Systems governed by (neutral) delay differential equations (DDEs) often come up in different fields of science and engineering. One of the most important mechanical application is the turning processes. For the simplest model of turning, the governing equation of motion is an autonomous DDE with a corresponding infinite dimensional state space. This fact results in an infinite number of characteristic roots, most of them having negative real parts referring to damped components of the vibration signals. There may be some finite number of characteristic roots that have positive real parts.

From the detailed introduction of mechanical models of turning processes in [59], we focus on a linear autonomous delay differential equation

$$m\xi''(t) + c\xi'(t) + k\xi(t) = -wh(\xi(t) - \xi(t-\tau)), \tag{2.44}$$

where m, c, k, w, h, τ are constants. For the meaning of every parameter, refer to [59].

Using the model parameters, equation (2.44) reads

$$\xi''(t) + 2\zeta w_n \xi'(t) + w_n^2 \xi(t) = -\frac{wh}{m}(\xi(t) - \xi(t-\tau)), \tag{2.45}$$

where $w_n = \sqrt{k/m}$, $\zeta = c/(2mw_n)$. Generally, $\zeta \approx 0.005 \sim 0.02$. Equation (2.45) is the standard linear delay differential equation model of the turning process.

Equation (2.45) can be even further simplified. Introduce the dimensionless time $\tilde{t}$ by $\tilde{t} = tw_n$, and by abuse of notation, drop the tilde immediately. This gives the dimensionless equation of motion

$$\xi''(t) + 2\zeta\xi'(t) + \xi(t) = -\tilde{w}(\xi(t) - \xi(t - w_n\tau)), \tag{2.46}$$

where $\tilde{w} = \frac{wh}{mw_n^2}$. In the following, we study the asymptotic stability of equation (2.46) by Theorem 2.5.2.

Equation (2.46) can be written as the following.

$$\xi''(t) + 2\zeta\xi'(t) + (1+\tilde{w})\xi(t) - \tilde{w}\xi(t - w_n\tau) = 0, \tag{2.47}$$

We denote $w_n\tau = r$. From (2.47), we have

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -2\zeta \end{pmatrix} \quad and \quad B = \begin{pmatrix} 0 & 0 \\ -\tilde{w} & 0 \end{pmatrix}.$$

The characteristic equation of A is

$$\lambda^2 + 2\zeta\lambda + 1 = 0.$$

The eigenvalues are

$$\lambda_1 = -\zeta + i\sqrt{1-\zeta^2}, \quad \lambda_2 = -\zeta - i\sqrt{1-\zeta^2},$$

so $|\lambda_1| = |\lambda_2| = 1$, $|\lambda_1 e^{\lambda_1(t-s)}| = |\lambda_1| e^{(Re\lambda_1)(t-s)}$ and $|\lambda_2 e^{\lambda_2(t-s)}| = |\lambda_2| e^{(Re\lambda_2)(t-s)}$.

If $\zeta \notin \{-1, 1\}$, the two different eigenvalues λ_1, λ_2 have eigenvectors V_1 and V_2, which are linearly independent. Suppose that $Q = (V_1, V_2)^{-1}$, then

$$QAQ^{-1} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \triangleq \Lambda.$$

Hence,

$$A = Q^{-1}\Lambda Q = (V_1, V_2)A(V_1, V_2)^{-1}, \quad e^{A(t-s)} = Q^{-1}e^{\Lambda(t-s)}Q.$$

Then we have

$$Ae^{A(t-s)} = Q^{-1}\Lambda QQ^{-1}e^{\Lambda(t-s)}Q = Q^{-1}\Lambda e^{\Lambda(t-s)}Q = Q^{-1}EQ,$$

where

$$E = \begin{pmatrix} \lambda_1 e^{\lambda_1(t-s)} & 0 \\ 0 & \lambda_2 e^{\lambda_2(t-s)} \end{pmatrix}.$$

Using the norm in Theorem 2.5.2, we have

$$\begin{aligned} |Ae^{A(t-s)}| &= \sup\{|Ez|_0 : |z|_0 = 1\} \\ &= \sup\left\{\left|x\lambda_1 e^{\lambda_1(t-s)}\right| + \left|y\lambda_2 e^{\lambda_2(t-s)}\right| : |x| + |y| = 1\right\} \\ &= \sup\left\{|x||\lambda_1|e^{(Re\lambda_1)(t-s)} + |y||\lambda_2|e^{(Re\lambda_2)(t-s)} : |x| + |y| = 1\right\} \\ &\leq 2e^{-\zeta(t-s)}. \end{aligned}$$

Hence,

$$\int_0^t |Ae^{A(t-s)}|\, ds \leq \int_0^t 2e^{-\zeta(t-s)}\, ds \leq 2\zeta^{-1}(1 - e^{-\zeta t}), \quad t \geq 0,$$

$$(-\tilde{w})r\left(1 + \int_0^t |Ae^{A(t-s)}|\, ds\right) \leq (-\tilde{w})r(1 + 2\zeta^{-1}(1 - e^{-\zeta t})) \leq (-\tilde{w})r(1 + 2\zeta^{-1}).$$

If $(-\tilde{w})r/\zeta < 1/3$, the conditions of Theorem 2.5.2 are satisfied, that is to say, if $\tilde{w} < 0$ very large or r very small, then every solution of (2.47) and its derivative tends to 0 as $t \to \infty$.

2.7 Notes and remarks

For more results on asymptotic behavior of autonomous delay differential equations, see the overview books by Hale and Verduyn Lunel [51], Diekmann, van Gils, Verduyn Lunel and Walther [29], Driver [35], papers by Driver [36], Philos and Purnaras [101, 102, 105, 106], Dix, Philos and Purnaras [33], Frasson [41, 40, 42], Frasson and Verduyn Lunel [39].

In this chapter, we used three methods to study asymptotic behavior of the solutions of functional differential equations, that is, ordinary differential equation (ODE) method, spectral method and fixed point method. The basic idea for the ODE method in Section 2.3 essentially originated in a very interesting asymptotic result due to Driver [37] concerning the solutions of linear differential systems with small delays. Motivated by the treatment in [37], Philos, Purnaras and many other authors [84, 101, 102, 103, 104, 105, 106, 137] have obtained some interesting results on the asymptotic behavior of the solutions to autonomous differential and difference equations with delays. For example, equations with neutral terms, equations with variable delays, periodic differential and difference equations with delays. Continuing the study for asymptotic behavior of a wide class of functional differential equations, Frasson and Verduyn Lunel [39] explored a new approach, the so-called spectral approach. Frasson [40, 41, 42] established some interesting results based on the results in [39]. Towards asymptotic behavior, a fixed point method (see Burton [13]) is introduced. This method is one of the main methods in this thesis. For more detailed information about the fixed point method, refer to Chapter 4, Chapter 5 and Chapter 6.

A paper based on the contents of this chapter has been submitted for publication ([17]).

Chapter 3

Asymptotic behavior of a class of nonautonomous neutral delay differential equations

In this chapter, asymptotic behavior of a class of nonautonomous neutral delay differential equations is studied. It should be emphasized that asymptotic behavior of nonautonomous equations is much more difficult than the case of autonomous equations. For instance, Frasson and Verduyn Lunel [39] studied the following linear periodic delay equation

$$x'(t) = a(t)x(t) + \sum_{j=1}^{k} b_j(t)x(t-\tau_j), \tag{3.1}$$

where $a(t+\omega) = a(t)$, $b_j(t+\omega) = b_j(t)$, $j = 1, 2, \cdots, k$, they considered a particular case where $\tau_j = j\omega$ (i.e. the delays are integer multiples of the period ω). However, it is very difficult to study general nonautonomous problems.

For a special class of nonautonomous problems, we can use an approach similar to the ODE method as we discussed in Chapter 2, which is based on the application of an appropriate solution of the generalized characteristic equation. For nonautonomous equations, solving the generalized characteristic equation becomes much harder: functional equation instead of algebraic equation. Our result can be applied in case the assumptions are satisfied, i.e., the generalized characteristic equation has a real solution.

3.1 Introduction and main result

For $r \geq 0$, let $\mathcal{C} = C([-r, 0], \mathbb{C})$ be the space of continuous functions taking $[-r, 0]$ into $\mathbb{C}$ with $\|\varphi\|$, $\varphi \in \mathcal{C}$, defined by $\|\varphi\| = \max_{-r\leq\theta\leq 0} |\varphi(\theta)|$. A delay differential equation of neutral type, or shortly, a neutral equation is a system of the form

$$\frac{d}{dt} Mx_t = L(t)x_t \quad t \geqslant t_0 \in \mathbb{R}, \tag{3.2}$$

where $x_t \in \mathcal{C}$ is defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$, $M : \mathcal{C} \to \mathbb{C}$ is continuous, linear and atomic at zero, (see [51] on page 255 for the concept of atomic at zero),

$$M\varphi = \varphi(0) - \int_{-r}^{0} \varphi(\theta)\, d\mu(\theta), \tag{3.3}$$

where $\mathrm{Var}_{[s,0]}\mu \to 0$, as $s \to 0$.

For (3.2), $L(t)$ denotes a family of bounded linear functionals on $\mathcal{C}$, and by the Riesz representation theorem, for each $t \geqslant t_0$, there exists a complex valued function of bounded variation $\eta(t,\cdot)$ on $[-r,0]$, normalized so that $\eta(t,0)=0$ and $\eta(t,\cdot)$ is continuous from the left in $(-r,0)$ such that

$$L(t)\varphi = \int_{-r}^{0} \varphi(\theta)\, d_\theta \eta(t,\theta). \tag{3.4}$$

For any $\varphi \in \mathcal{C}$, $\sigma \in [t_0,\infty)$, a function $x = x(\sigma,\varphi)$ defined on $[\sigma - r, \sigma + A)$ is said to be a solution of (3.2) on $(\sigma,\sigma+A)$ with initial φ at σ if x is continuous on $[\sigma - r, \sigma + A)$, $x_\sigma = \varphi$, Mx_t is continuously differentiable on $(\sigma, \sigma + A)$ and relation (3.2) is satisfied on $(\sigma, \sigma + A)$. For more information on this type of equations, see [51].

The initial-value problem (IVP) is

$$\begin{cases} \frac{d}{dt} M x_t = L(t)x_t \quad t \geq \sigma, \\ \\ x_\sigma = \varphi. \end{cases} \tag{3.5}$$

For $\mu = 0$ in (3.3), $M\varphi = \varphi(0)$ and equation (3.2) becomes a retarded functional differential equation,

$$x'(t) = L(t)x_t. \tag{3.6}$$

Consider the *generalized characteristic equation* of (3.6)

$$\lambda(t) = \int_0^r \exp\left(-\int_{t-\theta}^{t} \lambda(s)ds\right) d_\theta \eta(t,\theta) \tag{3.7}$$

which is obtained by looking for solutions to (3.6) of the form

$$x(t) = \exp\left(\int_0^t \lambda(s)\, ds\right). \tag{3.8}$$

By a solution of the generalized characteristic equation (3.7), we mean a continuous real-valued function $\lambda(\cdot)$ defined on $[t_0 - r, \infty)$ which satisfies (3.7).

Cuevas and Frasson [26] studied the asymptotic behavior of solutions of (3.6) with initial condition $x_\sigma = \varphi$, and obtained the following result.

Theorem 3.1.1. *Assume that $\lambda(t)$ is a real solution of (3.7) such that*

$$\limsup_{t\to\infty} \int_0^r \theta |e^{-\int_{t-\theta}^{t} \lambda(s)ds}| d_\theta |\eta|(t,\theta) < 1.$$

Then for each solution x of (3.6), we have that the limit

$$\lim_{t\to\infty} x(t) e^{-\int_{t_0}^{t} \lambda(s)ds}$$

exists, and

$$\lim_{t\to\infty} \left[x(t)e^{-\int_{t_0}^{t} \lambda(s)ds}\right]' = 0.$$

Furthermore,

$$\lim_{t\to\infty} x'(t)e^{-\int_{t_0}^{t} \lambda(s)ds} = \lim_{t\to\infty} \lambda(t)x(t)e^{-\int_{t_0}^{t} \lambda(s)ds},$$

if $\lim_{t\to\infty} \lambda(t)x(t)e^{-\int_{t_0}^{t} \lambda(s)ds}$ exists.

Motivated by the work of [26], we provide a generalization of [26], as it can be applied for instance for neutral delay differential equations with distributed delays or discrete delays, as far as the delays we considered are uniformly bounded. The method for the proof of the main result is similar to [26, 33].

For equation (3.2), the generalized characteristic equation is

$$\lambda(t) = \int_{-r}^{0} d\mu(\theta)\lambda(t+\theta)\exp\left(-\int_{t+\theta}^{t}\lambda(s)ds\right) + \int_{-r}^{0} d_\theta\eta(t,\theta)\exp\left(-\int_{t+\theta}^{t}\lambda(s)ds\right), \tag{3.9}$$

which is obtained by looking for solutions of (3.2) of the form (3.8) and the solutions of (3.9) are continuous functions defined in $[\sigma - r, \infty)$ satisfying (3.9). For autonomous neutral delay differential equations, the constant solutions of (3.9) are the roots of the so called characteristic equation. The following is our main result.

Theorem 3.1.2. *Assume that a real-valued function $\lambda(t)$ is a solution of (3.9) such that*

$$\limsup_{t\to\infty} \chi_{\lambda,t} < 1, \tag{3.10}$$

where

$$\chi_{\lambda,t} = \int_{-r}^{0} e^{-\int_{t+\theta}^{t}\lambda(s)\,ds}\, d|\mu|(\theta) + \int_{-r}^{0}(-\theta)e^{-\int_{t+\theta}^{t}\lambda(s)\,ds}\left(|\lambda(t+\theta)|\, d|\mu|(\theta) + d_\theta|\eta|(t,\theta)\right).$$

Then for each solution x of (3.5), we have that the limit

$$\lim_{t\to\infty} x(t)e^{-\int_{t_0}^{t}\lambda(s)\,ds} \tag{3.11}$$

exists, and

$$\lim_{t\to\infty}\left[x(t)e^{-\int_{t_0}^{t}\lambda(s)\,ds}\right]' = 0. \tag{3.12}$$

Furthermore,

$$\lim_{t\to\infty} x'(t)e^{-\int_{t_0}^{t}\lambda(s)\,ds} = \lim_{t\to\infty}\lambda(t)x(t)e^{-\int_{t_0}^{t}\lambda(s)\,ds} \tag{3.13}$$

if the limit at the right-hand side exists.

Remark 3.1.3. *The conditions in Theorem 3.1.2 are very strong and therefore the theorem is far from providing a general theory. However, it can be applied to deal with certain examples, see Section 3.3.*

3.2 Proof of Theorem 3.1.2

In this section, we prove Theorem 3.1.2. We start with some preparations.

From (3.10), we obtain that there exists $t_1 \geq t_0$, such that $\sup_{t\geq t_1}\chi_{\lambda,t} < 1$. Without loss of generality, we assume $t_1 = 0$ and define

$$\Gamma_\lambda := \sup_{t\geq 0}\chi_{\lambda,t} < 1.$$

For solutions x of (3.5), we set

$$y(t) = x(t)e^{-\int_0^t \lambda(s)\,ds}, \quad t \geqslant -r.$$

Then (3.5) becomes

$$\begin{aligned} y'(t) + \lambda(t)y(t) - \int_{-r}^{0} d\mu(\theta)y'(t+\theta)e^{-\int_{t+\theta}^{t} \lambda(s)\,ds} \\ = \int_{-r}^{0} y(t+\theta)e^{-\int_{t+\theta}^{t} \lambda(s)\,ds} \left(\lambda(t+\theta)\,d\mu(\theta) + \,d_\theta\eta(t,\theta)\right) \end{aligned} \tag{3.14}$$

and the initial condition is equivalent to

$$y(t) = \varphi(t)e^{-\int_0^t \lambda(s)\,ds}, \quad -r \leq t \leq 0. \tag{3.15}$$

Combining (3.15) with (3.9), for $t \geq -r$, we have

$$\begin{aligned} y'(t) &= \int_{-r}^{0} d\mu(\theta)y'(t+\theta)e^{-\int_{t+\theta}^{t} \lambda(s)\,ds} \\ &\quad - \int_{-r}^{0} e^{-\int_{t+\theta}^{t} \lambda(s)\,ds} \int_{-r}^{0} y'(s)\,ds\left(\lambda(t+\theta)\,d\mu(\theta) + \,d_\theta\eta(t,\theta)\right). \end{aligned} \tag{3.16}$$

From the definition of the solutions to (3.5), we know that $y'(t)$ is continuous, Let

$$M_{\varphi,\lambda_1} = \max\left\{\left|\varphi'(t)e^{-\int_0^t \lambda(s)\,ds} - \lambda(t)\varphi(t)e^{-\int_0^t \lambda(s)\,ds}\right| : -r \leq t \leq 0\right\}.$$

We shall show that M_{φ,λ_1} is also a bound of y' on the whole interval $[-r,\infty)$; i.e.,

$$|y'(t)| \leq M_{\varphi,\lambda_1}, \quad t \geq -r. \tag{3.17}$$

For this purpose, take $\varepsilon > 0$, then

$$|y'(t)| < M_{\varphi,\lambda_1} + \varepsilon \quad \text{for} \quad t \geq -r. \tag{3.18}$$

In fact, we suppose that there exists a point $t^* > 0$ such that

$$\begin{aligned} |y'(t)| &< M_{\varphi,\lambda_1} + \varepsilon \quad \text{for} \quad -r \leq t < t^*, \\ |y'(t^*)| &= M_{\varphi,\lambda_1} + \varepsilon. \end{aligned} \tag{3.19}$$

Then combining (3.16) and (3.19), we obtain

$$\begin{aligned} y'(t^*) &= M_{\varphi,\lambda_1} + \varepsilon \\ &\leq \left|\int_{-r}^{0} y'(t^*+\theta)e^{-\int_{t^*+\theta}^{t^*} \lambda(s)\,ds}\,d\mu(\theta)\right| \\ &\quad + \left|\int_{-r}^{0} e^{-\int_{t^*+\theta}^{t^*} \lambda(s)\,ds} \int_{-r}^{0} y'(s)\,ds\left(\lambda(t^*+\theta)\,d\mu(\theta) + \,d_\theta\eta(t^*,\theta)\right)\right| \\ &\leq (M_{\varphi,\lambda_1} + \varepsilon)\left\{\int_{-r}^{0} |e^{-\int_{t^*+\theta}^{t^*} \lambda(s)\,ds}|\,d|\mu|(\theta)\right. \\ &\quad \left. + \int_{-r}^{0} (-\theta)|e^{-\int_{t^*+\theta}^{t^*} \lambda(s)\,ds}|\left(|\lambda(t^*+\theta)|\,d|\mu|(\theta) + d_\theta|\eta|(t^*,\theta)\right)\right\} \\ &= (M_{\varphi,\lambda_1} + \varepsilon)\Gamma_\lambda < M_{\varphi,\lambda_1} + \varepsilon, \end{aligned} \tag{3.20}$$

which is a contradiction, so (3.18) holds. Since (3.18) holds for every $\varepsilon > 0$, it follows that $|y'(t)| \leq M_{\varphi,\lambda_0}$ for all $t \geq -r$.

We are now ready to prove Theorem 3.1.2.

Proof. By using (3.16) and (3.17), for $t \geq 0$, we have

$$\begin{aligned}
|y'(t)| &\leq \left|\int_{-r}^{0} y'(t+\theta)e^{-\int_{t+\theta}^{t}\lambda(s)ds}\,d\mu(\theta)\right| \\
&\quad + \left|\int_{-r}^{0} e^{-\int_{t+\theta}^{t}\lambda(s)\,ds}\int_{-r}^{0} y'(s)\,ds\,(\lambda(t+\theta)\,d\mu(\theta) + d_\theta\eta(t,\theta))\right| \\
&\leq M_{\varphi,\lambda_1}\left\{\int_{-r}^{0} |e^{-\int_{t+\theta}^{t}\lambda(s)\,ds}|\,d|\mu|(\theta)\right. \\
&\quad \left. + \int_{-r}^{0}(-\theta)|e^{-\int_{t+\theta}^{t}\lambda(s)\,ds}|\,(|\lambda(t+\theta)|\,d|\mu|(\theta) + d_\theta|\eta|(t,\theta))\right\} \\
&= M_{\varphi,\lambda_1}\Gamma_\lambda
\end{aligned} \tag{3.21}$$

which means $|y'(t)| \leq M_{\varphi,\lambda_1}\Gamma_{\lambda_1}$ for $t \geq 0$.

One can show by induction, that $y'(t)$ satisfies

$$|y'(t)| \leq M_{\varphi,\lambda_1}(\Gamma_\lambda)^n \quad \text{for } t \geq nr - r, \quad (n = 0, 1, 2, 3, \dots). \tag{3.22}$$

Since $0 \leq \chi_{\lambda,t} < 1$, it follows that $y'(t)$ tends to zero as $t \to \infty$. So we proved (3.12) and hence (3.13) holds. In the following, we will show (3.11) holds.

To prove that $\lim_{t\to\infty} y(t)$ exists, we consider (3.22). For an arbitrary $t \geq 0$, we set $n = [t/r]+1$ (the greatest integer less than or equal to $t/r+1$), then from $n = [t/r]+1 \leq t/r+1 \leq [t/r]+2 = n+1$, we have $t/r \leq n$. From (3.22),

$$|y'(t)| \leq M_{\varphi,\lambda_1}(\Gamma_\lambda)^n \leq M_{\varphi,\lambda_1}(\Gamma_\lambda)^{t/r} \quad \text{for } t \geq nr - r. \tag{3.23}$$

Now we use the Cauchy convergence criterion. For $t > T \geq 0$, from (3.23), we have

$$\begin{aligned}
|y(t) - y(T)| \leq \int_T^t |y'(s)|\,ds \leq \int_T^t M_{\varphi,\lambda_1}(\Gamma_\lambda)^{s/r}\,ds &= M_{\varphi,\lambda_1}\frac{r}{\ln\Gamma_\lambda}\left[(\Gamma_\lambda)^{s/r}\right]_{s=T}^{s=t} \qquad (3.24)\\
&= M_{\varphi,\lambda_1}\frac{r}{\ln\Gamma_\lambda}\left[(\Gamma_\lambda)^{t/r} - (\Gamma_\lambda)^{T/r}\right].
\end{aligned}$$

Let $T \to \infty$, we have $t \to \infty$, and by (3.25), we have

$$M_{\varphi,\lambda}\frac{r}{\ln\Gamma_\lambda}\left[(\Gamma_\lambda)^{t/r} - (\Gamma_\lambda)^{T/r}\right] \to 0;$$

and $\lim_{T\to\infty}|y(t) - y(T)| = 0$. The Cauchy convergence criterion implies the existence of $\lim_{t\to\infty} y(t)$. ☐

Remark 3.2.1. *Under the conditions of Theorem 3.1.2, a solution of (3.5) can not grow faster than exponential; i.e., there exists a constant $M > 0$, such that*

$$|x(t)| \leq Me^{\int_0^t \lambda(s)\,ds} \quad \textit{for } t \geq 0. \tag{3.25}$$

From (3.25), it is not difficult to show that:

(i) *Every solution of* (3.5) *is bounded if and only if* $\limsup_{t\to\infty} \int_0^t \lambda(s)\, ds < \infty$;

(ii) *Every solution of* (3.5) *tends to zero if and only if* $\limsup_{t\to\infty} \int_0^t \lambda(s)\, ds \to -\infty$.

Remark 3.2.2. *If the generalized characteristic equation* (3.9) *has a constant solution* $\lambda(t) = \lambda_0$, *then from Theorem* 3.1.2, $\lim_{t\to\infty} x(t)e^{-\lambda_0 t}$ *exists.*

3.3 Examples

Example 3.3.1. *Consider the linear differential equation with distributed delay*

$$x'(t) - \frac{1}{2}x'(t-1) = \int_{-1}^{0} \frac{x(t+\theta)}{2(t+\theta)}\, d\theta, \quad t > 1. \tag{3.26}$$

This equation can be written in the form (3.2) by setting $\mu(\theta) = -\frac{1}{2}$ for $\theta \le -1$, $\mu(\theta) = 0$ for $\theta > -1$, $\eta(t,\theta) = \ln t + \frac{1}{2}\ln(t+\theta)$ for $t > 1$ and $\theta \in [-1, 0]$. Since both $\theta \mapsto \eta(t,\theta)$ and $\theta \mapsto \mu(\theta)$ are increasing functions, $|\mu| = \mu, |\eta| = \eta$.

The generalized characteristic equation associated with (3.26) is

$$\lambda(t) = \frac{\lambda(t-1)}{2} \exp\left(-\int_{t-1}^{t} \lambda(s)\, ds\right) + \int_{-1}^{0} \frac{1}{2(t+\theta)} \exp\left(-\int_{t+\theta}^{t} \lambda(s)\, ds\right) d\theta,$$

which has a solution

$$\lambda(t) = 1/t. \tag{3.27}$$

For this $\lambda(t)$ and for $t > 1$, using the expression of $\chi_{\lambda,t}$, we obtain that

$$\begin{aligned} \chi_{\lambda,t} &= \frac{1}{2}\left(1 - \frac{1}{2t}\right) + \frac{1}{4t} + \int_{-1}^{0} \frac{-\theta}{2(t+\theta)} \exp\left[-\int_{t+\theta}^{t} \frac{ds}{s}\right] d\theta \\ &= \frac{1}{2} + \frac{1}{4(t)} \to \frac{1}{2} < 1 \quad \text{as } t \to \infty. \end{aligned}$$

Hence the hypothesis (3.10) of Theorem 3.1.2 is fulfilled. So we obtain that for each solution of (3.3.1)

$$\lim_{t\to\infty} \frac{x(t)}{t} \text{ exists}, \quad \lim_{t\to\infty} \left[\frac{x(t)}{t}\right]' = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{x'(t)}{t} = 0. \tag{3.28}$$

Example 3.3.2. *Consider the linear differential equation with distributed delay*

$$x'(t) - \frac{1}{p}x'(t-1) = \int_{-1}^{0} \frac{x(t+\theta)}{q(t+\varepsilon+\theta)}\, d\theta, \quad t > 1, \tag{3.29}$$

ε is any constant, p and q are positive constants such that $1/p + 1/q = 1$. This equation can be written in the form (3.2) by setting $\mu(\theta) = -\frac{1}{p}$ for $\theta \le -1$, $\mu(\theta) = 0$ for $\theta > -1$, $\eta(t,\theta) = \ln t + \frac{1}{q}\ln(t+\varepsilon+\theta)$ for $t > 1$ and $\theta \in [-1, 0]$. Since both $\theta \mapsto \eta(t,\theta)$ and $\theta \mapsto \mu(\theta)$ are increasing functions, $|\mu| = \mu, |\eta| = \eta$.

The generalized characteristic equation associated with (3.29) is

$$\lambda(t) = \frac{\lambda(t-1)}{p} \exp\left(-\int_{t-1}^{t} \lambda(s)\, ds\right) + \int_{-1}^{0} \frac{1}{q(t+\varepsilon+\theta)} \exp\left(-\int_{t+\theta}^{t} \lambda(s)\, ds\right) d\theta,$$

which has a solution

$$\lambda(t) = \frac{1}{t+\varepsilon}.$$

For this $\lambda(t)$ and for $t > 1$, using the expression of $\chi_{\lambda,t}$, we obtain that

$$\begin{aligned} \chi_{\lambda,t} &= \frac{1}{p}\left(1 - \frac{1}{2(t+\varepsilon)}\right) + \frac{1}{2p(t+\varepsilon)} + \int_{-1}^{0} \frac{-\theta}{2q(t+\varepsilon+\theta)} \exp\left[-\int_{t+\theta}^{t} \frac{ds}{s+\varepsilon}\right] d\theta \\ &= \frac{1}{p} + \frac{1}{2q(t+\varepsilon)} \to \frac{1}{p} < 1 \quad \text{as } t \to \infty. \end{aligned}$$

Hence the hypothesis (3.10) of Theorem 3.1.2 is fulfilled. So we obtain that for each solution of (3.3.1)

$$\lim_{t\to\infty} \frac{x(t)}{t} \text{ exists}, \quad \lim_{t\to\infty} \left[\frac{x(t)}{t}\right]' = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{x'(t)}{t} = 0. \tag{3.30}$$

Remark 3.3.3. *Note that if the generalized characteristic equation (3.9) has a solution is difficult to verify. Example 3.3.2 is an extension of Example 3.3.1, we added an ε, and the coefficients $1/2$ and $1/2$ changed to be $1/p$ and $1/q$, which has to be satisfied $1/p + 1/q = 1$.*

Example 3.3.4. *Consider the equation with variable delay*

$$x'(t) - \frac{2}{3}x'(t-1) = \frac{x(t-\tau(t))}{3(t+c-\tau(t))}, \quad t \geqslant t_0. \tag{3.31}$$

where $c \in \mathbb{R}$ and $\tau : [0,\infty) \to [-1,0]$ is a continuous function such that $t + c - \tau(t) > 0$ for $t \geqslant t_0$.

Equation (3.31) can be written in the form (3.2) by letting $\mu(\theta) = -\frac{2}{3}$ for $\theta \leq -1$, $\mu(\theta) = 0$ for $\theta > -1$, $\eta(t,\theta) = 0$ for $\theta < \tau(t)$, $\eta(t,\theta) = 1/3(t+c-\tau(t))$ for $\theta \geqslant \tau(t)$. Since both $\theta \mapsto \eta(t,\theta)$ and $\theta \mapsto \mu(\theta)$ are increasing functions, we have that $|\mu| = \mu$, $|\eta| = \eta$.

The generalzied characteristic equation associated with (3.31) is

$$\lambda(t) = \frac{2\lambda(t-1)}{3} \exp\left(-\int_{t-1}^{t} \lambda(s)ds\right) + \frac{1}{3(t+c-\tau(t))} \exp\left(-\int_{t-\tau(t)}^{t} \lambda(s)ds\right) \tag{3.32}$$

and we have that a solution of (3.32) is

$$\lambda(t) = \frac{1}{t+c}. \tag{3.33}$$

For (3.33), the left hand side of (3.10) reads

$$\begin{aligned} &\limsup_{t\to\infty} \left[\frac{2}{3}\left(1 - \frac{1}{t+c}\right) + \frac{1}{6(t+c)} + \int_{-1}^{0} (-\theta)|e^{-\int_{t-\theta}^{t} \lambda(s)ds}|d_\theta|\eta|(t,\theta)\right] \\ &= \limsup_{t\to\infty} \left[\frac{2}{3} - \frac{\tau(t)}{3(t+c)}\right] = \frac{2}{3} < 1. \end{aligned}$$

and hence hypothesis (3.10) of Theorem 3.1.2 is fulfilled and therefore, for each solution $x(t)$ of (3.31), we have that

$$\lim_{t\to\infty} \frac{x(t)}{t+c} \quad \text{exists}, \quad \text{and} \quad \lim_{t\to\infty} \left(\frac{x(t)}{t+c}\right)' = 0.$$

Manipulating further the limits in (3.31), we are able to establish that $x(t) = O(t)$ and $x'(t) = o(t)$ as $t \to \infty$.

3.4 Notes and remarks

A paper based on the contents of this chapter has been published in [15].

Dix et al. [32] studied the asymptotic behavior of solutions to a class of nonautonomous d-ifferential equation with discrete delays of the form

$$x'(t) = a(t)x(t) + \sum_{j=1}^{k} b_j(t)x(t-\tau_j), \quad t \geq 0$$

where the coefficients $a(t)$ and $b_j(t)$ are continuous real-valued functions on $[0, \infty)$, $\tau_j > 0$ for $j = 1, 2, \cdots, k$, by introducing the concept of the generalized characteristic equation and using an appropriate solution of this generalized characteristic equation. Existence of such a solution, however, is quite a restrictive condition. The basic idea in [32] is essentially originated in the work in Driver [37]. The extended results for asymptotic behavior of neutral delay differential equations can be found in Dix et al [33]. An asymptotic property of the solutions to second order linear nonautonomous delay differential equations is discussed in [107]. Cuevas and Frasson [26] provide a generalization of [32], as it can be applied for instance for retarded delay differential equations with distributed delays or discrete variable delays, as far as the delays are uniformly bounded. Our results in this chapter was motivated by the work in Cuevas and Frasson [26], we generalized the class of delay differential equations studied in Cuevas and Frasson [26] by adding a neutral term, the coefficient for the neutral term is restricted to be constant.

Chapter 4

A fixed point approach to stability of delay differential equations

In this chapter, we focus on stability of neutral delay differential equations that can have time dependent delays, mixed point delays and distributed delays, nonlinearities and impulsive effects. The approach we used in this chapter is based on a fixed point method. In Section 4.1, we consider four classes of equations of neutral type. In Section 4.2, we investigate the fixed point method for a class of equation that contains x' in a nonlinearity. In Section 4.3, we show that the fixed point method can be applied in a similar fashion to difference equations.

4.1 Stability results for nonlinear neutral delay differential equations

4.1.1 Introduction and main results

Liapunov's direct method provides simple geometric theorems for deciding the stability or instability of an equilibrium point of a differential equation. However, in the context of delay differential equations, Liapunov's direct method is not always as effective, in particular if the delay is unbounded or if the differential equation has unbounded terms. Therefore, it was recently proposed by Burton [13] and co-workers to use a fixed point method as an alternative. While Liapunov's direct method usually requires pointwise conditions, fixed point methods need conditions of an averaging nature, and, therefore, can handle various delays or unbounded terms more easily.

A typical stability result based on fixed point theory arguments follows a number of standard arguments adapted to the special structure of the equation under consideration. This leads to many different results in the literature for different classes of equations, for example, with time dependent delays, distributed delays, neutral terms, and certain nonlinearities, see [5, 6, 9, 11, 12, 13, 31, 34, 63, 64, 65, 110, 111, 112, 117, 118, 144, 145]. The aim of this section is to study the approach using fixed point theory in a systematic way and to unify recent results in the literature by considering four general classes of equations. For each of these classes of equations, we combine different techniques to prove new stability theorems. In addition, we present a number of examples to illustrate our results.

The first class consists of scalar neutral integro-differential equations of the form

$$x'(t) - c(t)x'(t - r_1(t)) = -a(t)x(t - r_2(t)) + \int_{t-r_3(t)}^{t} g(t, x(s))\, d\mu(t, s), \quad t \geq 0 \tag{4.1}$$

where the delays $r_j(t) : [0, \infty) \to [0, \infty)$ are continuous functions, the coefficients $a, c : [0, \infty) \to \mathbb{R}$ are continuous, where

$$r_0 = \min\left\{\inf_{t\geq 0}\{t - r_1(t)\}, \inf_{t\geq 0}\{t - r_2(t)\}, \inf_{t\geq 0}\{t - r_3(t)\}\right\}.$$

The kernal $\mu(t, s)$ is of bounded variation for each t and $g : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and for each t, $xg(t, x) > 0$ if $x \neq 0$ is sufficient small. We assume that g satisfies:

(G) $g(t, 0) = 0$, there exists an $l > 0$ such that g satisfies a Lipschitz condition with respect to x on $[0, \infty) \times [-l, l]$, that is, there exists a constant $L = 1$, such that

$$|g(t, x) - g(t, y)| \leq L|x - y| \quad \text{for} \quad t \geq 0 \quad \text{and} \quad x, y \in [-l, l].$$

A standard fixed point argument shows that the differential equation (4.1) provided with an initial condition

$$x(t) = \phi(t), \quad t \in [r_0, 0]. \tag{4.2}$$

where $\phi(s) \in C([r_0, 0], \mathbb{R})$ defines a well-posed initial-value problem and we denote by $x(t) := x(t, \phi)$ the solution of (4.1) with initial condition (4.2).

Definition 4.1.1. *The zero solution of* (4.1) *is said to be stable if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that for every initial function $\phi : [r_0, 0] \to (-\delta, \delta)$, we have that the corresponding solution satisfies $|x(t)| < \varepsilon$ for $t \geq 0$.*

Definition 4.1.2. *The zero solution of* (4.1) *is said to be asymptotically stable if it is stable and there exists a $\delta > 0$ such that for every initial function $\phi : [r_0, 0] \to (-\delta, \delta)$, the corresponding solution $x(t)$ tends to zero as $t \to \infty$.*

In our first result we obtain sufficient and necessary conditions for the asymptotic stability of (4.1) by introducing two auxiliary continuous functions $h_1(t)$ and $h_2(t)$ which will be used to define an appropriate map defined on a complete metric space so that we can apply a fixed point argument.

Theorem 4.1.3. *Consider the neutral integro-differential equation* (4.1) *and suppose that the following conditions are satisfied*

(i) *the delay $r_2(t)$ is differentiable, the delay $r_1(t)$ is twice differentiable with $r_1'(t) \neq 1$, and $t - r_j(t) \to \infty$ as $t \to \infty$, $j = 1, 2, 3$;*

(ii) *there exists a constant $\alpha \in (0, 1)$ and continuous functions $h_j : [r_0, \infty) \to \mathbb{R}$ (j=1,2) such that*

$$\begin{aligned}
&\left|\frac{c(t)}{1 - r_1'(t)}\right| + \sum_{j=1}^{2}\int_{t-r_j(t)}^{t}|h_j(s)|\,ds + \int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}\left(V_{[s-r_3(s),s]}(\mu(s,\cdot))\right)\,ds \\
&+ \int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}|h_2(s - r_2(s))(1 - r_2'(s)) - a(s)|\,ds \\
&+ \int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}|h_1(s - r_1(s))(1 - r_1'(s)) - k(s)|\,ds \\
&+ \sum_{j=1}^{2}\int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}\,|h_1(s) + h_2(s)|\left(\int_{s-r_j(s)}^{s}|h_j(u)|\,du\right)\,ds \leq \alpha,
\end{aligned}$$

where $k(s) = \big((1 - r_1'(s))^2\big)^{-1}\big([c(s)(h_1(u) + h_2(u)) + c'(s)](1 - r_1'(s)) + c(s)r_1''(s)\big)$ *and* $V_{[s-r_3(s),s]}(\mu(s,\cdot))$ *denotes the total variation of* $\mu(s,\cdot)$ *on* $[s - r_3(s), s]$*;*

(iii) *and such that*

$$\liminf_{t\to\infty} \int_0^t (h_1(s) + h_2(s))\, ds > -\infty.$$

Then the zero solution of (4.1) *is asymptotically stable if and only if*

(iv)

$$\int_0^t (h_1(s) + h_2(s))\, ds \to \infty \quad as \quad t \to \infty.$$

Remark 4.1.4. *Theorem* 4.1.3 *contains all the stability results for* (4.1) *discussed in* [8, 11, 12, 13, 31, 34, 63, 110, 144]. *In addition, in our result the delays can be unbounded and that the coefficients can change sign. See Example* 4.1.17 *and Example* 4.1.19.

A simple illustrative example is the scalar equation

$$x'(t) - c(t)x'(t - r(t)) = -a(t)x(t) + \int_{t-r(t)}^t k(t,s)x(s)\, ds, \quad t \geq 0,$$

where $r(t)$ is variable delay, $a, c : [0,\infty) \to \mathbb{R}$ are continuous functions, $k(t,s)$ is continuous with respect to its arguments.

The second class of delay differential equations that we will study in this section is of the form

$$x'(t) = -\int_{t-r(t)}^t a(t,s)g(s,x(s))\, ds. \tag{4.3}$$

where $r(t) : [0,\infty) \to [0,\infty)$, $a(t,s) : [0,\infty) \times [r_0,\infty) \to \mathbb{R}$ are continuous functions, g is a continuous function that satisfies Lipschitz condition with respect to x on $[r_0,\infty) \times [-l,l]$, where $r_0 = \inf_{t\geq 0}\{t - r(t)\}$.

Theorem 4.1.5. *Consider the functional differential equation* (4.3) *and suppose that the following conditions are satisfied,*

(i) $g(s,-x) = -g(s,x)$*;*

(ii) *there exists an* $l > 0$ *such that* g *satisfies a Lipschitz condition with respect to* x *on* $[r_0,\infty) \times [-l,l]$*, that is, there exists a constant* $L > 0$*, such that*

$$|g(s,x) - g(s,y)| \leq L|x - y| \quad for \quad s \geq r_0 \quad and \quad x,y \in [-l,l];$$

(iii) *there are functions* w *and* W *that are continuous, odd and strictly increasing on* $[-l,l]$ *such that* $w(x) \leq g(s,x) \leq W(x)$ *for* $x \in [0,l]$*;*

(iv) $x - w(x)$ *is non-decreasing on* $[0,l]$*;*

(v) $|x - g(s,x)| \leq l - w(l)$ *for* $x \in [-l,l]$*;*

(vi) *$v : [r_0, \infty) \to \mathbb{R}$ is a continuous function, $v(t) \geq 0$ for $t \geq 0$;*

(vii) *there exists a continuous function q such that*

$$\left| \int_t^u a(s,u)\,ds \right| \leq q(u) \quad \text{for} \quad t - r(t) \leq u \leq t;$$

(viii) *a positive number $\alpha < w(l)[W(l)]^{-1}$ exists such that*

$$\begin{aligned}
&\int_{t-r(t)}^t \left| v(u) + \int_t^u a(s,u)\,ds \right| du \\
&\quad + \int_0^t e^{-\int_s^t v(u)\,du} |v(s)| \int_{s-r(s)}^s \left| v(u) + \int_s^u a(s,u)\,ds \right| du\,ds \\
&\quad + \int_0^t e^{-\int_s^t v(u)\,du} \left| v(s-r(s)) + \int_s^{s-r(s)} a(u, s-r(s))\,du \right| |1 - r'(s)|\,ds \leq \alpha.
\end{aligned}$$

Then there exists a $\delta \in (0, l)$ such that, for each continuous initial function $\phi : [r_0, 0] \to (-\delta, \delta)$, there is a unique solution $x : [0, \infty) \to \mathbb{R}$ with $x(t) = \phi(t)$ on $[r_0, 0]$ of (4.3) such that $|x(t)|$ is bounded by l on $[r_0, \infty)$. This implies that the zero solution of (4.3) is stable.

Remark 4.1.6. *The proof is based on a generaliztion of some ideas of Jin and Luo [64] who discussed the case when $g(s,x) = g(x)$. We eliminate the condition that $t - r(t)$ is strictly increasing and obtain weaker conditions in Theorem 4.1.5 than those obtained in Theorem 4.1 of Becker and Burton [8]. See Example 4.1.22.*

A simple example is the scalar equation

$$x'(t) = -\int_{t-r(t)}^t a(s)x(s)\,ds, \quad t \geq 0,$$

where $r(t)$ is a variable delay, $r_0 = \inf_{t\geq 0}\{t - r(t)\}$, $a : [r_0, \infty)$ is a continuous function.

The third class consists of nonlinear delay differential equations of the form

$$x'(t) = -a(t)f(x(t - r_1(t))) + b(t)g(x(t - r_2(t))), \quad t \geq 0, \tag{4.4}$$

where $r_1, r_2 : [0, \infty) \to [0, \infty)$ are continuous functions, $r_0 = \min\{\inf_{t\geq 0}\{t - r_1(t)\}, \inf_{t\geq 0}\{t - r_2(t)\}\}$. The coefficients $a, b : [0, \infty) \to \mathbb{R}$ and $f, g : \mathbb{R} \to \mathbb{R}$ are continuous functions. We have the following result. Suppose, in addition, that $r_1(t)$ is differentiable, $t - r_j(t) \to \infty$ as $t \to \infty$, $j = 1, 2$, and that there exists a continuous function $\tilde{a} : [0, \infty) \to \mathbb{R}$ such that $a(t) = \tilde{a}(t)(1 - r_1'(t))$ and, finally, that the inverse function $h(t)$ of $t - r_1(t)$ exists. We then have the following result.

Theorem 4.1.7. *Consider the nonlinear delay differential equation (4.4) and suppose that*

(i) *$v(t) : [r_0, \infty) \to \mathbb{R}$ is a continuous function, $v(t) \geq 0$ as $t \geq 0$;*

(ii) *there exists a constant $l > 0$ such that $f(x), x - f(x)$, and $g(x)$ satisfy a Lipschitz condition with constant $L > 0$ on the interval $[-l, l]$;*

(iii) *the functions f and g are odd, increasing on $[0, l]$, $x - f(x)$ is nondecreasing on $[0, l]$;*

(iv) *there exists an $\alpha \in (0,1)$ with $\alpha g(l) < (1-\alpha) f(l)$ such that for $t \geq 0$,*

$$\begin{aligned}
&\int_0^t e^{-\int_s^t v(u)\,du} |\tilde{a}(h(s))|\,ds + \int_0^t e^{-\int_s^t v(u)\,du} |b(s)|\,ds \\
&+ \int_0^t e^{-\int_s^t v(u)\,du} |v(s - r_1(s))(1 - r_1'(s))|\,ds + \int_{t-r_1(t)}^t |\tilde{a}(h(s)) + v(s)|\,ds \\
&+ \int_0^t e^{-\int_s^t v(u)\,du} v(s) \int_{s-r_1(s)}^s |\tilde{a}(h(u)) + v(u)|\,du\,ds \leq \alpha.
\end{aligned}$$

Then the zero solution of (4.4) *is stable.*

Remark 4.1.8. *Burton* [13] *studied the special case when $b(t) \equiv 0$ and r_1 is a constant. Following the technique of Burton* [12]*, Ding and Li* [31] *studied stability properties of* (4.4) *as well. However, the condition* (iv) *in Ding and Li* [31] *is restrictive. By introducing a continuous function $v(t)$ for constructing a fixed point mapping argument, the alternative condition* (iv) *in Theorem* 4.1.7 *is obtained. Note that the condition that the functions $t - r_1(t)$ and $t - r_2(t)$ are strictly increasing is not needed in Theorem* 4.1.7*.*

A simple example is the scalar equation

$$x'(t) = -a(t)x(t - r_1(t)) + b(t)x(t - r_2(t)), \quad t \geq 0,$$

where $r_j(t)$, $j = 1, 2$, are variable delays, $a, b : [0, \infty) \to \mathbb{R}$ are continuous functions.

If we consider the impulsive effect on the solutions of equation (4.1), we come to our fourth class of equations

$$\begin{cases} x'(t) - c(t)x'(t - r_1(t)) = -b(t)x(t - r_2(t)) + \int_{t-r_3(t)}^t g(t, x(s))\,d\mu(t, s), \quad t \neq t_k, \\ x(t_k^+) - x(t_k) = d_k x(t_k), \quad k = 1, 2, \dots. \end{cases} \tag{4.5}$$

Suppose that the following conditions are satisfied

(H1) $0 \leq 0 < t_1 < t_2 \cdots < t_k < \cdots$ are fixed points with $t_k \to \infty$ as $k \to \infty$.

(H2) $b, c : [r_0, \infty) \to \mathbb{R}$, and $r_j(t) : [0, \infty) \to [0, \infty)$, $j = 1, 2, , 3$, are continuous functions, where $r_0 = \min\{\inf_{t \geq 0}\{t - r_1(t)\}, \inf_{t \geq 0}\{t - r_2(t)\}, \inf_{t \geq 0}\{t - r_3(t)\}\}$.

(H3) $\mu(t, s)$ is of bounded variation for each t, and $g : [r_0, \infty) \times \mathbb{R} \to \mathbb{R}$ is continuous function, $g(t, cx) = cg(t, x)$ for positive c, $xg(t, x) > 0$ if $x \neq 0$ is sufficient small. We assume that g satisfies:

(G) $g(t, 0) = 0$, there exists an $l > 0$ such that g satisfies Lipschitz condition with respect to x on $[r_0, \infty) \times [-l, l]$, that is,

$$|g(t, x) - g(t, y)| \leq |x - y| \quad for \quad t \geq r_0 \quad and \quad x, y \in [-l, l].$$

(H4) $d_k \in (-1, \infty)$ are constants for $k = 1, 2, \dots$.

(H5) $\lim_{t\to t_k^-} = x(t_k^-)$ and $\lim_{t\to t_k^+} = x(t_k^+)$ for $k = 1, 2, ...$

Definition 4.1.9. *For the initial function $\phi \in C([r_0, 0], \mathbb{R})$, we denote by $x(t) := x(t, \phi)$ the solution of (4.5) with initial condition (4.2), which satisfies the following conditions*

(i) *$x(t)$ is absolutely continuous on $[0, t_1)$ and each interval (t_k, t_{k+1});*

(ii) *$x(t_k^-)$ and $x(t_k^+)$ exist and $x(t_k^-) = x(t_k)$ for any $t_k \in [0, \infty)$;*

(iii) *$x(t)$ satisfies (4.5) almost everywhere in $[0, \infty)$, and may have a discontinuity of the first kind at t_k for $k = 1, 2, ...$*

Theorem 4.1.10. *Consider the impulsive nonlinear neutral integro-differential equation (4.5) and suppose that the following conditions are satisfied*

(i) *the delay $r_2(t)$ is differentiable, the delay $r_1(t)$ is twice differentiable with $r_1'(t) \neq 1$, and $t - r_j(t) \to \infty$ as $t \to \infty$, $j = 1, 2, 3$;*

(ii) *there exists a constant $\alpha \in (0, 1)$ and continuous functions $h_j : [r_0, \infty) \to \mathbb{R}$ $(j = 1, 2)$ such that*

$$\begin{aligned}
&\sum_{l=1}^{n} \left| \frac{d_l}{1+d_l} \prod_{t_l - r_1(t_l) \le t_k < t_l} (1+d_k)^{-1} e^{-\int_{t_l}^{t} (h_1(u)+h_2(u))\,du} \frac{c(t_l)}{1-r_1'(t_l)} \right| \\
&+ \left| \prod_{t-r_1(t) \le t_k < t} (1+d_k)^{-1} \frac{c(t)}{1-r_1'(t)} \right| + \sum_{j=1}^{2} \int_{t-r_j(t)}^{t} |h_j(u)|\,du \\
&+ \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} |h_1(s)+h_2(s)| \sum_{j=1}^{2} \int_{s-r_j(s)}^{s} |h_j(u)|\,du\,ds \\
&+ \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} \left| h_1(s-r_1(s))(1-r_1'(s)) - \prod_{s-r_1(s) \le t_k < s} (1+d_k)^{-1} k(s) \right| ds \\
&+ \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} \left| h_2(s-r_2(s))(1-r_2'(s)) - \prod_{s-r_2(s) \le t_k < s} (1+d_k)^{-1} b(s) \right| ds \\
&+ \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} \prod_{s-r_3(s) \le t_k < s} (1+d_k)^{-1} \left(V_{[s-r_3(s),s]}(\mu(s,\cdot)) \right) ds \le \alpha. \qquad (4.6)
\end{aligned}$$

where $k(s) = \left((1-r_1'(s))^2\right)^{-1}\left([c(s)(h_1(u)+h_2(u)) + c'(s)](1-r_1'(s)) + c(s)r_1''(s)\right)$ and $V_{[s-r_3(s),s]}(\mu(s,\cdot))$ denotes the total variation of $\mu(s,\cdot)$ on $[s - r_3(s), s]$;

(iii) *and such that*

$$\liminf_{t\to\infty} \int_0^t (h_1(s)+h_2(s))\,ds > -\infty.$$

(iv) *there exists a positive constant $M > 0$ such that $\prod_{0\le t_k<t}(1 + d_k) \le M$.*

Then the zero solution of (4.5) is asymptotically stable if and only if

(v)

$$\int_0^t (h_1(s) + h_2(s))\, ds \to \infty \quad as \quad t \to \infty.$$

Remark 4.1.11. *The proof is based on the ideas in* [144]. *Our result is a generalization of* [144].

The organization of this section is as follows. In Subsection 4.1.2, we present a proof of Theorem 4.1.3. The proof the Theorem 4.1.5 is presented in Subsection 4.1.3. The proof of Theorem 4.1.7 and the proof of 4.1.10 are given in Subsection 4.1.4 and Subsection 4.1.5, respectively.

4.1.2 Proof of Theorem 4.1.3

In this subsection, we will prove Theorem 4.1.3. We start with some preparation. First define

$$S_\phi^l = \Big\{x \mid x \in C([r_0, \infty), \mathbb{R}),\ \|x\| = \sup_{t \geq r_0} |x(t)| \leq l,\ x(t) = \phi(t) \text{ for } t \in [r_0, 0], \\ \text{and } x(t) \to 0 \text{ as } t \to \infty\Big\}.$$

If we define the metric $\rho(x, y) = \sup_{t \geq r_0}\{|x(t) - y(t)|\}$, then S_ϕ^l becomes a complete metric space.

If we multiply both sides of (4.1) by $e^{\int_0^t (h_1(s)+h_2(s))\, ds}$, integrate from 0 to t, and perform an integration by parts, we obtain

$$\begin{aligned} x(t) &= \Big\{\phi(0) - \frac{c(0)}{1 - r_1'(0)}\phi(-r_1(0)) - \sum_{j=1}^{2}\int_{-r_j(0)}^{0} h_j(s)\phi(s)\, ds\Big\} e^{-\int_0^t (h_1(s)+h_2(s))\, ds} \\ &\quad + \frac{c(t)}{1 - r_1'(t)} x(t - r_1(t)) + \sum_{j=1}^{2}\int_{t-r_j(t)}^{t} h_j(s)x(s)\, ds \\ &\quad + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\, du}[h_2(s - r_2(s))(1 - r_2'(s)) - a(s)]x(s - r_2(s))\, ds \\ &\quad + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\, du}[h_1(s - r_1(s))(1 - r_1'(s)) - k(s)]x(s - r_1(s))\, ds \\ &\quad + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\, du} \int_{s-r_3(s)}^{s} g(s, x(u))\, d\mu(s, u)\, ds \\ &\quad - \sum_{j=1}^{2}\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\, du}(h_1(s) + h_2(s)) \int_{s-r_j(s)}^{s} h_j(u)x(u)\, du\, ds. \end{aligned}$$

Lemma 4.1.12. *Let $\varphi \in S_\phi^l$ and define an operator by $P\varphi(t) = \phi(t)$ for $t \in [r_0, 0]$ and for $t \geq 0$,*

$$\begin{aligned} (P\varphi)(t) &= \Big\{\phi(0) - \frac{c(0)}{1 - r_1'(0)}\phi(-r_1(0)) - \sum_{j=1}^{2}\int_{-r_j(0)}^{0} h_j(s)\phi(s)\, ds\Big\} e^{-\int_0^t (h_1(s)+h_2(s))\, ds} \\ &\quad + \frac{c(t)}{1 - r_1'(t)}\varphi(t - r_1(t)) + \sum_{j=1}^{2}\int_{t-r_j(t)}^{t} h_j(s)\varphi(s)\, ds \end{aligned}$$

$$
\begin{aligned}
&+\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}[h_2(s-r_2(s))(1-r_2'(s))-a(s)]\varphi(s-r_2(s))\,ds \\
&+\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}[h_1(s-r_1(s))(1-r_1'(s))-k(s)]\varphi(s-r_1(s))\,ds \\
&+\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}\int_{s-r_3(s)}^{s} g(s,\varphi(u))\,d\mu(s,u)\,ds \\
&-\sum_{j=1}^{2}\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}(h_1(s)+h_2(s))\int_{s-r_j(s)}^{s} h_j(u)\varphi(u)\,du\,ds. \qquad (4.7)
\end{aligned}
$$

If conditions (i)-(iv) *in Theorem* 4.1.3 *are satisfied, then there exists* $\delta > 0$ *such that for any* $\phi : [r_0, 0] \to (-\delta, \delta)$*, we have that* $P : S_\phi^l \to S_\phi^l$ *and* P *is a contraction with respect to the metric defined on* S_ϕ^l*.*

Proof. Let $J = \sup_{t\geq 0}\left\{e^{-\int_0^t (h_1(s)+h_2(s))\,ds}\right\}$, by (iv), J is well defined. Suppose that (iv) holds.

It is clear that $P\varphi \in C([r_0,\infty),\mathbb{R})$. Hence, by (ii) and condition (G), we have

$$
\begin{aligned}
|(P\varphi)(t)| \leq\ & \|\phi\|\left(1+\left|\frac{c(0)}{1-r_1'(0)}\right|+\sum_{j=1}^{2}\int_{-r_j(0)}^{0}|h_j(s)|\,ds\right)e^{-\int_0^t (h_1(s)+h_2(s))\,ds} \\
&+l\Bigg\{\left|\frac{c(t)}{1-r_1'(t)}\right|+\sum_{j=1}^{2}\int_{t-r_j(t)}^{t}|h_j(s)|\,ds \\
&+\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}\left(V_{[s-r_3(s),s]}(\mu(s,\cdot))\right)ds \\
&+\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}|h_2(s-r_2(s))(1-r_2'(s))-a(s)|\,ds \\
&+\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}|h_1(s-r_1(s))(1-r_1'(s))-k(s)|\,ds \\
&+\sum_{j=1}^{2}\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}|h_1(s)+h_2(s)|\int_{s-r_j(s)}^{t}|h_j(u)|\,du\,ds\Bigg\} \\
\leq\ & \|\phi\|\left(1+\left|\frac{c(0)}{1-r_1'(0)}\right|+\sum_{j=1}^{2}\int_{-r_j(0)}^{0}|h_j(s)|\,ds\right)J+l\alpha.
\end{aligned}
$$

From this estimate, it follows that if

$$
\delta := \frac{(1-\alpha)l}{\left(1+\frac{|c(0)|}{|1-r_1'(0)|}+\sum_{j=1}^{2}\int_{-r_j(0)}^{0}|h_j(s)|\,ds\right)J},
$$

then $\|\phi\| \leq \delta$ implies that $|(P\varphi)(t)| \leq l$.

Next, we show that $(P\varphi)(t) \to 0$ as $t \to \infty$. Since $\varphi(t) \to 0$ and $t - r_j(t) \to \infty$ as $t \to \infty$, for each $\varepsilon > 0$, there exists a $T_1 > 0$ such that $t > T_1$ implies $|\varphi(t - r_j(t))| < \varepsilon$ for $j = 1, 2$. Thus for $t \geq T_1$,

$$\begin{aligned}
|I_2| &:= \left| \sum_{j=1}^{2} \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} (h_1(s)+h_2(s)) \left(\int_{s-r_j(s)}^{s} h_j(u)\varphi(u)\,du \right) ds \right| \\
&\leq \sum_{j=1}^{2} \int_0^{T_1} e^{-\int_s^t (h_1(u)+h_2(u))\,du} |h_1(s)+h_2(s)| \left(\int_{s-r_j(s)}^{s} |h_j(u)||\varphi(u)|\,du \right) ds \\
&\quad + \sum_{j=1}^{2} \int_{T_1}^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} |h_1(s)+h_2(s)| \left(\int_{s-r_j(s)}^{s} |h_j(u)||\varphi(u)|\,du \right) ds \\
&\leq l \sum_{j=1}^{2} \int_0^{T_1} e^{-\int_s^t (h_1(u)+h_2(u))\,du} |h_1(s)+h_2(s)| \left(\int_{s-r_j(s)}^{s} |h_j(u)|\,du \right) ds \\
&\quad + \varepsilon \sum_{j=1}^{2} \int_{T_1}^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} |h_1(s)+h_2(s)| \left(\int_{s-r_j(s)}^{s} |h_j(u)|\,du \right) ds. \qquad (4.8)
\end{aligned}$$

By the condition (iv), there exists $T_2 > T_1$ such that $t > T_2$ implies

$$l \sum_{j=1}^{2} \int_0^{T_1} e^{-\int_s^t (h_1(u)+h_2(u))\,du} |h_1(s)+h_2(s)| \left(\int_{s-r_j(s)}^{s} |h_j(u)|\,du \right) ds < \varepsilon.$$

Applying (ii), we have $|I_2| \to 0$ as $t \to \infty$.

Since $\varphi(t) \to 0$ and $t - r_3(t) \to \infty$ as $t \to \infty$, for each $\varepsilon > 0$, there exists a $T_3 > 0$ such that $t > T_3$ implies $|\varphi(t - r_3(t))| < \varepsilon$. Thus for $t \geq T_3$,

$$\begin{aligned}
|I_3| &:= \left| \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} \int_{s-r_3(s)}^{s} g(s,\varphi(u))\,d\mu(s,u)\,ds \right| \\
&\leq l \int_0^{T_3} e^{-\int_s^t (h_1(u)+h_2(u))\,du} \left(V_{[s-r_3(s),s]}(\mu(s,\cdot)) \right) ds \\
&\quad + \varepsilon \int_{T_3}^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} \left(V_{[s-r_3(s),s]}(\mu(s,\cdot)) \right) ds \qquad (4.9)
\end{aligned}$$

By the condition (iv), there exists $T_4 > T_3$ such that $t > T_4$ implies

$$l \int_0^{T_3} e^{-\int_s^t (h_1(u)+h_2(u))\,du} \left(V_{[s-r_3(s),s]}(\mu(s,\cdot)) \right) ds < \varepsilon.$$

Applying (ii), we have $|I_3| \to 0$ as $t \to \infty$.

Similarly, we can show that the rest terms in (4.7) approach zero as $t \to \infty$, which yields $(P\varphi)(t) \to 0$ as $t \to \infty$.

Finally, we show that P is a contraction mapping with contraction constant α. In fact, for

$\varphi, \eta \in S^l_\phi$,

$$\begin{aligned}
&|(P\varphi)(t) - (P\eta)(t)| \\
&\leq \Bigg\{ \left|\frac{c(t)}{1-r_1'(t)}\right| + \sum_{j=1}^{2}\int_{t-r_j(t)}^{t} |h_j(s)|\,ds + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} \left(V_{[s-r_3(s),s]}(\mu(s,\cdot))\right)\,ds \\
&\quad + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} |h_2(s-r_2(s))(1-r_2'(s)) - a(s)|\,ds \\
&\quad + \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} |h_1(s-r_1(s))(1-r_1'(s)) - k(s)|\,ds \\
&\quad + \sum_{j=1}^{2}\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} |(h_1(s)+h_2(s))| \int_{s-r_j(s)}^{t} |h_j(u)|\,du\,ds \Bigg\} \|\varphi - \eta\| \\
&\leq \alpha \|\varphi - \eta\|.
\end{aligned}$$

Thus, $P : S^l_\phi \to S^l_\phi$ and P is a contraction mapping. □

We are now ready to prove Theorem 4.1.3.

Proof. Let P be defined as in Lemma 4.1.12. By the contraction mapping principle, P has a unique fixed point x in S^l_ϕ which is by construction a solution of (4.1) with $x(t) = \phi(t)$ on $[r_0, 0]$ and $x(t) \to 0$ as $t \to \infty$.

Let $\varepsilon > 0$ be given, then we choose $m > 0$ so that $m < \min\{l, \varepsilon\}$. By considering S^m_ϕ, we obtain that there is a $\delta > 0$ such that $\|\phi\| < \delta$ implies that the unique solution of (4.1) with $x(t) = \phi(t)$ on $[r_0, 0]$ satifies $|x(t)| \leq m < \varepsilon$ for all $t \geq r_0$. This shows that the zero solution of (4.1) is asymptotically stable if (iv) holds.

Conversely, we suppose that condition (iv) fails. Then by (iii), there exists a sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty} \int_0^{t_n} (h_1(s) + h_2(s))\,ds = v$ for some $v \in \mathbb{R}$. We may choose a positive constant M such that

$$-M \leq \int_0^{t_n} (h_1(s) + h_2(s))\,ds \leq M, \quad \text{for all} \quad n \geq 1. \tag{4.10}$$

To simplify our expressions, we define

$$\begin{aligned}
w(s) \;:=\; & |h_2(s-r_2(s))(1-r_2'(s)) - a(s)| + |h_1(s-r_1(s))(1-r_1'(s)) - k(s)| \\
& + V_{[s-r_3(s),s]}(\mu(s,\cdot)) + |(h_1(s)+h_2(s))| \sum_{j=1}^{2}\int_{s-r_j(s)}^{t} |h_j(u)|\,du, \quad s \geq 0.
\end{aligned}$$

By (ii) we have

$$\int_0^{t_n} e^{-\int_s^{t_n} (h_1(u)+h_2(u))\,du} w(s)\,ds \leq \alpha \quad \text{for all} \quad n \geq 1. \tag{4.11}$$

Combining (4.10) and (4.11), we have

$$\int_0^{t_n} e^{\int_0^s (h_1(u)+h_2(u))\,du} w(s)\,ds \leq \alpha e^{\int_0^{t_n} (h_1(u)+h_2(u))\,du} \leq \alpha e^M \quad \text{for all} \quad n \geq 1,$$

which yields that the sequence $\int_0^{t_n} e^{\int_0^s (h_1(u)+h_2(u))\,du} w(s)\,ds$ is bounded. Therefore, there exists a convergent subsequence and without loss of generality, we can assume that

$$\lim_{k\to\infty}\int_0^{t_{n_k}} e^{\int_0^s (h_1(u)+h_2(u))\,du} w(s)\,ds = \gamma \quad \text{for} \quad \text{some} \quad \gamma\in\mathbb{R}^+.$$

We choose a positive integer $\overline{k}$ so large that

$$\lim_{k\to\infty}\int_{t_{n_{\overline{k}}}}^{t_{n_k}} e^{\int_0^s (h_1(u)+h_2(u))\,du} w(s)\,ds \le \frac{\delta_0}{4J}$$

for all $n_k > n_{\overline{k}}$, where $\delta_0 > 0$ satisfies $2\delta_0 J e^M + \alpha < 1$.

Now, we consider the solution $x(t) = x(t, t_{n_{\overline{k}}}, \phi)$ of (4.1) with $x(t_{n_{\overline{k}}}) = \delta_0$ and $x(s) \le \delta_0$ for $t_{n_{\overline{k}}} - r_0 \le s \le t_{n_{\overline{k}}}$, and we may choose ϕ such that $|x(t)| \le 1$ for $t \ge t_{n_{\overline{k}}}$ and

$$x(t_{n_{\overline{k}}}) - \frac{c(t_{n_{\overline{k}}})}{1-r_1'(t_{n_{\overline{k}}})}x(t_{n_{\overline{k}}} - r_1(t_{n_{\overline{k}}})) - \sum_{j=1}^{2}\int_{t_{n_{\overline{k}}}-r_j(t_{n_{\overline{k}}})}^{t_{n_{\overline{k}}}} h_j(s)x(s)\,ds \ge \frac{1}{2}\delta_0. \tag{4.12}$$

So, it follows from (4.12) with $x(t) = (Px)(t)$ that for $k \ge \overline{k}$,

$$\begin{aligned}
&\left| x(t_{n_k}) - \frac{c(t_{n_k})}{1-r_1'(t_{n_k})}x(t_{n_k} - r_1(t_{n_k})) - \sum_{j=1}^{2}\int_{t_{n_k}-r_j(t_{n_k})}^{t_{n_k}} h_j(s)x(s)\,ds \right| \\
&\ge \frac{1}{2}\delta_0 e^{-\int_{t_{n_{\overline{k}}}}^{t_{n_k}} (h_1(u)+h_2(u))\,du} - \int_{t_{n_{\overline{k}}}}^{t_{n_k}} e^{-\int_s^{t_{n_k}} (h_1(u)+h_2(u))\,du} w(s)\,ds \\
&\ge e^{-\int_{t_{n_{\overline{k}}}}^{t_{n_k}} (h_1(u)+h_2(u))\,du}\left(\frac{1}{2}\delta_0 - J\int_{t_{n_{\overline{k}}}}^{t_{n_k}} e^{\int_0^s (h_1(u)+h_2(u))\,du} w(s)\,ds\right) \\
&\ge \frac{1}{4}\delta_0 e^{-\int_{t_{n_{\overline{k}}}}^{t_{n_k}} (h_1(u)+h_2(u))\,du} \ge \frac{1}{4}\delta_0 e^{-2M} > 0.
\end{aligned} \tag{4.13}$$

On the other hand, suppose that $x(t) = x(t, t_{n_{\overline{k}}}, \phi) \to 0$ as $t\to\infty$. Since $t_{n_k} - r_j(t_{n_k}) \to \infty$ as $k\to\infty$, $j = 1, 2$ and (ii) holds, this would imply that

$$x(t_{n_k}) - \frac{c(t_{n_k})}{1-r_1'(t_{n_k})}x(t_{n_k} - r_1(t_{n_k})) - \sum_{j=1}^{2}\int_{t_{n_k}-r_j(t_{n_k})}^{t_{n_k}} h_j(s)x(s)\,ds \to 0 \quad as \quad k\to\infty,$$

which contradicts the estimate. Hence condition (iv) is necessary for the asymptotic stability of the zero solution of (4.1). □

Corollary 4.1.13. *Consider the equation*

$$x'(t) - c(t)x'(t-r(t)) = -a(t)x(t) + \int_{t-r(t)}^{t} g(t, x(s))\,d\mu(t,s). \tag{4.14}$$

Assume that $r(t)$ is twice differentiable, $r'(t)\neq 1$, $t - r(t)\to\infty$ as $t\to\infty$, g satisfies condition (G). *Suppose that there exists a constant $\alpha\in(0,1)$ and a continuous function $v : [r_0,\infty)\to\mathbb{R}$*

such that $\liminf_{t\to\infty}\int_0^t v(s)\,ds > -\infty$ *and*

$$\begin{aligned}&\int_{t-r(t)}^t |v(s)-a(s)|\,ds+\int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r(s)}^s |v(u)-a(u)|\,du\,ds\\&+\left|\frac{c(t)}{1-r'(t)}\right|+\int_0^t e^{-\int_s^t v(u)\,du}\left|[v(s-r(s))-a(s-r(s))](1-r'(s))-k(s)\right|\,ds\\&+\int_0^t e^{-\int_s^t v(u)\,du}V_{[s-r(s),s]}(\mu(s,\cdot))\,ds\le\alpha,\end{aligned}$$

where

$$k(s)=\frac{[c(s)v(s)+c'(s)](1-r'(s))+c(s)r''(s)}{(1-r'(s))^2}. \tag{4.15}$$

Then the zero solution of (4.14) *is asymptotically stable if and only if* $\int_0^t v(s)\,ds\to\infty$ *as* $t\to\infty$.

Corollary 4.1.14. *Consider the equation*

$$x'(t)-c(t)x'(t-r_1(t))=-a(t)x(t-r_2(t))+b(t)g(t,x(t-r_3(t))). \tag{4.16}$$

Assume that $r_2(t)$ *is differentiable,* $r_1(t)$ *is twice differentiable,* $r_1'(t)\neq 1$, $t-r_j(t)\to\infty$ *as* $t\to\infty$, $j=1,2,3$, g *satisfies condition* (G). *Suppose that there exists a constant* $\alpha\in(0,1)$ *and a continuous functions* $h_j:[r_0,\infty)\to\mathbb{R}$ *such that* $\liminf_{t\to\infty}\int_0^t(h_1(s)+h_2(s))\,ds>-\infty$, $j=1,2$, *and*

$$\begin{aligned}&\left|\frac{c(t)}{1-r_1'(t)}\right|+\sum_{j=1}^2\int_{t-r_j(t)}^t |h_j(s)|\,ds\\&+\int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}|h_2(s-r_2(s))(1-r_2'(s))-a(s)|\,ds\\&+\int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}\left[|h_1(s-r_1(s))(1-r_1'(s))-k(s)|+|b(s)|\right]\,ds\\&+\sum_{j=1}^2\int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}|h_1(s)+h_2(s)|\int_{s-r_j(s)}^s |h_j(u)|\,du\,ds\le\alpha,\end{aligned}$$

where $k(s)$ *and* $h_j(s)$ $(j=1,2)$ *are defined as in Theorem* 4.1.3. *Then the zero solution of* (4.16) *is asymptotically stable if and only if* $\int_0^t(h_1(s)+h_2(s))\,ds\to\infty$ *as* $t\to\infty$.

Corollary 4.1.15. *Consider the equation*

$$x'(t)-c(t)x'(t-r(t))=-a(t)x(t)+b(t)g(x(t-r(t))). \tag{4.17}$$

Assume that $r(t)$ *is twice differentiable,* $r'(t)\neq 1$, $t-r(t)\to\infty$ *as* $t\to\infty$. g *satisfies condition* (G). *Suppose that there exists a constant* $\alpha\in(0,1)$ *and a continuous function* $v:[r_0,\infty)\to\mathbb{R}$ *such that* $\liminf_{t\to\infty}\int_0^t v(s)\,ds>-\infty$ *and*

$$\begin{aligned}&\left|\frac{c(t)}{1-r'(t)}\right|+\int_{t-r(t)}^t |v(s)-a(s)|\,ds+\int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r(s)}^s |v(u)-a(u)|\,du\,ds\\&+\int_0^t e^{-\int_s^t v(u)\,du}\left[\left|(v(s-r(s))-a(s-r(s)))(1-r'(s))-k(s)\right|+|b(s)|\right]\,ds\le\alpha,\end{aligned}$$

where $k(s)$ is defined as in (4.15). Then the zero solution of (4.17) is asymptotically stable if and only if $\int_0^t v(s)\,ds \to \infty$ as $t \to \infty$.

Example 4.1.16. *Consider the neutral differential equation*

$$x'(t) = -\frac{1}{t+1}x(t) + \frac{1}{2t+2}x(t-0.05t) + 0.05x'(t-0.05t), \tag{4.18}$$

Define $a(t) = \frac{1}{t+1}$, $b(t) = \frac{1}{2t+2}$, $c(t) = 0.05$, $r(t) = 0.05t$ and $v(t) = \frac{2}{t+1}$. Then

$$\frac{|c(t)|}{|1-r'(t)|} = \frac{0.05}{1-0.05} = \frac{1}{19} \approx 0.0526.$$

Since $|v(s)-a(s)| = \frac{1}{s+1}$, $k(s) = \frac{2}{19(s+1)}$, we have

$$\begin{aligned}\int_{t-r(t)}^{t} |v(s)-a(s)|\,ds = \int_{0.95t}^{t} \frac{1}{s+1}\,ds \quad &< \quad 0.0513,\\ \int_0^t e^{-\int_s^t v(u)\,du}|v(s)| \int_{s-r(s)}^{s} |v(u)-a(u)|\,du\,ds \quad &< \quad 0.0513,\end{aligned}$$

$$\begin{aligned}&\int_0^t e^{-\int_s^t v(u)\,du}\Big\{ \big|[v(s-r(s)) - a(s-r(s))](1-r'(s)) - k(s) + b(s)\big| \Big\}\,ds\\ &= \int_0^t e^{-\int_s^t \frac{2}{u+1}\,du}\left|\frac{0.95}{0.95s+1} - \frac{2}{19(s+1)} + \frac{1}{2(s+1)}\right| ds \le \frac{17}{2\times 19} + \frac{1}{4} < 0.697.\end{aligned}$$

Hence, we have

$$\begin{aligned}&\left|\frac{c(t)}{1-r'(t)}\right| + \int_{t-r(t)}^{t} |v(s)-a(s)|\,ds + \int_0^t e^{-\int_s^t v(u)\,du}|v(s)| \int_{s-r(s)}^{s} |v(u)-a(u)|\,du\,ds\\ &\quad + \int_0^t e^{-\int_s^t v(u)\,du}\Big\{ \big|[v(s-r(s)) - a(s-r(s))](1-r'(s)) - k(s) + b(s)\big| \Big\}\,ds < 1,\end{aligned}$$

and since $\int_0^t v(s)\,ds = \int_0^t \frac{2}{s+1}\,ds = 2\ln(t+1)$, the conditions of Corollary 4.1.15 are satisfied. Therefore, the zero solution of (4.18) is asymptotically stable.

Example 4.1.17. *Consider the following differential equation*

$$x'(t) = -\frac{1}{32}\left(\frac{1}{4} - \frac{1}{3}\sin t + \varepsilon_1(t)\right) x\left(t - \left(1 - \frac{1}{3}\cos t + \varepsilon_2(t)\right)\right) + \frac{\cos t}{256}g(x(t-r_3(t))), \tag{4.19}$$

where $|\varepsilon_j(t)| < \varepsilon < \frac{2}{51}$, $|\varepsilon_j'(t)| < \varepsilon < \frac{2}{51}$, $j = 1, 2$, $r_3(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$ is an arbitrary continuous function which satisfies $t - r_3(t) \to \infty$ as $t \to \infty$, g satisfies condition (G).

Define $a(t) = \frac{1}{32}\left(\frac{1}{4} - \frac{1}{3}\sin t + \varepsilon_1(t)\right)$, $b(t) = \frac{\cos t}{256}$, $r_2(t) = 1 - \frac{1}{3}\cos t + \varepsilon_2(t)$, and $v(t) = \frac{1}{32}$. Then

$$\begin{aligned}\int_{t-r_2(t)}^{t} |v(s)|\,ds = \int_{t-1+\frac{1}{3}\cos t - \varepsilon_2(t)}^{t} \frac{1}{32}\,ds = \frac{1}{32}(1 - \frac{1}{3}\cos t + \varepsilon_2(t)) \quad &\le \quad \frac{1}{24} + \frac{\varepsilon}{32},\\ \int_0^t e^{-\int_s^t \frac{1}{32}\,du}\frac{1}{32}\int_{s-1+\frac{1}{3}\cos s + \varepsilon_2(s)}^{s} \frac{1}{32}\,du\,ds \quad &\le \quad \frac{1}{24} + \frac{\varepsilon}{32},\end{aligned}$$

$$\int_0^t e^{-\int_s^t v(u)\,du}\left(|v(s-r_2(s))(1-r_2'(s))-a(s)|+|b(s)|\right)ds$$
$$=\int_0^t e^{-\int_s^t \frac{1}{32}\,du}\left(\frac{1}{32}\times\frac{3}{4}+\frac{\varepsilon}{32}+\frac{1}{32}\times\frac{1}{8}\right)ds\le\frac{7}{8}+\varepsilon.$$

Hence,

$$\int_{t-r_2(t)}^t |v(s)|\,ds+\int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r_2(s)}^s |v(u)|\,du\,ds$$
$$+\int_0^t e^{-\int_s^t v(u)\,du}\left(|v(s-r_2(s))(1-r_2'(s))-a(s)|+|b(s)|\right)ds$$
$$\le\frac{1}{24}+\frac{\varepsilon}{32}+\frac{1}{24}+\frac{\varepsilon}{32}+\frac{7}{8}+\varepsilon=\frac{23}{24}+\frac{17\varepsilon}{16}<1,$$

and since $\int_0^t v(s)\,ds=\int_0^t\frac{1}{32}\,ds=\frac{1}{32}t$, the conditions of Corollary 4.1.14 are satisfied. Therefore, the zero solution of (4.19) is asymptotically stable.

Example 4.1.18. *Consider the following differential equation*

$$x'(t)=-a(t)x\left(t-1+\frac{1}{3}\cos t\right)+b(t)g(x(t-r_3(t))), \tag{4.20}$$

where $0<m_1\le a(t)$*,* $|b(t)|\le M_2$*,* $r_3(t)\in C(\mathbb{R}^+,\mathbb{R}^+)$ *is an arbitrary continuous function which satisfies* $t-r_3(t)\to\infty$ *as* $t\to\infty$*,* g *satisfies condition* (G)*.*

Define $r_2(t)=1-\frac{1}{3}\cos t$, if we choose $v(t)=v$ is a constant satisfying $v>\frac{4m_1}{5}$, we have

$$\int_{t-r_2(t)}^t |v(s)|\,ds=\int_{t-1+\frac{1}{3}\cos t}^t v\,ds=v(1-\frac{1}{3}\cos t)\ \le\ \frac{4}{3}v,$$
$$\int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r_2(s)}^s |v(u)|\,du\,ds\ \le\ \frac{4}{3}v,$$

$$\int_0^t e^{-\int_s^t v(u)\,du}\left(|v(s-r_2(s))(1-r_2'(s))-a(s)|+|b(s)|\right)ds$$
$$\le\int_0^t e^{-(t-s)v}(\frac{5v}{4}-m_1+M_2)\,ds\le\frac{5}{4}-\frac{m_1-M_2}{v}.$$

Hence,

$$\int_{t-r_2(t)}^t |v(s)|\,ds+\int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r_2(s)}^s |v(u)|\,du\,ds$$
$$+\int_0^t e^{-\int_s^t v(u)\,du}\left(|v(s-r_2(s))(1-r_2'(s))-a(s)|+|b(s)|\right)ds$$
$$\le\frac{8}{3}v+\frac{5}{4}-\frac{m_1-M_2}{v}.$$

Next, choose v such that $\frac{8}{3}v+\frac{5}{4}-\frac{m_1-M_2}{v}<1$, and since $\int_0^t v(s)\,ds=\int_0^t v\,ds=vt$, then the conditions of Corollary 4.1.14 are satisfied. Therefore, the zero solution of (4.20) is asymptotically stable.

For instance, if we choose $v=\frac{1}{32}$, $m_1=\frac{1}{32}$, $M_2=\frac{1}{64}$, we have $\frac{8}{3}v+\frac{5}{4}-\frac{m_1-M_2}{v}=\frac{5}{6}<1$.

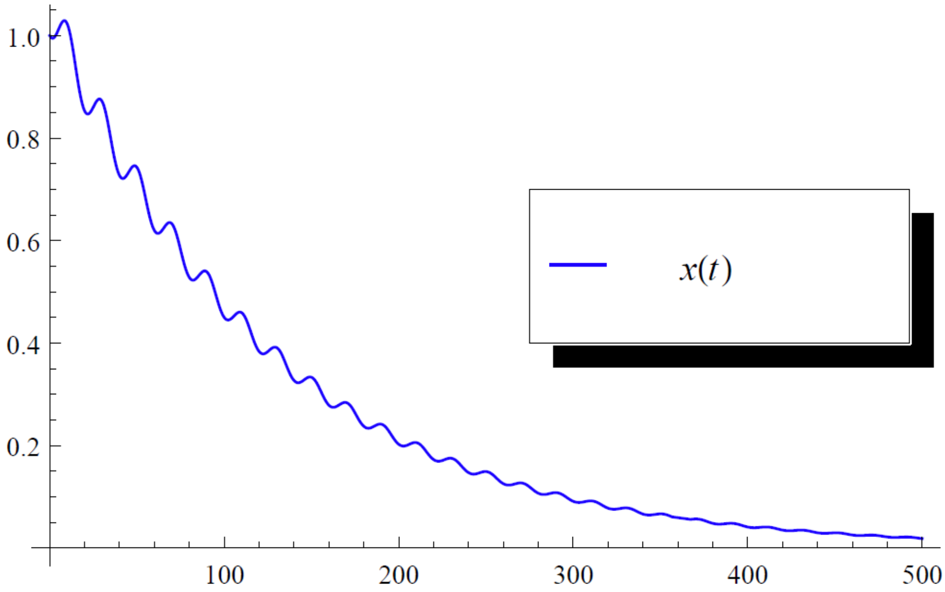

Figure 4.1: Numerical solution of (4.21).

Example 4.1.19. *Consider the following differential equation*

$$x'(t) = -\frac{1}{128}\left(1 - 2\sin\frac{5t}{16}\right)x(t-1), \tag{4.21}$$

Define $a(t) = \frac{1}{128}\left(1 - 2\sin\frac{5t}{16}\right), v(t) = \frac{1}{32}$, we obtain that

$$\int_{t-1}^{t} |v(s)|\,ds = \frac{1}{32}, \qquad \int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-1}^{s}|v(u)|\,du\,ds \leq \frac{1}{32},$$

$$\int_0^t e^{-\int_s^t v(u)\,du}\,|v(s-1) - a(s)|\;ds = \frac{3}{128}\int_0^t e^{-v(t-s)}\,ds + \frac{1}{64}\int_0^t e^{-v(t-s)}\sin\left(\frac{5}{16}s\right)\,ds < 0.104.$$

Hence,

$$\int_{t-1}^{t}|v(s)|\,ds + \int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-1}^{s}|v(u)|\,du\,ds$$
$$+ \int_0^t e^{-\int_s^t v(u)\,du}\,|v(s-1) - a(s)|\;ds < 0.9165.$$

Since $\int_0^t v(s)\,ds = \frac{t}{32} \to \infty$ as $t \to \infty$, the conditions of Corollary 4.1.14 are satisfied. Therefore, the zero solution of (4.21) is asymptotically stable.

Remark 4.1.20. *Zhao* [145] *investigated the case for which* $\left|\frac{c(t)}{1-\tau'(t)}\right| < 1$ *does not hold by considering the following neutral differential equation*

$$x'(t) = -b(t)x(t-\tau(t)) + c(t)x'(t-\tau(t)),$$

and presented a new criteria for asymptotic stability of the zero solution by employing an auxiliary function $p(t)$, but there seems to be a mistake in his computation for the transformations on page 6. We obtain that (4.12) in [145] actually should be

$$z'(t) = -\frac{p'(t)}{p(t)}z(t) - \frac{b(t)p(t-\tau(t)) - c(t)p'(t-\tau(t))}{p(t)}z(t-\tau(t)) + \frac{c(t)p(t-\tau(t))}{p(t)}z'(t-\tau(t)),$$

which is a special form of (4.17). By using the condition in Corollary 4.1.14*, the correct condition* (iii) *in* Theorem 3.1 *on page 6 of Zhao [145] should be*

$$\begin{aligned}&\left|\frac{p(t-\tau(t))c(t)}{p(t)(1-\tau'(t))}\right| + \int_{t-\tau(t)}^{t}\left|v(s) - \frac{p'(s)}{p(s)}\right| ds\\ &+ \int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-\tau(s)}^{s}\left|v(u) - \frac{p'(u)}{p(u)}\right| du\,ds\\ &+ \int_0^t e^{-\int_s^t v(u)\,du}\left\{\left|-\beta(s) + \left(v(s-\tau(s)) - \frac{p'(s-\tau(s))}{p(s-\tau(s))}\right)(1-\tau'(s)) - k(s)\right|\right\} ds \le \alpha,\end{aligned}$$

where

$$\beta(s) = \frac{b(s)p(s-\tau(s)) - c(s)p'(s-\tau(s))}{p(s)},$$

and

$$k(s) = \frac{[C(s)v(s) + C'(s)](1-\tau'(s)) + C(s)\tau''(s)}{(1-\tau'(s))^2}, C(s) = \frac{c(s)p(s-\tau(s))}{p(s)}.$$

4.1.3 Proof of Theorem 4.1.5

In this subsection, we will prove Theorem 4.1.5. We start with some preparations. First we write (4.3) in the following form

$$x'(t) = B(t, t-r(t))(1-r'(t))g(t-r(t), x(t-r(t))) + \frac{d}{dt}\int_{t-r(t)}^{t} B(t,s)g(s,x(s))\,ds, \quad (4.22)$$

where

$$B(t,s) := \int_t^s a(u,s)\,du, \quad \text{with} \quad B(t,t-r(t)) := \int_t^{t-r(t)} a(u, t-r(t))\,du. \quad (4.23)$$

If we multiply both sides of (4.22) by $e^{\int_0^t v(s)\,ds}$, then integrate from 0 to t, and then perform an integration by parts, then we obtain

$$\begin{aligned}x(t) \;=\; &\left\{\phi(0) - \int_{-r(0)}^{0}[v(s) + B(0,s)]g(s,\phi(s))\,ds\right\}e^{-\int_0^t v(s)\,ds}\\ &+ \int_{t-r(t)}^{t}[v(s) + B(t,s)]g(s,\phi(s))ds\\ &- \int_0^t e^{-\int_s^t v(u)\,du}v(s)\int_{s-r(s)}^{s}[v(u) + B(s,u)]g(u,x(u))\,du\,ds\\ &+ \int_0^t e^{-\int_s^t v(u)\,du}v(s)[x(s) - g(s,x(s))]\,ds\\ &+ \int_0^t e^{-\int_s^t v(u)\,du}[v(s-r(s)) + B(s,s-r(s))](1-r'(s))g(s-r(s),x(s-r(s)))\,ds.\end{aligned}$$

By (ii), we choose a common Lipschitz constant L for $g(s,x)$ and $x-g(s,x)$ on $[-l,l]$. For $t\in[r_0,\infty)$ and a constant $k>4$, we define

$$h(t)=kL\int_0^t[v(u)+q(u)+p(u)]\,du, \tag{4.24}$$

where q is as defined in (iv) of Theorem 4.1.5 and

$$p(u)=[v(u-r(u))+B(u,u-r(u))](1-r'(u)).$$

Now, let C be the space of all continuous functions $\varphi:[r_0,\infty)\to\mathbb{R}$ such that

$$|\varphi|_h:=\sup\left\{|\varphi(t)|e^{-h(t)}:t\in[r_0,\infty)\right\}<\infty,$$

where h is given by (4.24). Then $(C,|\cdot|_h)$ is a Banach space, which can be verified by Cauchy's criterion for uniform convergence. Thus (C,d) is a complete metric space, where d denotes the induced metric: $d(\varphi,\eta)=|\varphi-\eta|_h$ for $\varphi,\eta\in S$. Define

$$C_\phi^l=\left\{\varphi\mid\varphi\in C,\ \|\varphi\|=\sup_{t\geq r_0}|\varphi(t)|\leq l,\ \varphi(t)=\phi(t)\text{ for }t\in[r_0,0]\right\},$$

where $\phi:[r_0,0]\to[-l,l]$ is a given continuous initial function. Then C_ϕ^l is a closed subset of C and hence a complete metric space with the metric inherited from C.

Lemma 4.1.21. *Define the operator by $P\varphi(t)=\phi(t)$ for $t\in[r_0,0]$ and for $t\geq 0$,*

$$\begin{aligned}(P\varphi)(t)\quad=\quad&\left\{\phi(0)-\int_{-r(0)}^0[v(s)+B(0,s)]g(s,\phi(s))\,ds\right\}e^{-\int_0^t v(s)\,ds}\\&+\int_{t-r(t)}^t[v(s)+B(t,s)]g(s,\phi(s))ds\\&-\int_0^t e^{-\int_s^t v(u)\,du}v(s)\int_{s-r(s)}^s[v(u)+B(s,u)]g(u,x(u))\,du\,ds\\&+\int_0^t e^{-\int_s^t v(u)\,du}v(s)[x(s)-g(s,x(s))]\,ds\\&+\int_0^t e^{-\int_s^t v(u)\,du}[v(s-r(s))+B(s,s-r(s))](1-r'(s))g(s-r(s),x(s-r(s)))\,ds.\end{aligned}\tag{4.25}$$

If the conditions (i)-(viii) *in Theorem* 4.1.5 *are satisfied, then there exists $\delta>0$ such that for any $\phi:[r_0,0]\to(-\delta,\delta)$, we have that $P:C_\phi^l\to C_\phi^l$ and P is a contraction.*

Proof. First of all, given $\varphi\in C_\phi^l$ we show $P\varphi\in C_\phi^l$. Let $\phi:[r_0,0]\to(-\delta,\delta)$ be a continuous function, where $\delta>0$ satisfies

$$\delta+W(\delta)\int_{-r(0)}^0|v(u)+B(0,u)|\,du\leq w(l)-\alpha W(l). \tag{4.26}$$

Such δ exists since $W(0)=0$ and W is continuous on $[0,l]$. Note that $w(l)-\alpha W(l)>0$ by (iii) and (viii). By (iv), $w(l)\leq l$. For any $\varphi\in C_\phi^l$, by (4.26), we have $|(P\varphi)(t)|=|\phi(t)|<l$ for

$t \in [r_0, 0]$. Now we consider $(P\varphi)(t)$ for $t > 0$. By (i) and (iii), $|g(s,x)| \le W(l)$ for $x \in [-l, l]$ and $t \ge r_0$, thus using (iii) and (v), we obtain

$$\begin{aligned} |P(\varphi)(t)| &\le \delta + W(\delta)\int_{-r(0)}^{0} |v(u) + B(0,u)|\, du \\ &\quad + W(l)\int_{t-r(t)}^{t} |v(u) + B(t,u)|\, du \\ &\quad + W(l)\int_0^t e^{-\int_s^t v(u)\,du} v(s) \int_{s-r(s)}^{s} |v(u) + B(s,u)|\, du\, ds \\ &\quad + W(l)\int_0^t e^{-\int_s^t v(u)\,du} |v(s - r(s)) + B(s, s-r(s))||1 - r'(s)|\, ds \\ &\quad + (l - w(l))\int_0^t e^{-\int_s^t v(u)\,du} v(s)\, ds \\ &\le w(l) - \alpha W(l) + \alpha W(l) + l - w(l) = l. \end{aligned} \tag{4.27}$$

So $|P(\varphi)(t)| \le l$ for $t \in [r_0, \infty)$. Therefore, $P\varphi \in C_\phi^l$.

Next, we show that P is a contraction mapping on C_ϕ^l. Suppose that $\varphi, \eta \in C_\phi^l$,

$$\begin{aligned} &|P\varphi(t) - P\eta(t)|e^{-h(t)} \\ &\le \int_{t-r(t)}^{t} e^{-kL\int_u^t [v(s)+q(s)]\,ds} |v(u) + B(t,u)| L |\varphi(u) - \eta(u)| e^{-h(u)}\, du \\ &\quad + \int_0^t e^{-\int_s^t v(u)\,du} v(s) \int_{s-r(s)}^{s} e^{-kL\int_u^s [v(\theta)+q(\theta)]\,d\theta} |v(u) + B(s,u)| L |\varphi(u) - \eta(u)| e^{-h(u)}\, du\, ds \\ &\quad + \int_0^t e^{-kL\int_s^t p(u)\,du} p(s) L |\varphi(s - r(s)) - \eta(s - r(s))| e^{-h(s-r(s))}\, ds \\ &\quad + \int_0^t e^{-(kL+1)\int_s^t v(u)\,du} v(s) L |\varphi(s) - \eta(s)| e^{-h(s)}\, ds, \end{aligned}$$

since $|v(u) + B(t,u)| \le v(u) + q(u)$ for $t - r(t) \le u \le t$, we have

$$|P\varphi - P\eta|_h \le \left(\frac{1}{kL} + \frac{1}{kL} + \frac{1}{kL} + \frac{1}{kL}\right) L |\varphi - \eta|_h = \frac{4}{k}|\varphi - \eta|_h < |\varphi - \eta|_h,$$

since $k > 4$. Therefore, P is a contraction mapping. □

We are now ready to prove Theorem 4.1.5.

Proof. By contraction mapping principle, P has a unique fixed point $x \in C_\phi^l$, which is by construction a solution of (4.3) on $[0, \infty)$ and $|x(t)| \le l$ for $t \ge r_0$. Hence $x(t)$ is the only continuous function satisfying (4.3) for $t \ge 0$ with $x(t) = \phi(t)$ on $[r_0, 0]$.

Let $\varepsilon > 0$ be given and choose $m > 0$, such that $m < \min\{\varepsilon, l\}$, replacing l with m in (4.27), we see that there is $\delta > 0$ such that $|\varphi| \le m < \varepsilon$ for $t \ge r_0$. Hence, the zero solution of (4.3) is stable. □

Example 4.1.22. *Consider the following integro-differential equation*

$$x'(t) = -\int_{0.4635t}^{t} \frac{0.9}{s^2+1}\left(\frac{4}{5}+\frac{1}{10}\sin^2 s\right)x^3(s)\,ds. \tag{4.28}$$

We check the condition (vii) of Theorem 4.1 in Becker and Burton [8]. Since $f(t) = \frac{t}{0.4635}$, we obtain that

$$G(t,s) = \int_{t}^{s/0.4635} \frac{0.9}{s^2+1}\,du = \frac{0.9(s/0.4635-t)}{s^2+1}$$

for $t \geq 0$ and $0.4635t \leq s \leq t$. Consequently,

$$\lim_{t\geq 0}\left\{2\int_{0.4635t}^{t}|G(t,u)|\,du\right\} = 0.9\times 2\left(-\frac{\ln 0.4635+1}{0.4635}+1\right) = 0.9027.$$

Then there exists some $t_0 > 0$ such that for $t \geq t_0$, we have

$$2\int_{0.4635t}^{t}|G(t,u)|\,du > 0.9020.$$

Since $\frac{w(\frac{1}{2})}{W(\frac{1}{2})} = \frac{8}{9} = 0.8889 < 0.9020$. This implies that condition (vii) of Theorem 4.1 in Becker and Burton [8] does not hold. Thus Theorem 4.1 of Becker and Burton [8] can not be applied to equation (4.28). However, by (4.23),

$$B(t,s) = \int_{t}^{s} \frac{0.9}{s^2+1}\,du = \frac{0.9(s-t)}{s^2+1}.$$

Choosing $v(t) = \frac{0.9t}{t^2+1}$,

$$\begin{aligned}
\int_{t-r(t)}^{t}|v(u)+B(t,u)|\,du &= 0.9\times\int_{0.4635t}^{t}\left|\frac{2u-t}{u^2+1}\right|du\\
&= 0.9\times\int_{0.4635t}^{0.5t}\frac{t-2u}{u^2+1}\,du + 0.9\times\int_{0.5t}^{t}\frac{2u-t}{u^2+1}\,du\\
&= 0.9\times\Big[t\,(2\arctan 0.5t - \arctan t - \arctan 0.4635t)\\
&\quad + \ln(t^2+1) + \ln(0.4635^2t^2+1) - 2\ln(0.25t^2+1)\Big]\\
&:= w(t).
\end{aligned}$$

Since the function $w(t)$ is increasing on $[0,\infty)$ and

$$\lim_{t\to\infty} w(t) = 0.9\times(1/0.4635 - 3 + 2\ln 2 + 2\ln 0.927) = 0.3530,$$

we have

$$\int_{t-r(t)}^{t}|v(u)+B(t,u)|\,du < 0.3530,$$

$$\begin{aligned}
&\int_0^t e^{-\int_s^t v(u)\,du}|v(s-r(s))+B(s,s-r(s))||1-r'(s)|\,ds\\
&= (1/0.4635-2)\int_0^t e^{-\int_s^t \frac{0.9u}{u^2+1}\,du}\frac{0.9s}{s^2+1/0.4635^2}\,ds < 0.1575,
\end{aligned}$$

and $\int_0^t e^{-\int_s^t v(u)\,du} v(s) \int_{s-r(s)}^s |v(u) + B(s,u)|\,du\,ds < 0.3530$. Hence, we have

$$\begin{aligned}
&\int_{t-r(t)}^t |v(u) + B(t,u)|\,du + \int_0^t e^{-\int_s^t v(u)\,du} v(s) \int_{s-r(s)}^s |v(u) + B(s,u)|\,du\,ds \\
&+ \int_0^t e^{-\int_s^t v(u)\,du} |v(s-r(s)) + B(s, s-r(s))||1 - r'(s)|\,ds \\
&< 0.8635 < \frac{w(\frac{1}{2})}{W(\frac{1}{2})} = \frac{8}{9} = 0.8889.
\end{aligned}$$

By Theorem 4.1.5, the zero solution of (4.28) is stable. Together with Remark 4.1.6, this shows that our results extends the result of Becker and Burton [8].

4.1.4 Proof of Theorem 4.1.7

In this subsection, we will prove Theorem 4.1.7. We start with some preparation. Equation (4.4) can be written in the following equivalent form

$$x'(t) = -\tilde{a}(h(t))f(x(t)) + \frac{d}{dt}\int_{t-r_1(t)}^t \tilde{a}(h(s))f(x(s))\,ds + b(t)g(x(t - r_2(t))). \tag{4.29}$$

If we multiply both sides of (4.29) by $e^{\int_0^t v(s)\,ds}$, integrate from 0 to t, and perform an integration by parts, then we obtain

$$\begin{aligned}
x(t) \;=\; & \left\{\phi(0) - \int_{-r_1(0)}^0 [\tilde{a}(h(s)) + v(s)]f(\phi(s))\,ds\right\} e^{-\int_0^t v(s)\,ds} \\
&+ \int_0^t e^{-\int_s^t v(u)\,du} v(s)[x(s) - f(x(s))]\,ds - \int_0^t e^{-\int_s^t v(u)\,du} \tilde{a}(h(s))f(x(s))\,ds \\
&- \int_0^t e^{-\int_s^t v(u)\,du} v(s) \int_{s-r_1(s)}^s [\tilde{a}(h(u)) + v(u)]f(x(u))\,du\,ds \\
&+ \int_0^t e^{-\int_s^t v(u)\,du} v(s - r_1(s))(1 - r_1'(s))f(x(s - r_1(s)))\,ds \\
&+ \int_{t-r_1(t)}^t [\tilde{a}(h(s)) + v(s)]f(x(s))\,ds + \int_0^t e^{-\int_s^t v(u)\,du} b(s)g(x(s - r_2(s)))\,ds.
\end{aligned}$$

Let C be the weighted space of all continuous functions $\varphi : [r_0, \infty) \to \mathbb{R}$ with

$$|\varphi|_q := \sup\{|\varphi(t)|e^{-q(t)} : t \in [r_0, \infty)\} < \infty.$$

The weight function $q : [r_0, \infty) \to \mathbb{R}$ is here defined as follows

$$q(t) = \begin{cases} 1 & \text{for} \quad t \in [r_0, 0], \\ dl \int_0^t [v(s) + |\tilde{a}(h(s))| + |b(s)| + w(s)]\,ds & \text{for} \quad t \in [0, \infty), \end{cases}$$

where $d > 5$ is a constant, and

$$w(s) = \begin{cases} 0 & \text{for} \quad s \in [r_0, 0], \\ |v(s - r_1(s))(1 - r_1'(s))| & \text{for} \quad s \in [0, \infty). \end{cases}$$

The space $(C, |\cdot|_q)$ becomes a Banach space, which can be verified with Cauchy's criterion for uniform convergence. Define

$$C^l_\phi = \left\{\varphi \mid \varphi \in C, \quad \|\varphi\| = \sup_{t\geq r_0} |\varphi(t)| \leq l, \ \varphi(t) = \phi(t) \quad \text{for} \quad t \in [r_0, 0]\right\}$$

where $\phi : [r_0, 0] \to [-l, l]$ is a given continuous initial function. Then C^l_ϕ is a closed subset of $(C, |\cdot|)$ and hence a complete metric space with the metric inherited from C.

Lemma 4.1.23. *Let $\varphi \in C^l_\phi$. Define the operator by $P : C^l_\phi \to C^l_\phi$ by $(P\varphi)(t) = \phi(t), t \in [r_0, 0]$, and for $t \geq 0$,*

$$\begin{aligned}
(P\varphi)(t) \ &= \ \left\{\phi(0) - \int_{-r_1(0)}^{0} [\tilde{a}(h(s)) + v(s)]f(\phi(s))\, ds\right\} e^{-\int_0^t v(s)\, ds} \\
&+ \int_0^t e^{-\int_s^t v(u)\, du} v(s)[\varphi(s) - f(\varphi(s))]\, ds - \int_0^t e^{-\int_s^t v(u)\, du}\tilde{a}(h(s))f(\varphi(s))\, ds \\
&- \int_0^t e^{-\int_s^t v(u)\, du} v(s) \int_{s-r_1(s)}^{s} [\tilde{a}(h(u)) + v(u)]f(\varphi(u))\, du\, ds \\
&+ \int_0^t e^{-\int_s^t v(u)\, du} v(s - r_1(s))(1 - r_1'(s))f(\varphi(s - r_1(s)))\, ds \\
&+ \int_{t-r_1(t)}^{t} [\tilde{a}(h(s)) + v(s)]f(\varphi(s))\, ds + \int_0^t e^{-\int_s^t v(u)\, du} b(s)g(\varphi(s - r_2(s)))\, ds.
\end{aligned}$$

If the conditions (i)-(iv) *in Theorem 4.1.7 are satisfied, then there exists $\delta > 0$ such that for any $\phi : [r_0, 0] \to (-\delta, \delta)$, we have that $P : C^l_\phi \to C^l_\phi$ and P is a contraction with respect to the metric we defined on C^l_ϕ.*

Proof. Since f is odd and satisfies a Lipschitz condition on $[-l, l]$, and $f(0) = 0$, we choose a $\delta < l$ that satisfies

$$\delta + f(\delta) \int_{-r_1(0)}^{0} |\tilde{a}(h(s)) + v(s)|\, ds \leq (1 - \alpha)f(l) - \alpha g(l).$$

Let $\phi : [r_0, 0] \to (-\delta, \delta)$ be a continuous function. Thus $|\phi(t)| \leq l$ for $t \in [r_0, 0]$. Now we show for such ϕ, $P : C^l_\phi \to C^l_\phi$. In fact, for arbitrary $\varphi \in C^l_\phi$, it follows from the conditions in Theorem 4.1.7 that we have for $t > 0$,

$$\begin{aligned}
|(P\varphi)(t)| \ &\leq \ \delta + f(\delta) \int_{-r_1(0)}^{0} |\tilde{a}(h(s)) + v(s)|\, ds \\
&\quad + f(l)\left[\int_0^t e^{-\int_s^t v(u)\, du}|\tilde{a}(h(s))|\, ds + \int_{t-r_1(t)}^{t} |\tilde{a}(h(s)) + v(s)|\, ds\right. \\
&\quad + \int_0^t e^{-\int_s^t v(u)\, du}|v(s)| \int_{s-r_1(s)}^{s} |\tilde{a}(h(u)) + v(u)|\, du\, ds \\
&\quad \left. + \int_0^t e^{-\int_s^t v(u)\, du}|v(s - r_1(s))(1 - r_1'(s))|\, ds\right] \\
&\quad + g(l) \int_0^t e^{-\int_s^t v(u)\, du}|b(s)|\, ds + (l - f(l)) \int_0^t e^{-\int_s^t v(u)\, du} v(s)\, ds \\
&\leq \ (1 - \alpha)f(l) - \alpha g(l) + \alpha f(l) + \alpha g(l) + l - f(l) = l.
\end{aligned}$$

Hence $|(P\varphi)(t)| \leq l$ for $t \in [r_0, \infty)$. Therefore $P\varphi \in C^l_\phi$.

Next, we will show that P is a contraction mapping in C^l_ϕ. For $\varphi, \eta \in C^l_\phi$,

$$\begin{aligned}
&|(P\varphi)(t) - (P\eta)(t)|\, e^{-q(t)} \\
&\leq \int_0^t e^{-dL\int_s^t [v(u)+|\tilde{a}(h(u))|]\,du}\, [v(s) + |\tilde{a}(h(s))|]\, L|\varphi(s) - \eta(s)|e^{-q(s)}\, ds \\
&\quad + \int_0^t e^{-\int_s^t v(u)\,du} v(s) \int_{s-r_1(s)}^s e^{-dL\int_s^t [v(s)+|\tilde{a}(h(s))|]\,ds} [|\tilde{a}(h(u))| + v(u)] \\
&\quad \times L|\varphi(u) - \eta(u)|e^{-q(u)}\, du\, ds \\
&\quad + \int_0^t e^{-dL\int_s^t w(u)\,du} w(s) L|\varphi(s - r_1(s)) - \eta(s - r_1(s))|e^{-q(s-r_1(s))}\, ds \\
&\quad + \int_{t-r_1(t)}^t e^{-dL\int_s^t [v(u)+|\tilde{a}(h(u))|]\,du} [|\tilde{a}(h(s))| + v(s)] L|\varphi(s) - \eta(s)|e^{-q(s)}\, ds \\
&\quad + \int_0^t e^{-dL\int_s^t |b(u)|\,du} |b(s)| L|\varphi(s - r_2(s)) - \eta(s - r_2(s))|e^{-q(s-r_2(s))}\, ds.
\end{aligned}$$

So, we have

$$|(P\varphi)(t) - (P\eta)(t)|e^{-q(t)} \leq \left(\frac{1}{dL} + \frac{1}{dL} + \frac{1}{dL} + \frac{1}{dL} + \frac{1}{dL}\right) L|\varphi - \eta|_q \leq \frac{5}{d}|\varphi - \eta|_q$$

for all $t > 0$. Thus $|P\varphi - P\eta|_q \leq \frac{5}{d}|\varphi - \eta|_q$. Since $d > 5$, we conclude that P is a contraction on $(C^l_\phi, |\cdot|_q)$. □

We are now ready to prove Theorem 4.1.7.

Proof. By the contraction mapping principle, P has a unique fixed point x in C^l_ϕ, which is a solution of (4.4) with $x(t) = \phi(t)$ on $[r_0, 0]$ and $|x(t)| \leq l$.

Let $\varepsilon > 0$ be given. Then, we choose $m > 0$ so that $m < min\{l, \varepsilon\}$. By considering C^m_ϕ, we obtain existence of a $\delta > 0$ such that $\|\phi\| < \delta$ implies that the unique solution of (4.4) with $x(t) = \phi(t)$ on $[r_0, 0]$ satifies $|x(t)| \leq m < \varepsilon$ for all $t \geq r_0$. This shows that the zero solution of (4.4) is stable. This completes the proof of Theorem 4.1.7. □

Remark 4.1.24. *It is an open problem whether the zero solution of* (4.4) *is asymptotically stable. Our method of proof can not be used to solve this problem. The reason is that if we would add the condition to C^l_ϕ that $\varphi(t) \to 0$ as $t \to \infty$, then C^l_ϕ would no longer be complete under the weighted metric.*

4.1.5 Proof of Theorem 4.1.10

In this subsection, we will prove Theorem 4.1.10. We start with some preparations. First, we transform (4.5) into a neutral delay differential equation without impuses

$$\begin{aligned}
&z'(t) - \prod_{t-r_1(t)\leq t_k<t} (1 + d_k)^{-1} c(t)(1 - r_1'(t))z'(t - r_1(t)) \qquad (4.30)\\
&= -\prod_{t-r_2(t)\leq t_k<t} (1 + d_k)^{-1} b(t)z(t - r_2(t)) + \prod_{t-r_3(t)\leq t_k<t} (1 + d_k)^{-1} \int_{t-r_3(t)}^t g(t, z(s))\, d\mu(t, s)
\end{aligned}$$

for $t \geq 0$ with initial value $z(t) = \psi(t)$, $t \in [r_0, 0]$. By a solution of (4.30), we mean $z(t) \in C([r_0, \infty], \mathbb{R})$ satisfying (4.30).

Two fundamental results are esdtablished in the following lemmas.

Lemma 4.1.25. *Assume that* $(\mathrm{H}_1) - (\mathrm{H}_5)$ *hold.*

(i) *If* $z(t, \psi)$ *is a solution of* (4.30), *then* $x(t, \psi) = \prod_{0 \leq t_k < t}(1 + d_k) z(t, \psi)$ *is a solution of* (4.5).

(ii) *If* $x(t, \psi)$ *is a solution of* (4.5), *then* $z(t, \psi) = \prod_{0 \leq t_k < t}(1 + d_k)^{-1} x(t, \psi)$ *is a solution of* (4.30).

Proof. Denote by $z(t, \psi) := z(t)$ and $x(t, \psi) := x(t)$. First, we prove (i). It is clear that $x(t) = \prod_{0 \leq t_k < t}(1 + d_k) z(t, \psi)$ is absolutely continuous on each interval (t_k, t_{k+1}) and for any $t \neq t_k$, $k = 1, 2, \ldots$, we have that

$$
\begin{aligned}
& x'(t) - c(t)x'(t - r_1(t)) + b(t)x(t - r_2(t)) - \int_{t - r_3(t)}^{t} g(t, x(s))\, d\mu(t, s) \\
&= \prod_{0 \leq t_k < t}(1 + d_k) z'(t) - c(t) \prod_{0 \leq t_k < t - r_1(t)} (1 + d_k) z'(t - r_1(t)) \\
&\quad + b(t) \prod_{0 \leq t_k < t - r_2(t)} (1 + d_k) z(t - r_2(t)) - \int_{t - r_3(t)}^{t} \prod_{0 \leq t_k < s} (1 + d_k) g(t, z(s))\, d\mu(t, s) \\
&= \prod_{0 \leq t_k < t}(1 + d_k) \Bigg[z'(t) - \prod_{t - r_1(t) \leq t_k < t} (1 + d_k)^{-1} c(t) z'(t - r_1(t)) \\
&\quad + \prod_{t - r_2(t) \leq t_k < t} (1 + d_k)^{-1} b(t) z(t - r_2(t)) - \int_{t - r_3(t)}^{t} \prod_{s \leq t_k < t} (1 + d_k)^{-1} g(t, z(s))\, d\mu(t, s) \Bigg] \\
&= 0.
\end{aligned}
$$

On the other hand, for every t_k, $k = 1, 2, 3 \cdots$,

$$
x(t_k^+) = \lim_{t \to t_k^+} \prod_{0 \leq t_j < t} (1 + d_j) z(t) = \prod_{0 \leq t_j \leq t_k} (1 + d_j) z(t_k)
$$

and $x(t_k) = \prod_{0 \leq t_j < t_k}(1 + d_j) z(t_k)$. Hence, we obtain

$$
x(t_k^+) = (1 + d_k) x(t_k), \tag{4.31}
$$

for $k = 1, 2, \ldots,$. From (4.31) and (4.31), we have that $x(t)$ is the solution of (4.5).

Next, we prove (ii). Since $x(t)$ is absolutely continuous on each interval (t_k, t_{k+1}), it follows that, for any $k = 1, 2, \ldots$,

$$
z(t_k^+) = \prod_{0 \leq t_j \leq t_k} (1 + d_j)^{-1} x(t_k^+) = \prod_{0 \leq t_j < t_k} (1 + d_j)^{-1} x(t_k) = z(t_k)
$$

and

$$
z(t_k^-) = \prod_{0 \leq t_j \leq t_{k-1}} (1 + d_j)^{-1} x(t_k^-) = z(t_k)
$$

which implies that $z(t)$ is continuous on $[0,\infty)$. It is easy to find that $z(t)$ is also absolutely continuous on $[0,\infty)$. and we can easily to check that $z(t)$ is the solution of (4.30) corresponding to intial condition $z(t)=\psi(t)$, $t\in[r_0,0]$. The proof of Lemma 4.1.25 is complete.

Lemma 4.1.26. (Yan and Zhao [134]) *Assume that* $(\mathrm{H}_1)-(\mathrm{H}_5)$ *hold.*

(i) *Suppose that there exists a positive constant $M>0$ such that for any $t\geq 0$, $\prod_{0\leq t_k<t}(1+d_k)\leq M$. In addition, if the zero solution of (4.30) is stable, then the zero solution of (4.5) is also stable.*

(ii) *Suppose that there exists a positive constant $M>0$ such that for any $t\geq 0$, $\prod_{0\leq t_k<t}(1+d_k)\leq M$. In addition, if the zero solution of (4.30) is asymptotically stable, then the zero solution of (4.5) is also asymptotically stable.*

Proof. If we multiply both sides of (4.30) by $e^{\int_0^t(h_1(s)+h_2(s))\,ds}$, integrate from 0 to t, we obtain

$$\begin{aligned} z(t) &= \psi(0)e^{-\int_0^t(h_1(u)+h_2(u))\,du}+\int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}(h_1(s)+h_2(s))z(s)\,ds\\ &\quad+\int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}\prod_{s-r_1(s)\leq t_k<s}(1+d_k)^{-1}c(s)(1-r_1'(s))z'(s-r_1(s))\,ds\\ &\quad-\int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}\prod_{s-r_2(s)\leq t_k<s}(1+d_k)^{-1}b(s)z(s-r_2(s))\,ds\\ &\quad+\int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}\prod_{s-r_3(s)\leq t_k<s}(1+d_k)^{-1}\int_{s-r_3(s)}^s g(s,z(u))\,d\mu(s,u)\,ds. \end{aligned} \tag{4.32}$$

Defining

$$\begin{aligned} J_1(t) &:= \int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}\sum_{j=1}^2 h_j(s)z(s)\,ds\\ &= \sum_{j=1}^2\int_{t-r_j(t)}^t h_j(u)z(u)\,du-e^{-\int_0^t(h_1(u)+h_2(u))\,du}\sum_{j=1}^2\int_{-r_j(0)}^0 h_j(u)\psi(u)\,du\\ &\quad-\int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}(h_1(s)+h_2(s))\sum_{j=1}^2\int_{s-r_j(s)}^s h_j(u)z(u)\,du\,ds\\ &\quad+\int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}\sum_{j=1}^2 h_j(s-r_j(s))(1-r_j'(s))z(s-r_j(s))\,ds, \end{aligned} \tag{4.33}$$

we see that (4.32) can be written as

$$z(t) = \psi(0)e^{-\int_0^t(h_1(u)+h_2(u))\,du}+J_1(t)+\prod_{0\leq t_k<t}(1+d_k)^{-1}(J_2(t)+J_3(t)+J_4(t)),$$

where

$$\begin{aligned}
J_2(t) &= \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} \prod_{s\le t_k<t} (1+d_k)\frac{c(s)}{1-r_1'(s)}\,dx(s-r_1(s))\\
J_3(t) &= -\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} \prod_{s\le t_k<t} (1+d_k)b(s)x(s-r_2(s))\,ds\\
J_4(t) &= \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du} \prod_{s\le t_k<t} (1+d_k)\int_{s-r_3(s)}^{s} g(s,x(u))\,d\mu(s,u)\,ds. \qquad (4.34)
\end{aligned}$$

Define $n(t) := \max\{k \in \mathbb{Z}^+ : t_k < t\}$. Because of the discontinuity of $\prod_{0\le t_k<t}(1+d_k)$ at $t = t_k$, we obtain that $J_2(t)$ is given by the following

$$\begin{aligned}
J_2(t) &= \int_0^{t_1} e^{-\int_s^t (h_1(u)+h_2(u))\,du} \prod_{s\le t_k<t} (1+d_k)\frac{c(s)}{1-r_1'(s)}\,dx(s-r_1(s))\\
&\quad+\sum_{l=2}^{n(t)}\int_{t_{l-1}}^{t_l} e^{-\int_s^t (h_1(u)+h_2(u))\,du} \prod_{l\le k\le n} (1+d_k)\frac{c(s)}{1-r_1'(s)}\,dx(s-r_1(s))\\
&\quad+\int_{t_{n(t)}}^{t} e^{-\int_s^t (h_1(u)+h_2(u))\,du} \frac{c(s)}{1-r_1'(s)}\,dx(s-r_1(s)).
\end{aligned}$$

Performing an integration by parts, we have

$$\begin{aligned}
J_2(t) &= e^{-\int_0^t (h_1(u)+h_2(u))\,du}\\
&\quad\times\Bigg\{\prod_{l\le k\le n}(1+d_k)\left[e^{\int_0^{t_1} H(u)\,du}\frac{c(t_1)}{1-r_1'(t_1)}x(t_1-r_1(t_1))-\frac{c(0)x(-r_1(0))}{1-r_1'(0)}\right]\\
&\quad+\sum_{l=2}^{n(t)}\prod_{l\le k\le n}(1+d_k)\left[e^{\int_0^{t_l}(h_1(u)+h_2(u))\,du}\frac{c(t_l)}{1-r_1'(t_l)}x(t_l-r_1(t_l))\right.\\
&\quad\left.-e^{\int_0^{t_{l-1}}(h_1(u)+h_2(u))\,du}\frac{c(t_{l-1})}{1-r_1'(t_{l-1})}x(t_{l-1}-r_1(t_{l-1}))\right]\\
&\quad-e^{\int_0^{t_n}(h_1(u)+h_2(u))\,du}\frac{c(t_n)}{1-r_1'(t_n)}x(t_n-r_1(t_n))\Bigg\}+\frac{c(t)}{1-r_1'(t)}x(t-r_1(t))\\
&\quad-\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}\prod_{s\le t_k<t}(1+d_k)x(s-r_1(s))k(s)\,ds, \qquad (4.35)
\end{aligned}$$

where

$$k(s) = \frac{[c(s)(h_1(s)+h_2(s))+c'(s)](1-r_1'(s))+c(s)r_1''(s)}{(1-r_1'(s))^2}.$$

Combining (4.33), (4.34), (4.35) together with $x(t,\psi)=\prod_{0\le t_k<t}(1+d_k)z(t,\psi)$, we have

$$\begin{aligned}
z(t) &= e^{-\int_0^t(h_1(u)+h_2(u))\,du}\left[\psi(0)-\sum_{j=1}^{2}\int_{-r_j(0)}^{0}h_j(u)\psi(u)\,du-\frac{c(0)}{1-r_1'(0)}\psi(-r_1(0))+M(t)\right]\\
&\quad+\sum_{j=1}^{2}\int_{t-r_j(t)}^{t}h_j(u)z(u)\,du+\prod_{t-r_1(t)\le t_k<t}(1+d_k)^{-1}\frac{c(t)}{1-r_1'(t)}z(t-r_1(t))\\
&\quad+\int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}\left[h_1(s-r_1(s))(1-r_1'(s))-\prod_{s-r_1(s)\le t_k<s}(1+d_k)^{-1}k(s)\right]\\
&\quad\times z(s-r_1(s))\,ds\\
&\quad+\int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}\left[h_2(s-r_2(s))(1-r_2'(s))-\prod_{s-r_2(s)\le t_k<s}(1+d_k)^{-1}b(s)\right]\\
&\quad\times z(s-r_2(s))\,ds\\
&\quad-\int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}(h_1(s)+h_2(s))\sum_{j=1}^{2}\int_{s-r_j(s)}^{s}h_j(u)z(u)\,du\,ds\\
&\quad+\int_0^t e^{-\int_s^t(h_1(u)+h_2(u))\,du}\prod_{s-r_3(s)\le t_k<s}(1+d_k)^{-1}\int_{s-r_3(s)}^{s}g(s,z(u))\,d\mu(s,u)\,ds,
\end{aligned}$$

where

$$M(t)=\sum_{l=1}^{n(t)}\frac{d_l}{1+d_l}\prod_{t_l-r_1(t_l)\le t_k<t_l}(1+d_k)^{-1}e^{\int_0^{t_l}(h_1(u)+h_2(u))\,du}\frac{c(t_l)}{1-r_1'(t_l)}z(t_l-r_1(t_l)).$$

Define the space

$$S_\phi^l=\Big\{\varphi\mid\varphi\in C([r_0,\infty),\mathbb{R}),\ \|\varphi\|=\sup_{t\ge r_0}|\varphi(t)|\le l,\ \varphi(t)=\phi(t)\text{ for }t\in[r_0,0],\\ \varphi(t)\to 0\text{ as }t\to\infty\Big\}.$$

Then S_ϕ^l is a complete metric space with metric $\rho(x,y)=\sup_{t\ge r_0}\{|x(t)-y(t)|\}$.

Lemma 4.1.27. *Let $\varphi(t)\in S_\phi^l$ and define an operator by $P\varphi(t)=\phi(t)$ for $t\in[r_0,0]$ and for $t\ge 0$,*

$$P\varphi(t)=\sum_{i=1}^{8}I_i(t),$$

where

$$\begin{aligned}
I_1(t) &= e^{-\int_0^t(h_1(u)+h_2(u))\,du}\left[\psi(0)-\sum_{j=1}^{2}\int_{-r_j(0)}^{0}h_j(u)\psi(u)\,du-\frac{c(0)}{1-r_1'(0)}\psi(-r_1(0))\right],\\
I_2(t) &= \sum_{l=1}^{n(t)}\frac{d_l}{1+d_l}\prod_{t_l-r_1(t_l)\le t_k<t_l}(1+d_k)^{-1}e^{-\int_{t_l}^{t}(h_1(u)+h_2(u))\,du}\frac{c(t_l)}{1-r_1'(t_l)}\varphi(t_l-r_1(t_l)),
\end{aligned}$$

$$I_3(t) = \prod_{t-r_1(t)\le t_k<t}(1+d_k)^{-1}\frac{c(t)}{1-r_1'(t)}\varphi(t-r_1(t)), I_4(t) = \sum_{j=1}^{2}\int_{t-r_j(t)}^{t} h_j(u)\varphi(u)\,du,$$

$$I_5(t) = \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}\left[h_1(s-r_1(s))(1-r_1'(s)) - \prod_{s-r_1(s)\le t_k<s}(1+d_k)^{-1}k(s)\right] \times\varphi(s-r_1(s))\,ds,$$

$$I_6(t) = \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}\left[h_2(s-r_2(s))(1-r_2'(s)) - \prod_{s-r_2(s)\le t_k<s}(1+d_k)^{-1}b(s)\right] \times\varphi(s-r_2(s))\,ds,$$

$$I_7(t) = -\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}(h_1(s)+h_2(s))\sum_{j=1}^{2}\int_{s-r_j(s)}^{s} h_j(u)\varphi(u)\,du\,ds,$$

$$I_8(t) = \int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}\prod_{s-r_3(s)\le t_k<s}(1+d_k)^{-1}\int_{s-r_3(s)}^{s} g(s,\varphi(u))\,d\mu(s,u)\,ds.$$

If conditions (i)-(iv) *in Theorem* 4.1.3 *are satisfied, then there exists* $\delta > 0$ *such that for any* $\phi : [r_0, 0] \to (-\delta, \delta)$*, we have that* $P : S_\phi^l \to S_\phi^l$ *and* P *is a contraction with respect to the metric defined on* S_ϕ^l*.*

Proof. Fisrt, we prove that $P\varphi \le l$, for $\varphi \in S_\phi^l$. Indeed, we set

$$J = \sup_{t\ge 0}\left\{e^{-\int_0^t (h_1(s)+h_2(s))\,ds}\right\},$$

by (iv), J is well defined. Since $\varphi \in S_\phi^l$, we have

$$\|P\varphi\| \le \left[1 + \sum_{j=1}^{2}\int_{-r_j(0)}^{0}|h_j(u)|\,du + \left|\frac{c(0)}{1-r_1'(0)}\right|\right]\|\phi\|J + \alpha l.$$

Thus, we choose

$$\|\phi\| \le \delta := \frac{(1-\alpha)l}{\left(1+\sum_{j=1}^{2}\int_{-r_j(0)}^{0}|h_j(u)|\,du + \left|\frac{c(0)}{1-r_1'(0)}\right|\right)J},$$

and we obtain $|(P\varphi)(t)| \le l$.

Then, we prove that $(P\varphi)(t)$ is continuous. It is clear that $I_i(t)$ is continuous for $i = 1, 4, 7$ and $I_2(t), I_3(t), I_5(t), I_6(t)$ are continuous for $t \in (0, t_1)$ or $t \in (t_j, t_{j+1})$ for $j = 1, 2, \cdots$. It remains to prove that $I_2(t) + I_3(t)$, $I_5(t)$, $I_6(t)$ and $I_8(t)$ are continuous at $t = t_j$. Following the same discussion as [6] on page 7245, we have

$$\lim_{r\to 0^+}|I_2(t_j+r) - I_2(t_j) + I_3(t_j+r) - I_3(t_j)| = 0$$
$$\lim_{r\to 0^-}|I_2(t_j+r) - I_2(t_j) + I_3(t_j+r) - I_3(t_j)| = 0.$$

Take the $\lim_{r\to 0}$, we have

$$\begin{aligned}
&|I_5(t_j+r)-I_5(t_j)|\\
&\le \left|e^{-\int_{t_j}^{t_j+r}(h_1(u)+h_2(u))\,du}-1\right|\int_0^{t_j} e^{-\int_s^{t_j}(h_1(u)+h_2(u))\,du}\\
&\quad\times\left|h_1(s-r_1(s))(1-r_1'(s))-\prod_{s-r_1(s)\le t_k<s}(1+d_k)^{-1}k(s)\right||\varphi(s-r_1(s))|\,ds\\
&\quad+\int_{t_j}^{t_j+r} e^{-\int_s^{t_j+r}(h_1(u)+h_2(u))\,du}\left|h_1(s-r_1(s))(1-r_1'(s))-\prod_{s-r_1(s)\le t_k<s}(1+d_k)^{-1}k(s)\right|\\
&\quad\times|\varphi(s-r_1(s))|\,ds\to 0.
\end{aligned}$$

In the same way, we can prove that $I_6(t)$ and $I_8(t)$ are continuous at $t=t_j$. Therefore, $P\varphi$ is continuous.

Next, we prove that $(P\varphi)(t)\to 0$ as $t\to\infty$. Obviously, we have that $I_i(t)\to 0$ for $i=1,3,4$ since $\int_0^t(h_1(u)+h_2(u))\,du\to\infty$, $t-r_1(t)\to\infty$ and $\varphi(t)\to 0$ as $t\to\infty$.

In the following, we prove that $I_2(t)\to 0$ as $t\to\infty$. Since $\varphi(t)\to 0$ and $t-r_j(t)\to\infty$ as $t\to\infty$, for each $\varepsilon>0$, there exists a $N>0$ such that $n(t)\ge N$ implies $|\varphi(t_{n(t)}-r_j(t_{n(t)}))|<\varepsilon$, for $j=1,2$. Thus, we have

$$\begin{aligned}
|I_2(t)| &= \Bigg|e^{-\int_{t_N}^{t}(h_1(u)+h_2(u))\,du}\sum_{l=1}^{N}\frac{d_l}{1+d_l}\prod_{t_l-r_1(t_l)\le t_k<t_l}(1+d_k)^{-1}e^{-\int_{t_l}^{t_N}(h_1(u)+h_2(u))\,du}\\
&\quad\times\frac{c(t_l)}{1-r_1'(t_l)}\varphi(t_l-r_1(t_l))\\
&\quad+\sum_{l=N+1}^{n(t)}\frac{d_l}{1+d_l}\prod_{t_l-r_1(t_l)\le t_k<t_l}(1+d_k)^{-1}e^{-\int_{t_l}^{t}(h_1(u)+h_2(u))\,du}\frac{c(t_l)}{1-r_1'(t_l)}\varphi(t_l-r_1(t_l))\Bigg|\\
&\le \varepsilon+\alpha\varepsilon.
\end{aligned}$$

for t is large enough. In the same way, we can prove that $I_i(t)\to 0$ as $t\to\infty$ for $i=5,6,7,8$.

Finally, we prove that P is a contraction. In fact, for $\varphi,\eta\in S_\phi^l$,

$$\begin{aligned}
&|(P\varphi)(t)-(P\eta)(t)|\\
&\le\Bigg\{\sum_{l=1}^{n(t)}\left|\frac{d_l}{1+d_l}\prod_{t_l-r_1(t_l)\le t_k<t_l}(1+d_k)^{-1}e^{-\int_{t_l}^{t}(h_1(u)+h_2(u))\,du}\frac{c(t_l)}{1-r_1'(t_l)}\right|\\
&\quad+\left|\prod_{t-r_1(t)\le t_k<t}(1+d_k)^{-1}\frac{c(t)}{1-r_1'(t)}\right|+\sum_{j=1}^{2}\int_{t-r_j(t)}^{t}|h_j(u)|\,du
\end{aligned}$$

$$+\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}\left|h_1(s-r_1(s))(1-r_1'(s)) - \prod_{s-r_1(s)\le t_k<s}(1+d_k)^{-1}k(s)\right| ds$$

$$+\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}\left|h_2(s-r_2(s))(1-r_2'(s)) - \prod_{s-r_2(s)\le t_k<s}(1+d_k)^{-1}b(s)\right| ds$$

$$+\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}|h_1(s)+h_2(s)|\sum_{j=1}^{2}\int_{s-r_j(s)}^{s}|h_j(u)|\,du\,ds$$

$$\left.+\int_0^t e^{-\int_s^t (h_1(u)+h_2(u))\,du}\prod_{s-r_3(s)\le t_k<s}(1+d_k)^{-1}\left(V_{[s-r_3(s),s]}(\mu(s,\cdot))\right)\,ds\right\}\|\varphi-\eta\|$$

$$\le \alpha\|\varphi-\eta\|.$$

Thus, $P : S^l_\phi \to S^l_\phi$ is a contraction.

We are now ready to prove Theorem 4.1.10.

Proof. Let P be defined as in Lemma 4.1.27. By the contraction mapping principle, P has a unique fixed point x in S^l_ϕ which is a solution of (4.30) with $x(t) = \phi(t)$ on $[r_0, 0]$ and $x(t) \to 0$ as $t \to \infty$.

To prove stability at $t = 0$, let $\varepsilon > 0$ be given, then we choose $m > 0$ so that $m < \min\{l, \varepsilon\}$. By considering S^m_ϕ, we obtain there is a $\delta > 0$ such that $\|\phi\| < \delta$ implies that the unique solution of (4.30) with $z(t) = \phi(t)$ on $[r_0, 0]$ satifies $|z(t)| \le m < \varepsilon$ for all $t \ge r_0$. This shows that the zero solution of (4.30) is asymptotically stable if (v) holds. Combining this fact with Lemma 4.1.26, we obtain that the zero solution of (4.5) is asymptotically stable.
Conversely, we suppose that (v) fails. Then by (iii), there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty}\int_0^{t_n}(h_1(s)+h_2(s))\,ds = v$ for some $v \in \mathbb{R}$. We may choose a positive constant M such that

$$-M \le \int_0^{t_n}(h_1(s)+h_2(s))\,ds \le M \tag{4.36}$$

for all $n \ge 1$. To simplify our expressions, we define

$$\begin{aligned} w(s) \;:=\; & \left|h_1(s-r_1(s))(1-r_1'(s)) - \prod_{s-r_1(s)\le t_k<s}(1+d_k)^{-1}k(s)\right| \\ & + \left|h_2(s-r_2(s))(1-r_2'(s)) - \prod_{s-r_2(s)\le t_k<s}(1+d_k)^{-1}b(s)\right| \\ & + |h_1(s)+h_2(s)|\sum_{j=1}^{2}\int_{s-r_j(s)}^{s}|h_j(u)|\,du + \prod_{s-r_3(s)\le t_k<s}(1+d_k)^{-1}\left(V_{[s-r_3(s),s]}(\mu(s,\cdot))\right) \end{aligned}$$

for all $s \ge 0$. By (ii) we have

$$\int_0^{t_n} e^{-\int_s^{t_n}(h_1(u)+h_2(u))\,du} w(s)\,ds \le \alpha. \tag{4.37}$$

Combining (4.36) and (4.37), we have

$$\int_0^{t_n} e^{\int_0^s (h_1(u)+h_2(u))\,du} w(s)\,ds \le \alpha e^{\int_0^{t_n}(h_1(u)+h_2(u))\,du} \le \alpha e^M,$$

which yields that the sequence $\int_0^{t_n} e^{\int_0^s (h_1(u)+h_2(u))\,du} w(s)\,ds$ is bounded, there exists a convergent subsequence, we assume that

$$\lim_{k\to\infty}\int_0^{t_{n_k}} e^{\int_0^s (h_1(u)+h_2(u))\,du} w(s)\,ds = \gamma,$$

for some $\gamma \in \mathbb{R}^+$. We choose a positive integer $\overline{k}$ so large that

$$\lim_{k\to\infty}\int_{t_{n_{\overline{k}}}}^{t_{n_k}} e^{\int_0^s (h_1(u)+h_2(u))\,du} w(s)\,ds \le \frac{\delta_0}{4J}$$

for all $n_k > n_{\overline{k}}$, where $\delta_0 > 0$ satisfies $2\delta_0 J e^M + \alpha < 1$.

Now, we consider the solution $x(t) = x(t, t_{n_{\overline{k}}}, \psi)$ of (4.5) with $\psi(t_{n_{\overline{k}}}) = \delta_0$ and $\psi(s) \le \delta_0$ for $s \le t_{n_{\overline{k}}}$, and we may choose ψ such that $|x(t)| \le l$ for $t \ge t_{n_{\overline{k}}}$ and

$$\psi(t_{n_{\overline{k}}}) - \frac{c(t_{n_{\overline{k}}})}{1-r_1'(t_{n_{\overline{k}}})}\psi(t_{n_{\overline{k}}} - r_1(t_{n_{\overline{k}}})) - \sum_{j=1}^{2}\int_{t_{n_{\overline{k}}}-r_j(t_{n_{\overline{k}}})}^{t_{n_{\overline{k}}}} h_j(s)\psi(s)\,ds \ge \frac{1}{2}\delta_0.$$

So, it follows from the above inequality combining with $x(t) = (Px)(t)$ that for $k \ge \overline{k}$,

$$\begin{aligned}
&\left| z(t_{n_k}) - M(t)e^{-\int_0^{t_{n_k}}(h_1(u)+h_2(u))\,du} - \prod_{t_{n_k}-r_1(t_{n_k})\le t_i < t_{n_k}} (1+d_i)^{-1}\frac{c(t_{n_k})}{1-r_1'(t_{n_k})} z(t_{n_k} - r_1(t_{n_k})) \right.\\
&\quad \left. - \sum_{j=1}^{2}\int_{t_{n_k}-r_j(t_{n_k})}^{t_{n_k}} h_j(u)z(u)\,du \right|\\
&\ge \frac{1}{2}\delta_0 e^{-\int_{t_{n_{\overline{k}}}}^{t_{n_k}}(h_1(u)+h_2(u))\,du} - \int_{t_{n_{\overline{k}}}}^{t_{n_k}} e^{-\int_s^{t_{n_k}}(h_1(u)+h_2(u))\,du} w(s)\,ds\\
&= e^{-\int_{t_{n_{\overline{k}}}}^{t_{n_k}}(h_1(u)+h_2(u))\,du}\left(\frac{1}{2}\delta_0 - e^{-\int_0^{t_{n_{\overline{k}}}}(h_1(u)+h_2(u))\,du}\int_{t_{n_{\overline{k}}}}^{t_{n_k}} e^{-\int_0^s (h_1(u)+h_2(u))\,du} w(s)\,ds\right)\\
&\ge e^{-\int_{t_{n_{\overline{k}}}}^{t_{n_k}}(h_1(u)+h_2(u))\,du}\left(\frac{1}{2}\delta_0 - J\int_{t_{n_{\overline{k}}}}^{t_{n_k}} e^{\int_0^s (h_1(u)+h_2(u))\,du} w(s)\,ds\right)\\
&\ge \frac{1}{4}\delta_0 e^{\int_{t_{n_{\overline{k}}}}^{t_{n_k}}(h_1(u)+h_2(u))\,du} \ge \frac{1}{4}\delta_0 e^{-2M} > 0.
\end{aligned} \tag{4.38}$$

On the other hand, if the solution of (4.5) $x(t) = x(t, t_{n_{\overline{k}}}, \phi) \to 0$ as $t \to \infty$. Since $t_{n_k} - r_j(t_{n_k}) \to \infty$ as $k \to \infty$, $j = 1, 2$ and (ii) holds, we have

$$\begin{aligned}
&z(t_{n_k}) - M(t)e^{-\int_0^{t_{n_k}}(h_1(u)+h_2(u))\,du} - \prod_{t_{n_k}-r_1(t_{n_k})\le t_i < t_{n_k}} (1+d_i)^{-1}\frac{c(t_{n_k})}{1-r_1'(t_{n_k})} z(t_{n_k} - r_1(t_{n_k}))\\
&- \sum_{j=1}^{2}\int_{t_{n_k}-r_j(t_{n_k})}^{t_{n_k}} h_j(u)z(u)\,du \to 0, \quad \text{as} \quad k \to \infty,
\end{aligned}$$

which contradicts (4.38). Hence condition (iv) is necessary for the asymptotic stability of the zero solution of (4.5). □

Remark 4.1.28. *When $d_k = 0$, Theorem 4.1.10 is* Theorem 1.3 in [16] under the same sufficient conditions.

Corollary 4.1.29. *Consider the equation*

$$\begin{cases} x'(t) - c(t)x'(t-r(t)) = -b(t)x(t) + \int_{t-r(t)}^{t} g(t, x(s))\, d\mu(t,s), \quad t \neq t_k, \\ x(t_k^+) - x(t_k) = d_k x(t_k), \quad k = 1, 2, ..., \end{cases} \tag{4.39}$$

which can be transformed into a neutral delay differential equation without impuses

$$\begin{aligned} z'(t) - &\prod_{t-r(t)\le t_k<t} (1+d_k)^{-1} c(t)(1-r'(t))z'(t-r(t)) \\ &= -\prod_{t-r(t)\le t_k<t} (1+d_k)^{-1} b(t)z(t) + \prod_{t-r(t)\le t_k<t} (1+d_k)^{-1} \int_{t-r(t)}^{t} g(t, z(s))\, d\mu(t,s) \end{aligned} \tag{4.40}$$

Assume that the delay $r(t)$ is twice differentiable with $r'(t) \neq 1$, and $t - r(t) \to \infty$ as $t \to \infty$, g satisfies (G)*. Suppose that there exists a constant $\alpha \in (0,1)$ and a continuous function $v : [r_0, \infty) \to \mathbb{R}$ such that* $\liminf_{t\to\infty} \int_0^t v(s)\, ds > -\infty$ *and*

$$\begin{aligned} &\sum_{l=1}^{n(t)} \left| \frac{d_l}{1+d_l} \prod_{t_l - r(t_l)\le t_k<t_l} (1+d_k)^{-1} e^{-\int_{t_l}^{t} v(u)\,du} \frac{c(t_l)}{1-r'(t_l)} \right| \\ &+ \left| \prod_{t-r(t)\le t_k<t} (1+d_k)^{-1} \frac{c(t)}{1-r'(t)} \right| + \int_{t-r(t)}^{t} |v(s) - b(s)|\, ds \\ &+ \int_0^t e^{-\int_s^t v(u)\,du} |v(s)| \int_{s-r(s)}^{s} |v(u) - b(u)|\, du\, ds \\ &+ \int_0^t e^{-\int_s^t v(u)\,du} \left| (v(s-r(s)) - b(s-r(s)))\,(1-r'(s)) - \prod_{s-r(s)\le t_k<s} (1+d_k)^{-1} k(s) \right| ds \\ &+ \int_0^t e^{-\int_s^t v(u)\,du} \prod_{s-r(s)\le t_k<s} (1+d_k)^{-1} V_{[s-r(s),s]}(\mu(s,\cdot))\, ds \le \alpha, \end{aligned}$$

where

$$k(s) = \frac{[c(s)v(s) + c'(s)](1-r'(s)) + c(s)r''(s)}{(1-r'(s))^2}. \tag{4.41}$$

Then the zero solution of (4.40) *is asymptotically stable if and only if*

$$\int_0^t v(s)\, ds \to \infty \qquad \text{as} \qquad t \to \infty.$$

Remark 4.1.30. *Suppose that the conditions of Corollary 4.1.29 hold. If there exists a positive constant $M > 0$ such that for any $t \ge 0$, $\prod_{0\le t_k<t}(1+d_k) \le M$, then the zero solution of* (4.39) *is asymptotically stable. Furthermore, when $d_k = 0$, we obtain Corollary* 2.1 *in* [16].

Corollary 4.1.31. *Consider the equation*

$$\begin{cases} x'(t) - c(t)x'(t-r(t)) = -b(t)x(t) + a(t)g(x(t-r(t))), & t \neq t_k, \\ x(t_k^+) - x(t_k) = d_k x(t_k), & k = 1, 2, ..., \end{cases} \tag{4.42}$$

which can be transformed into a neutral delay differential equation without impuses

$$\begin{aligned} z'(t) &- \prod_{t-r(t)\le t_k<t} (1+d_k)^{-1} c(t)(1-r'(t))z'(t-r(t)) \\ &= -\prod_{t-r(t)\le t_k<t} (1+d_k)^{-1} b(t)z(t) + \prod_{t-r(t)\le t_k<t} (1+d_k)^{-1} a(t)g(z(t-r(t))) \end{aligned} \tag{4.43}$$

Assume that $r(t)$ is twice differentiable, $r'(t) \neq 1$, $t - r(t) \to \infty$ as $t \to \infty$. g satisfies (G). *Suppose that there exists a constant $\alpha \in (0,1)$ and a continuous function $v : [r_0, \infty) \to \mathbb{R}$ such that $\liminf_{t\to\infty} \int_0^t v(s)\, ds > -\infty$ and*

$$\begin{aligned} &\sum_{l=1}^{n(t)} \left| \frac{d_l}{1+d_l} \prod_{t_l-r(t_l)\le t_k<t_l} (1+d_k)^{-1} e^{-\int_{t_l}^t v(u)\,du} \frac{c(t_l)}{1-r'(t_l)} \right| \\ &+ \left| \prod_{t-r(t)\le t_k<t} (1+d_k)^{-1} \frac{c(t)}{1-r'(t)} \right| + \int_{t-r(t)}^t |v(s)-b(s)|\, ds \\ &+ \int_0^t e^{-\int_s^t v(u)\,du} |v(s)| \int_{s-r(s)}^s |v(u)-b(u)|\, du\, ds \\ &+ \int_0^t e^{-\int_s^t v(u)\,du} \left| (v(s-r(s)) - b(s-r(s)))\,(1-r'(s)) - \prod_{s-r(s)\le t_k<s} (1+d_k)^{-1} k(s) \right| ds \\ &+ \int_0^t e^{-\int_s^t v(u)\,du} \prod_{s-r(s)\le t_k<s} (1+d_k)^{-1} |a(s)|\, ds \le \alpha, \end{aligned} \tag{4.44}$$

where $k(s)$ is defined as (4.41). *Then the zero solution of* (4.43) *is asymptotically stable if and only if $\int_0^t v(s)\, ds \to \infty$ as $t \to \infty$.*

Remark 4.1.32. *Suppose the conditions of Corollary 4.1.31 hold. If there exists a positive constant $M > 0$ such that for any $t \ge 0$, $\prod_{0\le t_k<t}(1+d_k) \le M$, then the zero solution of* (4.42) *is asymptotically stable. Furthermore, when $d_k = 0$, we obtain Corollary 4.1.15.*

Remark 4.1.33. *For the case when $g(x) = x$, condition* (4.44) *becomes*

$$\begin{aligned} &\sum_{l=1}^{n(t)} \left| \frac{d_l}{1+d_l} \prod_{t_l-r(t_l)\le t_k<t_l} (1+d_k)^{-1} e^{-\int_{t_l}^t v(u)\,du} \frac{c(t_l)}{1-r'(t_l)} \right| + \left| \prod_{t-r(t)\le t_k<t} (1+d_k)^{-1} \frac{c(t)}{1-r'(t)} \right| \\ &+ \int_{t-r(t)}^t |v(s)-b(s)|\, ds + \int_0^t e^{-\int_s^t v(u)\,du} |v(s)| \int_{s-r(s)}^s |v(u)-b(u)|\, du\, ds \\ &+ \int_0^t e^{-\int_s^t v(u)\,du} \left| \prod_{s-r(s)\le t_k<s} (1+d_k)^{-1} \right| \\ &\times \left| a(s) + \prod_{s-r(s)\le t_k<s} (1+d_k)\,(v(s-r(s)) - b(s-r(s)))\,(1-r'(s)) - k(s) \right| ds \le \alpha. \end{aligned} \tag{4.45}$$

Remark 4.1.34. *In the proof of Theorem* 3.1 *of* [144]*, on page* 7243 *and page* 7244*, it seems to be some mistakes in their computations for* $J_4(t)$ *and* $z(t)$*, we obtain that on the first line of the representation of* $J_4(t)$ *(on page* 7243*) is*

$$\begin{aligned} J_4(t) &= e^{-\int_0^t (h_1(u)+h_2(u))\,du} \\ &\times \Bigg\{ \prod_{1\le k\le n} (1+d_k)\left[e^{\int_0^{t_1}(h_1(u)+h_2(u))\,du} \frac{c(t_1)}{1-\tau'(t_1)} x(t_1-\tau(t_1)) - \frac{c(0)x(-\tau(0))}{1-\tau'(0)} \right] \cdots, \end{aligned}$$

and on the third line of the representation of $z(t)$ *(on page* 7244*) is*

$$\int_0^t e^{-\int_s^t h(u)\,du}(1+d_k)^{-1} N(s) z(s-\tau(s))\,ds.$$

Furthermore, we have that $N(t)$ *(on page* 7244*) is*

$$N(t) = -b(t) + \prod_{t-\tau(t)\le t_k<t} (1+d_k) h(t-\tau(t))(1-\tau'(t)) - r(t).$$

Example 4.1.35.

$$\begin{cases} x'(t) = -b(t)x(t) + a(t)g(x(t-r(t))), \quad t \neq t_k, \\ x(t_k^+) - x(t_k) = d_k x(t_k), \quad k = 1, 2, ..., \end{cases} \tag{4.46}$$

for $t \ge \frac{1}{2}$*, where* $g(x) = \gamma_1 x$ *if* $x \ge 0$*,* $g(x) = \gamma_2 x$ *if* $x < 0$*,* $\gamma_1, \gamma_2 \in [0,1]$*,* $r(t) = 4$*,* $b(t) = \frac{\alpha}{t+1}$*,* $a(t) = \frac{\beta}{t+1}$*,* $\alpha, \beta > 0$*,* $\frac{5\beta}{\alpha} < 1$*,* $t_k = 3k\pi$*,* $1 + d_k = \frac{1}{5}$ *for all* $k = 1, 2, ...$
Then the zero solution of (4.46) is asymptotically stable.

Proof. It is not difficult to check that g satisfies the conditions in (H_3), $t_k = 2k\pi$ and $r(t) = 4$ implies that at most one impulse occurs at interval $[t-r(t), t)$ and hence $\prod_{t-r(t)\le t_k<t}(1+d_k)^{-1} \le 5$.

Choosing $v(t) = b(t) = \frac{\alpha}{t+1}$, from (4.44) of Corollary 4.1.31, we have

$$\int_0^t e^{-\int_s^t v(u)\,du} \prod_{s-r(s)\le t_k<s} (1+d_k)^{-1}|a(s)|\,ds \le 5\int_0^t e^{-\int_s^t \frac{\alpha}{u+1}\,du} \frac{\beta}{s+1}\,ds \le \frac{5\beta}{\alpha} < 1.$$

Since $v(t) = b(t)$, the left terms in (4.44) are all 0. Hence all the conditions of Corollary 4.1.31 hold. On the other hand, $\left|\prod_{\frac{1}{2}\le t_k<t}(1+d_k)\right| \le 1$. Hence, we obtain that the zero solution of (4.46) is asymptotically stable. □

Example 4.1.36.

$$\begin{cases} x'(t) - c(t)x'(t-r(t)) = -b(t)x(t) + a(t)x(t-r(t)), \quad t \neq t_k, \\ x(t_k^+) - x(t_k) = d_k x(t_k), \quad k = 1, 2, ... \end{cases} \tag{4.47}$$

for $t \ge \frac{1}{2}$*, where* $c(t) = \alpha \sin^3(\frac{t}{2})$*,* $\alpha < 1$*,* $r(t) = 2 + \sin t$*,* $b(t) = \frac{0.1}{t+1}$*,* $a(t) = k(t)$*,* $k(t)$ *is denoted as* (4.41)*,* $t_k = 2k\pi$*,* $1 + d_k = \frac{1}{2}$ *for all* $k = 1, 2, ...$
Then the zero solution of (4.47) is asymptotically stable.

Proof. $t_k = 2k\pi$ and $r(t) = 2 + \sin t < 3$ implies that at most one impulse occurs at interval $[t - r(t), t)$ and hence $\prod_{t-r(t)\le t_k<t}(1+d_k)^{-1} \le 2$. Choosing $v(t) = \frac{0.1}{t+1}$, from (4.45) of Remark 4.1.33, we have

$$\max_{t\in[0.5,\infty]}\left|\prod_{t-r(t)\le t_k<t}(1+d_k)^{-1}\frac{c(t)}{1-r'(t)}\right| \le \max_{t\in[0.5,\infty]}\left\{2\cdot\frac{\alpha}{2}\cdot\left|\sin\frac{t}{2}\right|\right\} \le \alpha,$$

$$\sum_{l=1}^{n(t)}\left|\frac{d_l}{1+d_l}\prod_{t_l-r(t_l)\le t_k<t_l}(1+d_k)^{-1}e^{-\int_{t_l}^{t}v(u)\,du}\frac{c(t_l)}{1-r'(t_l)}\right| = 0.$$

Since $v(t) = b(t)$ and $a(t) = k(t)$, the left terms in (4.45) are all 0. Hence all the conditions of Remark 4.1.33 hold. On the other hand, $\left|\prod_{\frac{1}{2}\le t_k<t}(1+d_k)\right| \le 1$. Hence, we obtain that the zero solution of (4.47) is asymptotically stable. □

4.2 A new criteria for stability of nonlinear functional differential equations based on a fixed point method

4.2.1 Introduction and main results

In this section, we study the stability of the following two classes of nonlinear neutral differential equations

$$x'(t) - c(t)x(t-r(t))x'(t-r(t)) = -a(t)x(t) + b(t)g(x(t-r(t))), \tag{4.48}$$

$$x'(t) - c(t)x(t-r(t))x'(t-r(t)) = -a(t)x(t) + \int_{t-r(t)}^{t}K(t,s)g(x(s))\,ds, \tag{4.49}$$

using a fixed point method, where $a, b : [0,\infty) \to \mathbb{R}$ are continuous functions, $c(t) : [0,\infty) \to \mathbb{R}$ is continuously differentiable function and $r(t) : [0,\infty) \to [0,\infty)$ is a continuous function, $K(t,s) : [0,\infty) \times [r_0,\infty) \to \mathbb{R}$ is a continuous function, where $r_0 = \inf\{t - r(t) : t \ge 0\}$, $g(x) = x^\gamma$ for $\gamma \ge 1$, then g satisfies a locally Lipschitz condition, that is, there exists $L > 0$ and $l > 0$ such that g satisfies $|g(x) - g(y)| \le L|x-y|$, for $x, y \in [-l, l]$.

Ahcene and Rabah [34] have studied the special case of (4.48) and (4.49) when $g(x) = x^2$. The results in [34] mainly dependent on the constraint $\left|\frac{c(t)}{1-r'(t)}\right| < 1$. However, there are interesting examples where the constraint is not satisfied. It is our aim in this section to remove this constraint condition and consider the stability of (4.48) and (4.49). We introduce an auxiliary continuous function $p(t)$ to define an appropriate mapping defined on a complete metric space so that we can apply a fixed point argument, and present new criteria for asymptotic stability of differential equations (4.48) and (4.49) which can be applied to the case $\left|\frac{c(t)}{1-r'(t)}\right| \ge 1$ as well. In addition, we present two examples to illustrate our main results.

A standard fixed point argument shows that the differential equation (4.48) provided with an initial condition

$$x(t) = \phi(t) \quad \text{for} \quad \mathrm{t} \in [\mathrm{r}_0, 0], \tag{4.50}$$

where $\phi \in C([r_0, 0], \mathbb{R})$ defines a well-posed initial-value problem and we denote by $x(t) := x(t, \phi)$ the solution of (4.48) with initial function (4.50).

Theorem 4.2.1. *Consider the nonlinear neutral delay differential equation (4.48) and suppose that*

(i) *the delay $r(t)$ is twice differentiable with $r'(t) \neq 1$, and $t - r(t) \to \infty$ as $t \to \infty$;*

(ii) *there exists a bounded positive function $p : [r_0, \infty) \to (0, \infty)$ with $p(0) = 1$ such that $p'(t)$ exists on $[r_0, \infty)$, and there exists a constant $\alpha \in (0, 1)$, a constant $l > 0$ and an arbitrary continuous functions $v : [r_0, \infty) \to \mathbb{R}$ such that*

$$\begin{aligned} &l\left\{\left|\frac{c(t)p^2(t-r(t))}{p(t)(1-r'(t))}\right| + \int_0^t |\overline{k}(s) - 2b_1(s)|e^{-\int_s^t v(u)\,du}\,ds\right\} \\ &+L\int_0^t e^{-\int_s^t v(u)\,du}\frac{|b(s)|p(s-r(s))^\gamma}{p(s)}\,ds + \int_{t-r(t)}^t \left|v(s) - a(s) - \frac{p'(s)}{p(s)}\right| ds \\ &+\int_0^t e^{-\int_s^t v(u)\,du}\left|v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\right| |1-r'(s)|\,ds \\ &+\int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r(s)}^s \left|v(u) - a(u) - \frac{p'(u)}{p(u)}\right| du\,ds \leq \alpha, \end{aligned} \tag{4.51}$$

where

$$\begin{aligned} \overline{k}(s) &= \frac{[\overline{c}(s)v(s) + \overline{c}'(s)](1-r'(s)) + \overline{c}(s)r''(s)}{(1-r'(s))^2}, \\ \overline{c}(s) &= \frac{c(s)p^2(s-r(s))}{p(s)}, \end{aligned} \tag{4.52}$$

$$b_1(s) = \frac{c(s)p(s-r(s))p'(s-r(s))}{p(s)}; \tag{4.53}$$

(iii) *and such that*

$$\liminf_{t\to\infty} \int_0^t v(s)\,ds > -\infty.$$

Then the zero solution $x(t, \phi)$ of (4.48) with a small continuous function $\phi(t)$ is asymptotically stable if and only if

(iv)

$$\int_0^t v(s)\,ds \to \infty \quad as \quad t \to \infty.$$

Remark 4.2.2. *The technique for constructing a mapping for fixed point argument comes from the idea in [145]. Our work extends and improves the results in [34, 145].*

Theorem 4.2.3. *Consider the nonlinear neutral delay differential equation (4.49) and suppose that*

(i) *the delay $r(t)$ is twice differentiable with $r'(t) \neq 1$, and $t - r(t) \to \infty$ as $t \to \infty$;*

(ii) *there exists a bounded function $p : [r_0, \infty) \to (0, \infty)$ with $p(0) = 1$ such that $p'(t)$ exists on $[r_0, \infty)$, and there exists a constant $\alpha \in (0, 1)$, a constant $l > 0$ and a continuous functions $v : [r_0, \infty) \to \mathbb{R}$ such that*

$$l\left\{\left|\frac{c(t)p^2(t-r(t))}{p(t)(1-r'(t))}\right| + \int_0^t |\overline{k}(s) - 2b_1(s)|e^{-\int_s^t v(u)\,du}\,ds\right\} \tag{4.54}$$
$$+L\int_0^t e^{-\int_s^t v(u)\,du}\int_{s-r(s)}^s \frac{|K(s,u)|p^\gamma(u)}{p(s)}\,du + \int_{t-r(t)}^t \left|v(s) - a(s) - \frac{p'(s)}{p(s)}\right| ds$$
$$+\int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r(s)}^s \left|v(u) - a(u) - \frac{p'(u)}{p(u)}\right| du\,ds$$
$$+\int_0^t e^{-\int_s^t v(u)\,du}\left|v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\right||1-r'(s)|\,ds \le \alpha,$$

where $\overline{k}(s)$ and $b_1(s)$ are defined as (4.52) and (4.53), respectively.

(iii)

$$\liminf_{t\to\infty}\int_0^t v(s)\,ds > -\infty.$$

Then the zero solution of $x(t, \psi)$ of (4.49) with a small continuous function $\psi(t)$ is asymptotically stable if and only if

(iv)

$$\int_0^t v(s)\,ds \to \infty \quad as \quad t \to \infty.$$

4.2.2 Proof of Theorem 4.2.1

In this subsection, we will prove Theorem 4.2.1. We start with some preparations. First define

$$S_\phi^l = \Big\{\varphi \mid \varphi \in C([r_0,\infty),\mathbb{R}),\ \|\varphi\| = \sup_{\varphi \ge r_0}|\varphi(t)| \le l,\ \varphi(t) = \phi(t) \text{ for } t \in [r_0, 0],$$
$$\varphi(t) \to 0 \text{ as } t \to \infty\Big\}.$$

Then S_ϕ^l is complete metric space with metric $\rho(x, y) = \sup_{t\ge r_0}\{|x(t) - y(t)|\}$.

Let $z(t) = \phi(t)$ on $[r_0, 0]$, and let

$$x(t) = p(t)z(t) \qquad \text{for} \quad t \ge 0. \tag{4.55}$$

Substituting (4.55) into (4.48), we obtain that

$$z'(t) = -\left(a(t) + \frac{p'(t)}{p(t)}\right)z(t) + \frac{c(t)p(t-r(t))p'(t-r(t))}{p(t)}z^2(t-r(t))$$
$$+\frac{c(t)p^2(t-r(t))}{p(t)}z(t-r(t))z'(t-r(t)) + \frac{b(t)p(t-r(t))^\gamma}{p(t)}g(z(t-r(t))). \tag{4.56}$$

Since $p(t)$ is a positive bounded function, we only have to prove that the zero solution of (4.56) is asymptotically stable.

If we multiply both sides of (4.56) by $e^{\int_0^t v(s)\,ds}$, integrate from 0 to t, and perform an integration by parts, we obtain

$$\begin{aligned} z(t) &= \left\{\phi(0) - \int_{-r(0)}^{0}\left[v(s) - a(s) - \frac{p'(s)}{p(s)}\right]\phi(s)\,ds - \frac{p^2(-r(0))}{2p(0)}\frac{c(0)}{1-r'(0)}\phi^2(-r(0))\right\} \\ &\qquad \times e^{-\int_0^t v(s)\,ds} + \frac{p^2(t-r(t))}{2p(t)}\frac{c(t)}{1-r'(t)}z^2(t-r(t)) \\ &\quad + \int_{t-r(t)}^{t}\left[v(s) - a(s) - \frac{p'(s)}{p(s)}\right]z(s)\,ds \\ &\quad - \int_0^t e^{-\int_s^t v(u)\,du}v(s)\int_{s-r(s)}^{s}\left[v(u) - a(u) - \frac{p'(u)}{p(u)}\right]z(u)\,du\,ds \\ &\quad + \int_0^t e^{-\int_s^t v(u)\,du}\left[v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\right](1-r'(s))z(s-r(s))\,ds \\ &\quad - \frac{1}{2}\int_0^t e^{-\int_s^t v(u)\,du}\left(\overline{k}(s) - 2b_1(s)\right)z^2(s-r(s))\,ds \\ &\quad + \int_0^t e^{-\int_s^t v(u)\,du}\frac{b(s)p(s-r(s))^\gamma}{p(s)}g(z(s-r(s)))\,ds, \end{aligned}$$

where $\overline{k}(s)$ and $b_1(s)$ are defined as in (4.52) and (4.53) respectively.

Lemma 4.2.4. *Let $\varphi(t) \in S_\phi^l$ and define an operator by $P\varphi(t) = \phi(t)$ for $t \in [r_0, 0]$ and for $t \geq 0$,*

$$(P\varphi)(t) = I_1(t) + I_2(t) + I_3(t) + I_4(t) + I_5(t) + I_6(t), \tag{4.57}$$

where

$$\begin{aligned} I_1(t) &= \left[\psi(0) - \int_{-r(0)}^{0}\left[v(s) - a(s) - \frac{p'(s)}{p(s)}\right]\phi(s)\,ds - \frac{p^2(-r(0))}{2p(0)}\frac{c(0)}{1-r'(0)}\phi^2(-r(0))\right] \\ &\qquad \times e^{-\int_0^t v(s)\,ds}, \\ I_2(t) &= \frac{p^2(t-r(t))}{2p(t)}\frac{c(t)}{1-r'(t)}\varphi^2(t-r(t)), I_3(t) = \int_{t-r(t)}^{t}\left[v(s) - a(s) - \frac{p'(s)}{p(s)}\right]\varphi(s)\,ds, \\ I_4(t) &= -\int_0^t e^{-\int_s^t v(u)\,du}v(s)\int_{s-r(s)}^{s}\left[v(u) - a(u) - \frac{p'(u)}{p(u)}\right]\varphi(u)\,du\,ds, \\ I_5(t) &= \int_0^t e^{-\int_s^t v(u)\,du}\left[v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\right](1-r'(s))\varphi(s-r(s))\,ds, \end{aligned}$$

$$I_6(t) = -\frac{1}{2}\int_0^t e^{-\int_s^t v(u)\,du}\Big(\overline{k}(s) - 2b_1(s)\Big)\varphi^2(s - r(s))\,ds,$$

$$I_7(t) = \int_0^t e^{-\int_s^t v(u)\,du}\frac{b(s)p(s-r(s))^\gamma}{p(s)}g(\varphi(s - r(s)))\,ds.$$

If conditions (i)-(iv) *in Theorem* 4.2.1 *are satisfied, then there exists* $\delta > 0$ *such that for any* $\phi : [r_0, 0] \to (-\delta, \delta)$*, we have that* $P : S^l_\phi \to S^l_\phi$ *and* P *is a contraction with respect to the metric defined on* S^l_ϕ*.*

Proof. Set $J = \sup_{t\geq 0}\left\{e^{-\int_0^t v(s)\,ds}\right\}$, by (iii), J is well defined. Suppose that (iv) holds.

Let ϕ be a given small bounded initial function with $\|\phi\| < \delta$, and let $\varphi \in S^l_\phi$, then $\|\varphi\| \leq l$ for l and α, we choose $\delta > 0$ such that

$$\left[\delta + \delta\int_{-r(0)}^0\left|v(s) - a(s) - \frac{p'(s)}{p(s)}\right|ds + \frac{p^2(-r(0))}{2p(0)}\frac{c(0)}{1 - r'(0)}\delta^2\right]J \leq (1-\alpha)l.$$

Since g satisfies locally Lipschitz condition, from (4.51) in Theorem 4.2.1, we have

$$|P\varphi(t)| \leq \left[\delta + \delta\int_{-r(0)}^0\left|v(s) - a(s) - \frac{p'(s)}{p(s)}\right|ds + \frac{p^2(-r(0))}{2p(0)}\frac{c(0)}{1 - r'(0)}\delta^2\right]J + \alpha l \leq l.$$

Thus, $\|P\varphi\| \leq l$.

Next, we show that $P\varphi \to 0$ as $t \to \infty$. It is clear that $I_i(t) \to 0$ for $i = 1, 2, 3, 4, 5, 7$, since $e^{\int_0^t v(s)\,ds} \to \infty$, $t - r(t) \to \infty$ and $\varphi \to 0$ as $t \to \infty$. Now, we prove that $I_6(t) \to 0$ as $t \to \infty$. For $t - r(t) \to \infty$ and $\varphi \to 0$, we obtain that for any $\varepsilon > 0$, there is a positive number $T_1 > 0$ such that for $t \geq T_1$, $\varphi(t - r(t)) < \varepsilon$, so we have

$$\begin{aligned}
|I_6(t)| &= \left|\frac{1}{2}\int_0^t e^{-\int_s^t v(u)\,du}\Big(\overline{k}(s) - 2b_1(s)\Big)\varphi^2(s - r(s))\,ds\right| \\
&\leq \frac{1}{2}e^{-\int_{T_1}^t v(u)\,du}\int_0^{T_1} e^{-\int_s^{T_1} v(u)\,du}\left|\overline{k}(s) - 2b_1(s)\right|\varphi^2(s - r(s))\,ds \\
&\quad + \frac{1}{2}\int_{T_1}^t e^{-\int_s^t v(u)\,du}\left|\overline{k}(s) - 2b_1(s)\right|\varphi^2(s - r(s))\,ds \\
&\leq \frac{1}{2}\left(\sup_{t\geq r_0}|\varphi(t)|\right)^2 e^{-\int_{T_1}^t v(u)\,du}\int_0^{T_1} e^{-\int_s^{T_1} v(u)\,du}\left|\overline{k}(s) - 2b_1(s)\right|ds \\
&\quad + \frac{1}{2}\varepsilon^2\int_{T_1}^t e^{-\int_s^t v(u)\,du}\left|\overline{k}(s) - 2b_1(s)\right|ds \\
&\leq \frac{\alpha}{2}l^2 e^{-\int_{T_1}^t v(u)\,du} + \alpha\varepsilon
\end{aligned}$$

By (iv), there exists $T_2 > T_1$ such that $t > T_2$ implies $\frac{\alpha}{2}l^2 e^{-\int_{T_1}^t v(u)\,du} < \varepsilon$, which implies $I_6(t) \to 0$ as $t \to \infty$. Hence, we have $(P\varphi)(t) \to 0$ as $t \to \infty$.

Finally, we show that P is a contraction mapping. In fact, for $\varphi, \eta \in S^l_\phi$, using condition (4.51) in Theorem 4.2.1, we obtain that

$$\begin{aligned}
&|(P\varphi)(t) - (P\eta)(t)| \\
&\le 2l\left|\frac{p^2(t-r(t))}{2p(t)}\frac{c(t)}{1-r'(t)}\right|\|\varphi-\eta\| + \int_{t-r(t)}^t \left|v(s)-a(s)-\frac{p'(s)}{p(s)}\right|\|\varphi-\eta\|\,ds \\
&\quad + \int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r(s)}^s \left|v(u)-a(u)-\frac{p'(u)}{p(u)}\right|\cdot\|\varphi-\eta\|\,du\,ds \\
&\quad + \int_0^t e^{-\int_s^t v(u)\,du}\left|v(s-r(s))-a(s-r(s))-\frac{p'(s-r(s))}{p(s-r(s))}\right||1-r'(s)|\|\varphi-\eta\|\,ds \\
&\quad + l\int_0^t e^{-\int_s^t v(u)\,du}\left|\overline{k}(s)-2b_1(s)\right|\|\varphi-\eta\|\,ds \\
&\quad + L\int_0^t e^{-\int_s^t v(u)\,du}\frac{|b(s)|p(s-r(s))^\gamma}{p(s)}\|\varphi-\eta\|\,ds \le \alpha\|\varphi-\eta\|.
\end{aligned}$$

Therefore, $P : S^l_\phi \to S^l_\phi$ is a contraction. □

We are now ready to prove Theorem 4.2.1.

Proof. Let P be defined as in Lemma 4.2.4. By the contraction mapping principle, P has a unique fixed point z in S^l_ϕ which is a solution of (4.56) with $z(t) = \phi(t)$ on $[r_0, 0]$ and $z(t) \to 0$ as $t \to \infty$.

To prove stability, let $\varepsilon > 0$ be given, then we choose $m > 0$ so that $m < \min\{L, \varepsilon\}$. Replacing L with m in S, we obtain there is a $\delta > 0$ such that $\|\phi\| < \delta$ implies that the unique solution of (4.56) with $z(t) = \phi(t)$ on $[r_0, 0]$ satifies $|z(t)| \le m < \varepsilon$ for all $t \ge r_0$. This shows that the zero solution of (4.56) is asymptotically stable if (iv) holds.

Conversely, we suppose that (iv) fails. Then by (iii), there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$ such that $\lim_{n\to\infty}\int_0^{t_n} v(s)\,ds = v$ for some $v \in \mathbb{R}$. We may choose a positive constant M such that

$$-M \le \int_0^{t_n} v(s)\,ds \le M \tag{4.58}$$

for all $n \ge 1$. To simplify our expressions, we define

$$\begin{aligned}
w(s) &= l|\overline{k}(s)-2b_1(s)| + |v(s)|\int_{s-r(s)}^s\left|v(u)-a(u)-\frac{p'(u)}{p(u)}\right|du + \frac{L|b(s)|p(s-r(s))^\gamma}{p(s)} \\
&\quad + \left|v(s-r(s))-a(s-r(s))-\frac{p'(s-r(s))}{p(s-r(s))}\right||1-r'(s)|
\end{aligned}$$

for all $s \ge 0$. By (ii) we have

$$\int_0^{t_n} e^{-\int_s^{t_n} v(u)\,du} w(s)\,ds \le \alpha. \tag{4.59}$$

Combining (4.58) and (4.59), we have

$$\int_0^{t_n} e^{\int_0^s v(u)\,du} w(s)\,ds \le \alpha e^{\int_0^{t_n} v(u)\,du} \le \alpha e^M,$$

which yields that the sequence $\int_0^{t_n} e^{\int_0^s v(u)\,du} w(s)\,ds$ is bounded, there exists a convergent subsequence, we assume that

$$\lim_{k\to\infty} \int_0^{t_{n_k}} e^{\int_0^s v(u)\,du} w(s)\,ds = \gamma$$

for some $\gamma \in \mathbb{R}^+$. We choose a positive integer k_1 so large that

$$\lim_{k\to\infty} \int_{t_{n_{k_1}}}^{t_{n_k}} e^{\int_0^s v(u)\,du} w(s)\,ds \le \frac{\delta_0}{4J}$$

for all $n_k > n_{k_1}$, where $\delta_0 > 0$ satisfies $2\delta_0 J e^M + \alpha < 1$.

Now, we consider the solution $z(t) = z(t, t_{n_{k_1}}, \phi)$ of (4.56) with $\phi(t_{n_{k_1}}) = \delta_0$ and $\phi(s) \le \delta_0$ for $s \le t_{n_{k_1}}$, and we may choose ϕ such that $|z(t)| \le 1$ for $t \ge t_{n_{k_1}}$ and

$$\begin{aligned}\phi(t_{n_{k_1}}) &- \int_{t_{n_{k_1}}-r(t_{n_{k_1}})}^{t_{n_{k_1}}} \left(v(s) - a(s) - \frac{p'(s)}{p(s)}\right)\phi(s)\,ds \\ &- \frac{p^2(t_{n_{k_1}} - r(t_{n_{k_1}}))}{2p(t_{n_{k_1}})} \frac{c(t_{n_{k_1}})}{1 - r'(t_{n_{k_1}})} \phi^2(t_{n_{k_1}} - r(t_{n_{k_1}})) \ge \frac{1}{2}\delta_0. \end{aligned} \tag{4.60}$$

So, it follows from (4.60) with $z(t) = (Pz)(t)$ that for $k \ge k_1$,

$$\begin{aligned} &\Bigg| z(t_{n_k}) - \frac{p^2(t_{n_k} - r(t_{n_k}))}{2p(t_{n_k})} \frac{c(t_{n_k})}{1 - r'(t_{n_k})} z^2(t - r(t_{n_k})) \\ &\quad - \int_{t_{n_k}-r(t_{n_k})}^{t_{n_k}} \left[v(s) - a(s) - \frac{p'(s)}{p(s)}\right] z(s)\,ds \Bigg| \\ &\ge \frac{1}{2}\delta_0 e^{-\int_{t_{n_{k_1}}}^{t_{n_k}} v(u)\,du} - \int_{t_{n_{k_1}}}^{t_{n_k}} e^{-\int_s^{t_{n_k}} v(u)\,du} w(s)\,ds \\ &= e^{-\int_{t_{n_{k_1}}}^{t_{n_k}} v(u)\,du} \left[\frac{1}{2}\delta_0 - e^{-\int_0^{t_{n_{k_1}}} v(u)\,du} \int_{t_{n_{k_1}}}^{t_{n_k}} e^{\int_0^s v(u)\,du} w(s)\,ds\right] \\ &\ge e^{-\int_{t_{n_{k_1}}}^{t_{n_k}} v(u)\,du} \left[\frac{1}{2}\delta_0 - J \int_{t_{n_{k_1}}}^{t_{n_k}} e^{\int_0^s v(u)\,du} w(s)\,ds\right] \\ &\ge \frac{1}{4}\delta_0 e^{-\int_{t_{n_{k_1}}}^{t_{n_k}} v(u)\,du} \ge \frac{1}{4}\delta_0 e^{-2M} > 0. \end{aligned} \tag{4.61}$$

On the other hand, if the solution of (4.56) $z(t) = z(t, t_{n_{k_1}}, \phi) \to 0$ as $t \to \infty$. Since $t_{n_k} - r(t_{n_k}) \to \infty$ as $k \to \infty$, and (ii) holds, we have

$$\begin{aligned} &z(t_{n_k}) - \frac{p^2(t_{n_k} - r(t_{n_k}))}{2p(t_{n_k})} \frac{c(t_{n_k})}{1 - r'(t_{n_k})} z^2(t - r(t_{n_k})) \\ &\quad - \int_{t_{n_k}-r(t_{n_k})}^{t_{n_k}} \left[v(s) - a(s) - \frac{p'(s)}{p(s)}\right] z(s)\,ds \to 0 \quad as \quad k \to \infty, \end{aligned}$$

which contradicts (4.61). Hence condition (iv) is necessary for the asymptotic stability of the zero solution of (4.56). □

Since $p(t)$ is a positive bounded function, from the above arguments we obtain that (iv) is necessary and sufficient condtion for the asymptotic stability of the zero solution of (4.48).

In case $g(x) = x^2$ in (4.48), we have the following result.

Corollary 4.2.5. *Suppose the following conditions are satisfied:*

(i) *the delay $r(t)$ is twice differentiable with $r'(t) \neq 1$, and $t - r(t) \to \infty$ as $t \to \infty$;*

(ii) *there exists a bounded function $p : [r_0, \infty) \to (0, \infty)$ with $p(0) = 1$ such that $p'(t)$ exists on $[r_0, \infty)$, and there exists a constant $\alpha \in (0, 1)$, a constant $l > 0$ and an arbitrary continuous functions $v : [r_0, \infty) \to \mathbb{R}$ such that*

$$\begin{aligned} & l\left\{\left|\frac{c(t)p^2(t-r(t))}{p(t)(1-r'(t))}\right| + \int_0^t |\overline{k}(s) - 2\overline{b}(s)| e^{-\int_s^t v(u)\,du}\,ds\right\} \\ & + \int_{t-r(t)}^t \left|v(s) - a(s) - \frac{p'(s)}{p(s)}\right| ds \\ & + \int_0^t e^{-\int_s^t v(u)\,du} |v(s)| \int_{s-r(s)}^s \left|v(u) - a(u) - \frac{p'(u)}{p(u)}\right| du\,ds \\ & + \int_0^t e^{-\int_s^t v(u)\,du} \left|v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\right| |1 - r'(s)|\,ds \le \alpha, \end{aligned} \tag{4.62}$$

where $\overline{k}(s)$ is defined as in (4.52),

$$\overline{b}(s) = \frac{b(s)p^2(s-r(s)) + c(s)p(s-r(s))p'(s-r(s))}{p(s)}; \tag{4.63}$$

(iii) *and such that*

$$\liminf_{t\to\infty} \int_0^t v(s)\,ds > -\infty.$$

Then the zero solution $x(t, \phi)$ of (4.48) with a small continuous function $\phi(t)$ is asymptotically stable if and only if

(iv)

$$\int_0^t v(s)\,ds \to \infty \quad as \quad t \to \infty.$$

Let $p(t) \equiv 1$ in Corollary 4.2.5, we have the following.

Corollary 4.2.6. *Suppose the following conditions are satisfied:*

(i) *the delay $r(t)$ is twice differentiable with $r'(t) \neq 1$, and $t - r(t) \to \infty$ as $t \to \infty$;*

(ii) *there exists a constant* $\alpha \in (0,1)$, *a constant* $l > 0$ *and a continuous functions* $v : [r_0, \infty) \to \mathbb{R}$ *such that*

$$l\left\{\left|\frac{c(t)}{1-r'(t)}\right| + \int_0^t |k(s) - 2b(s)| e^{-\int_s^t v(u)\,du}\, ds\right\} + \int_{t-r(t)}^t |v(s) - a(s)|\, ds$$
$$+ \int_0^t e^{-\int_s^t v(u)\,du} |v(s)| \int_{s-r(s)}^s |v(u) - a(u)|\, du\, ds$$
$$+ \int_0^t e^{-\int_s^t v(u)\,du}\, |v(s-r(s)) - a(s-r(s))|\, |1 - r'(s)|\, ds \le \alpha,$$

where

$$k(s) = \frac{[c(s)a(s) + c'(s)](1 - r'(s)) + c(s)r''(s)}{(1 - r'(s))^2};$$

(iii) *and such that*

$$\liminf_{t\to\infty} \int_0^t v(s)\, ds > -\infty.$$

Then the zero solution $x(t, \phi)$ *of* (4.48) *with a small continuous function* $\phi(t)$ *is asymptotically stable if and only if*

(iv)

$$\int_0^t v(s)\, ds \to \infty \quad as \quad t \to \infty.$$

4.2.3 Proof of Theorem 4.2.3

In this subsection, we will prove Theorem 4.2.3. We start with some preparations. First define

$$S_\phi^l = \Big\{\varphi \mid \varphi \in C([r_0,\infty),\mathbb{R}),\ \|\varphi\| = \sup_{\varphi \ge r_0} |\varphi(t)| \le l,\ \varphi(t) = \phi(t) \text{ for } t \in [r_0, 0],$$
$$\varphi(t) \to 0 \text{ as } t \to \infty\Big\}.$$

Then S_ϕ^l is complete metric space with metric $\rho(x,y) = \sup_{t\ge r_0}\{|x(t) - y(t)|\}$.

Let $z(t) = \phi(t)$ on $[r_0, 0]$, and let

$$x(t) = p(t)z(t), \quad \text{for} \quad t \ge 0. \tag{4.64}$$

If we substitute (4.64) into (4.49), we obtain

$$z'(t) = -\left(a(t) + \frac{p'(t)}{p(t)}\right) z(t) + \frac{c(t)p(t-r(t))p'(t-r(t))}{p(t)} z^2(t - r(t)) \tag{4.65}$$
$$+\frac{c(t)p^2(t-r(t))}{p(t)} z(t-r(t))z'(t-r(t)) + \int_{t-r(t)}^t \frac{p^\gamma(s)K(t,s)}{p(t)} g(z(s))\, ds.$$

Since $p(t)$ is bounded, we only need to prove that the zero solution of (4.65) is asymptotically stable.

If we multiply both sides of (4.65) by $e^{\int_0^t v(s)\,ds}$, integrate from 0 to t, and perform an integration by parts, we obtain

$$\begin{aligned} z(t) &= \left\{\phi(0) - \int_{-r(0)}^{0}\left[v(s) - a(s) - \frac{p'(s)}{p(s)}\right]\phi(s)\,ds - \frac{p^2(-r(0))}{2p(0)}\frac{c(0)}{1-r'(0)}\phi^2(-r(0))\right\} \\ &\quad \times e^{-\int_0^t v(s)\,ds} \\ &\quad + \frac{p^2(t-r(t))}{2p(t)}\frac{c(t)}{1-r'(t)}z^2(t-r(t)) + \int_{t-r(t)}^{t}\left[v(s) - a(s) - \frac{p'(s)}{p(s)}\right]z(s)\,ds \\ &\quad - \int_0^t e^{-\int_s^t v(u)\,du}v(s)\int_{s-r(s)}^{s}\left[v(u) - a(u) - \frac{p'(u)}{p(u)}\right]z(u)\,du\,ds \\ &\quad + \int_0^t e^{-\int_s^t v(u)\,du}\left[v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\right](1-r'(s))z(s-r(s))\,ds \\ &\quad - \frac{1}{2}\int_0^t e^{-\int_s^t v(u)\,du}\left[\overline{k}(s) - 2b_1(s)\right]z^2(s-r(s))\,ds \\ &\quad + \int_0^t e^{-\int_s^t v(u)\,du}\int_{s-r(s)}^{s}\frac{K(s,u)p^{\gamma}(u)}{p(s)}g(z(u))\,du\,ds \end{aligned}$$

where $\overline{k}(s)$ and $b_1(s)$ are defined as in (4.52) and (4.53) respectively.

Lemma 4.2.7. *Let $\varphi(t) \in S_{\phi}^{l}$ and define an operator by $P\varphi(t) = \phi(t)$ for $t \in [r_0, 0]$ and for $t \geq 0$,*

$$(P\varphi)(t) = \sum_{i=1}^{7} I_i(t), \tag{4.66}$$

where

$$\begin{aligned} I_1(t) &= \left\{\phi(0) - \int_{-r(0)}^{0}\left[v(s) - a(s) - \frac{p'(s)}{p(s)}\right]\phi(s)\,ds - \frac{p^2(-r(0))}{2p(0)}\frac{c(0)}{1-r'(0)}\phi^2(-r(0))\right\} \\ &\quad \times e^{-\int_0^t v(s)\,ds} \\ I_2(t) &= \frac{p^2(t-r(t))}{2p(t)}\frac{c(t)}{1-r'(t)}\varphi^2(t-r(t)), \\ I_3(t) &= \int_{t-r(t)}^{t}\left[v(s) - a(s) - \frac{p'(s)}{p(s)}\right]\varphi(s)\,ds, \\ I_4(t) &= -\int_0^t e^{-\int_s^t v(u)\,du}v(s)\int_{s-r(s)}^{s}\left[v(u) - a(u) - \frac{p'(u)}{p(u)}\right]\varphi(u)\,du\,ds, \end{aligned}$$

$$\begin{aligned}
I_5(t) &= \int_0^t e^{-\int_s^t v(u)\,du}\left[v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\right](1-r'(s))\varphi(s-r(s))\,ds,\\
I_6(t) &= -\frac{1}{2}\int_0^t e^{-\int_s^t v(u)\,du}\left(\overline{k}(s) - 2b_1(s)\right)\varphi^2(s-r(s))\,ds,\\
I_7(t) &= \int_0^t e^{-\int_s^t v(u)\,du}\int_{s-r(s)}^s \frac{K(s,u)p^\gamma(u)}{p(s)}g(\varphi(u))\,du\,ds.
\end{aligned}$$

If conditions (i)-(iii) *in Theorem* 4.2.3 *are satisfied, then there exists* $\delta > 0$ *such that for any* $\phi : [r_0, 0] \to (-\delta, \delta)$*, we have that* $P : S^l_\phi \to S^l_\phi$ *and* P *is a contraction with respect to the metric defined on* S^l_ϕ*.*

Proof. Set $J = \sup_{t\geq 0}\left\{e^{-\int_0^t v(s)\,ds}\right\}$, by (iii), J is well defined. Suppose that (iii) holds.

By using the similar arguments as in section 3.2.2, we obtain that $P\varphi \in S^l_\phi$. Now, we show that P is a contraction mapping. In fact, for $\varphi, \eta \in S$, by using condition (4.54) in Theorem 4.2.3, we obtain that

$$\begin{aligned}
&|(P\varphi)(t) - (P\eta)(t)|\\
&\leq \left|\frac{p^2(t-r(t))}{2p(t)}\frac{c(t)}{1-r'(t)}\right| 2l\,\|\varphi-\eta\| + \int_{t-r(t)}^t \left|v(s) - a(s) - \frac{p'(s)}{p(s)}\right|\cdot\|\varphi-\eta\|\,ds\\
&\quad + \int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r(s)}^s\left|v(u) - a(u) - \frac{p'(u)}{p(u)}\right|\cdot\|\varphi-\eta\|\,du\,ds\\
&\quad + \int_0^t e^{-\int_s^t v(u)\,du}\left|v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\right||1-r'(s)|\|\varphi-\eta\|\,ds\\
&\quad + \frac{1}{2}\int_0^t |\overline{k}(s) - 2b_1(s)|\cdot 2l\cdot\|\varphi-\eta\|\\
&\quad + L\int_0^t e^{-\int_s^t v(u)\,du}\int_{s-r(s)}^s\frac{|K(s,u)|p^\gamma(u)}{p(s)}\,du\cdot\|\varphi-\eta\|\,ds \leq \alpha\|\varphi-\eta\|.
\end{aligned}$$

Hence, we obtain that $P : S^l_\phi \to S^l_\phi$ is a contraction. □

We are now ready to prove Theorem 4.2.3.

Proof. Let P be defined as in Lemma 4.2.7. By the contraction mapping principle, P has a unique fixed point z in S^l_ϕ which is a solution of (4.49) with $z(t) = \phi(t)$ on $[r_0, 0]$ and $z(t) \to 0$ as $t \to \infty$.

Let $\varepsilon > 0$ be given, then we choose $m > 0$ so that $m < \min\{l, \varepsilon\}$. Replacing l with m in S^l_ϕ, we obtain there is a $\delta > 0$ such that $\|\phi\| < \delta$ implies that the unique solution of (4.49) with $z(t) = \phi(t)$ on $[r_0, 0]$ satifies $|z(t)| \leq m < \varepsilon$ for all $t \geq r_0$. This shows that the zero solution of (4.49) is asymptotically stable if (iv) holds.

Following the similar arguments as the proof of Theorem 4.2.1, we have that (v) is necessary for the asymptotic stability of the zero solution of (4.49). The proof is complete. □

In case $g(x) = x^2$, we have the following corollary.

Corollary 4.2.8. *Suppose the following conditions are satisfied:*

(i) *the delay $r(t)$ is twice differentiable, $r'(t) \neq 1$, $t - r(t) \to \infty$ as $t \to \infty$;*

(ii) *there exists a bounded function $p : [r_0, \infty) \to (0, \infty)$ with $p(0) = 1$ such that $p'(t)$ exists on $[r_0, \infty)$, and there exists a constant $\alpha \in (0,1)$, a constant $l > 0$ and a continuous functions $v : [r_0, \infty) \to \mathbb{R}$ such that*

$$l\left\{\left|\frac{c(t)p^2(t-r(t))}{p(t)(1-r'(t))}\right| + \int_0^t \left[|\overline{k}(s) - 2b_1(s)| + 2\int_{s-r(s)}^{s}\left|\frac{K(s,u)p^2(u)}{p(s)}\right| du\right] e^{-\int_s^t v(u)\,du}\,ds\right\} + \int_{t-r(t)}^{t}\left|v(s) - a(s) - \frac{p'(s)}{p(s)}\right| ds$$
$$+ \int_0^t e^{-\int_s^t v(u)\,du}\left|v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\right| |1 - r'(s)|\,ds$$
$$+ \int_0^t e^{-\int_s^t v(u)\,du} |v(s)| \int_{s-r(s)}^{s}\left|v(u) - a(u) - \frac{p'(u)}{p(u)}\right| du\,ds \leq \alpha, \tag{4.67}$$

where $\overline{k}(s)$ and $b_1(s)$ are defined as in (4.52) and (4.53);

(iii) *and such that*

$$\liminf_{t\to\infty} \int_0^t v(s)\,ds > -\infty.$$

Then the zero solution of $x(t,\phi)$ of (4.49) with a small continuous function $\phi(t)$ is asymptotically stable if and only if

(iv)

$$\int_0^t v(s)\,ds \to \infty \quad as \quad t \to \infty.$$

Let $p(t) \equiv 1$ in Corollary 4.2.8, we have the following result.

Corollary 4.2.9. *Suppose the following conditions are satisfied:*

(i) *the delay $r(t)$ is twice differentiable with $r'(t) \neq 1$, and $t - r(t) \to \infty$ as $t \to \infty$;*

(ii) *there exists a constant $\alpha \in (0,1)$ and a constant $l > 0$ and a continuous functions $v : [r_0, \infty) \to \mathbb{R}$ such that*

$$l\left\{\left|\frac{c(t)}{1-r'(t)}\right| + \int_0^t \left[|k(s)| + 2\int_{s-r(s)}^{s} |K(s,u)|\,du\right] e^{-\int_s^t v(u)\,du}\,ds\right\}$$
$$+ \int_{t-r(t)}^{t} |v(s) - a(s)|\,ds + \int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r(s)}^{s} |v(u) - a(u)|\,du\,ds$$
$$+ \int_0^t e^{-\int_s^t v(u)\,du}\,|v(s-r(s)) - a(s-r(s))|\,|1 - r'(s)|\,ds \leq \alpha, \tag{4.68}$$

where

$$k(s) = \frac{[c(s)a(s) + c'(s)](1 - r'(s)) + c(s)r''(s)}{(1 - r'(s))^2};$$

(iii) *and such that*

$$\liminf_{t\to\infty} \int_0^t v(s)\,ds > -\infty.$$

Then the zero solution of $x(t,\phi)$ of (4.49) with a small continuous function $\phi(t)$ is asymptotically stable if and only if

(iv)

$$\int_0^t v(s)\,ds \to \infty \quad as \quad t \to \infty.$$

4.2.4 Examples of the main results

Example 4.2.10. *Consider the following nonlinear neutral differential equation*

$$x'(t) - c(t)x(t - r(t))x'(t - r(t)) = -a(t)x(t) + b(t)x^2(t - r(t)) \tag{4.69}$$

for $t \geq 0$, where $a(t) = \frac{2}{t+1}$, $c(t) = 0.95$, $r(t) = 0.05t$, $l = 1$, $b(t)$ satisfies $|\bar{k}(s) - 2\bar{b}(s)| \leq \frac{0.3}{s+1}$, then the zero solution of (4.69) is asymptotically stable.

Proof. We check the condition (4.62) in Corollary 4.2.5, choosing $v(t) = \frac{1.5}{t+1}$ and $p(t) = \frac{1}{t+1}$,

$$\begin{aligned}
\left|\frac{c(t)p^2(t - r(t))}{p(t)(1 - r'(t))}\right| = \frac{t+1}{(0.95t+1)^2} &< 0.36,\\
\int_0^t e^{-\int_s^t v(u)\,du}|\bar{k}(s) - 2\bar{b}(s)|\,ds \leq \int_0^t e^{-\int_s^t \frac{1.5}{u+1}\,du}\frac{0.3}{s+1}\,ds &< 0.2,\\
\int_{t-r(t)}^t |v(s) - a(s) - \frac{p'(s)}{p(s)}|\,ds = \int_{0.95t}^t \frac{0.5}{s+1}\,ds &< 0.026,\\
\int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r(s)}^s |v(u) - a(u) - \frac{p'(u)}{p(u)}|\,du\,ds &< 0.026,
\end{aligned}$$

$$\begin{aligned}
&\int_0^t e^{-\int_s^t v(u)\,du}\left|v(s - r(s)) - a(s - r(s)) - \frac{p'(s - r(s))}{p(s - r(s))}\right||1 - r'(s)|\,ds\\
&= \int_0^t e^{-\int_s^t \frac{1.5}{u+1}\,du}\frac{0.5}{0.95s+1} \times 0.95\,ds < 0.33.
\end{aligned}$$

Hence, we have

$$\begin{aligned}
&l\left\{\left|\frac{c(t)p^2(t - r(t))}{p(t)(1 - r'(t))}\right| + \int_0^t e^{-\int_s^t v(u)\,du}|\bar{k}(s) - 2\bar{b}(s)|\,ds\right\}\\
&+ \int_{t-r(t)}^t \left|v(s) - a(s) - \frac{p'(s)}{p(s)}\right| ds + \int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r(s)}^s \left|v(u) - a(u) - \frac{p'(u)}{p(u)}\right| du\,ds\\
&+ \int_0^t e^{-\int_s^t v(u)\,du}\left|v(s - r(s)) - a(s - r(s)) - \frac{p'(s - r(s))}{p(s - r(s))}\right||1 - r'(s)|\,ds\\
&< 0.36 + 0.026 + 0.026 + 0.33 + 0.2 = 0.941 < 1,
\end{aligned}$$

and since $\int_0^t v(s)\,ds = \int_0^t \frac{1.5}{s+1}\,ds = 1.5\ln(t+1) \to \infty$ as $t \to \infty$, $p(t) \le 1$, so the conditions of Corollary 4.2.5 are satisfied. Therefore, the zero solution of (4.69) is asymptotically stable. □

However, $\left|\frac{c(t)}{1-r'(t)}\right| = 1$, the result in [34] is not applicable.

Example 4.2.11. *Consider the following nonlinear neutral Volterra integral equation*

$$x'(t) - c(t)x(t-r(t))x'(t-r(t)) = -a(t)x(t) + \int_{t-r(t)}^{t} K(t,s)x^2(s)\,ds \tag{4.70}$$

for $t \ge 0$, *where* $a(t) = \frac{2.5}{t+0.1}$, $c(t) = \frac{(0.95t+0.1)^2}{t+0.1}$, $r(t) = 0.05t$, $l = 1$, $K(t,s) = \frac{1}{t+0.1}$, *then the zero solution of* (4.70) *is asymptotically stable.*

Proof. We check the condition (4.67) in Corollary 4.2.8, choosing $v(t) = \frac{2}{t+0.1}$ and $p(t) = \frac{0.1}{t+0.1}$,

$$\begin{aligned}
\left|\frac{c(t)p^2(t-r(t))}{p(t)(1-r'(t))}\right| &= \frac{0.1}{0.95} < 0.106,\\
b_1(s) &= \frac{c(s)p(s-r(s))p'(s-r(s))}{p(s)} = -\frac{0.1}{0.95s+0.1},\\
\overline{c}(s) &= \frac{c(s)p^2(s-r(s))}{p(s)} = 0.1,\\
\overline{k}(s) &= \frac{[\overline{c}(s)v(s)+\overline{c}'(s)](1-r'(s))+\overline{c}(s)r''(s)}{(1-r'(s))^2} = \frac{0.2}{0.95(s+0.1)},
\end{aligned}$$

$$\begin{aligned}
&\int_0^t \left[|\overline{k}(s)-2b_1(s)| + 2\int_{s-r(s)}^{s}\left|\frac{K(s,u)p^2(u)}{p(s)}\right|du\right] e^{-\int_s^t v(u)\,du}\,ds\\
&\le \int_0^t e^{-\int_s^t \frac{2}{u+0.1}\,du}\left|\frac{0.2}{0.95(s+0.1)} + \frac{0.2}{0.95s+0.1}\right| ds\\
&\quad +2\int_0^t e^{-\int_s^t \frac{2}{u+0.1}\,du}\int_{0.95s}^{s}\frac{0.1}{(u+0.1)^2}\,du\,ds\\
&< \frac{0.1}{0.95} + \frac{0.1}{0.95} + 0.2\times\left(\frac{1}{0.95\times 2} + \frac{1}{2}\right) < 0.416,
\end{aligned}$$

$$\begin{aligned}
&\int_{t-r(t)}^{t}\left|v(s)-a(s)-\frac{p'(s)}{p(s)}\right| ds < 0.026,\\
&\int_0^t e^{-\int_s^t v(u)\,du}|v(s)|\int_{s-r(s)}^{s}\left|v(u)-a(u)-\frac{p'(u)}{p(u)}\right| du\,ds < 0.026,
\end{aligned}$$

$$\begin{aligned}
&\int_0^t e^{-\int_s^t v(u)\,du}\left|v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\right| |1-r'(s)|\,ds\\
&= \int_0^t e^{-\int_s^t \frac{2}{u+1}\,du}\frac{0.5}{0.95s+0.1}\times 0.95\,ds < 0.25.
\end{aligned}$$

Hence,

$$l\left\{\left|\frac{c(t)p^2(t-r(t))}{p(t)(1-r'(t))}\right| + \int_0^t \left[|\overline{k}(s) - 2b_1(s)|\right.\right.$$
$$\left.\left. +2\int_{s-r(s)}^s \left|\frac{K(s,u)p^2(u)}{p(s)}\right| du\right] e^{-\int_s^t v(u)\,du}\, ds\right\} + \int_{t-r(t)}^t \left|v(s) - a(s) - \frac{p'(s)}{p(s)}\right| ds$$
$$+ \int_0^t e^{-\int_s^t v(u)\,du}|v(s)| \int_{s-r(s)}^s \left|v(u) - a(u) - \frac{p'(u)}{p(u)}\right| du\, ds$$
$$+ \int_0^t e^{-\int_s^t v(u)\,du} \left|v(s-r(s)) - a(s-r(s)) - \frac{p'(s-r(s))}{p(s-r(s))}\right| |1-r'(s)|\, ds$$
$$< 0.106 + 0.416 + 0.026 + 0.026 + 0.25 = 0.824 < 1,$$

and since $\int_0^t v(s)\,ds = \int_0^t \frac{2}{s+0.1}\,ds = 2\ln(t+0.1) \to \infty$ as $t \to \infty$, $p(t) \le 1$, so the conditions of Corollary 4.2.8 are satisfied. Therefore, the zero solution of (4.70) is asymptotically stable. □

However, $\left|\frac{c(t)}{1-r'(t)}\right| = \frac{(0.95t+0.1)^2}{0.95(t+0.1)} \to \infty$ as $t \to \infty$, the result in [34] is not applicable.

4.3 Stability of nonlinear difference equations based on a fixed point method

4.3.1 Introduction and main results

In this section, we study the stability of zero solution of the nonlinear delay difference equation of the form

$$\Delta x(n) = -a(n)f(x(n-\tau(n))), \tag{4.71}$$

using a fixed point method, where Δ is a forward difference operator defined by $\Delta x(n) = x(n+1) - x(n)$, $a : \mathbb{Z}^+ \to \mathbb{R}$, $\tau : \mathbb{Z}^+ \to \mathbb{Z}^+$ with $n - \tau(n) \to \infty$ as $n \to \infty$, and we define $m(0) = \inf_{t\ge 0}\{n - \tau(n)\}$. Assume that f is continuous, locally Lipschitz, and odd, while $x - f(x)$ is nondecreasing and $f(x)$ is increasing on an interval $[0, l]$ for some $l > 0$. We say $x(n) = x(n, 0, \phi)$ is a solution of (4.71) if $x(n) = \phi(n)$ on $[m(0), 0] \cap \mathbb{Z}$ and $x(n)$ satisfies (4.71) for $n \in \mathbb{Z}^+$.

Definition 4.3.1. *The zero solution of (4.71) is said to be stable at $n = 0$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\phi : [m(0), 0] \cap \mathbb{Z} \to (-\delta, \delta)$ implies that $|x(n)| < \varepsilon$ for $n \in [m(0), \infty) \cap \mathbb{Z}$.*

Definition 4.3.2. *The zero solution of (4.71) is said to be asymptotically stable at $n = 0$ if it is stable at $n = 0$ and a $\delta > 0$ exists such that for any continuous function $\phi : [m(0), 0] \cap \mathbb{Z} \to (-\delta, \delta)$, the solution $x(n) = \phi(n)$ on $[m(0), 0] \cap \mathbb{Z}$ and tends to zero as $n \to \infty$.*

Some articles [65, 111, 135] have studied the the special cases of (4.71) by means of fixed point theory. Raffoul [111] studied the following linear difference equation with constant delay

$$\Delta x(n) = -a(n)x(n-\tau). \tag{4.72}$$

Jin and Luo [65] considered the generalized form of (4.72),

$$\Delta x(n) = -a(n)f(x(n-\tau)), \tag{4.73}$$

where f is continuous function, and obtained the following.

Theorem 4.3.3. (Jin and Luo [65]) *Let f be odd, increasing on $[0, l]$, satisfy a Lipschitz condition, and let $x - f(x)$ be nondecreasing on $[0, l]$. Suppose that $|a(n)| < 1$ and for each $l_1 \in (0, l]$ we have*

$$\begin{aligned}
&|l_1 - f(l_1)| \sup_{n\in\mathbb{Z}^+} \sum_{s=0}^{n-1} |a(s+\tau)| \prod_{k=s+1}^{n-1} [1 - a(k+\tau)] \\
&+ f(l_1) \sup_{n\in\mathbb{Z}^+} \sum_{s=0}^{n-1} |a(s+\tau)| \prod_{k=s+1}^{n-1} [1 - a(k+\tau)] \sum_{u=s-\tau}^{s-1} |a(u+\tau)| \\
&+ f(l_1) \sup_{n\in\mathbb{Z}^+} \sum_{s=n-\tau}^{n-1} |a(s+\tau)| < \alpha l_1
\end{aligned}$$

Then the zero solution of (4.73) is stable.

For the case when the delays are variable, Yankson [135] investigated the following delay difference equation

$$\Delta x(n) = -a(n)x(n - \tau(n))$$

and its generalization

$$\Delta x(n) = -\sum_{j=1}^{N} a_j(n)x(n - \tau_j(n)) \tag{4.74}$$

and obtained the following two theorems.

Theorem 4.3.4. (Yankson [135]) Suppose that the inverse function $g_j(n)$ of $n - \tau_j(n)$ exists, and assume that there exists a constant $\alpha \in (0, 1)$ such that

$$\begin{aligned}
&\sum_{s=0}^{n-1} \left[\left| \sum_{j=1}^{N} a_j(g_j(s)) \right| \left| \prod_{k=s+1}^{n-1} [1 - \sum_{j=1}^{N} a_j(g_j(k))] \right| \sum_{j=1}^{N} \sum_{u=s-\tau_j(s)}^{s-1} \left| a_j(g_j(u)) \right| \right] \\
&+ \sum_{j=1}^{N} \sum_{s=n-\tau_j(n)}^{n-1} |a_j(g_j(s))| \le \alpha.
\end{aligned}$$

Moreover, assume that there exists a positive constant M such that

$$\left| \prod_{s=0}^{n-1} \left[1 - \sum_{j=1}^{N} a_j(g_j(s)) \right] \right| \le M$$

Then the zero solution of (4.74) is stable.

Theorem 4.3.5. (Yankson [135]) *Assume that the hypotheses of Theorem 4.3.4 hold. Also assume that*

$$\prod_{k=0}^{n-1} \left[1 - \sum_{j=1}^{N} a_j(g_j(k)) \right] \to 0 \quad as \quad n \to \infty. \tag{4.75}$$

Then the zero solution of (4.74) is asymptotically stable.

It is our aim to study a class of nonlinear delay difference equations (4.71) using fixed point theory. This class of equations is more general than the equations considered in [65, 111, 135]. Our method of proof will be a fixed point approach similar to the one used in [65, 111, 135].

Theorem 4.3.6. *Consider the nonlinear delay difference equation (4.71) and suppose that the following conditions are satisfied*

(i) *the function f is odd, increasing on $[0, l]$;*

(ii) *assume that $f(x), x - f(x)$ satisfy a Lipschitz condition with constant $K > 0$ on an interval $[-l, l]$, $x - f(x)$ is nondecreasing on $[0, l]$;*

(iii) *suppose that the inverse function $g(n)$ of $n - \tau(n)$ exists and $|a(g(n))| < 1$;*

(iv) *there exists a constant $\alpha \in (0, 1)$ such that*

$$\sum_{s=0}^{n-1} |a(g(s))| \prod_{k=s+1}^{n-1} |1 - a(g(k))| + \sum_{s=n-\tau(n)}^{n-1} |a(g(s))| + \sum_{s=0}^{n-1} |a(g(s))| \prod_{k=s+1}^{n-1} |1 - a(g(k))| \sum_{u=s-\tau(s)}^{s-1} |a(g(u))| \leq \alpha. \tag{4.76}$$

Then the zero solution of (4.71) is stable.

Theorem 4.3.7. *Assume that the hypotheses of Theorem 4.3.6 hold. Also assume that*

$$\prod_{k=0}^{n-1} [1 - a(g(k))] \to 0 \quad as \quad n \to \infty. \tag{4.77}$$

Then the zero solution of (4.71) is asymptotically stable.

4.3.2 Proof of Theorem 4.3.6

In this subsection, we will prove Theorem 4.3.6. We start with some preparations. First we write (4.71) as the following form

$$\begin{aligned} \Delta x(n) &= -a(g(n))f(x(n)) + \Delta_n \sum_{s=n-\tau(n)}^{n-1} a(g(s))f(x(s)) \qquad (4.78) \\ &= -a(g(n))x(n) + a(g(n))[x(n) - f(x(n)] + \Delta_n \sum_{s=n-\tau(n)}^{n-1} a(g(s))f(x(s)) \end{aligned}$$

where Δ_n represents that the difference depends on n, (4.78) is equivalent to

$$x(n+1) = [1 - a(g(n))]x(n) + a(g(n))[x(n) - f(x(n)] + \Delta_n \sum_{s=n-\tau(n)}^{n-1} a(g(s))f(x(s)).$$

From variation of parameters followed by summation by parts, we have

$$\begin{aligned} x(n) &= x(0)\prod_{s=0}^{n-1}[1-a(g(s))] - \prod_{u=0}^{n-1}[1-a(g(u))]\sum_{s=-\tau(0)}^{-1} a(g(s))f(x(s)) \\ &\quad + \sum_{s=0}^{n-1} a(g(s)) \prod_{k=s+1}^{n-1}[1-a(g(k))][x(s)-f(x(s))] \\ &\quad + \sum_{s=n-\tau(n)}^{n-1} a(g(s))f(x(s)) - \sum_{s=0}^{n-1} a(g(s)) \prod_{k=s+1}^{n-1}[1-a(g(k))] \sum_{u=s-\tau(s)}^{s-1} a(g(u))f(x(u)) \end{aligned}$$

Let $D(0)$ the set of bounded sequences $\varphi : [m(0),\infty)\cap\mathbb{Z}\to\mathbb{R}$ with the supremum norm $\|\cdot\|$. Also, let $(C,\|\cdot\|)$ be the Banach space of real sequences $\varphi : [m(0),\infty)\cap\mathbb{Z}\to\mathbb{R}$ with the supremum norm $\|\cdot\|$. For any sequence φ with $\|\varphi\|\le l$ we define

$$\mathcal{S}_\phi^l = \{\varphi \mid \varphi\in C,\ \varphi(n)=\phi(n) \text{ for } n\in[m(0),0]\cap\mathbb{Z},\ |\varphi(n)|\le l\}.$$

Then $(\mathcal{S}_\phi^l,\|\cdot\|)$ is a complete space.

Lemma 4.3.8. *Let $\varphi\in\mathcal{S}_\phi^l$. Define an operator by $P\varphi(n)=\phi(n)$ for $n\in[m(0),0]\cap\mathbb{Z}$, and for $n\in\mathbb{Z}^+$,*

$$\begin{aligned} (P\varphi)(n) &= \left[\phi(0) - \sum_{s=-\tau(0)}^{-1} a(g(s))f(\phi(s))\right]\prod_{s=0}^{n-1}[1-a(g(s))] \\ &\quad + \sum_{s=0}^{n-1} a(g(s))\prod_{k=s+1}^{n-1}[1-a(g(k))][\varphi(s)-f(\varphi(s))] \\ &\quad + \sum_{s=n-\tau(n)}^{n-1} a(g(s))f(\varphi(s)) - \sum_{s=0}^{n-1} a(g(s))\prod_{k=s+1}^{n-1}[1-a(g(k))]\sum_{u=s-\tau(s)}^{s-1} a(g(u))f(\varphi(u)). \end{aligned} \tag{4.79}$$

If conditions (i)-(iv) *in Theorem 4.3.6 are satisfied, then there exists $\delta>0$ such that for any $\phi:[m(0),0]\to(-\delta,\delta)$, we have that $P: S_\phi^l\to S_\phi^l$ and P is a contraction mapping with respect to the metric defined on S_ϕ^l.*

Proof. For every $\varphi\in\mathcal{S}_\phi^l$, we have

$$\begin{aligned} |(P\varphi)(n)| &\le \|\phi\| + \sum_{s=-\tau(0)}^{-1}|a(g(s))|\|f(\phi)\| + f(l)\sum_{s=n-\tau(n)}^{n-1}|a(g(s))| \\ &\quad + (l-f(l))\sum_{s=0}^{n-1}|a(g(s))|\prod_{k=s+1}^{n-1}|1-a(g(k))| \\ &\quad + f(l)\left\{\sum_{s=0}^{n-1}|a(g(s))|\prod_{k=s+1}^{n-1}|1-a(g(k))|\sum_{u=s-\tau(s)}^{s-1}|a(g(u))|\right\} \\ &\le \|\phi\| + \sum_{s=-\tau(0)}^{-1}|a(g(s))|\|f(\phi)\| + (l-f(l))\alpha + f(l)\alpha \end{aligned}$$

$$= \|\phi\| + \sum_{s=-\tau(0)}^{-1} |a(g(s))| \|f(\phi)\| + l\alpha.$$

Choose $\delta > 0$ such that $\|\phi\| < \delta$, the Lipschitz constant K for f on $[0, l]$ implies that $\delta + K\delta \sum_{s=-\tau(0)}^{-1} |a(g(s))| \leq (1-\alpha)l$, then we have $|(P\varphi)(n)| \leq l$. Hence, $P\varphi \in \mathcal{S}_\phi^l$.

Suppose that $d > \max\{3, 1/K\}$. If we define a metric on $\mathcal{S}_\phi^l$ as follows,

$$|\varphi - \eta|_K := \sup_{n\in\mathbb{Z}^+} \prod_{j=0}^{n-1} \frac{1-|a(g(j))|}{dK[1+|a(g(j))|]} |\varphi(n) - \eta(n)|, \tag{4.80}$$

then $(\mathcal{S}_\phi^l, |\cdot|_K)$ is a complete metric space.

Next, we show that P is a contraction mapping on $\mathcal{S}_\phi^l$ with respect to the metric (4.80). For $\varphi, \eta \in \mathcal{S}_\phi^l$, we have

$$\begin{aligned} &|(P\varphi)(t) - (P\eta)(t)| \\ &\leq \sum_{s=0}^{n-1} |a(g(s))| \prod_{k=s+1}^{n-1} |1 - a(g(k))| |\varphi(s) - f(\varphi(s)) - \eta(s) + f(\eta(s))| \\ &\quad + \sum_{s=n-\tau(n)}^{n-1} |a(g(s))| |f(\varphi(s)) - f(\eta(s))| \\ &\quad + \sum_{s=0}^{n-1} |a(g(s))| \prod_{k=s+1}^{n-1} |1 - a(g(k))| \sum_{u=s-\tau(s)}^{s-1} |a(g(u))| |f(\varphi(u)) - f(\eta(u))|. \end{aligned} \tag{4.81}$$

Let $h(x) := x - f(x)$, then $h(x)$ satisfies a Lipschitz condition with constant $K > 0$ on an interval $[-l, l]$. If we multiply both sides of (4.81) by

$$\prod_{j=0}^{n-1} \frac{1-|a(g(j))|}{dK[1+|a(g(j))|]},$$

then the first term on the right-hand side of (4.81) becomes

$$\begin{aligned} &\prod_{j=0}^{n-1} \frac{1-|a(g(j))|}{dK[1+|a(g(j))|]} \sum_{s=0}^{n-1} |a(g(s))| \prod_{k=s+1}^{n-1} |1 - a(g(k))| |h(\varphi(s)) - h(\eta(s))| \\ &\leq K \sum_{s=0}^{n-1} \frac{|a(g(s))|[1-|a(g(s))|]}{dK[1+|a(g(s))|]} \prod_{j=0}^{s-1} \frac{1-|a(g(j))|}{dK[1+|a(g(j))|]} |\varphi(s) - \eta(s)| \\ &\quad \times \prod_{k=s+1}^{n-1} \frac{|1-a(g(k))|[1-|a(g(k))|]}{dK[1+|a(g(k))|]} \\ &\leq \frac{1}{d} |\varphi - \eta|_K \sum_{s=0}^{n-1} |a(g(s))| \prod_{k=s+1}^{n-1} [1 - |a(g(k))|] \leq \frac{1}{d} |\varphi - \eta|_K. \end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\prod_{j=0}^{n-1} \frac{1-|a(g(j))|}{dK[1+|a(g(j))|]} \sum_{s=n-\tau(n)}^{n-1} |a(g(s))||f(\varphi(s))-f(\eta(s))| \\
&\le K \sum_{s=n-\tau(n)}^{n-1} \frac{|a(g(s))|[1-|a(g(s))|]}{dK[1+|a(g(s))|]} \prod_{j=0}^{s-1} \frac{1-|a(g(j))|}{dK[1+|a(g(j))|]} |\varphi(s)-\eta(s)| \\
&\quad \times \prod_{k=s+1}^{n-1} \frac{1-|a(g(k))|}{dK[1+|a(g(k))|]} \\
&\le \frac{1}{d}|\varphi-\eta|_K \sum_{s=n-\tau(n)}^{n-1} |a(g(s))| \prod_{k=s+1}^{n-1} [1-|a(g(k))|] \le \frac{1}{d}|\varphi-\eta|_K,
\end{aligned}$$

and

$$\begin{aligned}
&\prod_{j=0}^{n-1} \frac{1-|a(g(j))|}{dK[1+|a(g(j))|]} \sum_{s=0}^{n-1} |a(g(s))| \prod_{k=s+1}^{n-1} |1-a(g(k))| \\
&\times \sum_{u=s-\tau(s)}^{s-1} |a(g(u))||f(\varphi(u))-f(\eta(u))| \\
&\le K \sum_{s=0}^{n-1} \frac{|a(g(s))|[1-|a(g(s))|]}{dK[1+|a(g(s))|]} \prod_{k=s+1}^{n-1} \frac{|1-a(g(k))|[1-|a(g(k))|]}{dK[1+|a(g(k))|]} \\
&\quad \times \sum_{u=s-\tau(s)}^{s-1} |a(g(u))| \prod_{j=0}^{u-1} \frac{1-|a(g(u))|}{dK[1+|a(g(u))|]} |\varphi(u)-\eta(u)| \prod_{j=u}^{s-1} \frac{1-|a(g(j))|}{dK[1+|a(g(j))|]} \\
&\le \frac{1}{d}|\varphi-\eta|_K \sum_{s=0}^{n-1} |a(g(s))| \prod_{k=s+1}^{n-1} [1-|a(g(k))|] \sum_{u=s-\tau(s)}^{s-1} |a(g(u))| \prod_{j=u}^{s-1} [1-|a(g(j))|] \\
&\le \frac{1}{d}|\varphi-\eta|_K.
\end{aligned}$$

Hence, $|P\varphi - P\eta|_K \le \frac{3}{d}|\varphi-\eta|_K$, since $d > 3$, we have that P is a contraction mapping on $\mathcal{S}_\phi^l$.

We are now ready to prove Theorem 4.3.6.

Proof. Let P be defined as in Lemma 4.3.8. By the contraction mapping principle, P has a unique fixed point in $\mathcal{S}_\phi^l$, which is a solution of (4.71) with $x(n) = \phi(n)$ on $[m(0), 0] \cap \mathbb{Z}$.

To prove stability at $n = 0$, let $\varepsilon > 0$ be given, then we choose $m > 0$ so that $m < \min\{l, \varepsilon\}$. By considering $\mathcal{S}_\phi^m$, we obtain there is a $\delta > 0$ such that $\|\phi\| < \delta$ implies that the unique solution of (4.71) with $x(n) = \phi(n)$ on $[m(0), 0] \cap \mathbb{Z}$ satifies $|x(n)| \le m < \varepsilon$ for all $n \ge m(0)$. This shows that the zero solution of (4.71) is stable. □

4.3.3 Proof of Theorem 4.3.7

In this subsection, we will prove Theorem 4.3.7.

Proof. From Theorem 4.3.6, the zero solution of (4.71) is stable. Let $\phi \in D(0)$ such that $\phi(n) < \delta$ and define the space

$$S_\phi^\varepsilon = \{\varphi \mid \varphi \in C,\ \varphi(n) = \phi(n) \text{ for } n \in [m(0), 0] \cap \mathbb{Z},\ |\varphi(n)| \le \varepsilon \text{ and } \varphi(n) \to 0 \text{ as } n \to \infty\}.$$

Then S_ϕ^ε is a complete metric space with respect to the metric (4.80). Define $P : S_\phi^\varepsilon \to S_\phi^\varepsilon$ by (4.79). From the proof of Theorem 4.3.6, the mapping P is a contraction and for every $\varphi \in S_\phi^\varepsilon$, $\|P\varphi\| \le \varepsilon$.

Next, we show that $(P\varphi)(n) \to 0$ as $n \to \infty$. The first term on the right side of (4.79) goes to zero because of condition (4.77). Consider the second term on the right side of (4.79),

$$\begin{aligned}
|I_2| &:= \left| \sum_{s=0}^{n-1} a(g(s)) \prod_{k=s+1}^{n-1} [1 - a(g(k))][\varphi(s) - f(\varphi(s))] \right| \\
&\le K \sum_{s=0}^{n-1} |a(g(s))| \prod_{k=s+1}^{n-1} |1 - a(g(k))||\varphi(s)| \le K\alpha\varepsilon,
\end{aligned}$$

which yields that $I_2 \to 0$ as $n \to \infty$. It is clear that
$I_3 := \sum_{s=n-\tau(n)}^{n-1} a(g(s))f(\varphi(s))$ goes to zero because of (4.76) and the fact $\varphi(n) \to 0\ as\ n \to \infty$. Now, we show that the last term on the right side of (4.79) goes to zero as $n \to \infty$. Since $\varphi(n) \to 0$ and $n - \tau(n) \to \infty$ as $n \to \infty$, for every $\varepsilon_1 > 0$, there exists $N_1 > 0$ such that $s > N_1$ implies $\varphi(n - \tau(n)) < \varepsilon_1$. Thus for $n \ge N_1$, the last term on the right side of (4.79) satisfies

$$\begin{aligned}
|I_4| &:= \left| \sum_{s=0}^{n-1} a(g(s)) \prod_{k=s+1}^{n-1} [1 - a(g(k))] \sum_{u=s-\tau(s)}^{s-1} a(g(u))f(\varphi(u)) \right| \\
&\le \sum_{s=0}^{N_1-1} |a(g(s))| \left| \prod_{k=s+1}^{n-1} [1 - a(g(k))] \right| \sum_{u=s-\tau(s)}^{s-1} |a(g(u))||f(\varphi(u))| \\
&\quad + \sum_{s=N_1}^{n-1} |a(g(s))| \left| \prod_{k=s+1}^{n-1} [1 - a(g(k))] \right| \sum_{u=s-\tau(s)}^{s-1} |a(g(u))||f(\varphi(u))| \\
&\le K \max_{0 \ge m(0)} |\varphi(0)| \sum_{s=0}^{N_1-1} |a(g(s))| \left| \prod_{k=s+1}^{n-1} [1 - a(g(k))] \right| \sum_{u=s-\tau(s)}^{s-1} \left| a(g(u)) \right| \\
&\quad + K\varepsilon_1 \sum_{s=N_1}^{n-1} |a(g(s))| \left| \prod_{k=s+1}^{n-1} [1 - a(g(k))] \right| \sum_{u=s-\tau(s)}^{s-1} |a(g(u))|.
\end{aligned}$$

Since $\varphi(n) \to 0$ as $n \to \infty$, there exisits $N_2 > N_1$ such that $n > N_2$ implies

$$\max_{0 \ge m(0)} |\varphi(0)| \sum_{s=0}^{N_1-1} |a(g(s))| \left| \prod_{k=s+1}^{n-1} [1 - a(g(k))] \right| \sum_{u=s-\tau(s)}^{s-1} |a(g(u))| < \varepsilon_1.$$

Applying (4.76) we obtain $|I_4| \le K\alpha\varepsilon_1 + K\varepsilon_1 < 2K\varepsilon_1$. Thus $I_4 \to 0$ as $n \to \infty$. Hence $(P\varphi)(n) \to 0$ as $n \to \infty$.

By the contraction mapping principle, P has a unique fixed point which is a solution of (4.71) and goes to zero as $n \to \infty$. Therefore, the zero solution is asymptotically stable. □

4.4 Notes and remarks

In 1997 several investigators began a systematic study towards a comprehensive stability method based on fixed point theory. Afterwards, a great deal of interesting results concerning stability theory are published by employing this method, see the overview papers by Burton [8, 9, 11, 12], Raffoul [110, 111, 112], Ardjouni and Djoudi [5, 6], Djoudi and Khemis [34], Zhang [141], Jin and Luo [63, 64, 65, 62], and Zhao [144, 145]. The book by Burton [13] offers a large collection of examples investigated by fixed point theory as a viable tool.

In this chapter, we study the appoarch based on fixed point theory to general classes of e-quations with delays. Section 4.1 includes the stability criteria of four classes of neutral delay differential equations, our results in Section 4.1 extend and improve the work in [8, 11, 12, 13, 31, 34, 63, 110, 144] by considering more general classes of neutral delay differential equations. Our results in Section 4.2 can be applied to the case when $\left|\frac{c(t)}{1-r'(t)}\right| \geq 1$, which improve the work in [34]. Our results in Section 4.3 concerns a class of nonlinear difference equations with variable delays, which is a generalization of the work in [65, 111, 135].

A paper [16] based the contents of Section 4.1 has been submitted to a journal for possible publication.

Chapter 5

Stability of neutral stochastic delay differential equations with impulses

This chapter concerns the stability of two classes of neutral stochastic delay differential equations with impulses.

In Section 5.1, asymptotic stability of a class of neutral stochastic delay differential equations with linear impulses is studied by means of fixed point methods. More specifically, two theorems for the asymptotic stability of the equations are presented by using two complete metric spaces which are defined by different types of norms.

In Section 5.2, exponential stability of a class of neutral stochastic partial differential equations with delays and impulses is investigated. The equation that will be considered in this section is an infinite dimensional stochastic differential equation with variable delays. The method by using an impulsive-integral inequality and the method based on fixed point theory, are applied to study exponential stability of mild solutions of the impulsive neutral stochastic partial delay differential equations, respectively.

5.1 Asymptotic stability of a class of neutral stochastic delay differential equations with linear impulses

5.1.1 Introduction and main results

A stochastic delay differential equation is a stochastic differential equation where the increment of the process depends on values of the process (and maybe other functions) of the past. These equations can be used to model processes with a memory.

Besides delay effect on the stochastic differential equations, impulsive effect is also a common phnomenon in a wide range of physical and engineering systems including the biological stocking or harvesting, the function of the heart, the change of an economy of a state, etc. For stochastic differential equations which include delay effects and impulsive effects are described as impulsive stochastic delay differential equations.

It was recently proposed by Luo [90] and Appleby [4] to use fixed point methods to deal with the stability problems for stochastic delay differential equations. However, to the best of our knowledge, the stochastic delay differential equations which have been studied by using fixed points method are those without impulses.

Very recently, Li, Sun and Shi [75, 62, 76] obtained some interesting results for the stability of zero solution of stochastic delay differential equations with impulsive effects by using the equivalent method. This method provides a way to study the stability of impulsive equations by constructing an equivalent relation between the stability of stochastic delay differential equations under impulses and that of a corresponding stochastic delay differential equations without impulses. As a consequence of this approach, the sufficient conditions for the stability of stochastic delay differential equations with impulses are obtained by using the existing stability results of the corresponding stochastic delay differential equations without impulses.

The aim of this section is to combine the approach based on fixed point methods and the equivalent method in order to study the stability of a general class of neutral stochastic delay differential equations with linear impulses. For the class of equations, we first transform the equations into the one without impulses, and then we use fixed point argument to obtain the sufficient conditions for asymptotic stability of the considered equations. In particular, we present two different sufficient conditions for the asymptotic stability of the equations by using two appropriate contraction mappings that are defined on different complete metric spaces. It turns out that our stability results for stochastic delay differential equations with impulses do not depend on the existing stability results for the corresponding stochastic delay differential equations without impulses.

The class of neutral stochastic differential equations that we will study in this section is of the form

$$\begin{cases} d[x(t)-q(t)x(t-\tau(t))] = [a(t)x(t)+b(t)x(t-\tau(t))]\,dt \\ \qquad\qquad +[c(t)x(t)+e(t)x(t-\delta(t))]\,dw(t), \quad t\neq t_k, \\ x(t_k{}^+)-x(t_k)=d_k x(t_k), \quad t=t_k, \quad k=1,2,3,\cdots, \end{cases} \tag{5.1}$$

where w is a one-dimensional standard Brownian motion on some filtrated probability space $\{\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P}\}$ which satisfies the usual conditions.

Denote by $\mathbb{R}=(-\infty,\infty)$, $\mathbb{R}^+=[0,\infty)$, $\mathbb{Z}^+=\{1,2,3,\cdots\}$. Define

$$\vartheta=\min\left\{\inf_{s\geq t_0}\{s-\tau(s)\},\inf_{s\geq t_0}\{s-\delta(s)\}\right\}$$

and $n(t)=\max\{k\in\mathbb{Z}^+ : t_k<t\}$. For simplicity, we define $\prod_{u\leq t_k<v}(\cdot)=\prod_{k\in\{k|k\in\mathbb{Z}^+,u\leq t_k<v\}}(\cdot)$ for all $u,v\in\mathbb{R}$. Here and in the sequel, we assume that a product equals unity if the number of factors is equal to zero.

A standard fixed point argument shows that the differential equation (5.1) provided with an initial condition

$$x(t)=\phi(t), \qquad t\in[\vartheta,0], \tag{5.2}$$

where $\phi(t)\in C([\vartheta,0],\mathbb{R})$ defines a well-posed initial-value problem, and we denote $x(t):=x(t,\phi)$ the solution of (5.1) with initial condition (5.2).

For equation (5.1) with initial condition (5.2), we suppose that the following conditions are satisfied

(H1) $0 \le t_0 < t_1 < t_2 < t_3 < \cdots < t_k < \cdots$ are fixed impulsive points such that $t_k \to \infty$ for $k \to \infty$;

(H2) $a(t), b(t), c(t), e(t) \in C(\mathbb{R}^+, \mathbb{R})$ and $\tau(t), \delta(t) \in C(\mathbb{R}^+, \mathbb{R}^+)$;

(H3) d_k is a real sequence such that $d_k \in (-1, \infty)$, $k = 1, 2, 3, \cdots$.

Definition 5.1.1. *For any $\phi \in C([\vartheta, 0], \mathbb{R})$, a function $x : [\vartheta, \infty) \to \mathbb{R}$ denoted by $x(t, 0, \phi)$ is said to be a solution of the system (5.1) on $[0, \infty)$ satisfying the initial value condition (5.2), if the following conditions are satisfied:*

(i) *$x : [\vartheta, \infty) \to L^2(\Omega, \mathbb{R})$ is continuous on $[\vartheta, \infty) \backslash \{t_k : k = 1, 2, 3, \cdots\}$ and adapted to $(\mathcal{F}_t)_{t \ge 0}$.*

(ii) *for any t_k, $k = 1, 2, 3, \cdots$, $x({t_k}^+) = \lim_{t \to {t_k}^+} x(t)$ and $x({t_k}^-) = \lim_{t \to {t_k}^-} x(t)$ exist in $L^2(\Omega, \mathbb{P})$ and $x({t_k}^-) = x(t)$, i.e. there exist $x({t_k}^+), x({t_k}^-) \in L^2(\Omega, \mathbb{P})$ such that $\lim_{t \to {t_k}^+} \mathbb{E}|x({t_k}^+) - x(t)|^2 = 0$, $\lim_{t \to {t_k}^-} \mathbb{E}|x({t_k}^-) - x(t)|^2 = 0$ and $x({t_k}^-) = x(t)$ $\mathbb{P}$-a.e.*

(iii) *$x(t)$ satisfies the differential equation in (5.1) in the sense that for every k and every $s_0, s_1 \in (t_k, t_{k+1}]$ with $s_0 \le s_1$, $\mathbb{P}$-almost surely,*

$$\begin{aligned}&(x(s_1) - q(s_1)x(s_1 - \tau(s_1))) - (x(s_0) - q(s_0)x(s_0 - \tau(s_0))) \\ &\quad = \int_{s_0}^{s_1} [a(s)x(s) + b(s)x(s - \tau(s))]\, ds + \int_{s_0}^{s_1} [c(s)x(s) + e(s)x(s - \delta(s))]\, dw(s).\end{aligned}$$

Remark 5.1.2. *Note that continuity of $x : (t_k, t_{k+1}) \to L^2(\Omega, \mathbb{P})$ implies that x is measurable with respect to $B_{(t_k, t_{k+1})} \otimes \mathcal{F}$, where $B_{(t_k, t_{k+1})}$ is the Borel-σ-algebra, and that $\iint |x(t, \omega)|^2 \, d\mathbb{P}\, dt \le \sup_{t \in (t_k, t_{k+1})} \int |x(t, \omega)|^2 \, d\mathbb{P} < \infty$.*

Denote

$$C^b_{\mathcal{F}_0}(\delta) = \left\{ \phi \mid \phi \in C^b_{\mathcal{F}_0}([\vartheta, 0], \mathbb{R}), \ \sup_{\vartheta \le s \le 0} \mathbb{E}|\phi(s)|^2 \le \delta \right\},$$

where $\mathbb{E}$ denotes expectation with respect to the probability measure $\mathbb{P}$, refer to Section 1.3 for more detailed information.

Definition 5.1.3. *The zero solution of the system (5.1) is said to be*

(i) *mean square stable if for any $\varepsilon > 0$, there is a scalar $\delta = \delta(\varepsilon) > 0$ such that for every initial function $\phi \in C^b_{\mathcal{F}_0}(\delta)$ we have that for corresponding solution $x(t, 0, \phi)$ satisfies $\mathbb{E}|x(t, 0, \phi)|^2 < \varepsilon$ for $t \ge 0$.*

(ii) *mean square asymptotically stable if it is stable and for any $\varepsilon > 0$, there exists a scalar $\delta = \delta(\varepsilon) > 0$ such that for every initial function $\phi \in C^b_{\mathcal{F}_0}(\delta)$, the corresponding solution $x(t, 0, \phi)$ satisfies $\lim_{t \to \infty} \mathbb{E}|x(t, 0, \phi)|^2 = 0$ for $t \ge 0$.*

(iii) *exponentially stable in mean square if there is a pair of positive constants* λ *and* K *such that*

$$\mathbb{E}|x(t,0,\phi)|^2 \leq K\|\phi\|_{L^2}^2 e^{-\lambda t},$$

holds for any $\phi \in C^b_{\mathcal{F}_0}([\vartheta,0],\mathbb{R})$*, here* λ *is called the exponential convergence rate.*

Denote by $\mathcal{S}_\phi$ the space of all $\mathcal{F}$-adapted processes $\varphi(t,\omega) : [\vartheta,\infty)\times\Omega \to \mathbb{R}$ such that $t \mapsto \varphi(t) : [\vartheta,\infty) \mapsto L^2(\Omega,\mathbb{R})$ is continuous in $t \neq t_k$ $(k = 1,2,\cdots)$, $\lim_{t\to t_k^-}\varphi(t)$ and $\lim_{t\to t_k^+}\varphi(t)$ exist, and $\lim_{t\to t_k^-}\varphi(t) = \varphi(t_k)$ in $L^2(\Omega;\mathbb{P})$. Moreover, $\varphi(t,\cdot) = \phi(t)$ for $t \in [\vartheta,0]$, and $\mathbb{E}|\varphi(t)|^2 \to 0$ as $t \to \infty$. If we define a norm as

$$\|\varphi\|^2 := \sup_{t\geq\vartheta}\left(\mathbb{E}|\varphi(t)|^2\right), \tag{5.3}$$

then $\mathcal{S}_\phi$ is a complete metric space with respect to the norm (5.3). Using a contraction mapping defined on the space $\mathcal{S}_\phi$, we come to our first result, which is proved in Subsection 5.1.2.

Theorem 5.1.4. *Consider the impulsive neutral stochastic differential equation* (5.1) *and suppose that the following conditions are satisfied*

(i) *the delay* $\tau(t)$ *is differentiable and* $t-\tau(t)\to\infty$ *as* $t\to\infty$, $t-\delta(t)\to\infty$ *as* $t\to\infty$*;*

(ii) *there exist a constant* $\alpha\in(0,1)$ *and a continuous function* $h(t)$ *such that for* $t\geq 0$,

$$\begin{aligned} H_1(t) \;:=\; 2\Bigg\{&\sum_{l=1}^{n(t)}\left|\frac{d_l}{1+d_l}\prod_{t_l-\tau(t_l)\leq t_k<t_l}(1+d_k)^{-1}e^{-\int_{t_l}^t h(u)\,du}q(t_l)\right| \\ &+\left|\prod_{t-\tau(t)<t_k<t}(1+d_k)^{-1}q(t)\right| + \int_{t-\tau(t)}^t |a(s)+h(s)|\,ds \\ &+\int_0^t e^{-\int_s^t h(u)\,du}|h(s)|\int_{s-\tau(s)}^s |a(u)+h(u)|\,du\,ds \\ &+\int_0^t e^{-\int_s^t h(u)\,du}\Bigg|(a(s-\tau(s))+h(s-\tau(s)))(1-\tau'(s)) \\ &+\prod_{s-\tau(s)\leq t_k<s}(1+d_k)^{-1}b(s) - \prod_{s-\tau(s)\leq t_k<s}(1+d_k)^{-1}h(s)q(s)\Bigg|\,ds\Bigg\}^2 \\ &+2\int_0^t e^{-2\int_s^t h(u)\,du}\left(|c(s)| + \left|\prod_{s-\delta(s)\leq t_k<s}(1+d_k)^{-1}e(s)\right|\right)^2 ds \leq \alpha, \end{aligned}$$

(iii) *and such that* $\liminf_{t\to\infty}\int_0^t h(s)\,ds > -\infty$*;*

(iv) *there exists a constant* $M>0$ *such that for any* $t\geq 0$,

$$\left|\prod_{0\leq t_k<t}(1+d_k)\right| \leq M.$$

Then the zero solution of (5.1) *is mean square asmptotically stable if*

(v)

$$\int_0^t h(s)\,ds \to \infty \quad \text{as} \quad \text{t} \to \infty.$$

Remark 5.1.5. *Note that if $\varphi \in C([a,b]; L^2(\Omega,\mathbb{R}))$, then $\varphi \in L^2([a,b]; L^2(\Omega,\mathbb{R}))$ and the latter space is known to be naturally isometrically isomorphic to $L^2([a,b]\times\Omega;\mathbb{R})$ (see [30] for detail), so φ is joint measurable and $\int_a^b\int_\Omega |\varphi(t,\omega)|^2\,dt\,d\mathbb{P}(\omega) < \infty$.*

Remark 5.1.6. *Note that if $-1 < d_k \le 0$ for all k, then (iv) in Theorem 5.1.4 is always satisfied. If $d_k = 0$, Theorem 5.1.4 is Theorem 2.1 in [90] under sufficient conditions.*

Remark 5.1.7. *Impulses are small perturbation, it can stabilize or destabilize a system. As an example, consider the equation*

$$\begin{cases} x'(t) = ax(t) & a > 0, \\ x(k^+) = (1+d_k)x(k), & k \in \mathbb{N}, \end{cases} \tag{5.4}$$

From (5.4), we obtain that

(1) *if $d_k = 0$ for all $k \in \mathbb{N}$, $x(t) = e^{at}x(0)$;*

(2) *if $d_k \neq 0$ for all $k \in \mathbb{N}$,*

$$x(1) = e^a x(0) \qquad x(1^+) = (1+d_1)e^a x(0)$$

$$x(1+t) = e^{at}x(1^+) = e^{at}(1+d_1)e^a x(0), \quad 0 < t \le 1,$$

$$x(2) = (1+d_1)e^{2a}x(0) \qquad x(2^+) = (1+d_1)(1+d_2)e^{2a}x(0)$$

$$x(2+t) = e^{at}x(2^+) = e^{at}(1+d_1)(1+d_2)e^{2a}x(0), \quad 0 < t \le 1,$$

$$\vdots$$

$$x(n) = (1+d_1)(1+d_2)\cdots(1+d_{n-1})e^{na}x(0).$$

$$x(n^+) = (1+d_1)(1+d_2)\cdots(1+d_n)e^{na}x(0).$$

If $d_k = d$ $(k = 1,2,3,\cdots)$ is a constant, then

$$x(n^+) = (1+d)^n e^{na}x(0) = [(1+d)e^a]^n x(0),$$

If $(1+d)e^a < 1$, then $x(n^+) \to 0$; If $(1+d)e^a > 1$, then $x(n^+) \to \infty$.

Remark 5.1.8. *The condition (ii) in Theorem 5.1.4 can not be changed to be $H_1(t) < 1$. For example, if $H_1(t) = \int_0^t e^{-s}\,ds$ for all t, then $H_1(t) = 1 - e^{-t} < 1$. However, $H_1(t) \to 1$ as $t \to \infty$. Hence, the condition $H_1(t) < 1$ can not imply that there exsits a constant $\alpha \in (0,1)$ such that $H_1(t) \le \alpha < 1$.*

For the case when the delays $\tau(t)$ and $\delta(t)$ are bounded by a positive constant τ, we define another complete metric space as the following.

Denote by $\mathcal{C}_\phi$ the space of all $\mathcal{F}$-adapted processes $\varphi(t,\omega) : [-\tau,\infty)\times\Omega \to \mathbb{R}$, which is almost surely continuous in $t \neq t_k$ $(k=1,2,\cdots)$ for fixed $\omega\in\Omega$, $\lim_{t\to t_k^-}\varphi(t)$ and $\lim_{t\to t_k^+}\varphi(t)$ exist, and $\lim_{t\to t_k^-}\varphi(t) = \varphi(t_k)$. Moreover, $\varphi(t,\cdot) = \phi(t)$ for $t\in[-\tau,0]$ and for $t\to\infty$, $\mathbb{E}\left(\sup_{t-\tau\le s\le t}|\varphi(s)|^2\right)\to 0$. If we define a norm as

$$\|\varphi\|^2 := \sup_{t\ge 0}\mathbb{E}\left(\sup_{t-\tau\le s\le t}|\varphi(s)|^2\right), \tag{5.5}$$

then $\mathcal{C}_\phi$ is a complete metric space. Based on the space $\mathcal{C}_\phi$, we come to our second result, which is proved in Subsection 5.1.3.

Theorem 5.1.9. *Consider the impulsive neutral stochastic differential equation* (5.1) *and suppose that the following conditions are satisfied*

(i) *the delay $\tau(t)$ is differentiable;*

(ii) *there exist a constant $\alpha\in(0,1)$ and a continuous function $h(t)$ such that for $t\ge 0$,*

$$\begin{aligned}
H_2(t) \;:=\; & 2\Bigg\{\sum_{l=1}^{n(t)}\left|\frac{d_l}{1+d_l}\prod_{t_l-\tau(t_l)\le t_k<t_l}(1+d_k)^{-1}e^{-\int_{t_l}^t h(u)\,du}q(t_l)\right| \\
& +\left|\prod_{t-\tau(t)<t_k<t}(1+d_k)^{-1}q(t)\right| + \int_{t-\tau(t)}^t |a(s)+h(s)|\,ds \\
& +\int_0^t e^{-\int_s^t h(u)\,du}|h(s)|\int_{s-\tau(s)}^s |a(u)+h(u)|\,du\,ds \\
& +\int_0^t e^{-\int_s^t h(u)\,du}\Bigg|(a(s-\tau(s))+h(s-\tau(s)))(1-\tau'(s)) \\
& +\prod_{s-\tau(s)\le t_k<s}(1+d_k)^{-1}b(s) - \prod_{s-\tau(s)\le t_k<s}(1+d_k)^{-1}h(s)q(s)\Bigg|\,ds\Bigg\}^2 \\
& +8\int_0^t\left(\sup_{t-\tau\le r\le t}e^{-2\int_s^r h(u)\,du}\right)\left(|c(s)|+\left|\prod_{s-\delta(s)\le t_k<s}(1+d_k)^{-1}e(s)\right|\right)^2 ds \\
& \le \alpha;
\end{aligned}$$

where $n(t) := \sup\{k\in\mathbb{Z}^+ : t_k<t\}$.

(iii) *and such that $\liminf_{t\to\infty}\int_0^t h(s)\,ds > -\infty$;*

(iv) *there exists a constant $M>0$ such that for any $t\ge 0$, $\left|\prod_{0\le t_k<t}(1+d_k)\right|\le M$.*

Then the zero solution of (5.1) *is asymptotically stable if*

(v)

$$\int_0^t h(s)\,ds\to\infty \quad \text{as} \quad \mathrm{t}\to\infty.$$

5.1.2 Proof of Theorem 5.1.4

In this subsection, we will prove Theorem 5.1.4. We start with some preparations. First we transform (5.1) into a neutral stochastic delay differential equation without impulses:

$$\begin{aligned} d\left[y(t)-\tilde{q}(t)y(t-\tau(t))\right] &= \left[a(t)y(t)+\tilde{b}(t)y(t-\tau(t))\right]dt \\ &\quad + \left[c(t)y(t)+\tilde{e}(t)y(t-\delta(t))\right]dw(t), \end{aligned} \tag{5.6}$$

where

$$\begin{aligned} \tilde{q}(t) &= \prod_{t-\tau(t)\le t_k<t}(1+d_k)^{-1}q(t), \quad \tilde{b}(t)=\prod_{t-\tau(t)\le t_k<t}(1+d_k)^{-1}b(t), \\ \tilde{e}(t) &= \prod_{t-\delta(t)\le t_k<t}(1+d_k)^{-1}e(t). \end{aligned}$$

By a solution of the system (5.6) and (5.2) we mean a continuous function $y(t)$ on $[\vartheta,\infty)$ satisfying equation (5.6) almost everywhere for $t\ge 0$ and satisfies equation (5.2).

We start with two fundamental lemmas. The lemmas allow us to reduce the problem of stability of an impulsive neutral stochastic delay differential equation to problem of stability of a neutral stochastic delay differential equation.

Lemma 5.1.10. *Assume that* $(\mathrm{H}_1)-(\mathrm{H}_3)$ *hold, then*

(i) *if $y(t)$ is a solution of (5.6), then $x(t)=\prod_{0<t_k<t}(1+d_k)y(t)$ is a solution of (5.1);*

(ii) *if $x(t)$ is a solution of (5.1), then $y(t)=\prod_{0<t_k<t}(1+d_k)^{-1}x(t)$ is a solution of (5.6).*

Proof. First we prove (i). Let $y(t)$ be a possible solution of the problem (5.6), it is easy to see that $x(t)=\prod_{0<t_j<t}(1+d_k)y(t)$ is continuous on $(0,t_1]$ and on each interval $(t_k,t_{k+1}]$, $k\in\mathbb{N}$ and for any $t\ne t_k$, $j\in\mathbb{N}$,

$$\begin{aligned} dx(t) &= d\left[\prod_{0<t_k<t}(1+d_j)y(t)\right] = \prod_{0<t_k<t}(1+d_k)dy(t) \\ &= \prod_{0<t_k<t}(1+d_k)\Big\{d\left[\tilde{q}(t)y(t-\tau(t))\right]+\Big(a(t)y(t)+\tilde{b}(t)y(t-\tau(t))\Big)dt \\ &\qquad + \left(c(t)y(t)+\tilde{e}(t)y(t-\delta(t))\right)dw(t)\Big\} \\ &= d[q(t)x(t-\tau(t))]+[a(t)x(t)+b(t)x(t-\tau(t))]dt+[c(t)x(t)+e(t)x(t-\delta(t))]dw(t). \end{aligned}$$

Thus $x(t)$ satisfies the differential equation (5.1) for almost everywhere in $[0,\infty)$ when $t\ne t_k$, $k\in\mathbb{N}$, On the other hand, for every $t=t_k$, $k\in\mathbb{N}$,

$$\begin{aligned} x(t_k^+) &= \lim_{t\to t_k^+}\prod_{0<t_j<t}(1+d_j)y(t)=\prod_{0<t_j\le t_k}(1+d_j)y(t_k^+) \\ &= (1+d_k)\prod_{0<t_j<t_k}(1+d_j)y(t_k)=(1+d_k)x(t_k) \end{aligned}$$

and

$$x(t_k^-)=\lim_{t\to t_k^-}\prod_{0<t_j<t}(1+d_j)y(t)=\prod_{0<t_j<t_k}(1+d_j)y(t_k^-)=x(t_k).$$

Therefore, we have that $x(t)$ is a solution of equation (5.1) corresponding to the initial condition (5.2). In fact, if $y(t)$ is a solution of (5.6) with initial condition (5.2), then $x(t) = \prod_{0<t_k<t}(1+d_k)y(t) = y(t) = \phi(t)$ on $[\vartheta, 0]$.

Next, we prove (ii). If $x(t)$ is a solution of (5.1), then $x(t)$ is continuous on $(0, t_1]$ and on each interval $(t_k, t_{k+1}]$, $j \in \mathbb{N}$. Therefore, $y(t) = \prod_{0<t_k<t}(1+d_k)^{-1}x(t)$ is continuous on $(0, t_1]$ and on each interval $(t_k, t_{k+1}]$, $k \in \mathbb{N}$. Using the similar way as the proof for (i), we can easily check that $y(t) = \prod_{0<t_k<t}(1+d_k)^{-1}x(t)$ is the solution of (5.6) on $[\vartheta, \infty)$ corresponding to the initial condition (5.2). On the other hand, for every $t = t_k$, $k \in \mathbb{N}$,

$$\begin{aligned} y(t_k^+) &= \lim_{t\to t_k^+} \prod_{0<t_j<t}(1+d_j)^{-1}x(t) = \prod_{0<t_j\le t_k}(1+d_j)^{-1}x(t_k^+) \\ &= (1+d_k)^{-1}\prod_{0<t_j<t_k}(1+d_j)^{-1}x(t_k^+) = \prod_{0<t_j<t_k}(1+d_j)^{-1}x(t_k) = y(t_k) \end{aligned}$$

and

$$y(t_k^-) = \lim_{t\to t_k^-} \prod_{0<t_j<t}(1+d_j)^{-1}x(t) = \prod_{0<t_j<t_k}(1+d_j)^{-1}x(t_k^-) = \prod_{0<t_j<t_k}(1+d_j)^{-1}x(t_k) = y(t_k).$$

This completes the proof of the lemma. □

Lemma 5.1.11. *Assume that* $(\mathrm{H}_1) - (\mathrm{H}_3)$ *hold, and there exists a positive constant* M *such that, for any* $t > 0$,

$$\left| \prod_{0<t_k<t}(1+d_k) \right| \le M. \tag{5.7}$$

(i) *If the zero solution of equation* (5.6) *is mean square stable, then the zero solution of equation* (5.1) *is also mean square stable.*

(ii) *If the zero solution of equation* (5.6) *is mean square asymptotically stable, then the zero solution of equation* (5.1) *is also mean square asymptotically stable.*

(iii) *If the zero solution of equation* (5.6) *is mean square exponentially stable, then the zero solution of equation* (5.1) *is also mean square exponentially stable.*

Proof. First, we prove (i). Let $x(t)$ and $y(t)$ be the solution of equations (5.1) and (5.6) corresponding to initial conditions (5.2). If the zero solution of equation (5.6) is mean square stable, from the definition of mean square stable, we have that, for any $\varepsilon > 0$, there exists $\delta > 0$ such that the initial function $\phi \in C^b_{\mathcal{F}_0}(\delta)$ implies

$$\mathbb{E}|y(t)|^2 < \frac{\varepsilon}{M^2}, \quad \text{for} \quad \mathrm{t} > 0. \tag{5.8}$$

From Lemma 5.1.10, we obatin that $x(t) = \prod_{0<t_k<t}(1+d_k)y(t)$ is a solution of (5.1) on $[\vartheta, \infty)$, and combining with (5.8), we have that

$$\mathbb{E}|x(t)|^2 = \mathbb{E}\left| \prod_{0<t_k<t}(1+d_k)y(t) \right|^2 \le \left| \prod_{0<t_k<t}(1+d_k) \right|^2 \mathbb{E}|y(t)|^2 < M^2\frac{\varepsilon}{M^2} = \varepsilon,$$

which implies that the zero solution of (5.1) is mean square stable.

(ii)and (iii) can be proved similarly as (i). □

If we multiply both sides of (5.6) by $e^{\int_0^t h(s)\,ds}$, integrate from 0 to t, and perform an integration by parts, we obtain

$$\begin{aligned}
(Py)(t) &= \left[\phi(0) - \int_{\tau(0)}^{0} (a(s)+h(s))\phi(s)\,ds - q(0)\phi(-\tau(0)) + M(t)\right] e^{-\int_0^t h(u)\,du} \\
&\quad + \prod_{t-\tau(t)\le t_k<t} (1+d_k)^{-1} q(t) y(t-\tau(t)) + \int_{t-\tau(t)}^{t} (a(s)+h(s))y(s)\,ds \\
&\quad - \int_0^t e^{-\int_s^t h(u)\,du} h(s) \int_{s-\tau(s)}^{s} (a(u)+h(u))y(u)\,du\,ds \\
&\quad + \int_0^t e^{-\int_s^t h(u)\,du} \Bigg[(a(s-\tau(s)) + h(s-\tau(s)))(1-\tau'(s)) \\
&\quad + \prod_{s-\tau(s)\le t_k<s} (1+d_k)^{-1} b(s) - \prod_{s-\tau(s)\le t_k<s} (1+d_k)^{-1} h(s) q(s) \Bigg] y(s-\tau(s))\,ds \\
&\quad + \int_0^t e^{-\int_s^t h(u)\,du} \left[c(s)y(s) + \prod_{s-\delta(s)\le t_k<s} (1+d_k)^{-1} e(s) y(s-\delta(s)) \right] dw(s).
\end{aligned}$$

Lemma 5.1.12. *Let $\varphi \in \mathcal{S}_\phi$. Define an operator by $(P\varphi)(t) = \phi(t)$ for $t \in [\vartheta, 0]$ and for $t \ge 0$,*

$$(P\varphi)(t) = \sum_{i=1}^{5} I_i(t), \tag{5.9}$$

where

$$\begin{aligned}
M(t) &= \sum_{l=1}^{n(t)} \frac{d_l}{1+d_l} \prod_{t_l-\tau(t_l)\le t_k<t_l} (1+d_k)^{-1} e^{\int_0^{t_l} h(u)\,du} q(t_l)\varphi(t_l-\tau(t_l)), \\
I_1(t) &= \left[\phi(0) - \int_{\tau(0)}^{0} (a(s)+h(s))\phi(s)\,ds - q(0)\phi(-\tau(0))\right] e^{-\int_0^t h(u)\,du}, \\
I_2(t) &= \sum_{l=1}^{n(t)} \frac{d_l}{1+d_l} \prod_{t_l-\tau(t_l)\le t_k<t_l} (1+d_k)^{-1} e^{-\int_{t_l}^{t} h(u)\,du} q(t_l)\varphi(t_l-\tau(t_l)) \\
&\quad + \prod_{t-\tau(t)\le t_k<t} (1+d_k)^{-1} q(t)\varphi(t-\tau(t)), \\
I_3(t) &= \int_{t-\tau(t)}^{t} (a(s)+h(s))\varphi(s)\,ds - \int_0^t e^{-\int_s^t h(u)\,du} h(s) \int_{s-\tau(s)}^{s} (a(u)+h(u))\varphi(u)\,du\,ds,
\end{aligned}$$

$$
\begin{aligned}
I_4(t) &= \int_0^t e^{-\int_s^t h(u)\,du}\Bigg[(a(s-\tau(s))+h(s-\tau(s)))(1-\tau'(s)) \\
&\quad + \prod_{s-\tau(s)\le t_k<s}(1+d_k)^{-1}b(s) - \prod_{s-\tau(s)\le t_k<s}(1+d_k)^{-1}h(s)q(s)\Bigg]\varphi(s-\tau(s))\,ds, \\
I_5(t) &= \int_0^t e^{-\int_s^t h(u)\,du}\Bigg[c(s)y(s) + \prod_{s-\delta(s)\le t_k<s}(1+d_k)^{-1}e(s)\varphi(s-\delta(s))\Bigg]\,dw(s). \qquad (5.10)
\end{aligned}
$$

If conditions (i)-(iv) *in Theorem* 5.1.4 *are satisfied, then there exists* $\delta > 0$ *such that for any* $\phi : [\vartheta, 0] \to (-\delta, \delta)$, *we have that* $P : \mathcal{S}_\phi \to \mathcal{S}_\phi$ *and* P *is a contraction with respect to the norm* (5.3).

Proof. We first verify the continuity of $P\varphi$ on $[0,\infty)$ to $L^2(\Omega,\mathcal{F},\mathbb{P}).I_2(t)$ is continuous if $t \in (0,t_1)$ or $t \in (t_j,t_{j+1})$ for $j = 1,2,3,\cdots$. It remains to prove that $I_2(t)$ is continuous at $t = t_j$. Let $\varphi \in S$, take the limit $r \to 0^+$, we have

$$
\begin{aligned}
&\mathbb{E}|I_2(t_j+r) - I_2(t_j)|^2 \\
&= \mathbb{E}\Bigg|\left(e^{-\int_{t_j}^{t_j+r} h(u)\,du} - 1\right)\sum_{l=1}^{j-1}\frac{d_l}{1+d_l}\prod_{t_l-\tau(t_l)\le t_k<t_l}(1+d_k)^{-1}e^{-\int_{t_l}^{t_j} h(u)\,du}q(t_l)\varphi(t_l-\tau(t_l)) \\
&\quad + \frac{d_j}{1+d_j}\prod_{t_j-\tau(t_j)\le t_k<t_j}(1+d_k)^{-1}e^{-\int_{t_j}^{t_j+r} h(u)\,du}q(t_j)\varphi(t_j-\tau(t_j)) \\
&\quad + \prod_{t_j+r-\tau(t_j+r)\le t_k<t_j+r}(1+d_k)^{-1}q(t_j+r)\varphi(t_j+r-\tau(t_j+r)) \\
&\quad - \prod_{t_j-\tau(t_j)\le t_k<t_j}(1+d_k)^{-1}q(t_j)\varphi(t_j-\tau(t_j))\Bigg|^2 \\
&\le 2\mathbb{E}\left|e^{-\int_{t_j}^{t_j+r} h(u)\,du} - 1\right|^2\left|\sum_{l=1}^{j}\frac{d_l}{1+d_l}\prod_{t_l-\tau(t_l)\le t_k<t_l}(1+d_k)^{-1}e^{-\int_{t_l}^{t_j} h(u)\,du}q(t_l)\varphi(t_l-\tau(t_l))\right|^2 \\
&\quad + 2\mathbb{E}\Bigg[\left|\frac{1}{1+d_j}\prod_{t_j-\tau(t_j)\le t_k<t_j}(1+d_k)^{-1}\right| \\
&\qquad \times |q(t_j+r)\varphi(t_j+r-\tau(t_j+r)) - q(t_j)\varphi(t_j-\tau(t_j))| \\
&\quad + \left|\prod_{t_j+r-\tau(t_j+r)\le t_k<t_j+r}(1+d_k)^{-1} - \frac{1}{1+d_j}\prod_{t_j-\tau(t_j)\le t_k<t_j}(1+d_k)^{-1}\right| \\
&\qquad \times |q(t_j+r)\varphi(t_j+r-\tau(t_j+r))|\Bigg]^2 \longrightarrow 0,
\end{aligned}
$$

since

$$\lim_{r\to 0^+} \prod_{t_j+r-\tau(t_j+r)\le t_k<t_j+r} (1+d_k)^{-1} = \frac{1}{1+d_j} \prod_{t_j-\tau(t_j)\le t_k<t_j} (1+d_k)^{-1}.$$

Take the limit $r \to 0^-$, we have

$$\begin{aligned}
&\mathbb{E}|I_2(t_j+r) - I_2(t_j)|^2 \\
&= \mathbb{E}\Bigg| \left(e^{-\int_{t_j}^{t_j+r} h(u)\,du} - 1\right) \sum_{l=1}^{j-1} \frac{d_l}{1+d_l} \prod_{t_l-\tau(t_l)\le t_k<t_l} (1+d_k)^{-1} e^{-\int_{t_l}^{t_j} h(u)\,du} q(t_l)\varphi(t_l-\tau(t_l)) \\
&\quad + \prod_{t_j+r-\tau(t_j+r)\le t_k<t_j+r} (1+d_k)^{-1} q(t_j+r)\varphi(t_j+r-\tau(t_j+r)) \\
&\quad - \prod_{t_j-\tau(t_j)\le t_k<t_j} (1+d_k)^{-1} q(t_j)\varphi(t_j-\tau(t_j))\Bigg|^2 \\
&\le 2\mathbb{E}\left|e^{-\int_{t_j}^{t_j+r} h(u)\,du} - 1\right|^2 \left|\sum_{l=1}^{j-1} \frac{d_l}{1+d_l} \prod_{t_l-\tau(t_l)\le t_k<t_l} (1+d_k)^{-1} e^{-\int_{t_l}^{t_j} h(u)\,du} q(t_l)y(t_l-\tau(t_l))\right|^2 \\
&\quad + 2\mathbb{E}\Bigg| \prod_{t_j+r-\tau(t_j+r)\le t_k<t_j+r} (1+d_k)^{-1} q(t_j+r)\varphi(t_j+r-\tau(t_j+r)) \\
&\quad - \prod_{t_j-\tau(t_j)\le t_k<t_j} (1+d_k)^{-1} q(t_j)\varphi(t_j-\tau(t_j))\Bigg|^2 \to 0,
\end{aligned}$$

since

$$\lim_{r\to 0^-} \prod_{t_j+r-\tau(t_j+r)\le t_k<t_j+r} (1+d_k)^{-1} = \prod_{t_j-\tau(t_j)\le t_k<t_j} (1+d_k)^{-1}.$$

All the other terms continuous because of the following (a) and (b).

(a) If $\sup_{s\in[0,t]} \left[\mathbb{E}g(s)^2\right] < \infty$ for each t, then $t \mapsto \int_0^t g(s)\,ds : [0,\infty) \mapsto L^2(\Omega, \mathcal{F}_0, \mathbb{P})$ is continuous.

(b) If $\sup_{s\in[0,t]} \left[\mathbb{E}g(s)^2\right] < \infty$ for each t, then $t \mapsto \int_0^t g(s)\,dw : [0,\infty) \mapsto L^2(\Omega, \mathcal{F}_0, \mathbb{P})$ is continuous.

Thus P is continuous on $[0,\infty)$.

Next, We prove that $P(\mathcal{S}_\phi) \subseteq \mathcal{S}_\phi$. For $\varphi \in \mathcal{S}_\phi$, we consider the first term of $I_3(t)$,

$$\begin{aligned}
&\mathbb{E}\left(\int_{t-\tau(t)}^{t}(a(s)+h(s))\varphi(s)\,ds\right)^2\\
&=\mathbb{E}\left(\int_{t-\tau(t)}^{t}(a(s)+h(s))\varphi(s)\,ds\int_{t-\tau(t)}^{t}(a(v)+h(v))\varphi(v)\,dv\right)\\
&=\int_{t-\tau(t)}^{t}\int_{t-\tau(t)}^{t}(a(s)+h(s))(a(v)+h(v))\mathbb{E}\varphi(s)\varphi(v)\,ds\,dv\\
&\le\int_{t-\tau(t)}^{t}\int_{t-\tau(t)}^{t}|a(s)+h(s)||a(v)+h(v)|\,ds\,dv\sup_{s,v\in[t-\tau(t),t]}\mathbb{E}\varphi(s)\varphi(v)\\
&=\left(\int_{t-\tau(t)}^{t}|a(s)+h(s)|\,ds\right)^2\sup_{s,v\in[t-\tau(t),t]}\left(\mathbb{E}\varphi(s)^2\right)^{1/2}\left(\mathbb{E}\varphi(v)^2\right)^{1/2}.
\end{aligned}$$

Since $\mathbb{E}\varphi(t)^2 \to 0$ and $t-\tau(t)\to\infty$ as $t\to\infty$, so for every $\varepsilon>0$, there exists $T_0>0$ such that $t>T_0$ implies $\mathbb{E}\varphi(t)^2<\varepsilon$ and $\mathbb{E}\varphi(t-\tau(t))^2<\varepsilon$. Hence, for $t>T_0$, we have that

$$\sup_{s,v\in[t-\tau(t),t]}\left(\mathbb{E}\varphi(s)^2\right)^{1/2}\left(\mathbb{E}\varphi(v)^2\right)^{1/2}<\varepsilon,$$

Combining with condition (ii) in Theorem 5.1.4, we obtain

$$\mathbb{E}\left(\int_{t-\tau(t)}^{t}(a(s)+h(s))\varphi(s)\,ds\right)^2\to 0\quad\text{as}\quad t\to\infty.$$

Following the similar estimation as above, we obtain that $\mathbb{E}|I_i(s)|^2\to 0$ as $t\to\infty$, $i=1,2,3,4$. It follows from the last term $I_5(s)$ in (5.9) that

$$\begin{aligned}
&\mathbb{E}|I_5(t)|^2\\
&=\mathbb{E}\left|\int_0^t e^{-\int_z^t h(u)\,du}\left[c(z)\varphi(z)+\prod_{z-\delta(z)\le s_k<z}(1+d_k)^{-1}e(z)\varphi(z-\delta(z))\right]dw(z)\right|^2\\
&\le\mathbb{E}\left|\int_0^t e^{-\int_z^t h(u)\,du}\left[c(z)\varphi(z)+\prod_{z-\delta(z)\le s_k<z}(1+d_k)^{-1}e(z)\varphi(z-\delta(z))\right]dw(z)\right|^2\\
&\le\mathbb{E}\int_0^t e^{-2\int_z^t h(u)\,du}\left[c(z)\varphi(z)+\prod_{z-\delta(z)\le s_k<z}(1+d_k)^{-1}e(z)\varphi(z-\delta(z))\right]^2dz.
\end{aligned}$$

Since $\mathbb{E}|\varphi(t)|^2\to 0$ and $t-\delta(t)\to 0$ as $t\to\infty$, then for any $\varepsilon>0$, there exists $T>0$ such that $t>T$ implies $\mathbb{E}|\varphi(t)|^2<\varepsilon$ and $\mathbb{E}|\varphi(t-\delta(t))|^2<\varepsilon$. Hence, for $t>T$, we have that

$$\begin{aligned}
&\mathbb{E}\int_0^t e^{-2\int_z^t h(u)\,du}\left[c(z)\varphi(z)+\prod_{z-\delta(z)\le s_k<z}(1+d_k)^{-1}e(z)\varphi(z-\delta(z))\right]^2dz\\
&\quad<\left\{\int_0^t e^{-2\int_z^t h(u)\,du}\left[|c(z)|+\left|\prod_{z-\delta(z)\le s_k<z}(1+d_k)^{-1}e(z)\right|\right]^2dz\right\}\varepsilon,
\end{aligned}$$

which yields $\mathbb{E}|I_5(s)|^2 \to 0$ as $t \to \infty$. Hence, we obtain that $P(\mathcal{S})_\phi \subseteq \mathcal{S}_\phi$.

Finally, We show that P is contraction mapping. For $\varphi, \psi \in \mathcal{S}_\phi$, we have that

$$
\begin{aligned}
&\sup_{s\geq\vartheta}\left\{\mathbb{E}|(P\varphi)(s)-(P\psi)(s)|^2\right\}\\
&=\sup_{s\geq\vartheta}\mathbb{E}\Bigg|\sum_{l=1}^{n(t)}\frac{d_l}{1+d_l}\prod_{s_l-\tau(s_l)\leq s_k<s_l}(1+d_k)^{-1}e^{-\int_{s_l}^{s}h(u)\,du}q(s_l)\\
&\qquad\times(\varphi(s_l-\tau(s_l))-\psi(s_l-\tau(s_l)))\\
&\quad+\prod_{s-\tau(s)\leq s_k<s}(1+d_k)^{-1}q(s)(\varphi(s-\tau(s))-\psi(s-\tau(s)))\\
&\quad+\int_{s-\tau(s)}^{s}(a(z)+h(z))(\varphi(z)-\psi(z))\,dz\\
&\quad-\int_0^s e^{-\int_z^s h(u)\,du}h(z)\int_{z-\tau(z)}^{z}(a(u)+h(u))(\varphi(u)-\psi(u))\,du\,dz\\
&\quad+\int_0^s e^{-\int_z^s h(u)\,du}\Bigg[(a(z-\tau(z))+h(z-\tau(z)))(1-\tau'(z))\\
&\quad+\prod_{z-\tau(z)\leq s_k<z}(1+d_k)^{-1}b(z)-\prod_{z-\tau(z)\leq s_k<z}(1+d_k)^{-1}h(z)q(z)\Bigg]\\
&\qquad\times(\varphi(z-\tau(z))-\psi(z-\tau(z)))\,dz+\int_0^s e^{-\int_z^s h(u)\,du}\Bigg[c(z)(\varphi(z)-\psi(z))\\
&\quad+\prod_{z-\delta(z)\leq s_k<z}(1+d_k)^{-1}e(z)(\varphi(z-\delta(z))-\psi(z-\delta(z)))\Bigg]\,dw(z)\Bigg|^2\\
&\leq\sup_{s\geq\vartheta}\Bigg\{2\Bigg[\sum_{l=1}^{n(t)}\Bigg|\frac{d_l}{1+d_l}\prod_{s_l-\tau(s_l)\leq s_k<s_l}(1+d_k)^{-1}e^{-\int_{s_l}^{s}h(u)\,du}q(s_l)\Bigg|\\
&\quad+\Bigg|\prod_{s-\tau(s)<s_k<s}(1+d_k)^{-1}q(s)\Bigg|+\int_{s-\tau(s)}^{s}|a(z)+h(z)|\,dz\\
&\quad+\int_0^s e^{-\int_z^s h(u)\,du}|h(z)|\int_{z-\tau(z)}^{z}|a(u)+h(u)|\,du\,dz\\
&\quad+\int_0^s e^{-\int_z^s h(u)\,du}\Bigg|(a(z-\tau(z))+h(z-\tau(z)))(1-\tau'(z))\\
&\quad+\prod_{z-\tau(z)\leq s_k<z}(1+d_k)^{-1}b(z)-\prod_{z-\tau(z)\leq s_k<z}(1+d_k)^{-1}h(z)q(z)\Bigg|\,dz\Bigg]^2\\
&\quad+2\int_0^s e^{-2\int_z^s h(u)\,du}\Bigg(c(z)+\prod_{z-\delta(z)\leq s_k<z}(1+d_k)^{-1}e(z)\Bigg)^2\,dz\Bigg\}\\
&\qquad\times\sup_{s\geq\vartheta}\left\{\mathbb{E}|\varphi(s)-\psi(s)|^2\right\}\leq\alpha\sup_{s\geq\vartheta}\left\{\mathbb{E}|\varphi(s)-\psi(s)|^2\right\}.
\end{aligned}
$$

Therefore, $P : \mathcal{S}_\phi \to \mathcal{S}_\phi$ is a contraction mapping with respect to the norm (5.3). □

We are now ready to prove Theorem 5.1.4.

Proof. Let P be defined as in Lemma 5.1.12. By a contraction mapping principle, P has a fixed point $y \in \mathcal{S}_\phi$, which is a solution of (5.6) with initial function (5.2) and $\mathbb{E}|y(s)|^2 \to 0$ as $t \to \infty$.

To obtain mean square asymptotic stability, we need to show that the zero solution of (5.6) is mean square stable. Let $\varepsilon > 0$ be given and choose $\delta > 0$ ($\delta < \varepsilon$) satisfying

$$2\delta \left(1 + \int_{\tau(0)}^{0} |a(s) + h(s)|\, ds + |q(0)|\right)^2 e^{-2\int_0^{t^*} h(u)\, du} + 2\varepsilon\alpha < \varepsilon,$$

If $y(t, 0, \phi)$ is a solution of (5.6) with $\|\phi\|^2 < \delta$, then $y(t) = (Py)(t)$ as defined in (5.9). We claim that $\mathbb{E}|y(t)|^2 < \varepsilon$ for all $t \geq 0$. Notice that $\mathbb{E}|y(t)|^2 < \varepsilon$ on $t \in [\vartheta, 0]$. Suppose there exists $t^* > 0$ such that $\mathbb{E}|y(t^*)|^2 = \varepsilon$ and $\mathbb{E}|y(s)|^2 < \varepsilon$ for all $\vartheta \leq s < t^*$, it follows from (5.9),

$$\begin{aligned}
\mathbb{E}|y(t^*)|^2 \;\leq\;& 2\|\phi\|^2 \left(1 + \int_{\tau(0)}^{0} |a(s) + h(s)|\, ds + |q(0)|\right)^2 e^{-2\int_0^{t^*} h(u)\, du} \\
&+ 2\varepsilon \left\{ 2 \left[\sum_{l=1}^{n(t)} \left| \frac{d_l}{1+d_l} \prod_{t_l - \tau(t_l) \leq t_k < t_l} (1+d_k)^{-1} e^{\int_{t^*}^{t_l} h(u)\, du} q(t_l) \right| \right.\right. \\
&+ \left| \prod_{t^* - \tau(t^*) < t_k < t^*} (1+d_k)^{-1} q(s) \right| + \int_{t^* - \tau(t^*)}^{t^*} |a(z) + h(z)|\, dz \\
&+ \int_0^{t^*} e^{-\int_z^{t^*} |h(u)|\, du} h(z) \int_{z-\tau(z)}^{z} |a(u) + h(u)|\, du\, dz \\
&+ \int_0^{t^*} e^{-\int_z^{t^*} h(u)\, du} \Big| (a(z - \tau(z)) + h(z - \tau(z)))(1 - \tau'(z)) \\
&\left. + \prod_{z - \tau(z) \leq s_k < z} (1+d_k)^{-1} b(z) - \prod_{z-\tau(z) \leq s_k < z} (1+d_k)^{-1} h(z) q(z) \Big|\, dz \right]^2 \\
&\left. + 2\int_0^{t^*} e^{-2\int_z^{t^*} h(u)\, du} \left(|c(z)| + \left| \prod_{z - \delta(z) \leq s_k < z} (1+d_k)^{-1} e(z) \right| \right)^2 dz \right\} \\
\leq\;& 2\delta \left(1 + \int_{\tau(0)}^{0} |a(s) + h(s)|\, ds + |q(0)|\right)^2 e^{-2\int_0^{t^*} h(u)\, du} + 2\varepsilon\alpha < \varepsilon,
\end{aligned}$$

which contradicts the definition of t^*. Thus, the zero solution of (5.6) is mean square stable. It follows that the zero solution of (5.6) is mean square asymptotically stable if (iii) holds.

Combining Lemma 5.1.11 and Theorem 5.1.4, we obtain that the zero solution of (5.1) is mean square asymptotically stable. □

Corollary 5.1.13. *Suppose that the conditions of* Theorem 5.1.4 *hold. Moreover,*

(i) *there exists a constant* $0 < \rho \leq 1$ *such that* $1 + d_k \leq \rho$,

(ii) *there exist constants* $\zeta > 0$ *such that* $t_k \leq \zeta k$.

If there exists a constant $M > 0$ *such that* $\left|\prod_{0<t_k<t}(1+d_k)\right| \leq M$, *then the zero solution of* (5.1) *is mean square exponentially stable.*

Proof. From (i) and (ii), we obtain that

$$\begin{aligned}\mathbb{E}|x(s)|^2 = \mathbb{E}\left|\prod_{0<t_k<s}(1+d_k)y(s)\right|^2 &\leq \rho^{2n}\mathbb{E}|y(s)|^2 = \frac{1}{\rho^2}e^{-2(n+1)|\ln\rho|}\mathbb{E}|y(s)|^2 \\ &\leq \frac{1}{\rho^2}e^{-\frac{t_{n+1}}{\zeta}|2\ln\rho|}\mathbb{E}|y(s)|^2 \leq \frac{1}{\rho^2}e^{-\frac{2|\ln\rho|}{\zeta}t}\mathbb{E}|y(s)|^2\end{aligned}$$

the proof of Theorem 5.1.4 indicates that for any $\varepsilon > 0$ and $\sigma \geq 0$, there exists a $\delta = \delta(\varepsilon, \sigma) > 0$ such that $\|\phi\|^2 < \delta$ implies $\mathbb{E}|y(t)|^2 < \varepsilon$ for $t \geq 0$. Therefore, the zero solution of (5.1) is mean square exponentially stable. □

If $q(t) \equiv 0$ in (5.1), then we come to the following stochastic delay differential equation

$$\begin{cases} dx(t) = [a(t)x(t) + b(t)x(t-\tau(t))]dt + [c(t)x(t) + e(t)x(t-\delta(t))]dw(t), & t \neq t_k, \\ x(t_k{}^+) - x(t_k) = d_k x(t_k), & t = t_k, \end{cases} \tag{5.11}$$

Corollary 5.1.14. *Suppose that the following conditions are satisfied*

(i) *the delay* $\tau(t)$ *is differentiable and* $t - \tau(t) \to \infty$ *as* $t \to \infty$, $t - \delta(t) \to \infty$ *as* $t \to \infty$;

(ii) *there exist a constant* $\alpha \in (0,1)$ *and a continuous function* $h(t)$ *such that for* $t \geq 0$,

$$\begin{aligned} 2\Bigg[&\int_{t-\tau(t)}^{t}|a(s)+h(s)|\,ds + \int_0^t e^{-\int_s^t h(u)\,du}|h(s)|\int_{s-\tau(s)}^{s}|a(u)+h(u)|\,du\,ds \\ &+ \int_0^t e^{-\int_s^t h(u)\,du}\Bigg|(a(s-\tau(s)) + h(s-\tau(s)))(1-\tau'(s)) \\ &+ \prod_{s-\tau(s)\leq t_k<s}(1+d_k)^{-1}b(s)\Bigg|\,ds\Bigg]^2 \\ &+2\int_0^t e^{-2\int_s^t h(u)\,du}\left(|c(s)| + \left|\prod_{s-\delta(s)\leq t_k<s}(1+d_k)^{-1}e(s)\right|\right)^2 ds \leq \alpha; \end{aligned}$$

(iii) *and such that* $\liminf_{t\to\infty}\int_0^t h(s)\,ds > -\infty$;

(iv) *there exists a constant* $M > 0$ *such that*

$$\left|\prod_{0<t_k<t}(1+d_k)\right| \leq M.$$

Then the zero solution of (5.11) *is mean square asmptotically stable if*

(v)

$$\int_0^t h(s)\,ds \to \infty \quad \text{as} \quad \text{t} \to \infty.$$

5.1.3 Proof of Theorem 5.1.9

Lemma 5.1.15. *Let $\varphi \in \mathcal{C}_\phi$. Define an operator by $(P\varphi)(t) = \phi(t)$ for $t \in [-\tau, 0]$ and for $t \geq 0$,*

$$(P\varphi)(t) = \sum_{i=1}^{5} I_i(t),$$

where $I_i(t)$, $I = 1, 2, 3, 4, 5$, is denoted as in (5.10).

If conditions (i)-(iv) in Theorem 5.1.9 are satisfied, then there exists $\delta > 0$ such that for any $\phi : [-\tau, 0] \to (-\delta, \delta)$, we have that $P : \mathcal{C}_\phi \to \mathcal{C}_\phi$ and P is a contraction with respect to the norm (5.5).

Proof. First, following the proof of Theorem 5.1.4, we note that P is continuous on $[0, \infty)$.

Next, We prove that $P(\mathcal{C}_\phi) \subseteq \mathcal{C}_\phi$. For $\varphi \in \mathcal{C}_\phi$, it is easy to check that

$$\mathbb{E}\left(\sup_{t-\tau\leq s\leq t} |I_i(s)|^2\right) \to 0 \quad \text{as} \quad t \to \infty, \quad i = 1, 2, 3, 4.$$

Note that for every r,

$$\int_0^s e^{-\int_z^r h(u)\,du}\left[c(z)\varphi(z) + \prod_{z-\delta(z)\leq s_k<z} (1+d_k)^{-1} e(z)\varphi(z-\delta(z))\right] dw(z),$$

is a martingale, then

$$\sup_{t-\tau\leq r\leq t}\left|\int_0^s e^{-\int_z^r h(u)\,du}\left[c(z)\varphi(z) + \prod_{z-\delta(z)\leq s_k<z} (1+d_k)^{-1} e(z)\varphi(z-\delta(z))\right] dw(z)\right|$$

is a submartingle, by Doob's inequality, we have that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t-\tau\leq s\leq t} |I_5(s)|^2\right] \\
&= \mathbb{E}\Bigg\{\sup_{t-\tau\leq s\leq t}\Bigg|\int_0^s e^{-\int_z^s h(u)\,du} \\
&\quad \times\left[c(z)y(z) + \prod_{z-\delta(z)\leq s_k<z} (1+d_k)^{-1} e(z)\varphi(z-\delta(z))\right] dw(z)\Bigg|^2\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
&\le \mathbb{E}\Bigg\{ \sup_{t-\tau\le s\le t}\sup_{t-\tau\le r\le t}\Bigg| \int_0^s e^{-\int_z^r h(u)\,du} \\
&\qquad \times \Bigg[c(z)\varphi(z) + \prod_{z-\delta(z)\le s_k<z} (1+d_k)^{-1} e(z)\varphi(z-\delta(z))\Bigg] dw(z)\Bigg|^2 \Bigg\} \\
&\le 4\mathbb{E}\Bigg\{ \sup_{t-\tau\le r\le t}\Bigg| \int_0^t e^{-\int_z^r h(u)\,du} \\
&\qquad \times \Bigg[c(z)\varphi(z) + \prod_{z-\delta(z)\le s_k<z} (1+d_k)^{-1} e(z)\varphi(z-\delta(z))\Bigg] dw(z)\Bigg|^2 \Bigg\} \\
&\le 4\mathbb{E}\int_0^t \Bigg[\sup_{t-\tau\le r\le t} e^{-2\int_z^r h(u)\,du}\Bigg] \Bigg[c(z)y(z) + \prod_{z-\delta(z)\le s_k<z} (1+d_k)^{-1} e(z)\varphi(z-\delta(z))\Bigg]^2 dz \\
&\le 4\int_0^t \Bigg[\sup_{t-\tau\le r\le t} e^{-2\int_z^r h(u)\,du}\Bigg] \Bigg[c(z) + \prod_{z-\delta(z)\le s_k<z} (1+d_k)^{-1} e(z)\Bigg]^2 dz \\
&\qquad \times \sup_{t\ge 0}\mathbb{E}\Bigg[\sup_{t-\tau\le s\le t} |\varphi(s)|^2\Bigg],
\end{aligned} \tag{5.12}
$$

since $\mathbb{E}\left(\sup_{t-\tau\le s\le t}|\varphi(s)|^2\right) \to 0$ as $t\to\infty$, then for any $\varepsilon>0$, there exists $T>0$ such that $t>T$ implies $\mathbb{E}\sup_{t-\tau\le s\le t}|\varphi(s)|^2<\varepsilon$. Hence, for $t>T$, we have that

$$
\begin{aligned}
&4\int_0^t \Bigg[\sup_{t-\tau\le r\le t} e^{-2\int_z^r h(u)\,du}\Bigg] \Bigg[c(z) + \prod_{z-\delta(z)\le s_k<z} (1+d_k)^{-1} e(z)\Bigg]^2 dz \sup_{t\ge 0}\mathbb{E}\Bigg[\sup_{t-\tau\le s\le t} |\varphi(s)|^2\Bigg] \\
&\quad < \Bigg\{ 4\int_0^t \Bigg[\sup_{t-\tau\le r\le t} e^{-2\int_z^r h(u)\,du}\Bigg] \Bigg[c(z) + \prod_{z-\delta(z)\le s_k<z} (1+d_k)^{-1} e(z)\Bigg]^2 dz\Bigg\}\varepsilon,
\end{aligned}
$$

which yields $\mathbb{E}\left[\sup_{t-\tau\le s\le t}|I_5(s)|^2\right]\to 0 \quad \text{as} \quad \mathrm{t}\to\infty$. Hence, we obtain that $P(\mathcal{C}_\phi)\subseteq\mathcal{C}_\phi$.

Finally, We show that P is contraction mapping. For $\varphi,\psi\in\mathcal{C}_\phi$, we have that

$$
\begin{aligned}
&\sup_{t\ge 0}\Bigg\{ \mathbb{E}\Bigg[\sup_{t-\tau\le s\le t} |(P\varphi)(s)-(P\psi)(s)|^2\Bigg]\Bigg\} \\
&= \sup_{t\ge 0}\mathbb{E}\Bigg\{ \sup_{t-\tau\le s\le t}\Bigg| \sum_{l=1}^{n(t)} \frac{d_l}{1+d_l} \prod_{s_l-\tau(s_l)\le s_k<s_l} (1+d_k)^{-1} e^{-\int_{s_l}^s h(u)\,du} q(s_l) \\
&\qquad \times(\varphi(s_l-\tau(s_l))-\psi(s_l-\tau(s_l))) \\
&\qquad + \prod_{s-\tau(s)\le s_k<s} (1+d_k)^{-1} q(s)(\varphi(s-\tau(s))-\psi(s-\tau(s))) \\
&\qquad + \int_{s-\tau(s)}^s (a(z)+h(z))(\varphi(z)-\psi(z))\,dz \\
&\qquad - \int_0^s e^{-\int_z^s h(u)\,du} h(z)\int_{z-\tau(z)}^z (a(u)+h(u))(\varphi(u)-\psi(u))\,du\,dz
\end{aligned}
$$

$$
\begin{aligned}
&+\int_0^s e^{-\int_z^s h(u)\,du}\Bigg[(a(z-\tau(z))+h(z-\tau(z)))(1-\tau'(z)) \\
&+\prod_{z-\tau(z)\le s_k<z}(1+d_k)^{-1}b(z)-\prod_{z-\tau(z)\le s_k<z}(1+d_k)^{-1}h(z)q(z)\Bigg] \\
&\qquad\times(\varphi(z-\tau(z))-\psi(z-\tau(z)))\,dz+\int_0^s e^{-\int_z^s h(u)\,du}\Bigg[c(z)(\varphi(z)-\psi(z)) \\
&+\prod_{z-\delta(z)\le s_k<z}(1+d_k)^{-1}e(z)(\varphi(z-\delta(z))-\psi(z-\delta(z)))\Bigg]\,dw(z)\Bigg|^2\Bigg\} \\
\le \sup_{s\ge 0}&\Bigg\{2\Bigg[\sum_{l=1}^{n(t)}\Bigg|\frac{d_l}{1+d_l}\prod_{s_l-\tau(s_l)\le s_k<s_l}(1+d_k)^{-1}e^{-\int_{s_l}^s h(u)\,du}q(s_l)\Bigg| \\
&+\Bigg|\prod_{s-\tau(s)<s_k<s}(1+d_k)^{-1}q(s)\Bigg|+\int_{s-\tau(s)}^s|a(z)+h(z)|\,dz \\
&+\int_0^s e^{-\int_z^s h(u)\,du}|h(z)|\int_{z-\tau(z)}^z|a(u)+h(u)|\,du\,dz \\
&+\int_0^s e^{-\int_z^s h(u)\,du}\Bigg|(a(z-\tau(z))+h(z-\tau(z)))(1-\tau'(z)) \\
&+\prod_{z-\tau(z)\le s_k<z}(1+d_k)^{-1}b(z)-\prod_{z-\tau(z)\le s_k<z}(1+d_k)^{-1}h(z)q(z)\Bigg|\,dz\Bigg]^2 \\
&+8\int_0^s\left[\sup_{s-\tau\le r\le s}e^{-2\int_z^r h(u)\,du}\right]\left[c(z)+\prod_{z-\delta(z)\le s_k<z}(1+d_k)^{-1}e(z)\right]^2dz\Bigg\} \\
&\qquad\times\sup_{t\ge 0}\mathbb{E}\left[\sup_{t-\tau\le s\le t}|\varphi(s)-\psi(s)|^2\right]\le\alpha\sup_{t\ge 0}\mathbb{E}\left[\sup_{t-\tau\le s\le t}|\varphi(s)-\psi(s)|^2\right].
\end{aligned}
$$

Therefore, P is a contraction mapping with respect to the norm (5.5).

We are now ready to prove Theorem 5.1.9.

Proof. Let P be defined as in Lemma 5.1.15. By a contraction mapping principle, P has a fixed point $y\in S$, which is a solution of (5.6) with initial function (5.2) and $\mathbb{E}\sup_{t-\tau\le s\le t}|y(s)|^2\to 0$ as $t\to\infty$.

To obtain mean square asymptotic stability, we need to show that the zero solution of (5.6) is mean square stable. Let $\varepsilon>0$ be given and choose $\delta>0$ ($\delta<\varepsilon$) satisfying

$$2\delta\left(1+\int_{\tau(0)}^0|a(s)+h(s)|\,ds+|q(0)|\right)^2e^{-2\int_0^{t^*}h(u)\,du}+2\varepsilon\alpha<\varepsilon,$$

If $y(t,0,\phi)$ is a solution of (5.6) with $\|\phi\|^2<\delta$, then $y(t)=(Py)(t)$ as defined in (5.9). We claim that $\mathbb{E}|y(t)|^2<\varepsilon$ for all $t\ge 0$. Notice that $\mathbb{E}|y(t)|^2<\varepsilon$ on $t\in[-\tau,0]$. Suppose there exists $t^*>0$ such that $\mathbb{E}|y(t^*)|^2=\varepsilon$ and $\mathbb{E}|y(s)|^2<\varepsilon$ for all $-\tau\le s<t^*$, it follows from (5.9),

$$\begin{aligned}
\mathbb{E}|y(t^*)|^2 \;\le\;& 2\|\phi\|^2\left(1+\int_{\tau(0)}^{0}|a(s)+h(s)|\,ds+|q(0)|\right)^2 e^{-2\int_0^{t^*}h(u)\,du}\\
&+2\varepsilon\Bigg\{2\Bigg[\sum_{l=1}^{n(t)}\left|\frac{d_l}{1+d_l}\prod_{t_l-\tau(t_l)\le t_k<t_l}(1+d_k)^{-1}e^{\int_{t^*}^{t_l}h(u)\,du}q(t_l)\right|\\
&+\left|\prod_{t^*-\tau(t^*)<t_k<t^*}(1+d_k)^{-1}q(s)\right|+\int_{t^*-\tau(t^*)}^{t^*}|a(z)+h(z)|\,dz\\
&+\int_0^{t^*}e^{-\int_z^{t^*}|h(u)|\,du}h(z)\int_{z-\tau(z)}^{z}|a(u)+h(u)|\,du\,dz\\
&+\int_0^{t^*}e^{-\int_z^{t^*}h(u)\,du}\Big|(a(z-\tau(z))+h(z-\tau(z)))(1-\tau'(z))\\
&+\prod_{z-\tau(z)\le s_k<z}(1+d_k)^{-1}b(z)-\prod_{z-\tau(z)\le s_k<z}(1+d_k)^{-1}h(z)q(z)\Big|\,dz\Bigg]^2\\
&+8\int_0^{t^*}e^{-2\int_z^{t^*}h(u)\,du}\left(|c(z)|+\left|\prod_{z-\delta(z)\le s_k<z}(1+d_k)^{-1}e(z)\right|\right)^2dz\Bigg\}\\
\le\;& 2\delta\left[1+\int_{\tau(0)}^{0}|a(s)+h(s)|\,ds+|q(0)|\right]^2 e^{-2\int_0^{t^*}h(u)\,du}+2\varepsilon\alpha<\varepsilon,
\end{aligned}$$

which contradicts the definition of t^*. Thus, the zero solution of (5.6) is mean square stable. It follows that the zero solution of (5.6) is mean square asymptotically stable if (iii) holds.

Combining Lemma 5.1.11 and Theorem 5.1.9, we have that the zero solution of (5.1) is asymptotically stable. □

Remark 5.1.16. *In Theorem 5.1.9, we obtain* $\lim_{t\to\infty}\mathbb{E}\sup_{t-\tau\le s\le t}|y(s,0,\phi)|^2=0$*, that is, for any function* $s\mapsto y_t(s,0,\phi)$*, we have* $\lim_{t\to\infty}\mathbb{E}|y_t(\cdot,0,\phi)|^2_{C[-\tau,0]}=0$*, which implies* $\lim_{t\to\infty}\mathbb{E}|y(t,0,\phi)|^2=0$.

Remark 5.1.17. *In some papers, see, for example, [89, 90, 131, 132], the norm is defined by*

$$\|\psi\|_{[0,t]}=\left\{\mathbb{E}\left(\sup_{s\in[0,t]}|\psi(s,\omega)|^2\right)\right\}^{1/2}.$$

As in [90], to show $P(\mathcal{S})\subseteq\mathcal{S}$*, we need to estimate* $\mathbb{E}\sup_{s\in[0,t]}|I_5(s)|^2$*, where*

$$I_5(s)=\int_0^s e^{-\int_z^s h(u)\,du}\,[c(z)x(z)+e(z)x(z-\delta(z))]\,dw(z).$$

However, $I_5(s)$ *is not a local martingale, see Lemma 1.3.40 in Section 1.3 of Chapter* **??** *for the explanation. Hence, Burkholder-Davis-Gundy Inequality can not be applied directly.*

5.1.4 Examples

Example 5.1.18. *Consider the following linear stochastic delay differential equation with linear impulsive effects*

$$\begin{cases} dx(t) = [ax(t) + b(t)x(t-\tau(t))]dt + e(t)x(t-\tau(t))dw(t), \quad t \neq t_k \\ x(t_k{}^+) - x(t_k) = d_k x(t_k), \quad t = t_k, \end{cases} \tag{5.13}$$

where $t_k = 2k\pi$, $1 + d_k = \frac{1}{2}$ *for all* $k = 1, 2, \cdots$, $0 \leq \tau(t) < 2\pi$ *is a continuous function such that* $t - \tau(t) \to \infty$ *as* $t \to \infty$, $a < 0$ *is a constant,* $c(t), e(t)$ *are bounded continuous function such that* $|b(t)| \leq \bar{b}$, $|e(t)| \leq \bar{e}$, *where* $\bar{b}, \bar{e}$ *are positive constants.*

Since $t_k = 2k\pi$ and $0 \leq \tau(t) < 2\pi$, we have that at most one impulse occurs at interval $[t - \tau(t), t)$, and hence $\prod_{t-\tau(t)<t_k<t}(1 + d_k)^{-1} \leq 2$. Choosing $h(t) \equiv -a$ in Corollary 5.1.14,

$$\int_{t-\tau(t)}^{t} |a(s) + h(s)|\, ds + \int_0^t e^{-\int_s^t |h(u)|\, du} h(s) \int_{s-\tau(s)}^{s} |a(u) + h(u)|\, du\, ds = 0,$$

$$2\left[\int_0^t e^{-\int_s^t h(u)\, du}\left|(a(s-\tau(s)) + h(s-\tau(s)))(1-\tau'(s)) + \prod_{s-\tau(s)\leq t_k<s}(1+d_k)^{-1}b(s)\right| ds\right]^2$$
$$\leq 2\left[\int_0^t e^{\int_s^t a\, du}\left|\prod_{s-\tau(s)\leq t_k<s}(1+d_k)^{-1}\bar{b}\right| ds\right]^2 \leq \frac{8\bar{b}^2}{a^2},$$

$$2\int_0^t e^{-2\int_s^t h(u)\, du}\left|\prod_{s-\tau(s)\leq t_k<s}(1+d_k)^{-1}e(s)\right|^2 ds$$
$$\leq 2\int_0^t e^{2\int_s^t a\, du}\left|\prod_{s-\tau(s)\leq t_k<s}(1+d_k)^{-1}e(s)\right|^2 ds \leq \frac{4\bar{e}^2}{-a}.$$

Hence, we obtain that $H_1(t) \leq \frac{8\bar{b}^2 - 4a\bar{e}^2}{a^2}$. On the other hand, $\left|\prod_{0\leq t_k<t}(1+d_k)\right| \leq 1$. From Corollary 5.1.14, we obtan that the zero solution of (5.13) is mean square asymptotically stable if

$$\frac{8\bar{b}^2 - 4a\bar{e}^2}{a^2} < 1. \tag{5.14}$$

Now, we check the conditions in Theorem 5.1.9,

$$8\int_0^t \left[\sup_{t-\tau\leq r\leq t} e^{-2\int_s^r h(u)\, du}\right]\left[\left|\prod_{s-\delta(s)\leq t_k<s}(1+d_k)^{-1}e(s)\right|\right]^2 ds \leq \frac{16\bar{e}^2}{-a}e^{-4a\pi}.$$

Hence, we obtain that $H_2(t) \leq \frac{8\bar{b}^2 - 16a\bar{e}^2 e^{-4a\pi}}{a^2}$. From Theorem 5.1.9, the zero solution of (5.13) is mean square asymptotically stable if

$$\frac{8\bar{b}^2 - 16a\bar{e}^2 e^{-4a\pi}}{a^2} < 1. \tag{5.15}$$

It is not difficult to find that condition (5.14) is weaker than condition (5.15).

Example 5.1.19. *Consider the following linear neutral stochastic delay differential equations with impulsive effects*

$$\begin{cases} d\left[x(t) - qx\left(t - \frac{\sin t}{2}\right)\right] = -2x(t)dt + \left[cx(t) + ex(t-r)\right]dw(t), \quad t \neq t_k \\ x(t_k{}^+) - x(t_k) = d_k x(t_k), \quad t = t_k, \end{cases} \tag{5.16}$$

where $d_k = \frac{1}{2^k}$,$1+d_k = 1+\frac{1}{2^k}$ *for all* $k = 1, 2, \cdots$, $0 \le r < 1$ *is a constant,* $\inf_{k\in\mathbb{N}}\{t_{k+1}-t_k\} = 1$. *Suppose that* q, c, e *are positive constants.*

Since $\inf_{k\in\mathbb{N}}\{t_{k+1} - t_k\} = 1$ and $\tau(t) = \frac{\sin t}{2} \le \frac{1}{2}$, we have that at most one impulse occurs at interval $[t - \tau(t), t)$, and hence $\prod_{t-\tau(t)<t_k<t}(1+d_k)^{-1} \le \frac{2}{3}$. Choosing $h(t) \equiv 2$ in Theorem 5.1.4,

$$\sum_{l=1}^{n(t)} \left| \frac{d_l}{1+d_l} \prod_{t_l-\tau(t_l)\le t_k<t_l} (1+d_k)^{-1} e^{-\int_{t_l}^t h(u)\,du} q(t_l) \right| \le \frac{2q}{3} \sum_{l=1}^{n(t)} \frac{1}{1+2^l} < \frac{2qn}{9},$$

$$\left| \prod_{t-\tau(t)<t_k<t} (1+d_k)^{-1} q(t) \right| = \frac{2q}{3},$$

$$\int_{t-\tau(t)}^t |a(s)+h(s)|\,ds + \int_0^t e^{-\int_s^t |h(u)|\,du} h(s) \int_{s-\tau(s)}^s |a(u)+h(u)|\,du\,ds = 0,$$

$$\begin{aligned} &\int_0^t e^{-\int_s^t h(u)\,du} \Bigg| (a(s-\tau(s)) + h(s-\tau(s)))(1-\tau'(s)) + \prod_{s-\tau(s)\le t_k<s} (1+d_k)^{-1} b(s) \\ &- \prod_{s-\tau(s)\le t_k<s} (1+d_k)^{-1} h(s) q(s) \Bigg| \,ds \le \int_0^t e^{-\int_s^t 2\,du} \frac{4q}{3}\,ds \le \frac{2q}{3}, \end{aligned}$$

$$\begin{aligned} 2\int_0^t e^{-2\int_s^t h(u)\,du} \left[|c(s)| + \left| \prod_{s-r\le t_k<s} (1+d_k)^{-1} e(s) \right| \right]^2 ds \quad &\le \quad 2\int_0^t e^{-2\int_s^t 2\,du} \left(c + \frac{2e}{3}\right)^2 ds \\ &\le \quad \frac{1}{2}\left(c + \frac{2e}{3}\right)^2. \end{aligned}$$

Hence, we obtain that $H_1(t) \le 2\left(\frac{2nq}{9} + \frac{4q}{3}\right)^2 + \frac{1}{2}\left(c + \frac{2e}{3}\right)^2$. Hence, from Theorem 5.1.4, we obtain that the zero solution of (5.13) is mean square asymptotically stable if

$$2\left(\frac{2nq}{9} + \frac{4q}{3}\right)^2 + \frac{1}{2}\left(c + \frac{2e}{3}\right)^2 < 1. \tag{5.17}$$

Now, we check the conditions in Theorem 5.1.9,

$$\begin{aligned} &8\int_0^t \left[\sup_{t-\tau\le r\le t} e^{-2\int_s^r h(u)\,du} \right] \left[|c(s)| + \left| \prod_{s-\delta(s)\le t_k<s} (1+d_k)^{-1} e(s) \right| \right]^2 ds \\ &\le 2e^2 \left(c + \frac{2d}{3}\right)^2 (1 - e^{-4t}) < 2e^2 \left(c + \frac{2e}{3}\right)^2. \end{aligned}$$

Hence, we obtain that $H_2(t) < 2\left(\frac{2nq}{9}+\frac{4q}{3}\right)^2 + 2e^2\left(c+\frac{2e}{3}\right)^2$.

On the other hand, $\left|\prod_{0\le t_k<t}(1+d_k)\right| \le 2$. From Theorem 5.1.9, the zero solution of (5.13) is mean square asymptotically stable if

$$2\left(\frac{2nq}{9}+\frac{4q}{3}\right)^2 + 2e^2\left(c+\frac{2d}{3}\right)^2 < 1. \tag{5.18}$$

It is not difficult to find that condition (5.17) is weaker than condition (5.18).

Example 5.1.20. *Consider the following linear neutral stochastic delay differential equations with impulsive effects*

$$\begin{cases} dx(t) = -a(t)x(t)dt + bx(t-r)dw(t), & t \ne t_k \\ x(t_k{}^+) - x(t_k) = d_k x(t_k), & t = t_k, \end{cases} \tag{5.19}$$

where $0 \le r < 2\pi$, $t_k = 2k\pi$, $1 + d_k = c$ *for all* $k = 1, 2, \cdots$ *for some* $0 < c \le 1$, $a(t)$ *is a continuous function such that* $\int_0^t a(s)\,ds = \infty$ *and*

$$\sup_{t\ge 0}\left\{\frac{b^2}{c^2}\int_0^t e^{-2\int_s^t a(u)\,du}\,ds\right\} < 1,$$

then the zero solution of (5.19) *is mean square asymptotically stable.*

Since $t_k = 2k\pi$ and $0 \le r < 2\pi$, we have that at most one impulse occurs at interval $[t-r, t)$, and hence $\prod_{t-r<t_k<t}(1+d_k)^{-1} \le \frac{1}{c}$. If we choose $h(t) \equiv a(t)$ in Theorem 5.1.4, then by Theorem 5.1.4, we have that the zero solution of (5.19) is mean square asymptotically stable. Since $1 + d_k = c \le 1$ and $t_k = 2k\pi$, from Theorem 5.1.13, we have that the zero solution of (5.19) is also mean square exponentially stable.

If $c = 1$ for all $k = 1, 2, \cdots$, the condition reduces to the condition (3.6) in [90] for equation (5.19) without impulses.

5.2 Exponential stability of a class of impulsive neutral stochastic partial differential equations with variable delays and Poisson jumps

5.2.1 Introduction and preliminaries

The classical technique applied in the study of stability of stochastic delay differential equations is based on a stochastic version of Liapunov's direct method. The success of Liapunov's direct method depending on finding good Liapunov functionals, which may be difficult, especially for equations with unbounded delays or unbounded terms. Recently, Chen [21] used an appropriate impulsive-integral inequality to establish sufficient conditions for exponential stability of impulsive stochastic partial delay differential equations, and it turns out that it is a convenient way to study exponential stability of mild solutions of impulsive stochastic delay differential equations. Sakthivel and Luo [117, 118] have discussed the asymptotic stability for mild solutions of impulsive stochastic partial delay differential equations by using fixed point methods. This powerful method is also an effective tool to deal with exponential stability for mild solutions to stochastic partial differential equations with delays, see, for example, [91] and Cui et al. [27]. However, to the best of our knowledge, there is no result about exponential stability of mild solutions of impulsive stochastic partial differential equations with variable delays and Poisson jumps.

The aim of this section is to study the impulsive effects to a class of stochastic partial differential equations with variable delays and Poisson jumps by using two methods, the method by using an appropriate impulsive-integral inequality and fixed point methods.

Let $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\}$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. right continuous and $\mathcal{F}_0$ containing all P-null sets). Let X, Y be two real separable Hilbert spaces which are both equipped with a norm denoted by $\|\cdot\|$. Let $\mathcal{L}(Y, X)$ denote the space of all bounded linear operators from Y into X.

Suppose $\{p(t), t \geq 0\}$ is a σ-finite stationary $\mathcal{F}_t$-adapted Possion point process taking values in measurable space $(U, \mathcal{B}(U))$. The random measure N_p defined by $N_p((0,t]\times\Lambda) := \sum_{s\in(0,t]} 1_\Lambda(p(s))$ for $\Lambda \in \mathcal{B}(U)$ is called the Poisson random measure induced by $p(\cdot)$, thus, we can define the measure $\tilde{N}$ by $\tilde{N}(dt, dy) = N_p(dt, dy) - \nu(dy)dt$, where ν is the characteristic measure of N_p, which is called the compensated Poisson random measure. Let $w = (w(t))_{t\geq 0}$, independent of the Poisson point process, be a Y-valued Wiener process defined on $\{\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\}$ with covariance operator Q, that is

$$E\langle w(t), x\rangle_Y \langle w(s), y\rangle_Y = (t\wedge s)\langle Qx, y\rangle_Y, \quad x, y \in Y,$$

where Q is a positive, self-adjoint, trace class operator on Y. Furthermore, $\mathcal{L}_2^0(Y, X)$ denotes the space of all Q-Hilbert-Schmidt operators from Y to X with the norm

$$\|\xi\|^2_{\mathcal{L}_2^0} := tr(\xi Q \xi^*) < \infty, \quad \xi \in \mathcal{L}_2^0(Y, X).$$

For the construction of a stochastic integral in a Hilbert space, see Da Prato and Zabczyk [108].

For Borel set $Z \in \mathcal{B}(U\setminus\{0\})$, we consider the following impulsive neutral stochastic delay differential equation with Poisson jumps

$$\begin{cases} d[x(t) + u(t, x(t-\tau(t)))] = [Ax(t)dt + f(t, x(t-\delta(t)))]dt + g(t, x(t-\rho(t))dw(t) \\ \qquad\qquad\qquad\qquad + \int_Z h(t, x(t-\sigma(t)), y)\, \tilde{N}(dt, dy), \quad t \geq 0, \quad t \neq t_k, \\ \Delta x(t_k) = I_k(x(t_k^-)), \quad t = t_k, \quad k = 1, 2, \cdots, \\ x_0(\theta) = \phi, \quad \theta \in [-\tau, 0], \quad a.s. \end{cases} \tag{5.20}$$

where $\phi \in PC$ and the functions $\tau(t), \delta(t), \rho(t), \sigma(t) : [0,\infty) \to [0,\tau]$ ($\tau > 0$ is a constant) are continuous functions, where $PC \equiv PC([-\tau, 0]; X)$ is the space of all almost surely bounded $\mathcal{F}_0$-measurable functions from $[-\tau, 0]$ into X that are continuous everywhere except for a finite number of points s at which the left and right limits $\phi(s-)$ and $\phi(s+)$ exist and $\phi(s^+) = \phi(s)$ as usual, equipped with the supremum norm $\|\phi\|_0 = \text{essup}_{\omega\in\Omega} \sup_{t\in[-\tau,0]} \|\phi(t)(\omega)\|$; $-A$ is a closed, densely defined linear operator generating an analytic semigroup $S(t)(t \geq 0)$ on the

Hilbert space X; then it is possible under some circumstances (we refer the readers to [100] for a detailed presentations of the definition and relevant properties of $(-A)^\alpha$) to define the fractional power $(-A)^\alpha : D((-A)^\alpha) \to X$ which is a closed linear operator with its domain $D((-A)^\alpha)$, for $\alpha \in (0,1]$. Moreover, the fixed moments of time t_k satisfy $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k\to\infty} t_k = \infty$; $x(t_k^-)$ and $x(t_k^+)$ represent the left and right limits of $x(t)$ at time $t = t_k$, $k = 1, 2, \cdots$, respectively. $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ denotes the jump in the state x at time t_k with $I_k(\cdot) : X \to X (k = 1, 2, \cdots)$ determining the size of the jump; $u, f : [0,\infty) \times X \to X$, $g : [0,\infty) \times X \to \mathcal{L}_2^0(Y,X)$, $h : [0,\infty) \times X \times U \to X$ are given functions to be specified later.

Definition 5.2.1. (Chen [21]) *Let $\sigma \in L(Y,X)$, and define*

$$\|\sigma\|^2_{\mathcal{L}_2^0} := tr(\sigma Q \sigma^*) = \sum_{n=1}^{\infty} \|\sqrt{\lambda_n}\sigma e_n\|^2$$

If $\|\sigma\|^2_{\mathcal{L}_2^0} < +\infty$, then σ is called a Q-Hilbert-Schmidt operator and $\mathcal{L}_2^0(Y,X)$ denotes the space of all Q-Hilbert-Schmidt operators $\sigma : Y \to X$.

Note that equation (5.20) could be the infinite dimensional formulation of a partial differential equation with delays, (see, e.g. Da Prato and Zabczyk [108]).

Definition 5.2.2. *An X-valued stochastic process $x(t)$, $t \in [0,+\infty)$, is called a mild solution of (5.20) if*

(i) *$x(t)$ is adapted to $\mathcal{F}_t$, $t \geq 0$;*

(ii) *$t \mapsto x(t)$ has* càdlàg *paths on $[0,+\infty)$ almost surely, and for $t \in [0,+\infty)$, $x(t)$ satisfies the following integral equation*

$$\begin{aligned} x(t) \;=\; & S(t)(\phi(0) + u(0,\phi)) - u(t, x(t-\tau(t))) \\ & - \int_0^t AS(t-s)u(s, x(s-\tau(s)))\,ds + \int_0^t S(t-s)f(s, x(s-\delta(s)))\,ds \\ & + \int_0^t \int_Z S(t-s)h(s, x(s-\theta(s)), y)\,\tilde{N}(\,ds,\,dy) \\ & + \int_0^t S(t-s)g(s, x(s-\rho(s)))\,dw(s) + \sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-)), \end{aligned} \quad (5.21)$$

where $x_0(\cdot) = \phi$, a.s.

Definition 5.2.3. *Let $p > 0$. Equation (5.20) is said to be exponentially stable in pth moment, if for any initial value $\phi \in PC$, there exists a pair of positive constants γ and C such that*

$$\mathbb{E}\|x(t)\|^p \leq C\|\phi\|_0^p e^{-\gamma t}, \quad t \geq 0,$$

where $\mathbb{E}$ denotes expectation with respect to the probability measure $\mathbb{P}$ and x is the mild solution of (5.20).

To obtain our main results, we impose the following assumptions:

(A1) A is the infinitesimal generator of an analytic semigroup of bounded linear operators $\{S(t), t \geq 0\}$ in X such that $0 \in \rho(-A)$, the resolvent set of $-A$, and $S(t)$ is uniformly bounded,

$$\|S(t)\| \leq Me^{-\gamma t}, \quad t \geq 0,$$

for some constants $\gamma, M > 0$;

(A2) the mappings $f(t,\cdot), \sigma(t,\cdot)$ and $h(t,\cdot)$ satisfy Lipschitz conditions, that is, there exist $L_1, L_2, L_3 > 0$ such that for any $x, y \in H$ and $t \geq 0$,

$$\begin{aligned} \|f(t,x) - f(t,y)\| &\leq L_1\|x-y\|, \quad L_1 > 0, \\ \|g(t,x) - g(t,y)\| &\leq L_2\|x-y\|, \quad L_2 > 0, \\ \int_z \|h(t,x,z) - h(t,y,z)\|^2 \nu(dz) &\leq L_3^2\|x-y\|^2, \quad L_3 > 0, \end{aligned}$$

we further assume that $f(t,0) = g(t,0) = h(t,0,z) = 0$ for all $t \geq 0$, $z \in Z$. Hence, (5.20) has a trivial solution $x = 0$ when $\phi = 0$.

(A3) The mapping $(-A)^\alpha u(t,\cdot)$ satisfies a uniformly Lipschitz condition: there exists a positive constant $K > 0$ such that for any $x, y \in X$,

$$\|(-A)^\alpha u(t,x) - (-A)^\alpha u(t,y)\| \leq K\|x-y\|, \quad u(t,0) = 0, \quad t \geq 0,$$

for $\alpha \in (1/p, 1]$ (for some $p \geq 2$) and $u(t,\cdot) \in D((-A)^\alpha)$. Moreover, for $\alpha \in (1/p, 1]$, $\kappa = \|(-A)^{-\alpha}\|K < 1$.

(A4) $I_k \in C(X,X)$ and there exists a positive constant q_k such that $\|I_k(x) - I_k(y)\| \leq q_k\|x-y\|$ and $I_k(0) = 0, k = 1, 2, 3, \cdots$, for each $x, y \in X$.

Remark 5.2.4. *Under the conditions* (A1)-(A4) *and suppose that the mappings $f(t,\cdot), \sigma(t,\cdot)$ and $h(t,\cdot)$ satisfy linear growth conditions, the existence and uniqueness of the system* (5.20) *can be shown by using Picard iterative method (see, Anguraj and Vinodkumar* [2]*).*

Lemma 5.2.5. (Theorem 6.13, [100]) *Suppose that the assumption* (A1) *holds, then for any $\beta \in (0,1]$, we have that*

(i) *for each $x \in \mathcal{D}((-A)^\beta)$,*

$$S(t)(-A)^\beta x = (-A)^\beta S(t)x;$$

(ii) *there exist positive constant $M_\beta > 0$ such that*

$$\left\|(-A)^\beta S(t)\right\| \leq M_\beta t^{-\beta} e^{-\gamma t}, \quad t > 0.$$

Lemma 5.2.6. (Da Prato and Zabczyk [108]) *For any $p \geq 2$ and for an arbitrary $\mathcal{L}_2^0$-valued adapted caglad process $\Phi(\cdot)$,*

$$\sup_{s\in[0,t]} \mathbb{E}\left\|\int_0^s \Phi(u)dw(u)\right\|^p \leq c_p \left(\int_0^t \left(\mathbb{E}\|\Phi(s)\|^p_{\mathcal{L}_2^0}\right)^{2/p} ds\right)^{p/2}.$$

where $c_p = (p(p-1)/2)^{p/2}$.

Lemma 5.2.7. (Mao [96]) *Let $p \in [1,\infty)$ and $\nu \in (0,1)$. For any two real positive numbers $a, b > 0$,*

$$(a+b)^p \leq \nu^{1-p}a^p + (1-\nu)^{1-p}b^p.$$

5.2.2 Exponential stability by an impulsive-integral inequality

In this subsection, we study exponential stability in *pth* moment of mild solution of the system (5.20) by using an impulsive-integral inequality.

Theorem 5.2.8. *Consider the neutral stochastic partial differential equations* (5.20)*, let* $p \geq 2$ *and suppose that the conditions* (A1)-(A4) *are satisfied. Then* (5.20) *is exponentially stable in pth moment, if the inequality*

$$\begin{aligned}
&9^{p-1}(1-\kappa)^{-p}\left[M_{1-\alpha}^p K^p \gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} + M^p L_3^p \gamma^{1-p}\right] \\
&\quad +3^{p-1}(1-\kappa)^{-p}\left[M^p L_1^p \gamma^{1-p} + c_p M^p L_2^p \left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\right] \\
&\quad +9^{p-1}\gamma(1-\kappa)^{-p} M^p \left(\sum_{t_k<t} q_k\right)^p < \gamma
\end{aligned} \tag{5.22}$$

and $\sum_{t_k<t} q_k < +\infty$ *hold, where* $c_p = (p(p-1)/2)^{p/2}$, $\kappa = \|(-A)^{-\alpha}\| K < 1$.

We start with an impulsive-integral inequality lemma, which is essential to the proof of Theorem 5.2.8.

Lemma 5.2.9. *If* $\gamma > 0$*, and* $\lambda_0, \lambda, \lambda^*, \lambda_k (k = 1, 2, \cdots)$ *are positive constants such that* $\frac{\lambda^*}{\gamma} + \lambda + \sum_{k=1}^{\infty} \lambda_k < 1$*, and a function* $y : [-\tau, \infty) \to [0, \infty)$ *such that the inequality*

$$y(t) \leq \begin{cases} \lambda_0 e^{-p\gamma t} + \lambda^* \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} y(s+\theta)\, ds \\ \quad +\lambda \sup_{\theta\in[-\tau,0]} y(t+\theta) + \sum_{t_k<t} \lambda_k e^{-p\gamma(t-t_k)} y(t_k^-), \quad t \geq 0, \\ \lambda_0 e^{-p\gamma t}, \quad t \in [-\tau, 0], \end{cases} \tag{5.23}$$

holds, then we have that $y(t) \leq M_2 e^{-p\mu t} (t \geq -\tau)$*, where* μ *is a positive root of the algebraic equation* $\left(\lambda + \frac{\lambda^*}{\gamma - p\mu}\right) e^{p\mu\tau} + \sum_{k=1}^{\infty} \lambda_k = 1$ *and* $M_2 = \max\left\{\frac{\lambda_0(\gamma-p\mu)}{\lambda^* e^{p\mu\tau}}, \lambda_0\right\} > 0$.

Proof. Let $F(\mu) = \left(\lambda + \frac{\lambda^*}{\gamma-p\mu}\right) e^{p\mu\tau} + \sum_{k=1}^{\infty} \lambda_k - 1$. We have $F(0)F(\gamma^-/p) < 0$, hence, there exists constant $\mu \in (0, \gamma)$ such that $F(\mu) = 0$. For any $\varepsilon > 0$, let

$$C_\varepsilon = \max\left\{\frac{(\varepsilon + \lambda_0)(\gamma - p\mu)}{\lambda^* e^{p\mu\tau}}, \varepsilon + \lambda_0\right\}.$$

We claim that (5.23) implies

$$y(t) \leq C_\varepsilon e^{-p\mu t}, \quad t \geq -\tau. \tag{5.24}$$

It is easily shown that (5.24) holds for $t \in [-\tau, 0]$. Assume that there exists $t_1^* > 0$ such that

$$y(t) < C_\varepsilon e^{-p\mu t}, \quad t \in [-\tau, t_1^*), \qquad y(t_1^*) = C_\varepsilon e^{-p\mu t_1^*}. \tag{5.25}$$

Combining with (5.23), we have

$$\begin{aligned}
y(t_1^*) &\leq \lambda_0 e^{-p\gamma t_1^*} + \lambda C_\varepsilon \sup_{\theta\in[-\tau,0]} e^{-p\mu(t_1^*+\theta)} \\
&\quad + \lambda^* C_\varepsilon \int_0^{t_1^*} e^{-\gamma(t_1^*-s)} \sup_{\theta\in[-\tau,0]} e^{-p\mu(s+\theta)}\,ds + C_\varepsilon \sum_{t_k<t_1^*} \lambda_k e^{-p\gamma(t_1^*-t_k)} e^{-p\mu t_k} \\
&\leq \lambda_0 e^{-p\gamma t_1^*} + \lambda C_\varepsilon e^{-p\mu t_1^*} e^{p\mu\tau} - \frac{\lambda^* C_\varepsilon e^{p\mu\tau}}{\gamma - p\mu} e^{-\gamma t_1^*} + \frac{\lambda^* e^{p\mu\tau}}{\gamma - p\mu} C_\varepsilon e^{-p\mu t_1^*} + \left(\sum_{k=1}^{\infty} \lambda_k\right) C_\varepsilon e^{-p\mu t_1^*} \\
&= \lambda_0 e^{-p\gamma t_1^*} - \frac{\lambda^* C_\varepsilon e^{p\mu\tau}}{\gamma - p\mu} e^{-\gamma t_1^*} + \left[\left(\lambda + \frac{\lambda^*}{\gamma - p\mu}\right) e^{p\mu\tau} + \sum_{k=1}^{\infty} \lambda_k\right] C_\varepsilon e^{-p\mu t_1^*}.
\end{aligned}$$

From the definition of C_ε, we have

$$\lambda_0 e^{-p\gamma t_1^*} - \frac{\lambda^* C_\varepsilon e^{p\mu\tau}}{\gamma - p\mu} e^{-\gamma t_1^*} \leq \lambda_0 e^{-p\gamma t_1^*} - \frac{\lambda^* e^{p\mu\tau}}{\gamma - p\mu} e^{-\gamma t_1^*} \cdot \frac{(\varepsilon + \lambda_0)(\gamma - p\mu)}{\lambda^* e^{p\mu\tau}} < 0.$$

Then, together with the definition of μ, we obtain that $y(t_1^*) < C_\varepsilon e^{-p\mu t_1^*}$, which contradicts (5.25), so (5.24) holds. As $\varepsilon > 0$ is arbitrarily small, in view of (5.24), it follows that $y(t) \leq M_2 e^{-p\mu t}$, for $t \geq -\tau$, where $M_2 = \max\left\{\frac{\lambda_0(\gamma - p\mu)}{\lambda^* e^{p\mu\tau}}, \lambda_0\right\}$. □

We are now ready to prove Theorem 5.2.8.

Proof. Based on an elementary inequality and Lemma 5.2.7, for any real numbers a, b, c, d, e and f, we have

$$\begin{aligned}
&(a+b+c+d+e+f)^p \\
&\quad \leq 3^{p-1}(a+b+c+d)^p + 3^{p-1}e^p + 3^{p-1}f^p \\
&\quad \leq 3^{p-1}\left[\left(1+\frac{1}{\varepsilon}\right)^{p-1} a^p + (1+\varepsilon)^{p-1}(b+c+d)^p\right] + 3^{p-1}e^p + 3^{p-1}f^p \\
&\quad \leq 3^{p-1}\left(1+\frac{1}{\varepsilon}\right)^{p-1} a^p + 9^{p-1}(1+\varepsilon)^{p-1}(b^p + c^p + d^p) + 3^{p-1}e^p + 3^{p-1}f^p. \quad (5.26)
\end{aligned}$$

From Lemma 5.2.7 and (5.21), we obtain

$$\begin{aligned}
&\mathbb{E}\|x(t)\|^p \\
&\quad \leq \kappa^{1-p}\mathbb{E}\,\|u(t, x(t-\tau(t)))\|^p \\
&\qquad + (1-\kappa)^{1-p}\mathbb{E}\Bigg\| S(t)(\phi(0) + u(0,\phi)) - \int_0^t AS(t-s)u(s, x(s-\tau(s)))\,ds \\
&\qquad + \int_0^t S(t-s)f(s, x(s-\delta(s)))\,ds + \int_0^t S(t-s)g(s, x(s-\rho(s)))\,dw(s) \\
&\qquad + \int_0^t \int_Z S(t-s)h(s, x(s-\sigma(s)), y)\,\tilde{N}(ds, dy) + \sum_{0<t_k<t} S(t-t_k) I_k(x(t_k^-))\Bigg\|^p
\end{aligned}$$

$$\begin{aligned}
&\le \kappa^{1-p}\mathbb{E}\left\|u(t,x(t-\tau(t)))\right\|^p + 3^{p-1}(1-\kappa)^{1-p}\left(1+\frac{1}{\varepsilon}\right)^{p-1}\mathbb{E}\left\|S(t)(\phi(0)+u(0,\phi))\right\|^p \\
&\quad +9^{p-1}(1-\kappa)^{1-p}(1+\varepsilon)^{p-1}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,x(s-\tau(s)))\,ds\right\|^p \\
&\quad +9^{p-1}(1-\kappa)^{1-p}(1+\varepsilon)^{p-1}\mathbb{E}\left\|\int_0^t\int_Z S(t-s)h(s,x(s-\sigma(s)),y)\,\tilde{N}(ds,dy)\right\|^p \\
&\quad +9^{p-1}(1-\kappa)^{1-p}(1+\varepsilon)^{p-1}\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)I_k(x(t_k^-))\right\|^p \\
&\quad +3^{p-1}(1-\kappa)^{1-p}\mathbb{E}\left\|\int_0^t S(t-s)f(s,x(s-\delta(s)))\,ds\right\|^p \\
&\quad +3^{p-1}(1-\kappa)^{1-p}\mathbb{E}\left\|\int_0^t S(t-s)g(s,x(s-\rho(s)))\,dw(s)\right\|^p. \qquad (5.27)
\end{aligned}$$

Now, we estimate the right-hand side of (5.27). By assumption (A3), we obtain

$$\begin{aligned}
\mathbb{E}\left\|u(t,x(t-\tau(t)))\right\|^p &\le \|(-A)^{-\alpha}\|^p\mathbb{E}\|(-A)^{\alpha}u(t,x(t-\tau(t)))\|^p \\
&\le K^p\|(-A)^{-\alpha}\|^p E\|x(t-\tau(t))\|^p. \qquad (5.28)
\end{aligned}$$

Applying Lemma 5.2.5, Hölder inequality and assumption (A3), we obtain

$$\begin{aligned}
&\mathbb{E}\left\|\int_0^t AS(t-s)u(s,x(s-\tau(s)))\,ds\right\|^p \qquad (5.29)\\
&\le M_{1-\alpha}^p K^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\int_0^t e^{-\gamma(t-s)}E\|x(s-\tau(s))\|^p\,ds.
\end{aligned}$$

Similarly, we obtain that

$$\begin{aligned}
&\mathbb{E}\left\|\int_0^t S(t-s)f(s,x(s-\delta(s)))\,ds\right\|^p \\
&\quad\le \mathbb{E}\left[\int_0^t \|S(t-s)f(s,x(s-\delta(s)))\|\,ds\right]^p \\
&\quad\le \mathbb{E}\left[\int_0^t Me^{-\gamma(t-s)}\|f(s,x(s-\delta(s)))\|\,ds\right]^p \\
&\quad\le M^pL_1^p\mathbb{E}\left[\int_0^t e^{-\gamma(t-s)}\|x(s-\delta(s))\|\,ds\right]^p \\
&\quad= M^pL_1^p\mathbb{E}\left[\int_0^t e^{-(\gamma(p-1)/p)(t-s)}e^{-(\gamma/p)(t-s)}\|x(s-\delta(s))\|\,ds\right]^p \\
&\quad\le M^pL_1^p\left[\int_0^t e^{-\gamma(t-s)}\,ds\right]^{p-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\delta(s))\|^p\,ds \\
&\quad\le M^pL_1^p\gamma^{1-p}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\delta(s))\|^p\,ds; \qquad (5.30)
\end{aligned}$$

$$
\begin{aligned}
&\mathbb{E}\left\|\int_0^t S(t-s)g(s,x(s-\rho(s)))\,dw(s)\right\|^p \\
&\quad\le c_pM^p\left[\int_0^t\left(e^{-\gamma p(t-s)}\mathbb{E}\|g(s,x(s-\rho(s)))\|^p\right)^{2/p}ds\right]^{p/2} \\
&\quad\le c_pM^pL_2^p\left[\int_0^t\left(e^{-\gamma p(t-s)}\mathbb{E}\|x(s-\rho(s))\|^p\right)^{2/p}ds\right]^{p/2} \\
&\quad= c_pM^pL_2^p\left[\int_0^t\left(e^{-\gamma(p-1)(t-s)}e^{-\gamma(t-s)}\mathbb{E}\|x(s-\rho(s))\|^p\right)^{2/p}ds\right]^{p/2} \\
&\quad\le c_pM^pL_2^p\left[\int_0^t e^{-\left(\frac{2(p-1)}{p-2}\right)\gamma(t-s)}ds\right]^{p-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\rho(s))\|^p\,ds \\
&\quad\le c_pM^pL_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\rho(s))\|^p\,ds
\end{aligned}
\tag{5.31}
$$

$$
\begin{aligned}
&\mathbb{E}\left\|\int_0^t\int_Z S(t-s)h(s,x(s-\sigma(s)),z)\,\tilde{N}(ds,dz)\right\|^p \\
&\quad\le c_p\mathbb{E}\left[\int_0^t\int_Z\|S(t-s)h(s,x(s-\sigma(s)),z)\|^2\,ds\nu(dz)\right]^{p/2} \\
&\quad\le c_pM^p\mathbb{E}\left[\int_0^t\int_Z e^{-2\gamma(t-s)}\|h(s,x(s-\sigma(s)),z)\|^2\,ds\nu(dz)\right]^{p/2} \\
&\quad\le c_pM^p\mathbb{E}\left[\int_0^t e^{-2\gamma(t-s)}\int_Z\|h(s,x(s-\sigma(s)),z)\|^2\nu(dz)\,ds\right]^{p/2} \\
&\quad\le c_pM^pL_3^p\left[\int_0^t e^{-2\gamma(t-s)}\mathbb{E}\|x(s-\sigma(s))\|^2\,ds\right]^{p/2} \\
&\quad= c_pM^pL_3^p\left[\int_0^t e^{-\frac{2(p-1)}{p}\gamma(t-s)}e^{-\frac{2}{p}\gamma(t-s)}\mathbb{E}\|x(s-\sigma(s))\|^2\,ds\right]^{p/2} \\
&\le c_pM^pL_3^p\left(\int_0^t e^{-\frac{2(p-1)}{p}\cdot\frac{p}{p-2}\gamma(t-s)}ds\right)^{(p-2)/2}e^{-\gamma(t-s)}\mathbb{E}\|x(s-\sigma(s))\|^p\,ds \\
&\le c_pM^pL_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|x(s-\sigma(s))\|^p\,ds.
\end{aligned}
\tag{5.32}
$$

Furthermore, we obtain that

$$
\mathbb{E}\left\|\sum_{0<t_k<t}S(t-t_k)I_k(x(t_k^-))\right\|^p\le M^p\left(\sum_{t_k<t}q_k\right)^{p-1}\sum_{t_k<t}q_ke^{-\gamma p(t-t_k)}\mathbb{E}\|x(t_k^-)\|^p. \tag{5.33}
$$

Hence, from (5.27)-(5.33), we obtain

$$
\begin{aligned}
&\mathbb{E}\|x(t)\|^p \\
&\quad\le \kappa \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(t+\theta)\|^p + 3^{p-1}(1-\kappa)^{1-p}\left(1+\frac{1}{\varepsilon}\right)^{p-1} \\
&\qquad\quad \times M^p e^{-\gamma p t}(1+K\|(-A)^{-\alpha}\|)^p \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(\theta)\|^p \\
&\qquad +9^{p-1}(1-\kappa)^{1-p}(1+\varepsilon)^{p-1}M_{1-\alpha}^p K^p \gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} \\
&\qquad\quad \times \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(s+\theta(s))\|^p\, ds \\
&\qquad +9^{p-1}(1-\kappa)^{1-p}(1+\varepsilon)^{p-1}c_p M^p L_3^p \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2} \\
&\qquad\quad \times \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(s+\theta(s))\|^p\, ds \\
&\qquad +9^{p-1}(1-\kappa)^{1-p}(1+\varepsilon)^{p-1}M^p \left(\sum_{t_k<t} q_k\right)^{p-1} \sum_{t_k<t} q_k e^{-\gamma p(t-t_k)} \mathbb{E}\|x(t_k^-)\|^p \\
&\qquad +3^{p-1}(1-\kappa)^{1-p}M^p L_1^p \gamma^{1-p} \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(s+\theta(s))\|^p\, ds \qquad (5.34)\\
&\qquad +3^{p-1}(1-\kappa)^{1-p}c_p M^p L_2^p \left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2} \int_0^t e^{-\gamma(t-s)} \sup_{\theta\in[-\tau,0]} \mathbb{E}\|x(s+\theta(s))\|^p\, ds.
\end{aligned}
$$

From inequality (5.22), (5.34) and the condition (A3), we can always find a number $\varepsilon>0$ small enough such that

$$
\begin{aligned}
&\kappa + 9^{p-1}\gamma^{-1}(1-\kappa)^{1-p}(1+\varepsilon)^{p-1} \\
&\quad \times \left[M_{1-\alpha}^p K^p \gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} + c_p M^p L_3^p \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\right] \\
&\quad +3^{p-1}\gamma^{-1}(1-\kappa)^{1-p}\left[M^p L_1^p \gamma^{1-p} + c_p M^p L_2^p \left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\right] \\
&\quad +9^{p-1}(1-\kappa)^{1-p}(1+\varepsilon)^{p-1}M^p \left(\sum_{t_k<t} q_k\right)^p < 1.
\end{aligned}
$$

Hence, from Lemma 5.2.9, we obtain that $E\|x(t)\|^p \le \tilde{M} e^{-p\mu t}(\tilde{M}>0, \mu\in(0,\gamma))$. Therefore, the mild solution of the system (5.20) is exponentially stable in pth moment. □

If $h\equiv 0$ or $\nu\equiv 0$, system (5.20) becomes

$$
\begin{cases}
d[x(t)+u(t,x(t-\tau(t)))] = [Ax(t)dt + f(t,x(t-\delta(t)))]dt \\
\qquad\qquad\qquad\qquad +g(t,x(t-\rho(t))dw(t), \quad t\ge 0, \quad t\ne t_k, \\
\Delta x(t_k) = I_k(x(t_k^-)), \quad t=t_k, \quad k=1,2,\cdots, \\
x_0(\cdot) = \phi \in PC,
\end{cases} \qquad (5.35)
$$

Corollary 5.2.10. *Consider the impulsive stochastic partial differential equation* (5.35) *and suppose that the conditions* (A1)-(A4) *are satisfied. Then the mild solution of* (5.35) *is exponential stability in pth moment, if the following inequality*

$$
\begin{aligned}
&6^{p-1}(1-\kappa)^{-p}M_{1-\alpha}^{p}K^{p}\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\\
&\quad+3^{p-1}(1-\kappa)^{-p}\left[M^{p}L_{1}^{p}\gamma^{1-p}+c_{p}M^{p}L_{2}^{p}\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\right]\\
&\quad+6^{p-1}\gamma(1-\kappa)^{-p}M^{p}\left(\sum_{t_k<t}q_k\right)^{p}<\gamma
\end{aligned}
$$

holds, where $c_p=(p(p-1)/2)^{p/2}$.

Remark 5.2.11. *If* $I_k(\cdot)\equiv 0$ $(k=1,2,\cdots)$ *in* (5.35), Theorem 3.2 *in* [21] *can be obtained from* Corollary 5.2.10.

If $h\equiv 0$ and the impulsive effects $I_k(\cdot)\equiv 0$ $(k=1,2,\cdots)$, system (5.20) becomes

$$
\begin{cases}
dx(t)=[Ax(t)dt+f(t,x(t-\delta(t)))]dt+g(t,x(t-\rho(t))dw(t) & t\geq 0,\\
x_0(\cdot)=\phi\in PC.
\end{cases}
\tag{5.36}
$$

Corollary 5.2.12. *Consider the stochastic partial differential equation* (5.36) *and suppose that the conditions* (A1)-(A4) *are satisfied. Then the mild solution of* (5.36) *is exponential stability in pth moment, if the following inequality*

$$
3^{p-1}M^{p}\left[L_{1}^{p}\gamma^{1-p}+c_{p}L_{2}^{p}\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\right]<\gamma \tag{5.37}
$$

holds, where $c_p=(p(p-1)/2)^{p/2}$.

Remark 5.2.13. *Corollary* 5.2.12 *is consistent with the result in Luo* [90] *which was studied by using fixed point methods.*

Example 5.2.14. *Consider the following neutral stochastic partial differential equation with delays and poisson jumps of the form*

$$
\begin{aligned}
d&\left[x(t,\xi)+\frac{\alpha_3}{M_{1-\alpha}\|(-A)^{\alpha}\|}x(t-\tau(t)),\xi)\right]\\
&=\left[\frac{\partial^2}{\partial\xi^2}x(t,\xi)dt+\alpha_1x(t-\delta(t),\xi)\right]dt+\alpha_2x(t-\rho(t),\xi)dw(t)\\
&\quad+\int_Z\alpha_4yx(t-\theta(t),\xi)\,\tilde{N}(dt,dy),\quad t\geq 0,\quad t\neq t_k,
\end{aligned}
\tag{5.38}
$$

and for $t=t_k,\quad k=1,2,\cdots m$,

$$
\Delta x(t_k,\xi)=d_kx(t_k,\xi), \tag{5.39}
$$

where $d_k\geq 0,\sum_{k=1}^{m}d_k<\infty$, $x(t,0)=x(t,\pi)=0$, $\alpha_i>0,i=1,2,3,\quad 0<\tau(t),\delta(t),\rho(t),\theta(t)<\tau$, $x(s,\xi)=\phi(s,\xi)$, $\phi(\cdot,\xi)\in C$, $\phi(s,\cdot)\in\mathcal{L}^2[0,\pi]$, $-\tau\leq s\leq 0$, $0\leq\xi\leq\pi$, $\tau\geq 0$, $t\geq 0$, $w(t)$ *is a standard one-dimensional Wiener process and* $\|\phi\|_C<+\infty$ *a.s., and* $M_{1-\alpha}\geq 1$ $(\alpha\in(1/2,1])$.

Take $X = L^2[0,\pi]$,$Y = R^1$, define $A : X \to X$ by $-A = \frac{\partial^2}{\partial \xi^2}$ with domain

$$D(-A) = \left\{ \omega \in X : \quad \omega, \quad \frac{\partial^2 \omega}{\partial \xi} \quad \text{are} \quad \text{continuous}, \frac{\partial^2 \omega}{\partial \xi^2} \in X, \quad \omega(0) = \omega(\pi) = 0 \right\}.$$

Then

$$(-A)\omega = \sum_{n=1}^{\infty} n^2 (\omega, \omega_n)\omega_n, \quad \omega \in D(-A),$$

where $\omega_n(\xi) = \sqrt{2/\pi}\sin n\xi$, $n = 1, 2, 3, \cdots$, is orthonormal set of eigenvector of $-A$. It is well known that A is the infinitesimal generator of an analytic semigroup $S(t)(t \geq 0)$ in X and is given (see Pazy[100], Page 70) by

$$S(t)\omega = \sum_{n=1}^{\infty} \exp(-n^2 t)(\omega, \omega_n)\omega_n, \quad \omega \in X,$$

that satisfies $\|S(t)\| \leq \exp\left(-\pi^2 t\right)$, $t \geq 0$, and hence is a contraction semigroup. Let

$$\begin{aligned}
u(t, x(t-\tau(t),\xi)) &= \frac{\alpha_3}{M_{1-\alpha}\|(-A)^\alpha\|} x(t-\tau(t)),\xi), \\
f(t, x(t-\delta(t),\xi)) &= \alpha_1 x(t-\delta(t),\xi), \quad g(t, x(t-\rho(t),\xi) = \alpha_2 x(t-\rho(t),\xi), \\
h(t, x(t-\theta(t),\xi), z) &= \alpha_4 z x(t-\theta(t),\xi).
\end{aligned}$$

We obtain that

$$\begin{aligned}
&\|f(t, x(t-\delta(t),\xi)) - f(t, y(t-\delta(t),\xi))\| \\
&\quad \leq \alpha_1 \|x(t-\delta(t),\xi) - y(t-\delta(t),\xi)\|, \quad f(t,0) = 0.
\end{aligned}$$

$$\begin{aligned}
&\|g(t, x(t-\rho(t),\xi) - g(t, y(t-\rho(t),\xi)\| \\
&\quad \leq \alpha_2 \|x(t-\delta(t),\xi) - y(t-\delta(t),\xi)\|, \quad g(t,0) = 0.
\end{aligned}$$

$$\begin{aligned}
&\int_Z \|h(t, x(t-\theta(t),\xi), z) - h(t, y(t-\theta(t),\xi), z)\| \, \nu(dz) \\
&\quad \leq \alpha_4 \int_Z z \, \nu(dz) \|x(t-\theta(t),\xi) - y(t-\theta(t),\xi)\|, \quad h(t,0,z) = 0.
\end{aligned}$$

$$\begin{aligned}
&\|(-A)^\alpha u(t, x(t-\tau(t),\xi)) - (-A)^\alpha u(t, y(t-\tau(t),\xi))\| \\
&\quad \leq \frac{\alpha_3}{M_{1-\alpha}} \|x(t-\tau(t)),\xi) - y(t-\tau(t)),\xi)\|, \quad (-A)^\alpha u(t,0,\xi)) = 0.
\end{aligned}$$

From the definition of $(-A)^{-\alpha}$, we obtain

$$\|(-A)^{-\alpha}\| \leq \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} \|S(t)\| \, dt \leq \frac{1}{\pi^{2\alpha}}.$$

Thus, when $\alpha_3 < M_{1-\alpha}\pi^{2\alpha}(\alpha \in (1/2,1])$. By Theorem 5.2.8, the mild solution of (5.20) is exponential stability in mean square provide that

$$9\alpha_3^2\pi^{2-4\alpha}\Gamma(2\alpha-1) + 9\alpha_4^2\pi^{-2}\left(\int_Z z\,\nu(dz)\right)^2 + 3\left(\alpha_1^2\pi^{-2}+\alpha_2^2\right)$$
$$+9\pi^2\left(\sum_{k=1}^m q_k\right)^2 < \left(\pi - \frac{\alpha_3}{M_{1-\alpha}\pi^{2\alpha-1}}\right)^2, \quad \alpha \in \left(\frac{1}{2},1\right].$$

5.2.3 Exponential stability by using fixed point methods

In this subsection, we study existence and exponential stability in *pth* moment of mild solution of the system (5.20) by using fixed point methods.

Denote by $\mathcal{S}_\phi$ the space of all $\mathcal{F}$-adapted càdlàg processes: $\varphi(t,\omega) : [-\tau,\infty)\times\Omega \to X$ such that $\phi(s,\cdot) = \phi(s)$ for $s \in [-\tau,0]$ and $e^{\eta t}\mathbb{E}\|\varphi(t)\|^p \to 0$ as $t\to\infty$, where $0<\eta<\gamma$. If we define the metric as

$$\|\varphi\|_{\mathcal{S}_\phi} := \sup_{s\geq -\tau}\mathbb{E}\|\varphi(s)\|^p, \tag{5.40}$$

then $\mathcal{S}_\phi$ is a complete metric space with respect to (5.40). Using a contraction mapping defined on the space $\mathcal{S}_\phi$ and applying a contraction mapping principle, we obtain the following result.

Theorem 5.2.15. *Suppose that the assumptions* (A1)-(A4) *hold for some* $\alpha \in (1/p,1]$, $p\geq 2$, *and the following conditions also hold,*

(i) *there exists a constant* $\tilde{q}$ *such that* $q_k \leq \tilde{q}(t_k - t_{k-1})$, $k = 1,2,\cdots$,

(ii) *and such that*

$$6^{p-1}\|(-A)^{-\alpha}\|^pK^p + 6^{p-1}M_{1-\alpha}^pK^p\gamma^{-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}$$
$$+6^{p-1}M^pL_1^p\gamma^{-p} + 6^{p-1}c_pM^pL_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\gamma^{-1}$$
$$+6^{p-1}c_pM^pL_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\gamma^{-1} + 6^{p-1}M^p\tilde{q}^p\gamma^{-p} < 1,$$

where $\Gamma(\cdot)$ *is the Gamma function,* $M_{1-\alpha}$ *is the corresponding constant as in Lemma* 5.2.5,

then the mild solution of the system (5.20) *is exponentially stable.*

Remark 5.2.16. *In our results, we do not require the monotone decreasing behavior of the delays, i.e.* $\tau'(t)\leq 0$, $\delta'(t)\leq 0$, $\rho'(t)\leq 0$, $\theta'(t)\leq 0$ *for* $t\geq 0$, *which is needed in* [14].

Remark 5.2.17. *Sakthivel and Luo* [117, 118] *and Jiang and Shen* [61] *studied asymptotic stability of special cases of the system* (5.20) *by using fixed point theory. In* [61, 117, 118], *the estimate of the impulsive term is*

$$\sup_{t\in[0,T]}\mathbb{E}\left\|\sum_{0<t_k<t}S(t-t_k)(I_k(x(t_k^-)) - I_k(y(t_k^-)))\right\|^p \leq M^pe^{-\gamma pT}\mathbb{E}\left(\sum_{k=1}^m q_k^p\right)\sup_{t\in[0,T]}\mathbb{E}\|x(t)-y(t)\|^p$$

which seems to be a mistake. It should be

$$\begin{aligned}\sup_{t\geq 0}\mathbb{E}&\left\|\sum_{0<t_k<t} S(t-t_k)(I_k(x(t_k^-))-I_k(y(t_k^-)))\right\|^p\\ &\leq M^p\sup_{t\geq 0}\mathbb{E}\left[\sum_{0<t_k<t} e^{-\gamma(t-t_k)}q_k\|x(t_k^-)-y(t_k^-)\|\right]^p.\end{aligned}$$

To estimate the impulsive effects to the system (5.20)*, we consider the case which satisfy condition* (i) *in Theorem* 5.2.15*. Note that k in condition* (i) *can be equal to infinity.*

To prove Theorem 5.2.15, we start with a lemma.

Lemma 5.2.18. *Define an operator by $(\pi\varphi)(t)=\phi(t)$ for $t\in[-\tau,0]$, and for $t\geq 0$,*

$$\begin{aligned}(\pi\varphi)(t) &= S(t)(\phi(0)+u(0,\phi))-u(t,\varphi(t-\tau(t)))-\int_0^t AS(t-s)u(s,\varphi(s-\tau(s)))\,ds\\ &\quad+\int_0^t S(t-s)f(s,\varphi(s-\delta(s)))\,ds+\int_0^t S(t-s)g(s,\varphi(s-\rho(s)))\,dw(s)\\ &\quad+\int_0^t\int_Z S(t-s)h(s,\varphi(s-\sigma(s)),y)\,\tilde{N}(ds,dy)+\sum_{0<t_k<t}S(t-t_k)I_k(\varphi(t_k^-))\\ &:= \sum_{i=1}^{7}J_i(t).\end{aligned}\tag{5.41}$$

Suppose that the assumptions (A1) – (A4) *hold. If the conditions (i) and (ii) in Theorem* 5.2.15 *are satisfied, then $\pi:\mathcal{S}_\phi\to\mathcal{S}_\phi$ and π is a contraction mapping.*

Proof. First, we prove the continuity in pth moment of π on $[0,\infty)$. Let $\varphi\in\mathcal{S}_\phi$, $t_1\geq 0$, and $|r|$ be sufficiently small, from (5.41), we have

$$\mathbb{E}\|(\pi\varphi)(t_1+r)-(\pi\varphi)(t_1)\|^p\leq 7^{p-1}\sum_{i=1}^{7}\mathbb{E}\|J_i(t_1+r)-J_i(t_1)\|^p.\tag{5.42}$$

It is easily to check that

$$\mathbb{E}\|J_i(t_1+r)-J_i(t_1)\|^p\to 0\quad\text{as}\quad r\to 0, i=1,2,3,4,7.$$

Further, by using Hölder inequality and Lemma 5.2.6, we obtain

$$\begin{aligned}&\mathbb{E}\|J_5(t_1+r)-J_5(t_1)\|^p\\ &\leq 2^{p-1}\mathbb{E}\left\|\int_0^{t_1}(S(t_1+r-s)-S(t_1))g(s,\varphi(s-\rho(s)))\,dw(s)\right\|^p\\ &\quad+2^{p-1}\mathbb{E}\left\|\int_{t_1}^{t_1+r}S(t_1+r-s)g(s,\varphi(s-\rho(s)))\,dw(s)\right\|^p\\ &\leq 2^{p-1}c_p\left(\int_0^{t_1}(\mathbb{E}\|(S(t_1+r-s)-S(t_1))g(s,\varphi(s-\rho(s)))\|^p)^{2/p}\,ds\right)^{p/2}\\ &\quad+2^{p-1}c_p\left(\int_{t_1}^{t_1+r}(\mathbb{E}\|S(t_1+r-s)g(s,\varphi(s-\rho(s)))\|^p)^{2/p}\,ds\right)^{p/2}\to 0\quad\text{as}\quad r\to 0.\end{aligned}$$

Similarly, we can verify that $\mathbb{E}\|J_6(t_1+r)-J_6(t_1)\|^p \to 0 \quad \text{as} \quad r\to 0$. Thus, π is indeed continuous in pth moment on $[0,\infty)$.

Next, We show that $\pi(\mathcal{S}_\phi)\subseteq\mathcal{S}_\phi$. It follows form (5.41) that

$$\begin{aligned} e^{\eta t}\mathbb{E}\|(\pi\varphi)(t)\|^p \leq\ & 7^{p-1}e^{\eta t}\mathbb{E}\|S(t)(\phi(0)+u(0,\phi))\|^p + 7^{p-1}e^{\eta t}\mathbb{E}\|u(t,\varphi(t-\tau(t)))\|^p \\ & +7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,\varphi(s-\tau(s)))\,ds\right\|^p \\ & +7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t S(t-s)f(s,\varphi(s-\delta(s)))\,ds\right\|^p \\ & +7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t S(t-s)g(s,\varphi(s-\rho(s)))\,dw(s)\right\|^p \\ & +7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t\int_Z S(t-s)h(s,\varphi(s-\sigma(s)),y)\,\tilde{N}(ds,dy)\right\|^p \\ & +7^{p-1}e^{\eta t}\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)I_k\varphi(t_k^-)\right\|^p. \end{aligned} \tag{5.43}$$

Now, we estimate the terms on the right-hand side of (5.43). By the condition (A1) and (A3), we obtain

$$7^{p-1}e^{\eta t}\mathbb{E}\|S(t)(\phi(0)+u(0,\phi))\|^p \leq M^p e^{(\eta-p\gamma)t}\mathbb{E}\|\phi(0)+u(0,\phi)\|^p \to 0 \quad \text{as} \quad t\to\infty \tag{5.44}$$

and

$$\begin{aligned} 7^{p-1}e^{\eta t}\mathbb{E}\|u(t,\varphi(t-\tau(t)))\|^p \leq\ & 7^{p-1}e^{\eta t}\|(-A)^{-\alpha}\|^p E\|(-A)^{\alpha}u(t,\varphi(t-\tau(t)))\|_H^p \\ \leq\ & 7^{p-1}e^{\eta\tau}K^p\|(-A)^{-\alpha}\|^p e^{\eta(t-\tau(t))}E\|\varphi(t-\tau(t))\|^p \\ & \to 0 \quad \text{as} \quad t\to\infty. \end{aligned} \tag{5.45}$$

Using Hölder inequality, we obtain

$$\begin{aligned} & 7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,\varphi(s-\tau(s)))\,ds\right\|^p \\ & \leq 7^{p-1}e^{\eta t}M_{1-\alpha}^p K^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\int_0^t e^{-\gamma(t-s)}E\|\varphi(s-\tau(s))\|^p\,ds \\ & \leq 7^{p-1}M_{1-\alpha}^p K^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} \\ & \quad \times e^{\eta\tau}\int_0^t e^{-(\gamma-\eta)(t-s)}e^{\eta(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p\,ds. \end{aligned} \tag{5.46}$$

For any $\varphi\in\mathcal{S}_\phi$ and $\varepsilon>0$, there exists $t_1>0$ such that $e^{\eta(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p<\varepsilon$ for $t\geq t_1$.

Thus, from (5.46), we obtain

$$
\begin{aligned}
&7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,\varphi(s-\tau(s)))\,ds\right\|^p \\
&\quad\le 7^{p-1}M_{1-\alpha}^pK^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} \\
&\qquad\quad\times e^{\eta\tau}\int_0^{t_1} e^{-(\gamma-\alpha)(t-s)}e^{\eta(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p\,ds \\
&\qquad +7^{p-1}M_{1-\alpha}^pK^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} \\
&\qquad\quad\times e^{\eta\tau}\int_{t_1}^{t} e^{-(\gamma-\alpha)(t-s)}e^{\eta(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p\,ds \\
&\quad\le 7^{p-1}M_{1-\alpha}^pK^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} \\
&\qquad\quad\times e^{\eta\tau}\int_0^{t_1} e^{-(\gamma-\eta)(t-s)}e^{\eta(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p\,ds \\
&\qquad +7^{p-1}M_{1-\alpha}^pK^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\,(e^{\eta\tau}/(\gamma-\eta))\,\varepsilon. \qquad (5.47)
\end{aligned}
$$

Since $e^{-(\gamma-\eta)t}\to 0$ as $t\to\infty$, then there exists $t_2\ge t_1$ such that for $t\ge t_2$, we have

$$
\begin{aligned}
&7^{p-1}M_{1-\alpha}^pK^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1} \\
&e^{\eta\tau}\int_0^{t_1} e^{-(\gamma-\eta)(t-s)}e^{\eta(s-\tau(s))}E\|\varphi(s-\tau(s))\|^p\,ds \\
&\quad\le \varepsilon-7^{p-1}M_{1-\alpha}^pK^p\gamma^{1-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}\,(e^{\eta\tau}/(\gamma-\eta))\,\varepsilon. \qquad (5.48)
\end{aligned}
$$

So, from (5.47)and (5.48), we obtain that for any $t\ge t_2$,

$$
7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,\varphi(s-\tau(s)))\,ds\right\|^p<\varepsilon. \qquad (5.49)
$$

Hence

$$
7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t AS(t-s)u(s,\varphi(s-\tau(s)))\,ds\right\|^p\to 0\quad\text{as}\quad t\to\infty.
$$

As for the fourth term on the right-hand side of (5.43), for any $\varphi\in\mathcal{S}_\phi$, we have that for $p>2$,

$$
\begin{aligned}
&7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t S(t-s)f(s,\varphi(s-\delta(s)))\,ds\right\|^p \\
&\quad\le 7^{p-1}e^{\eta t}E\left[\int_0^t Me^{-\gamma(t-s)}\|f(s,\varphi(s-\delta(s)))\|\,ds\right]^p \\
&\quad\le 7^{p-1}e^{\eta t}M^pL_1^pE\left[\int_0^t e^{-\gamma(t-s)}\|\varphi(s-\delta(s))\|\,ds\right]^p \\
&\quad\le 7^{p-1}e^{\eta t}M^pL_1^p\left[\int_0^t e^{-\gamma(t-s)}\,ds\right]^{p-1}\int_0^t e^{-\gamma(t-s)}E\|\varphi(s-\delta(s))\|^p\,ds \\
&\quad\le 7^{p-1}e^{\eta\tau}M^pL_1^p\gamma^{1-p}\int_0^t e^{-(\gamma-\eta)(t-s)}e^{\eta(s-\delta(s))}E\|\varphi(s-\delta(s))\|^p\,ds. \qquad (5.50)
\end{aligned}
$$

From Lemma 5.2.6 and Hölder inequality, we obtain

$$
\begin{aligned}
&7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t S(t-s)g(s,\varphi(s-\rho(s)))\,dw(s)\right\|^p \\
&\le 7^{p-1}e^{\eta t}c_pM^p\left[\int_0^t \left(e^{-\gamma p(t-s)}\mathbb{E}\|g(s,\varphi(s-\rho(s)))\|^p\right)^{2/p} ds\right]^{p/2} \\
&\le 7^{p-1}e^{\eta t}c_pM^pL_2^p\left[\int_0^t \left(e^{-\gamma p(t-s)}\mathbb{E}\|\varphi(s-\rho(s))\|^p\right)^{2/p} ds\right]^{p/2} \\
&\le 7^{p-1}e^{\eta t}c_pM^pL_2^p\left[\int_0^t e^{-\left(\frac{2(p-1)}{p-2}\right)\gamma(t-s)}\,ds\right]^{p-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|\varphi(s-\rho(s))\|^p\,ds \\
&\le 7^{p-1}e^{\eta t}c_pM^pL_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|\varphi(s-\rho(s))\|^p\,ds \qquad (5.51)\\
&\le 7^{p-1}c_pM^pL_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}e^{\eta\tau}\int_0^t e^{-(\gamma-\eta)(t-s)}e^{\eta(s-\rho(s))}\mathbb{E}\|\varphi(s-\rho(s))\|^p\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
&7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t\int_Z S(t-s)h(s,\varphi(s-\sigma(s)),z)\,\tilde{N}(ds,dz)\right\|^p \\
&\le 7^{p-1}e^{\eta t}c_p\mathbb{E}\left[\int_0^t\int_Z \|S(t-s)h(s,\varphi(s-\sigma(s)),z)\|^2\,ds\nu(dz)\right]^{p/2} \\
&\le 7^{p-1}e^{\eta t}c_pM^p\mathbb{E}\left[\int_0^t\int_Z e^{-2\gamma(t-s)}\|h(s,\varphi(s-\sigma(s)),z)\|^2\,ds\nu(dz)\right]^{p/2} \\
&\le 7^{p-1}e^{\eta t}c_pM^p\mathbb{E}\left[\int_0^t e^{-2\gamma(t-s)}\int_Z \|h(s,\varphi(s-\sigma(s)),z)\|^2\nu(dz)\,ds\right]^{p/2} \\
&\le 7^{p-1}e^{\eta t}c_pM^pL_3^p\left[\int_0^t e^{-2\gamma(t-s)}\mathbb{E}\|\varphi(s-\sigma(s))\|^2\,ds\right]^{p/2} \\
&\le 7^{p-1}e^{\eta t}c_pM^pL_3^p\left(\int_0^t e^{-\frac{2(p-1)}{p}\cdot\frac{p}{p-2}\gamma(t-s)}\,ds\right)^{(p-2)/2} e^{-\gamma(t-s)}\mathbb{E}\|\varphi(s-\sigma(s))\|^p\,ds \\
&\le 7^{p-1}e^{\eta t}c_pM^pL_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|\varphi(s-\sigma(s))\|^p\,ds \qquad (5.52)\\
&\le 7^{p-1}c_pM^pL_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}e^{\eta\tau}\int_0^t e^{-(\gamma-\eta)(t-s)}e^{\eta(s-\sigma(s))}\mathbb{E}\|\varphi(s-\sigma(s))\|^p\,ds,
\end{aligned}
$$

where $c_p=(p(p-1)/2)^{p/2}$. We remark that if $p=2$, the inequality (5.51) also holds with $0^0:=1$. Similar to the proof of (5.46), from estimate (5.50), (5.51) and (5.52), we obtain that

$$
\begin{aligned}
&7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t S(t-s)f(s,\varphi(s-\delta(s)))\,ds\right\|^p \to 0 \quad \text{as} \quad t\to\infty,\\
&7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t S(t-s)g(s,\varphi(s-\rho(s)))\,dw(s)\right\|^p \to 0 \quad \text{as} \quad t\to\infty,\\
&7^{p-1}e^{\eta t}\mathbb{E}\left\|\int_0^t\int_Z S(t-s)h(s,\varphi(s-\sigma(s)),z)\,\tilde{N}(ds,dz)\right\|^p \to 0 \quad \text{as} \quad t\to\infty. \qquad (5.53)
\end{aligned}
$$

Now, we estimate the impulsive term, from the condition (i), we obtain

$$
\begin{aligned}
&7^{p-1}e^{\eta t}\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)I_k\varphi(t_k^-)\right\|^p \\
&\le 7^{p-1}e^{\eta t}\mathbb{E}\left(\sum_{0<t_k<t} Me^{-\gamma(t-t_k)}q_k\|\varphi(t_k^-)\|\right)^p \\
&\le 7^{p-1}e^{\eta t}\mathbb{E}\left(\sum_{0<t_k<t} Me^{-\gamma(t-t_k)}\tilde{q}\|\varphi(t_k^-)(t_k-t_{k-1})\|\right)^p \\
&\le 7^{p-1}e^{\eta t}\mathbb{E}\left(\int_0^t Me^{-\gamma(t-s)}\tilde{q}\|\varphi(s)\|\,ds\right)^p \\
&\le 7^{p-1}M^p\tilde{q}^p\left(\int_0^t e^{-\gamma(t-s)}\,ds\right)^{p-1}\int_0^t e^{-\gamma(t-s)}\mathbb{E}\|\varphi(s)\|^p\,ds \\
&\le 7^{p-1}M^p\tilde{q}^p\gamma^{1-p}\int_0^t e^{-(\gamma-\eta)(t-s)}e^{\eta s}\mathbb{E}\|\varphi(s)\|^p\,ds. \qquad (5.54)
\end{aligned}
$$

From (5.54), we obtain

$$
7^{p-1}e^{\eta t}\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)I_k\varphi(t_k^-)\right\|^p \to 0 \quad \text{as} \quad t\to\infty. \qquad (5.55)
$$

Hence, from (5.43), (5.53) and (5.55), we obtain that

$$
e^{\eta t}\mathbb{E}\,\|(\pi\varphi)(t)\|^p \to 0 \quad \text{as} \quad t\to\infty.
$$

Therefore, we conclude that $\pi(\mathcal{S}_\phi)\subseteq\mathcal{S}_\phi$.

Finally, we show that π is a contraction mapping. For any $\varphi,\psi\in\mathcal{S}_\phi$, using the estimate (5.27)-(5.33), we obtain

$$
\begin{aligned}
&\sup_{t\ge-\tau}\mathbb{E}\|(\pi\varphi)(t)-(\pi\psi)(t)\|^p \\
&\le 6^{p-1}\sup_{t\ge-\tau}\mathbb{E}\|u(t,\varphi(t-\tau(t)))-u(t,\psi(t-\tau(t)))\|^p \\
&\quad+6^{p-1}\sup_{t\ge-\tau}\mathbb{E}\left\|\int_0^t AS(t-s)(u(s,\varphi(s-\tau(s)))-u(s,\psi(s-\tau(s))))\,ds\right\|^p \\
&\quad+6^{p-1}\sup_{t\ge-\tau}\mathbb{E}\left\|\int_0^t S(t-s)(f(s,\varphi(s-\delta(s)))-f(s,\psi(s-\delta(s))))\,ds\right\|^p \\
&\quad+6^{p-1}\sup_{t\ge-\tau}\mathbb{E}\left\|\int_0^t S(t-s)(g(s,\varphi(s-\rho(s)))-g(s,\psi(s-\rho(s)))))\,dw(s)\right\|^p \\
&\quad+6^{p-1}\sup_{t\ge-\tau}\mathbb{E}\left\|\int_0^t\int_Z S(t-s)(h(s,\varphi(s-\sigma(s)),y)-h(s,\psi(s-\sigma(s)),y))\,\tilde{N}(ds,dy)\right\|^p \\
&\quad+6^{p-1}\sup_{t\ge-\tau}\mathbb{E}\left\|\sum_{0<t_k<t} S(t-t_k)(I_k(\varphi(t_k^-))-I_k(\psi(t_k^-)))\right\|^p
\end{aligned}
$$

$$\leq \Bigg[6^{p-1}\|(-A)^{-\alpha}\|^p K^p + 6^{p-1}M_{1-\alpha}^p K^p \gamma^{-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}$$
$$+6^{p-1}M^p L_1^p \gamma^{-p} + 6^{p-1}c_p M^p L_2^p \left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\gamma^{-1}$$
$$+6^{p-1}c_p M^p L_3^p \left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\gamma^{-1} + 6^{p-1}M^p\tilde{q}^p\gamma^{-p}\Bigg]\sup_{t\geq 0}\mathbb{E}\|\varphi(t)-\psi(t)\|^p.$$

Thus, by the condition (ii), we know that π is a contraction mapping.

Hence, using a contraction mapping principle, π has a unique fixed point $x(t)$ in $\mathcal{S}_\phi$, which is a solution of the system (5.20) with $x(s)=\phi(s)$ on $[-\tau,0]$ and $e^{\eta t}\mathbb{E}\|x(t)\|^p \to 0$ as $t\to\infty$. □

If the impulsive effects $I_k(\cdot)\equiv 0$ $(k=1,2,\cdots)$, system (5.20) reduces to the following neutral stochastic partial differential equations with delays

$$\begin{cases} d[x(t)+u(t,x(t-\tau(t)))] = [Ax(t)dt + f(t,x(t-\delta(t)))]dt + g(t,x(t-\rho(t))dw(t) \\ \qquad\qquad + \int_Z h(t,x(t-\theta(t)),y)\,\tilde{N}(dt,dy), \quad t\geq 0 \\ x_0(\cdot)=\phi\in PC. \end{cases} \tag{5.56}$$

Corollary 5.2.19. *Consider the stochastic partial differential equation* (5.56) *and suppose that the conditions* (A1)-(A4) *are satisfied. Then the mild solution of* (5.56) *is exponential stability in pth moment, if the following inequality*

$$5^{p-1}\|(-A)^{-\alpha}\|^p K^p + 5^{p-1}M_{1-\alpha}^p K^p \gamma^{-p\alpha}(\Gamma(1+p(\alpha-1)/(p-1)))^{p-1}$$
$$+5^{p-1}M^p L_1^p\gamma^{-p} + 5^{p-1}c_p M^p L_2^p\left(\frac{2\gamma(p-1)}{p-2}\right)^{1-p/2}\gamma^{-1}$$
$$+5^{p-1}c_p M^p L_3^p\left(\frac{p-2}{2(p-1)\gamma}\right)^{(p-2)/2}\gamma^{-1} < 1$$

holds, where $c_p=(p(p-1)/2)^{p/2}$.

Example 5.2.20. Consider a neutral stochastic partial differential equation with delays and Poisson jumps of the form

$$\begin{cases} d\left[x(t,\xi)+\alpha_0 x(t-\tau(t),\xi)\right] = \left[\frac{\partial^2}{\partial\xi^2}x(t,\xi)dt + \alpha_1 x(t-\delta(t),\xi)\right]dt \\ \qquad\qquad +\alpha_2 x(t-\rho(t),\xi)dw(t), \quad t\neq t_k, \\ \Delta x(t_k,\xi) = b_k x(t_k,\xi), \quad t=t_k, \quad k=1,2,\cdots, \\ x(s,\xi)=\phi(s,\xi), \quad \phi(\cdot,\xi)\in C, \quad \phi(s,\cdot)\in\mathcal{L}^2[0,\pi], \quad -\tau\leq s\leq 0, \quad 0\leq\xi\leq\pi, \end{cases} \tag{5.57}$$

for $t\geq 0$, where $\tau\geq 0$ is a constant, $w(t)$ is a standard one-dimensional Wiener process and $\|\phi\|_C < +\infty$ a.s., and $\alpha\in(1/p,1]$, $p\geq 2$. We suppose that $x(t,0)=x(t,\pi)=0$ and $0\leq\tau(t),\delta(t),\rho(t)\leq\tau$, and we suppose that α_0, α_1 and α_2 are positive constants. Further, we suppose that there exists a constant b such that $b_k\leq b(t_k-t_{k-1})$.

Take $X = L^2[0,\pi]$,$Y = R^1$, define $A : X \to X$ by $A = \frac{\partial^2}{\partial \xi^2}$ with domain

$$D(A) = \left\{x \in X \mid x,\ \ \partial x/\partial\xi \ \text{ are continuous, } \ \partial^2 x/\partial\xi^2 \in X,\ \ x(0) = x(\pi) = 0\right\}.$$

Then $Ax = \sum_{n=1}^{\infty} n^2 \langle x, e_n\rangle e_n, \quad x \in D(A)$, where $e_n(\xi) = \sqrt{2/\pi}\sin n\xi$, $n = 1, 2, 3, \cdots$, is an orthonormal set of eigenvectors of $-A$. It is well known that A is the infinitesimal generator of an analytic semigroup $S(t)$ in X and is given (see [100], Page 70) by

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2 t}\langle x, e_n\rangle e_n, \quad x \in X.$$

Hence, $\|S(t)\| \le e^{-\pi^2 t}$, $t \ge 0$. We define A^α (actually $|A|^\alpha$) for the self-adjoint operator A by the classical spectral theorem,

$$|A|^\alpha S(t)x = \sum_{n=1}^{\infty} \left(n^2\right)^\alpha e^{-n^2 t}\langle x, e_n\rangle e_n.$$

From the arguments in [125], we have that for $0 < m < 1$ and $M_\alpha = \left(\frac{\alpha}{1-m}\right)^\alpha$,

$$\|(-A)^\alpha S(t)x\| \le M_\alpha e^{-mt} t^{-\alpha}\|x\|, \quad t > 0$$

holds for all $x \in X$. Let $u(t, x(t-\tau(t), \xi)) = \alpha_0 x(t-\tau(t), \xi)$, $f(t, x(t-\delta(t), \xi)) = \alpha_1 x(t-\delta(t), \xi)$, $g(t, x(t-\rho(t), \xi)) = \alpha_2 x(t-\rho(t), \xi)$. It is clear that $u(t, \cdot)$, $f(t, \cdot)$ and $g(t, \cdot)$ satisfy conditions (A2) and (A3). Further, from the definition of $(-A)^{-\alpha}$, we have that

$$\|(-A)^{-\alpha}\| \le \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1}\|S(t)\|\, dt \le \frac{1}{\pi^{2\alpha}}.$$

Thus, by Theorem 5.2.15, the mild solution of (5.57) is exponentially stable in pth moment provided that

$$\left(\frac{\alpha_0}{\pi^{2\alpha}}\right)^p + \left(\frac{1-\alpha}{1-m}\right)^{1-\alpha} \alpha_0^p \pi^{-2p\alpha} \left(\Gamma\left(1 + \frac{p(\alpha-1)}{p-1}\right)\right)^{p-1} + \alpha_1^p \pi^{-2p}$$
$$+ c_p \alpha_2^p \pi^{-2} \left(\frac{2\pi^2(p-1)}{p-2}\right)^{1-p/2} + b^p \pi^{-2p} < 6^{1-p},$$

where $\Gamma(\cdot)$ is the Gamma function, $\alpha \in (1/p, 1]$, $p \ge 2$, $0 < m < 1$, $c_p = (p(p-1)/2)^{p/2}$.

5.3 Notes and remarks

Luo [90] and Appleby [4] have applied fixed point method to deal with the stability problems for stochastic differential equations with stochastic effects. Following the ideas of [4, 90], by employing a contraction mapping principle and stochastic integral technique, many other investigators [89, 131, 132, 147] considered its generalization.

Many methods which are used frequently to investigate the stability problems for stochastic partial differential equations are ineffective to study the exponential stability of mild solutions for impulsive stochastic delay differential equations with delays, see, for example, the comparison theorem in Govindan [48, 49], the Gronwall inequality in Caraballo and Liu [14], the analytic technique in Liu and Truman [87], Taniguchi [121] and the semigroup method in Taniguchi et al.[123], the methods proposed in Caraballo et al.[122], Ichikawa [57, 58], Liu and Mao [86], Wan and Duan [130], Liu [85, 88] and Taniguchi [124].

Based on the contents of Section 5.2, a paper [18] has been submitted to a journal for possible publication.

CHAPTER 6

Stochastic delayed neural networks

This chapter presents stability properties of a class of stochastic delayed neural networks without impulses and a class of stochastic delayed neural networks with impulses.

In Section 6.1, we present new conditions for asymptotic stability and exponential stability of a class of stochastic recurrent neural networks with discrete and distributed time varying delays. Our approaches are based on the method using fixed point theory and the method using an appropriate integral inequality, which do not resort to any Liapunov function or Liapunov functional. Our results neither require the boundedness, monotonicity and differentiability of the activation functions nor differentiability of the time varying delays. In particular, a class of neural networks without stochastic perturbations is also considered by using the two approaches.

In Section 6.2, we consider the impulsive effects on the class of stochastic delayed recurrent neural networks that is discussed in Section 6.1. New sufficient conditions for asymptotic stability and exponential stability of the class of impulsive stochastic delayed recurrent neural networks are presented by using fixed point methods. In particular, as in Section 6.1, a class of impulsive neural networks without stochastic perturbations is also considered.

6.1 Stability of stochastic delayed neural networks

6.1.1 Introduction and main results

During the past few decades, neural networks such as Hopfield neural networks [53], Cellular neural networks [24, 25], Cohen-Grossberg neural networks [136] and bidirectional associative memory neural networks (BAM Networks) [68, 69, 70] have been well investigated since they play an important role in many areas such as combinatorial optimization, signal processing and pattern recognition.

Due to the finite switching speed of neurons and amplifiers, time delays which may lead to instability and bad performance in neural processing and signal transmission are commonly encountered in both biological and artificial neural networks. In addition, neural networks usually have a spatial extent due to the presence of a multitude of parallel pathways with a variety of axon sizes and lengths [128]. Thus there will be a distribution of conduction velocities along these pathways and a distribution of propagation delays [146]. In these circumstances the signal propagation is not instantaneous and may not be suitably modeled with discrete delays. Therefore, a more appropriate way which incorporates continuously distributed delays in neural network models has been used. Further, due to random fluctuations and probabilistic causes in the network, noises do exist in a neural network. Thus, it is necessary and rewarding to study stochastic effects to the stability property of neural networks.

Liapunov's direct method has long been viewed the main classical method of studying stability problems in many areas of stochastic delay differential equations. The success of Lyapunov's direct method depends on finding a suitable Liapunov function or Liapunov functional. However, it may be difficult to look for a good Liapunov functional for some classes of stochastic delay differential equations. Therefore, an alternative may be explored to overcome such difficulties.

It was proposed by Burton [13] and his co-workers to use fixed point methods to study the stability problem for deterministic systems. Luo [90] and Appleby [4] have applied this method to deal with the stability problems for stochastic delay differential equations, and afterwards, a great number of classes of stochastic delay differential equations are discussed by using fixed point methods, see, for example, [34, 91, 92, 117, 118]. It turns out that the fixed point method is a powerful technique in dealing with stability problems for deterministic and stochastic differential equations with delays. Moreover, it has an advantage that it can yield the existence, uniqueness and stability criteria of the considered system in one step. Chen [21, 23] has applied an appropriate integral inequality to study exponential stability of some classes of stochastic delay differential equations, and it turns out that it is a convenient way to discuss exponential stability of a system.

The aim of this section is to study a general class of stochastic neural networks by using fixed point methods and the method by employing an appropriate integral inequality. Indeed, we consider the following class of stochastic neural networks with varying discrete and distributed delays which is described by

$$\begin{aligned} dx_i(t) &= \Big[- c_i x_i(t) + \sum_{j=1}^{n} a_{ij} f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(t-\tau(t))) \\ &\quad + \sum_{j=1}^{n} l_{ij} \int_{t-r(t)}^{t} h_j(x_j(s))\, ds \Big] dt + \sum_{j=1}^{n} \sigma_{ij}(t, x_j(t), x_j(t-\tau(t)))\, dw_j(t), \end{aligned} \tag{6.1}$$

or

$$\begin{aligned} dx(t) &= \Big[- Cx(t) + Af(x(t)) + Bg(x(t-\tau(t))) + W \int_{t-r(t)}^{t} h(x(s))\, ds \Big] dt \\ &\quad + \sigma(t, x(t), x(t-\tau(t)))\, dw(t) \end{aligned}$$

for $i = 1, 2, 3, \cdots, n$, where $x(t) = (x_1(t), x_2(t), \cdots, x_n(t))^T \in \mathbb{R}^n$ is the state vector associated with the neurons; $C = \operatorname{diag}(c_1, c_2, \cdots, c_n) > 0$ where $c_i > 0$ represents the rate with which the ith unit will reset its potential to the resting state in isolation when disconnected from the network and the external stochastic perturbations; $A = (a_{ij})_{n\times n}$, $B = (b_{ij})_{n\times n}$ and $W = (l_{ij})_{n\times n}$ represent the connection weight matrix, delayed connection weight matrix and distributed delayed connection weight matrix, respectively; f_j, g_j, h_j are activation functions, $f(x(t)) = (f_1(x(t)), f_2(x(t)), \cdots, f_n(x(t)))^T \in \mathbb{R}^n$, $g(x(t)) = (g_1(x(t)), g_2(x(t)), \cdots, g_n(x(t)))^T \in \mathbb{R}^n$, $h(x(t)) = (h_1(x(t)), h_2(x(t)), \cdots, h_n(x(t)))^T \in \mathbb{R}^n$. Moreover, $w(t) = (w_1(t), w_2(t), \cdots, w_n(t))^T \in \mathbb{R}^n$ is an n-dimensional Brownion motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with natural complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$ (i.e. $\mathcal{F}_t$ = completion of $\sigma\{\omega(s) : 0 \leq s \leq t\}$) and $\sigma : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\sigma = (\sigma_{ij})_{n\times n}$ is the diffusion coefficient matrix. $\tau(t)$ and $r(t)$ denote a discrete time varying delay and the bound of a distributed time varying delay, respectively. Denote $\vartheta = \inf_{t\geq 0}\{t - \tau(t), t - r(t)\}$.

The initial condition for the system (6.1) is given by

$$x(t) = \phi(t), \qquad t \in [\vartheta, 0], \tag{6.2}$$

where $t \mapsto \phi(t) = (\phi_1(t), \phi_2(t), \cdots, \phi_n(t))^T \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$ with the norm defined as

$$\|\phi\|^p = \sup_{\vartheta \le t \le 0} \left(\mathbb{E} \sum_{i=1}^{n} |\phi_i(t)|^p \right),$$

where $\mathbb{E}$ denotes expectation with respect to the probability measure $\mathbb{P}$ and $p \geq 2$.

To obtain our main results, we suppose the following conditions are satisfied:

(A1) the delays $\tau(t), r(t)$ are continuous functions such that $t - \tau(t) \to \infty$ and $t - r(t) \to \infty$ as $t \to \infty$;

(A2) $f_j(x)$, $g_j(x)$, and $h_j(x)$ satisfy Lipschitz conditions. That is, for each $j = 1, 2, 3, \cdots, n$, there exist constants α_j, β_j, γ_j such that for every $x, y \in \mathbb{R}^n$,

$$|f_j(x) - f_j(y)| \le \alpha_j |x - y|, \quad |g_j(x) - g_j(y)| \le \beta_j |x - y|, \quad |h_j(x) - h_j(y)| \le \gamma_j |x - y|;$$

(A3) Assume that $f(0) \equiv 0$, $g(0) \equiv 0$, $h(0) \equiv 0$, $\sigma(t, 0, 0) \equiv 0$;

(A4) $\sigma(t, x, y)$ satisfies a Lipschitz condition. That is, there are nonnegative constants μ_i and ν_i such that $\forall\ i, j$,

$$(\sigma_{ij}(t, x, y) - \sigma_{ij}(t, u, v))^2 \le \mu_j (x_j - u_j)^2 + \nu_j (y_j - v_j)^2.$$

It follows from [43, 98] that under the hypotheses $(A1), (A2), (A3)$ and $(A4)$, system (6.1) with initial condition (6.2) has one unique global solution which is denoted by $x(t, \phi)$ or $x(t)$ such that $t \mapsto x(t, \phi) : [0, \infty) \to L^p(\Omega; \mathbb{R}^n)$ is adapted and continuous and $\mathbb{E}[\sup_{0 \le s \le t} \|x(s, 0, \phi)\|^p] < \infty$ for $t > 0$. Clearly, system (6.1) admits the trivial solution $x(t, 0, 0) \equiv 0$.

Definition 6.1.1. *The trivial solution of system* (6.1) *is said to be stable in pth* ($p \geq 2$) *moment if for arbitrary given* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $\|\phi\|^p < \delta$ *yields that*

$$\mathbb{E}\|x(t, \phi)\|^p < \varepsilon, \quad t \geq 0.$$

where $\phi \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$. *In particular, when* $p = 2$, *the trivial solution is said to be mean square stable.*

Definition 6.1.2. *The trivial solution of system* (6.1) *is said to be asymptotically stable in pth* ($p \geq 2$) *moment if it is stable in pth moment and there exists a* $\delta > 0$, *such that* $\|\phi\|^p < \delta$ *implies*

$$\lim_{t \to \infty} \mathbb{E}\|x(t, \phi)\|^p = 0.$$

where $\phi \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$.

Definition 6.1.3. *The trivial solution of system* (6.1) *is said to be pth* ($p \geq 2$) *moment exponentially stable if there exists a pair of constants* $\lambda, C > 0$ *such that*

$$\mathbb{E}\|x(t,\phi)\|^p \leq C\mathbb{E}\|\phi\|^p e^{-\lambda t}, \quad t \geq 0,$$

holds for $\phi \in C\left([\vartheta, 0], L^p_{\mathcal{F}_0}(\Omega; \mathbb{R}^n)\right)$. *Especially, when* $p = 2$, *we speak of exponentially stable in mean square.*

Different choices of norms can be considered on spaces of stochastic processes. The norms we choose should be such that the space under consideration is complete and the equation yields a contraction with respect to the norm. For the system (6.1) with initial condition (6.2), we consider the following two different complete spaces which are defined by using two types of norms.

Define $\mathcal{S}_\phi$ the space of all $\mathcal{F}_t$-adapted processes $\varphi(t,\omega) : [\vartheta, \infty) \times \Omega \to \mathbb{R}^n$ such that $\varphi \in C\left([\vartheta, \infty), L^p_{\mathcal{F}_t}(\Omega; \mathbb{R}^n)\right)$. Moreover, we require $\varphi(t,\cdot) = \phi(t)$ for $t \in [\vartheta, 0]$ and $\mathbb{E}\sum_{i=1}^n |\varphi_i(t)|^p \to 0$ as $t \to \infty$, $i = 1, 2, \cdots, n$. If we define the norm

$$\|\varphi\|^p := \sup_{t \geq \vartheta}\left(\mathbb{E}\sum_{i=1}^n |\varphi_i(t)|^p\right), \tag{6.3}$$

then $\mathcal{S}_\phi$ is a complete metric space. Using a contraction mapping defined on the space $\mathcal{S}_\phi$ and applying a contraction mapping principle, we obtain our first result. Its proof is given in Subsection 6.1.2.

Theorem 6.1.4. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the function* $r(t)$ *is bounded by a constant* r ($r > 0$)*;*

(ii) *and such that*

$$\begin{aligned}\alpha \triangleq\ & 5^{p-1}\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n |a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 5^{p-1}\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n |b_{ij}|^q|\beta_j|^q\right)^{p/q} \\ & + 5^{p-1}\sum_{i=1}^n\left(\frac{r}{c_i}\right)^p\left(\sum_{j=1}^n |l_{ij}|^q|\gamma_j|^q\right)^{p/q} + 5^{p-1}n^{p-1}\sum_{i=1}^n c_i^{-p/2}\left(\mu^{p/2} + \nu^{p/2}\right) < 1,\end{aligned} \tag{6.4}$$

where $\mu = \max\{\mu_1, \mu_2, \cdots, \mu_n\}$, $\nu = \max\{\nu_1, \nu_2, \cdots, \nu_n\}$*;*

then the trivial solution of (6.1) *is pth moment asymptotically stable.*

Consider a case when both the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ. Let $\phi \in L^p_{\mathcal{F}_0}(\Omega, C([\vartheta, 0], \mathbb{R}^n))$, define $\mathcal{C}_\phi$ to be the space of all $\mathcal{F}_t$-adapted processes $\varphi(t,\omega) : [-\tau, \infty) \times \Omega \to \mathbb{R}^n$ such that $\varphi \in L^p(\Omega, C([\vartheta, \infty), \mathbb{R}^n))$. Moreover, we set $\varphi(t,\cdot) = \phi(t)$ for $t \in [\vartheta, 0]$, $\varphi(t,\cdot) = \phi(\vartheta)$ for $t \in [-\tau, \vartheta]$ (in case $-\tau < \vartheta$), and for $t \to \infty$, $\sum_{i=1}^n \mathbb{E}\sup_{t-\tau \leq s \leq t}|\varphi_i(s)|^p \to 0$. The norm on $\mathcal{C}_\phi$ is defined as

$$\|\varphi\|^p = \sup_{t \geq 0}\left[\sum_{i=1}^n \mathbb{E}\left(\sup_{t-\tau \leq s \leq t}|\varphi_i(s)|^p\right)\right], \tag{6.5}$$

then $\mathcal{C}_\phi$ is a complete metric space. Using a contraction mapping defined on the space $\mathcal{C}_\phi$ and applying a contraction mapping principle, we obtain our second result, which is proved in Subsection 6.1.3.

Theorem 6.1.5. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ $(\tau > 0)$;*

(ii) *and such that*

$$\begin{aligned}\alpha \triangleq\ & 5^{p-1}e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 5^{p-1}e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q} \\ & +5^{p-1}\tau^p e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q} \\ & +5^{p-1}K_p n^p e^{pc\tau}q^p c^{1-p/2}(2c)^{-1}\left(\mu^{p/2}+\nu^{p/2}\right) < 1, \end{aligned} \tag{6.6}$$

where $c = \min\{c_1, c_2, \cdots c_n\}$, $\mu = \max\{\mu_1, \mu_2, \cdots \mu_n\}$, $\nu = \max\{\nu_1, \nu_2, \cdots \nu_n\}$;

then the trivial solution of (6.1) *is pth moment asymptotically stable. More than that, for every* $\varepsilon > 0$, *there exists a* $\delta > 0$ *such that* $\|\phi\| < \delta$ *implies* $\sum_{i=1}^n \mathbb{E}\sup_{t-\tau\le s\le t}|x_i(s)|^p < \varepsilon$ *and* $\lim_{t\to\infty}\left\{\mathbb{E}\left[\sup_{t-\tau\le s\le t}\|x(s,0,\phi)\|^p\right]\right\} = 0.$

Remark 6.1.6. *In some papers, see, for example,* [89, 90, 131, 132], *the norm for the space of stochastic process is defined as*

$$\|\varphi\|_{[0,t]} = \left[\mathbb{E}\left(\sup_{s\in[0,t]}|\varphi(s)|^2\right)\right]^{1/2}.$$

As in [90], *in order to show* $P(\mathcal{S}) \subseteq \mathcal{S}$, *we need to estimate* $\mathbb{E}\sup_{s\in[0,t]}|I_5(s)|^2$, *where*

$$I_5(s) = \int_0^s e^{-\int_z^s h(u)\,du}\left[c(z)x(z) + e(z)x(z-\delta(z))\right]d\omega(z).$$

However, $I_5(s)$ *is not a local martingale* (*see Section* 1.4 *for its proof*). *Hence, Burkholder-Davis-Gundy Inequality can not be applied directly.*

Using an appropriate integral inequality, we obtain sufficient conditions for exponential stability of (6.1) with initial condition (6.2), which is our third result. For its proof, see Subsection 6.1.4.

Theorem 6.1.7. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ;*

(ii) *and such that*

$$5^{p-1}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+5^{p-1}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q} \tag{6.7}$$
$$+5^{p-1}\left(\frac{\tau}{c}\right)^p\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}+5^{p-1}n^pc^{-p/2}(\mu^{p/2}+\nu^{p/2})<1,$$

where $c=\min\{c_1,c_2,\cdots,c_n\}$, $\mu=\max\{\mu_1,\mu_2,\cdots,\mu_n\}$, $\nu=\max\{\nu_1,\nu_2,\cdots,\nu_n\}$*;*

then the trivial solution of (6.1) *is exponentially stable in pth moment,*

Remark 6.1.8. *The stability criteria we provided in our main results are only in terms of the system parameters* c_i, a_{ij}, b_{ij}, l_{ij}, *etc. Hence, these criteria can usually be verified easily in applications.*

Remark 6.1.9. *Many articles, see, for example,* [116, 120] *have studied stochastic neural network* (6.1) *and special cases of* (6.1)*. However, they impose the following condition on the delays*

(H) *the discrete delay* $\tau(t)$ *is differentiable function and* $r(t)$ *in the distributed delay is nonnegative and bounded, that is, there exist constants* τ_M,ζ,r_M *such that*

$$0\le\tau(t)\le\tau_M,\quad \tau'(t)\le\zeta,\quad r(t)\le r_M. \tag{6.8}$$

In our results, condition (H) *is replaced by other assumptions, which may be satisfied when* (H) *is not.*

Theorem 6.1.7 can, for example, be applied to establish exponential stability in *pth* moment of a two dimensional stochastically perturbed Hopfield neural network with time-varying delay, the delay is bounded but not differentiable, see Example 6.1.31 for details.

Consider a case when there are no stochastic effects in the system (6.1), which then comes down to the neural network described by

$$\frac{dx_i(t)}{dt}=-c_ix_i(t)+\sum_{j=1}^{n}a_{ij}f_j(x_j(t))+\sum_{j=1}^{n}b_{ij}g_j(x_j(t-\tau(t)))+\sum_{j=1}^{n}d_{ij}\int_{t-r(t)}^{t}h_j(x_j(s))\,ds, \tag{6.9}$$
$$i=1,2,3,\cdots,n,$$

or

$$\frac{dx(t)}{dt}=-Cx(t)+Af(x(t))+Bg(x-\tau(t))+D\int_{t-r(t)}^{t}h(x(s))\,ds, \tag{6.10}$$

where $x(\cdot)=(x_1(\cdot),x_2(\cdot),\cdots,x_n(\cdot))^T$ is the neuron state vector of the transformed system (6.9).

The initial condition for the system (6.9) is

$$x(t)=\phi(t),\qquad t\in[\vartheta,0], \tag{6.11}$$

where ϕ is a continuous function with the norm defined by $\|\phi\|=\sup_{\vartheta\le t\le 0}\sum_{i=1}^{n}|\phi_i(t)|$.

Assume that $(A1)-(A3)$ are satisfied, then (6.9) admits a trivial solution $x=0$. Denote by $x(t,\phi)=(x_1(t,\phi_1),\cdots,x_n(t,\phi_n))^T\in\mathbb{R}^n$ the solution of (6.9) with initial condition (6.11).

Definition 6.1.10. *For the system* (6.9) *with initial condition* (6.11)*, we have that*

(i) *the trivial solution of* (6.9) *is said to be stable if for any* $\varepsilon > 0$*, there exists* $\delta > 0$ *such that for any initial condition* $\phi \in C([\vartheta, 0], \mathbb{R}^n)$ *satisfying* $\|\phi\| < \delta$*, we have for the corresponding solution that* $\|x(t, \phi)\| < \varepsilon$ *for* $t \geq 0$*;*

(ii) *the trivial solution of* (6.9) *is said to be asymptotically stable if it is stable and for any initial condition* $\phi \in C([\vartheta, 0], \mathbb{R}^n)$ *we have for the corresponding solution that* $\lim_{t\to\infty} \|x(t, \phi)\| = 0$*;*

(iii) *the trivial solution of* (6.9) *is said to be globally exponentially stable if there exist scalars* $\lambda > 0$ *and* $C > 0$ *such that for any initial condition* $\phi \in C([\vartheta, 0], \mathbb{R}^n)$*, we have for the corresponding solution that* $\|x(t, \phi)\| \leq Ce^{-\lambda t}\|\phi\|$ *for* $t \geq 0$*.*

Define $\mathcal{H}_\phi = \mathcal{H}_{1\phi} \times \mathcal{H}_{2\phi} \times \cdots \times \mathcal{H}_{n\phi}$, where $\mathcal{H}_{i\phi}$ is the space consisting of continuous functions $\varphi_i(t) : [\vartheta, \infty) \to \mathbb{R}$ such that $\varphi_i(\theta) = \phi(\theta)$ for $\vartheta \leq \theta \leq 0$ and $\varphi_i(t) \to 0$ as $t \to \infty$, $i = 1, 2 \cdots, n$. For any $\varphi(t) = (\varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t)) \in \mathcal{H}_\phi$ and $\eta(t) = (\eta_1(t), \eta_2(t), \cdots, \eta_n(t)) \in \mathcal{H}_\phi$, if we define the metric as $d(\varphi, \eta) = \sup_{t\geq\vartheta} \sum_{i=1}^n |\varphi_i(t) - \eta_i(t)|$, then $\mathcal{H}_\phi$ becomes a complete metric space.

Using a contraction mapping defined on the space H_ϕ and applying a contraction mapping principle, we obtain our fourth result, which is proved in Subsection 6.1.5.

Theorem 6.1.11. *Suppose that the assumptions* (A1)-(A3) *hold. If the following conditions are satisfied,*

(i) *the function* $r(t)$ *is bounded by a constant* r $(r > 0)$*;*

(ii) *and such that*

$$\alpha \triangleq \sum_{i=1}^n \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^n \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \sum_{i=1}^n \frac{r}{c_i} \max_{j=1,2,\cdots,n} |d_{ij}\gamma_j| < 1; \quad (6.12)$$

then the trivial solution of (6.9) *is asymptotically stable.*

Remark 6.1.12. *Theorem* 6.1.11 *is an extension and improvement of the result in Lai and Zhang* [74].

By establishing an appropriate integral inequality, we obtain sufficient conditions for exponential stability of (6.9), which is our fifth result. Its proof is given in Subsection 6.1.6.

Theorem 6.1.13. *Suppose that the assumptions* (A1)-(A3) *hold. If the following conditions are satisfied,*

(i) *the discrete delay* $\tau(t)$ *and* $r(t)$ *in the distributed delay are bounded by a constant* τ $(\tau > 0)$*;*

(ii) *and such that*

$$\frac{1}{c} \sum_{i=1}^n \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \frac{1}{c} \sum_{i=1}^n \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \frac{1}{c} \sum_{i=1}^n \tau \max_{j=1,2,\cdots,n} |d_{ij}\gamma_j| < 1, \quad (6.13)$$

where $c = \min\{c_1, c_2, \cdots, c_n\}$*;*

then the trivial solution of (6.9) *with initial condition* (6.11) *is exponentially stable.*

Remark 6.1.14. *Several exponential stability results* [77, 126, 127] *were provided for the system* (6.9)*, by constructing an appropriate Liapunov functional and employing linear matrix inequality* (LMI) *method, and their results depends on the condition that the delays are satisfied* (H)*. From our main results, we provide other assumptions. The delays in our results are required to be bounded.*

Remark 6.1.15. *From Theorem* 6.1.11 *and Theorem* 6.1.13*, we find that the terms with* f, g, h *in equation* (6.10) *can be viewed as perturbations of the stable equation* $dx(t)/dt = -Cx(t)$*. Condition* (ii) *in Theorem* 6.1.11 *and condition* (ii) *in Theorem* 6.1.13 *require the perturbation to be small relative to the stabilizing force of* C*.*

Theorem 6.1.13 can, for example, be applied to establish exponential stability of a two dimensional cellular neural network with time-varying delay, see Example 6.1.29 for details.

The rest of this section is organized as follows. In Subsection 6.1.2, we present a proof of Theorem 6.1.4. The proof of Theorem 6.1.5 is presented in Subsection 6.1.3 and the proof of Theorem 6.1.7 is given in Section 6.1.4. we present the proofs of Theorem 6.1.11 and Theorem 6.1.13 in Subsection 6.1.5 and Subsection 6.1.6, respectively. Some examples are given to illustrate our main results in Subsection 6.1.7.

6.1.2 Proof of Theorem 6.1.4

In this subsection, we prove Theorem 6.1.4. We start with some preparations.

Lemma 6.1.16. ([96, 129]) *If* $w(t) = (w_1(t), w_2(t), \cdots, w_n(t))^T$ *(*$t \geq 0$*) is a* n*-dimensional Brownian motion defined on a complete probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, then for each* $t \geq 0$*, we have the following formula*

$$\mathbb{E}\left(\int_0^t f_i(s)\,dw_i(s)\int_0^t f_j(s)\,dw_j(s)\right) = \mathbb{E}\int_0^t f_i(s)f_j(s)\,d\langle w_i, w_i\rangle_s,$$

where $\langle w_i, w_i\rangle_s = \delta_{ij}s$ *are the cross-variations, and* δ_{ij} *is the correlation coefficient,* f_i *is adapted and* $f_i \in L^2(\Omega \times [0,t])$*,* $i, j = 1, 2, \cdots, n$*.*

If we multiply both sides of (6.1) by $e^{c_i t}$ and integrate from 0 to t, we obtain

$$\begin{aligned} x_i(t) \;=\;& e^{-c_i t}x_i(0) + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n a_{ij}f_j(x_j(s))\,ds + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n b_{ij}g_j(x_j(s-\tau(s)))\,ds \\ &+ \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n l_{ij}\int_{s-r(s)}^s h_j(x_j(u))\,du\,ds \\ &+ \int_0^t e^{-c_i(t-s)}\sum_{j=1}^n \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\,dw_j(s) \qquad (6.14) \end{aligned}$$

for $t \geq 0$, $i = 1, 2, 3, \cdots, n$.

Lemma 6.1.17. *Define an operator by* $(Q\varphi)(t) = \phi(t)$ *for* $t \in [\vartheta, 0]$*, and for* $t \geq 0$*,* $i = 1, 2, 3, \cdots, n$*,*

$$\begin{aligned}(Q\varphi)_i(t) &= e^{-c_i t}\varphi_i(0) + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n a_{ij} f_j(\varphi_j(s))\, ds \\ &\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n b_{ij} g_j(\varphi_j(s-\tau(s)))\, ds \\ &\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(\varphi_j(u))\, du\, ds \\ &\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n \sigma_{ij}(s, \varphi_j(s), \varphi_j(s-\tau(s)))\, dw_j(s). \qquad (6.15)\end{aligned}$$

Suppose that the assumption (A1)-(A4) *holds. If conditions* (i) *and* (ii) *in Theorem* 6.1.4 *are satisfied, then* $Q : \mathcal{S}_\phi \to \mathcal{S}_\phi$ *and* Q *is a contraction mapping.*

Proof. Denote $(Q\varphi)_i(t) := J_{1i}(t) + J_{2i}(t) + J_{3i}(t) + J_{4i}(t) + J_{5i}(t)$, where

$$\begin{aligned}J_{1i}(t) &= e^{-c_i t}\varphi_i(0), \qquad J_{2i}(t) = \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n a_{ij} f_j(\varphi_j(s))\, ds, \\ J_{3i}(t) &= \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n b_{ij} g_j(\varphi_j(s-\tau(s)))\, ds, \\ J_{4i}(t) &= \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(\varphi_j(u))\, du\, ds, \\ J_{5i}(t) &= \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n \sigma_{ij}(s, \varphi_j(s), \varphi_j(s-\tau(s)))\, dw_j(s).\end{aligned}$$

Step1. From the definition of the metric space $\mathcal{S}_\phi$, we have that $\mathbb{E}\sum_{i=1}^n |\varphi_i(t)|^p < \infty$ for all $t \geq 0$, $\varphi \in \mathcal{S}_\phi$.

Step2. We prove the continuity in *pth* moment of Qx on $[0, \infty)$ for $x \in \mathcal{S}_\phi$. Let $x \in \mathcal{S}_\phi$, $t_1 \geq 0$, let $r \in \mathbb{R}$ with $|r|$ sufficiently small and $r > 0$ if $t_1 = 0$, we have

$$\begin{aligned}\mathbb{E}\sum_{i=1}^n |J_{2i}(t_1+r) - J_{2i}(t_1)|^p &= \mathbb{E}\sum_{i=1}^n \Big| \int_0^{t_1} \left(e^{-c_i(t_1+r-s)} - e^{-c_i(t_1-s)}\right) \sum_{j=1}^n a_{ij} f_j(x_j(s))\, ds \\ &\quad + \int_{t_1}^{t_1+r} e^{-c_i(t_1+r-s)} \sum_{j=1}^n a_{ij} f_j(x_j(s))\, ds \Big|^p \to 0 \quad \text{as} \quad r \to 0.\end{aligned}$$

Similarly, we have that

$$\mathbb{E}\sum_{i=1}^n |J_{3i}(t_1+r) - J_{3i}(t_1)|^p \to 0 \;\text{ as }\; r \to 0, \quad \mathbb{E}\sum_{i=1}^n |J_{4i}(t_1+r) - J_{4i}(t_1)|^p \to 0 \;\text{ as }\; r \to 0.$$

In the following, we check the continuity of $J_{5i}(t)$.

$$
\begin{aligned}
&\mathbb{E}\sum_{i=1}^{n}|J_{5i}(t_1+r)-J_{5i}(t_1)|^p \\
&= \mathbb{E}\sum_{i=1}^{n}\bigg|\int_0^{t_1}\Big(e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}\Big)\sum_{j=1}^{n}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s) \\
&\qquad +\int_{t_1}^{t_1+r}e^{-c_i(t_1+r-s)}\sum_{j=1}^{n}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s)\bigg|^p \\
&\le n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\bigg|\int_0^{t_1}\Big(e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}\Big)\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s) \\
&\qquad +\int_{t_1}^{t_1+r}e^{-c_i(t_1+r-s)}\sum_{j=1}^{n}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s)\bigg|^p \\
&\le (2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\bigg|\int_0^{t_1}\Big(e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}\Big)\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s)\bigg|^p \\
&\qquad +(2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\bigg|\int_{t_1}^{t_1+r}e^{-c_i(t_1+r-s)}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s)\bigg|^p \\
&= (2n)^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\bigg\{\mathbb{E}\bigg[\int_0^{t_1}\Big(e^{-c_i(t_1+r-s)}-e^{-c_i(t_1-s)}\Big)^2\sigma_{ij}^2(s,x_j(s),x_j(s-\tau(s)))\,ds\bigg]^{p/2} \\
&\qquad +\mathbb{E}\bigg[\int_{t_1}^{t_1+r}e^{-2c_i(t_1+r-s)}\sigma_{ij}^2(s,x_j(s),x_j(s-\tau(s)))\,ds\bigg]^{p/2}\bigg\}\to 0 \quad \text{as} \quad r\to 0.
\end{aligned}
$$

Thus, Qx is indeed continuous in pth moment on $[0,\infty)$.

Step3. We prove that $Q(\mathcal{S}_\phi)\subseteq\mathcal{S}_\phi$.

$$
\mathbb{E}\sum_{i=1}^{n}|(Q\varphi)_i(t)|^p=\mathbb{E}\sum_{i=1}^{n}\bigg|\sum_{j=1}^{5}J_{ji}(t)\bigg|^p\le 5^{p-1}\sum_{j=1}^{5}\mathbb{E}\sum_{i=1}^{n}|J_{ji}(t)|^p. \tag{6.16}
$$

Now, we estimate the terms on the right-hand side of the above inequality.

$$
\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|J_{2i}(t)|^p &\le \sum_{i=1}^{n}\mathbb{E}\bigg[\int_0^t e^{-\frac{c_i(t-s)}{q}}e^{-\frac{c_i(t-s)}{p}}\sum_{j=1}^{n}|a_{ij}||f_j(\varphi_j(s))|\,ds\bigg]^p \\
&\le \sum_{i=1}^{n}\mathbb{E}\bigg[\bigg(\int_0^t e^{-c_i(t-s)}\,ds\bigg)^{p/q}\int_0^t e^{-c_i(t-s)}\bigg(\sum_{j=1}^{n}|a_{ij}||f_j(\varphi_j(s))|\bigg)^p ds\bigg] \\
&\le \sum_{i=1}^{n}c_i^{-p/q}\mathbb{E}\bigg[\int_0^t e^{-c_i(t-s)}\bigg(\sum_{j=1}^{n}|a_{ij}||\alpha_j||\varphi_j(s)|\bigg)^p ds\bigg] \\
&\le \sum_{i=1}^{n}c_i^{-p/q}\bigg(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\bigg)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(s)|^p\bigg)ds. \qquad (6.17)
\end{aligned}
$$

Since $\varphi \in \mathcal{S}_\phi$, we have that $\lim_{t\to\infty} \mathbb{E}\sum_{i=1}^n |\varphi_i(t)|^p = 0$. Thus for any $\varepsilon > 0$, there exists $T_1 > 0$ such that $t \geq T_1$ implies $\mathbb{E}\sum_{i=1}^n |\varphi_i(t)|^p < \varepsilon$, combining with (6.17), we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^n |J_{2i}(t)|^p &\leq \sum_{i=1}^n c_i^{-p/q}\left(\sum_{j=1}^n |a_{ij}|^q|\alpha_j|^q\right)^{p/q} \int_0^{T_1} e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^n |\varphi_j(s)|^p\right) ds \\
&\quad + \sum_{i=1}^n c_i^{-p/q}\left(\sum_{j=1}^n |a_{ij}|^q|\alpha_j|^q\right)^{p/q} \int_{T_1}^{t} e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^n |\varphi_j(s)|^p\right) ds \\
&< \sum_{i=1}^n c_i^{-p} e^{-c_i t}(e^{c_i T_1} - 1)\left(\sum_{j=1}^n |a_{ij}|^q|\alpha_j|^q\right)^{p/q} \sup_{0\leq s\leq T_1}\left[\mathbb{E}\left(\sum_{j=1}^n |\varphi_j(s)|^p\right)\right] \\
&\quad + \varepsilon\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n |a_{ij}|^q|\alpha_j|^q\right)^{p/q}.
\end{aligned}$$

Hence, from the fact that $c_i > 0$ $(i = 1, 2, \cdots, n)$, we obtain that $\mathbb{E}\sum_{i=1}^n |J_{2i}(t)|^p \to 0$ as $t \to \infty$.

With the similar computation as (6.17), we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^n |J_{3i}(t)|^p &\leq \sum_{i=1}^n c_i^{-p/q}\left(\sum_{j=1}^n |b_{ij}|^q|\beta_j|^q\right)^{p/q} \int_0^{t} e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^n |\varphi_j(s-\tau(s)))|^p\right) ds \\
\mathbb{E}\sum_{i=1}^n |J_{4i}(t)|^p &\leq \sum_{i=1}^n c_i^{-p/q}\left(\sum_{j=1}^n |l_{ij}|^q|\gamma_j|^q\right)^{p/q} \int_0^{t} e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^n \left|\int_{s-r(s)}^{s} \varphi_j(u)\,du\right|^p\right] ds \\
&\leq \sum_{i=1}^n \left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^n |l_{ij}|^q|\gamma_j|^q\right)^{p/q} \int_0^{t} e^{-c_i(t-s)} \int_{s-r(s)}^{s} \mathbb{E}\left[\sum_{j=1}^n |\varphi_j(u)|^p\right] du\,ds.
\end{aligned} \tag{6.18}$$

Using Lemma 6.1.16, we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^n |J_{5i}(t)|^p &= \sum_{i=1}^n \mathbb{E}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n \sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,dw_j(s)\right|^p \\
&\leq n^{p-1}\sum_{i=1}^n\sum_{j=1}^n \mathbb{E}\left\{\left[\int_0^t e^{-c_i(t-s)}|\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))|\,dw_j(s)\right]^2\right\}^{p/2} \\
&= n^{p-1}\sum_{i=1}^n\sum_{j=1}^n \mathbb{E}\left[\int_0^t e^{-2c_i(t-s)}\sigma_{ij}^2(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,ds\right]^{p/2} \\
&\leq n^{p-1}\sum_{i=1}^n\sum_{j=1}^n \mathbb{E}\left[\int_0^t e^{-2c_i(t-s)}\left(\mu_j\varphi_j^2(s) + \nu_j\varphi_j^2(s-\tau(s))\right)\,ds\right]^{p/2}
\end{aligned}$$

$$
\begin{aligned}
&\le n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\Bigg[\Big(\int_0^t e^{-2c_i(t-s)}\mu_j\varphi_j^2(s)\,ds\Big)^{p/2} \\
&\quad+\Big(\int_0^t e^{-2c_i(t-s)}\nu_j\varphi_j^2(s-\tau(s))\,ds\Big)^{p/2}\Bigg] \\
&\le n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\Bigg[\bigg(\int_0^t e^{-2c_i(t-s)}\,ds\bigg)^{p/2-1}\int_0^t e^{-2c_i(t-s)}\mu_j^{p/2}|\varphi_j(s)|^p\,ds\Bigg] \\
&\quad+n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\Bigg\{\bigg(\int_0^t e^{-2c_i(t-s)}\,ds\bigg)^{p/2-1}\int_0^t e^{-2c_i(t-s)}\nu_j^{p/2}|\varphi_j(s-\tau(s))|^p\,ds\Bigg\} \\
&\le n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\Bigg[\mu^{p/2}\int_0^t e^{-2c_i(t-s)}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(s)|^p\bigg)\,ds \\
&\quad+\nu^{p/2}\int_0^t e^{-2c_i(t-s)}E\bigg(\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\bigg)\,ds\Bigg] \\
&\le n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\Bigg[\mu^{p/2}\int_0^t e^{-c_i(t-s)}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(s)|^p\bigg)\,ds \\
&\quad+\nu^{p/2}\int_0^t e^{-c_i(t-s)}E\bigg(\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\bigg)\,ds\Bigg]. \qquad (6.19)
\end{aligned}
$$

Since $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|^p \to 0$, $t-\tau(t)\to\infty$ and $t-r(t)\to\infty$ as $t\to\infty$, for each $\varepsilon>0$, there exists $T_2>0$ such that $t\ge T_2$ implies $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t-\tau(s))|^p<\varepsilon$ and $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t-r(t))|^p<\varepsilon$. From (6.18), we obtain that

$$
\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t)|^p \;\le\; & \sum_{i=1}^{n}c_i^{-p/q}\bigg(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\bigg)^{p/q}\int_0^{T_2}e^{-c_i(t-s)}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\bigg)\,ds \\
&+\sum_{i=1}^{n}c_i^{-p/q}\bigg(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\bigg)^{p/q}\int_{T_2}^{t}e^{-c_i(t-s)}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\bigg)\,ds \\
<\; & \sum_{i=1}^{n}\bigg(\frac{1}{c_i}\bigg)^{p/q}e^{-c_it}\int_0^{T_2}e^{c_is}\,ds\bigg(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\bigg)^{p/q} \\
&\times\sup_{\vartheta\le s\le T_2}\bigg\{\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\bigg)\bigg\}+\varepsilon\sum_{i=1}^{n}c_i^{-p}\bigg(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\bigg)^{p/q} \\
\mathbb{E}\sum_{i=1}^{n}|J_{4i}(t)|^p \;\le\; & \sum_{i=1}^{n}\bigg(\frac{r}{c_i}\bigg)^{p/q}\bigg(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\bigg)^{p/q}\int_0^{T_2}e^{-c_i(t-s)}\int_{s-r(s)}^{s}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(u)|^p\bigg)\,du\,ds \\
&+\sum_{i=1}^{n}\bigg(\frac{r}{c_i}\bigg)^{p/q}\bigg(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\bigg)^{p/q}\int_{T_2}^{t}e^{-c_i(t-s)}\int_{s-r(s)}^{s}\mathbb{E}\bigg(\sum_{j=1}^{n}|\varphi_j(u)|^p\bigg)\,du\,ds
\end{aligned}
$$

$$
\begin{aligned}
&< \sum_{i=1}^{n} re^{-c_i t}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\sup_{\vartheta\le u\le T_2}\left\{\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)|^p\right)\right\}\frac{(e^{c_iT_2}-1)}{c_i}\\
&\quad+\sum_{i=1}^{n}\frac{\varepsilon r}{c_i}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}.
\end{aligned}
$$

Further, from (6.19), we obtain

$$
\begin{aligned}
\mathbb{E}\sum_{i=1}^{n}|J_{5i}(t)|^p &\le n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\Bigg[\mu^{p/2}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds\\
&\quad+\nu^{p/2}\int_0^t e^{-c_i(t-s)}E\left(\sum_{j=1}^{n}|\varphi_j(s-\tau(s))|^p\right)ds\Bigg]\\
&< n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\Bigg\{\mu^{p/2}\sup_{0\le s\le T_2}\left[\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)\right]\\
&\quad+\nu^{p/2}\sup_{\vartheta\le s\le T_2}\left[E\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)\right]\Bigg\}\frac{e^{-c_it}\left(e^{c_iT_2}-1\right)}{c_i}\\
&\quad+n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\left(\frac{\varepsilon(\mu^{p/2}+\nu^{p/2})}{c_i}\right).
\end{aligned}
$$

Hence, let $t\to\infty$, from the fact that $c_i>0$ $(i=1,2,\cdots,n)$, we obtain that

$$
\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t)|^p\to 0,\quad \mathbb{E}\sum_{i=1}^{n}|J_{4i}(t)|^p\to 0,\quad\text{and}\quad \mathbb{E}\sum_{i=1}^{n}|J_{5i}(t)|^p\to 0.
$$

Thus, combining with (6.16), we obtain that $\mathbb{E}\sum_{i=1}^{n}|(Q\varphi)_i(t)|^p\to 0$ as $\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|^p\to 0$. Therefore, $Q:\mathcal{S}_\phi\to\mathcal{S}_\phi$.

Step4. We prove that Q is a contraction mapping. For any $\varphi,\psi\in\mathcal{S}_\phi$, from (6.17)-(6.19), we obtain

$$
\begin{aligned}
&\sup_{s\ge\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}|Q\varphi_i(s)-Q\psi_i(s)|^p\right\}\\
&\le 4^{p-1}\sup_{s\ge\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}a_{ij}\Big(f_j(x_j(u))-f_j(y_j(u))\Big)du\right|^p\right\}\\
&\quad+4^{p-1}\sup_{s\ge\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}b_{ij}\Big(g_j(x_j(u-\tau(u)))-g_j(y_j(u-\tau(u)))\Big)du\right|^p\right\}\\
&\quad+4^{p-1}\sup_{s\ge\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}\Big(h_j(\varphi_j(v))-h_j(\psi_j(v))\Big)dv\,du\right|^p\right\}
\end{aligned}
$$

$$+4^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}\Big(\sigma_{ij}(s,x_j(s),x_j(u-\tau(u)))\right.\right.$$
$$\left.\left.-\sigma_{ij}(s,y_j(s),y_j(s-\tau(u)))\Big)\,dw_j(u)\right|^p\right\}$$
$$\leq 4^{p-1}\left\{\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\right.$$
$$\left.+\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^p\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}+n^{p-1}\sum_{i=1}^{n}c_i^{-p/2}\big(\mu^{p/2}+\nu^{p/2}\big)\right\}$$
$$\times\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{j=1}^{n}|\varphi_j(s)-\psi_j(s)|^p\right\}=\alpha\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{j=1}^{n}|\varphi_j(s)-\psi_j(s)|^p\right\}.$$

From (6.4), we obtain that $Q:\mathcal{S}_\phi\to\mathcal{S}_\phi$ is a contraction mapping. □

We are now ready to prove Theorem 6.1.4.

Proof. From Lemma 6.1.17, by a contraction mapping principle, we obtain that Q has a unique fixed point $x(t)$, which is a solution of (6.1) with $x(t)=\phi(t)$ as $t\in[\vartheta,0]$ and $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p\to 0$ as $t\to\infty$.

Now, we prove that the trivial solution of (6.1) is pth moment stable. Let $\varepsilon>0$ be given and choose $\delta>0$ ($\delta<\varepsilon$) such that $5^{p-1}\delta<(1-\alpha)\varepsilon$.

If $x(t)=(x_1(t),x_2(t),\cdots,x_n(t))^T$ is a solution of (6.1) with the initial condition satisfying $\mathbb{E}\sum_{i=1}^{n}|\phi_i(t)|^p<\delta$, then $x(t)=(Qx)(t)$ defined in (6.15). We claim that $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p<\varepsilon$ for all $t\geq 0$. Notice that $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p<\varepsilon$ for $t\in[\vartheta,0]$, we suppose that there exists $t^*>0$ such that $\mathbb{E}\sum_{i=1}^{n}|x_i(t^*)|^p=\varepsilon$ and $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p<\varepsilon$ for $\vartheta\leq t<t^*$, then it follows from (6.4), we obtain that

$$\mathbb{E}\sum_{i=1}^{n}|x_i(t^*)|^p$$
$$\leq 5^{p-1}\mathbb{E}\sum_{i=1}^{n}e^{-pc_it^*}|x_i(0)|^p$$
$$+5^{p-1}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^{t^*}e^{-c_i(t^*-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s)|^p\right)ds$$
$$+5^{p-1}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^{t^*}e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s-\tau(s)))|^p\right)ds$$
$$+5^{p-1}\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^{t^*}e^{-c_i(t^*-s)}\int_{s-r(s)}^{s}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(u)|^p\right)du\,ds$$

$$
\begin{aligned}
&+5^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{1-p/2}\Bigg[\mu^{p/2}\int_0^{t^*} e^{-c_i(t^*-s)}\mathbb{E}\Bigg(\sum_{j=1}^{n}|x_j(s)|^p\Bigg)\,ds \\
&\quad +\nu^{p/2}\int_0^{t^*} e^{-c_i(t-s)}\mathbb{E}\Bigg(\sum_{j=1}^{n}|x_j(s-\tau(s))|^p\Bigg)\,ds\Bigg] \\
&\le \Bigg[5^{p-1}\sum_{i=1}^{n} c_i^{-p}\Bigg(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\Bigg)^{p/q} + 5^{p-1}\sum_{i=1}^{n} c_i^{-p}\Bigg(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\Bigg)^{p/q} \\
&\quad +5^{p-1}\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^p\Bigg(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\Bigg)^{p/q} + 5^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)\Bigg]\varepsilon + 5^{p-1}\delta \\
&< (1-\alpha)\varepsilon + \alpha\varepsilon = \varepsilon,
\end{aligned}
$$

which is a contradiction. Therefore, the trivial solution of (6.1) is asymptotically stable in *pth* moment. □

Corollary 6.1.18. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the function $r(t)$ is bounded by a constant r ($r > 0$),*

(ii) *and such that*

$$
\begin{aligned}
&5\sum_{i=1}^{n} c_i^{-2}\Bigg(\sum_{j=1}^{n} a_{ij}^2\alpha_j^2\Bigg) + 5\sum_{i=1}^{n} c_i^{-2}\Bigg(\sum_{j=1}^{n} b_{ij}^2\beta_j^2\Bigg) + 5\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^2\Bigg(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\Bigg) \\
&+20n\sum_{i=1}^{n} c_i^{-1}(\mu+\nu) < 1,
\end{aligned}
$$

where c, μ, ν are defined as in Theorem 6.1.4,

then the trivial solution of (6.1) *is asymptotically stable in mean square.*

Consider the stochastic neural networks without distributed delays

$$
\begin{aligned}
dx_i(t) \;=\; &\Bigg[-c_ix_i(t) + \sum_{j=1}^{n} a_{ij}f_j(x_j(t)) + \sum_{j=1}^{n} b_{ij}g_j(x_j(t-\tau(t)))\Bigg]\,dt \\
&+\sum_{j=1}^{n}\sigma_{ij}(t, x_j(t), x_j(t-\tau(t)))\,dw_j(t)
\end{aligned}
\tag{6.20}
$$

for $i = 1, 2, 3, \cdots, n$.

Corollary 6.1.19. *Suppose that the assumptions* (A1)-(A4) *hold. The trivial solution of* (6.20) *is asymptotically stable in pth moment if the following inequality holds,*

$$
\begin{aligned}
&4^{p-1}\sum_{i=1}^{n} c_i^{-p}\Bigg(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\Bigg)^{p/q} + 4^{p-1}\sum_{i=1}^{n} c_i^{-p}\Bigg(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\Bigg)^{p/q} \\
&+4^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right) < 1,
\end{aligned}
\tag{6.21}
$$

where μ, ν are defined as in Theorem 6.1.4. Note that the discrete delay $\tau(t)$ can be unbounded.

Remark 6.1.20. *Condition* (A4) *can be relaxed. In fact, if* $p = 2$*, then*

$$\text{(A4}') \quad \forall i, \quad \sum_{j=1}^{n} (\sigma_{ij}(t,x,y) - \sigma_{ij}(t,u,v))^2 \le \sum_{j=1}^{n} \mu_j (x_j - u_j)^2 + \nu_j (y_j - v_j)^2 \tag{6.22}$$

is sufficient, as can be easily observed from the proof of Theorem 6.1.4. If $p \ge 2$*, then* (A4) *can also be replaced by* (A4′)*, but the factor* n^{p-1} *in front of the last term in* (6.4) *has to be repalced by* $n^{(3p/2)-2}$*. This can be seen from the proof of Theorem 6.1.4 with the aid of a few more application of the Hölder inequality.*

6.1.3 Proof of Theorem 6.1.5

In this subsection, we prove Theorem 6.1.5. We start with some preparations.

Lemma 6.1.21. *Define an operator by* $(P\varphi)(t) = \phi(t)$ *for* $t \in [-\tau, 0]$*, and for* $t \ge 0$*,* $(P\varphi)(t)$ *is defined as the right hand side of* (6.15)*. If the conditions* (i) *and* (ii) *in Theorem 6.1.5 are satisfied, then* $P : \mathcal{C}_\phi \to \mathcal{C}_\phi$ *is a contraction mapping.*

Proof. Observe that all terms at the right hand side of (6.15) have continuous paths, almost surely. Now, we prove that $P(\mathcal{C}_\phi) \subseteq \mathcal{C}_\phi$.

$$\sum_{i=1}^{n} \mathbb{E}\left[\sup_{t-\tau\le s\le t} |(P\varphi)_i(s)|^p\right] = \sum_{i=1}^{n} \mathbb{E}\left[\sup_{t-\tau\le s\le t} \left|\sum_{j=1}^{5} J_{ji}(s)\right|^p\right] \le 5^{p-1} \sum_{j=1}^{5}\sum_{i=1}^{n} \mathbb{E}\left[\sup_{t-\tau\le s\le t} |J_{ji}(s)|^p\right].$$

Estimating the terms on the right-hand side of the above inequality. Let $c = \min\{c_1, c_2, c_3, \cdots, c_n\}$, and let q be such that $1/p + 1/q = 1$,

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^{n} \sup_{t-\tau\le s\le t} |J_{2i}(s)|^p\right] \\
&= \mathbb{E}\left[\sum_{i=1}^{n} \sup_{t-\tau\le s\le t} \left|\int_0^s e^{-c_i(s-u)} \sum_{j=1}^{n} a_{ij} f_j(\varphi_j(u))\, du\right|^p\right] \\
&\le c^{-p/q} \mathbb{E}\left\{\sum_{i=1}^{n} \sup_{t-\tau\le s\le t} \left[\int_0^s e^{-c(s-u)} \left(\sum_{j=1}^{n} |a_{ij}| \alpha_j |\varphi_j(u)|\right)^p du\right]\right\} \\
&\le c^{-p/q} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^q |\alpha_j|^q\right)^{p/q} \mathbb{E}\left\{\sup_{t-\tau\le s\le t} \left[\int_0^s e^{-c(s-u)} \left(\sum_{j=1}^{n} |\varphi_j(u)|^p\right) du\right]\right\} \\
&\le c^{-p/q} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^q |\alpha_j|^q\right)^{p/q} \sum_{j=1}^{n} \mathbb{E}\left\{\sup_{t-\tau\le s\le t} \left[\int_0^s e^{-c(s-u)} |\varphi_j(u)|^p\, du\right]\right\} \\
&\le c^{-p/q} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^q |\alpha_j|^q\right)^{p/q} \sum_{j=1}^{n} \mathbb{E}\left\{\sup_{t-\tau\le s\le t} \left[\int_0^s e^{-c(s-u)} \left(\sup_{u-\tau\le v\le u} |\varphi_j(v)|^p\right) du\right]\right\} \\
&\le e^{c\tau} c^{-p/q} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^q |\alpha_j|^q\right)^{p/q} \sum_{j=1}^{n} \mathbb{E}\left[\int_0^t e^{-c(t-u)} \left(\sup_{u-\tau\le v\le u} |\varphi_j(v)|^p\right) du\right]
\end{aligned} \tag{6.23}$$

Since $\sum_{j=1}^n \mathbb{E}\sup_{t-\tau\le s\le t}|\varphi_j(s)|^p \to 0$ as $t\to\infty$, then for any $\varepsilon>0$, there exists $T_1\ge 0$ such that $t\ge T_1$ implies

$$\sum_{j=1}^n \mathbb{E}\left(\sup_{t-\tau\le s\le t}|\varphi_j(s)|^p\right) < \varepsilon,$$

which yields that

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t e^{-c(t-u)}\left(\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\right)du\right]\\
&= \int_0^{T_1} e^{-c(t-u)}\mathbb{E}\left(\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\right)du + \int_{T_1}^t e^{-c(t-u)}\mathbb{E}\left(\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\right)du\\
&\le \int_0^{T_1} e^{-c(t-u)}\left(\sup_{\vartheta\le v\le T_1}|\varphi_j(v)|^p\right)du + \frac{\varepsilon}{c}.
\end{aligned}$$

Then combining with (6.23), we obatin that $\mathbb{E}\sum_{i=1}^n \sup_{t-\tau\le s\le t}|J_{2i}(s)|^p \to 0$ as $t\to\infty$. Similarly, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^n \sup_{t-\tau\le s\le t}|J_{3i}(s)|^p\right]\\
&\le c^{-p/q}\sum_{i=1}^n\left(\sum_{j=1}^n |b_{ij}|^q|\beta_j|^q\right)^{p/q}\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}\left(\sum_{j=1}^n|\varphi_j(u-\tau(u))|^p\right)du\right]\right\}\\
&\le c^{-p/q}\sum_{i=1}^n\left(\sum_{j=1}^n |b_{ij}|^q|\beta_j|^q\right)^{p/q}\sum_{j=1}^n\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}|\varphi_j(u-\tau(u))|^p\,du\right]\right\}\\
&\le c^{-p/q}\sum_{i=1}^n\left(\sum_{j=1}^n |b_{ij}|^q|\beta_j|^q\right)^{p/q}\sum_{j=1}^n\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\,du\right]\right\}\\
&\le e^{c\tau}c^{-p/q}\sum_{i=1}^n\left(\sum_{j=1}^n |b_{ij}|^q|\beta_j|^q\right)^{p/q}\sum_{j=1}^n\mathbb{E}\left[\int_0^t e^{-c(t-u)}\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\,du\right] \qquad (6.24)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^n \sup_{t-\tau\le s\le t}|J_{4i}(s)|^p\right]\\
&\le c^{-p/q}\sum_{i=1}^n\left(\sum_{j=1}^n |l_{ij}|^q|\gamma_j|^q\right)^{p/q}\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}\sum_{j=1}^n\left|\int_{u-r(u)}^u\varphi_j(v)\,dv\right|^p du\right]\right\}\\
&\le c^{-p/q}\sum_{i=1}^n\left(\sum_{j=1}^n |l_{ij}|^q|\gamma_j|^q\right)^{p/q}\sum_{j=1}^n\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}\left|\int_{u-r(u)}^u\varphi_j(v)\,dv\right|^p du\right]\right\}\\
&\le \tau^p c^{-p/q}\sum_{i=1}^n\left(\sum_{j=1}^n |l_{ij}|^q|\gamma_j|^q\right)^{p/q}\sum_{j=1}^n\mathbb{E}\left\{\sup_{t-\tau\le s\le t}\left[\int_0^s e^{-c(s-u)}\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\,du\right]\right\}\\
&\le \tau^p e^{c\tau}c^{-p/q}\sum_{i=1}^n\left(\sum_{j=1}^n |l_{ij}|^q|\gamma_j|^q\right)^{p/q}\sum_{j=1}^n\mathbb{E}\left[\int_0^t e^{-c(t-u)}\sup_{u-\tau\le v\le u}|\varphi_j(v)|^p\,du\right]. \qquad (6.25)
\end{aligned}$$

Let $\mu = \max\{\mu_1, \mu_2, \cdots, \mu_n\}$, $\nu = \max\{\nu_1, \nu_2, \cdots, \nu_n\}$. Due to the fact that

$$\left| \int_0^s e^{-c_i(s-u)} \sigma_{ij}(u, \varphi_j(u), \varphi_j(u-\tau(u)))\, dw_j(u) \right|^p$$

is a submartingale and the supremum of submartingale is also a submartingale, using Doob's inequality for positive submartingale, we obtain that

$$\begin{aligned}
&\mathbb{E}\left[\sum_{i=1}^n \sup_{t-\tau\le s\le t} |J_{5i}(s)|^p\right] \\
&\le n^{p-1} \sum_{i=1}^n \sum_{j=1}^n \mathbb{E}\left[\sup_{t-\tau\le s\le t} \left| \int_0^s e^{-c_i(s-u)} \sigma_{ij}(u, \varphi_j(u), \varphi_j(u-\tau(u)))\, dw_j(u) \right|^p\right] \\
&\le n^{p-1} \sum_{i=1}^n \sum_{j=1}^n \mathbb{E}\left\{\sup_{t-\tau\le s\le t} \left[\sup_{t-\tau\le r\le t} \left| \int_0^s e^{-c(r-u)} \sigma_{ij}(u, \varphi_j(u), \varphi_j(u-\tau(u)))\, dw_j(u) \right|^p\right]\right\} \\
&\le n^{p-1} \sum_{i=1}^n \sum_{j=1}^n q^p \sup_{t-\tau\le s\le t} \left\{\mathbb{E}\left[\sup_{t-\tau\le r\le t} \left| \int_0^s e^{-c(r-u)} \sigma_{ij}(u, \varphi_j(u), \varphi_j(u-\tau(u)))\, dw_j(u) \right|^p\right]\right\} \\
&\le n^{p-1} e^{pc\tau} \sum_{i=1}^n \sum_{j=1}^n q^p \sup_{t-\tau\le s\le t} \left[\mathbb{E}\left| \int_0^s e^{-c(t-u)} \sigma_{ij}(u, \varphi_j(u), \varphi_j(u-\tau(u)))\, dw_j(u) \right|^p\right] \\
&\le K_p n^{p-1} e^{pc\tau} \sum_{i=1}^n \sum_{j=1}^n q^p \sup_{t-\tau\le s\le t} \left[\mathbb{E}\left(\int_0^s e^{-2c(t-u)} \sigma_{ij}^2(u, \varphi_j(u), \varphi_j(u-\tau(u)))\, du \right)^{p/2}\right] \\
&\le K_p n^{p-1} e^{pc\tau} \sum_{i=1}^n \sum_{j=1}^n q^p 2^{p/2-1} \sup_{t-\tau\le s\le t} \mathbb{E}\left[\left(\int_0^s e^{-2c(t-u)} \left(\mu_j \varphi_j^2(u)\right) du \right)^{p/2}\right] \\
&\quad + K_p n^{p-1} e^{pc\tau} \sum_{i=1}^n \sum_{j=1}^n q^p 2^{p/2-1} \sup_{t-\tau\le s\le t} \left[\mathbb{E}\left(\int_0^s e^{-2c(t-u)} \left(\nu_j \varphi_j^2(u-\tau(u))\right) du \right)^{p/2}\right] \\
&\le K_p n^{p-1} e^{pc\tau} \sum_{i=1}^n \sum_{j=1}^n q^p 2^{p/2-1} \sup_{t-\tau\le s\le t} \left\{\mathbb{E}\left[\left(\int_0^s e^{-2c(t-u)}\, du \right)^{p/2-1}\right.\right. \\
&\qquad \left.\left.\times \left(\int_0^s e^{-2c(t-u)} \mu_j^{p/2} |\varphi_j(u)|^p\, du + \int_0^s e^{-2c(t-u)} \nu_j^{p/2} |\varphi_j(u-\tau(u))|^p\, du \right)\right]\right\} \\
&\le K_p n^p e^{pc\tau} q^p c^{1-p/2} (\mu^{p/2} + \nu^{p/2}) \int_0^t e^{-2c(t-u)} \sum_{j=1}^n \mathbb{E}\left[\sup_{u-\tau\le v\le u} |\varphi_j(v)|^p\, du\right].
\end{aligned} \tag{6.26}$$

Using the similar arguments as for the term (6.23) and combining with (6.24), (6.25) and (6.26), we obtain that $\sum_{i=1}^n \mathbb{E}\left[\sup_{t-\tau\le s\le t} |(P\varphi)_i(s)|^p\right] \to 0$ as $t \to \infty$. Thus, $P(\mathcal{C}_\phi) \subseteq \mathcal{C}_\phi$.

Finally, we prove that P is a contraction mapping. For any $\varphi, \psi \in \mathcal{C}_\phi$, from (6.23)-(6.26), we obtain that

$$\begin{aligned}
&\sup_{t\geq 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}|(P\varphi)_i(s)-(P\psi)_i(s)|^p\right]\right\}\\
&\leq 4^{p-1}\sup_{t\geq 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}a_{ij}\Big(f_j(\varphi_j(u))-f_j(\psi_j(u))\Big)\,du\right|^p\right]\right\}\\
&\quad+4^{p-1}\sup_{t\geq 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}b_{ij}\right.\right.\right.\\
&\qquad\left.\left.\left.\times\Big(g_j(\varphi_j(u-\tau(u)))-g_j(\psi_j(u-\tau(u)))\Big)\,du\right|^p\right]\right\}\\
&\quad+4^{p-1}\sup_{t\geq 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}\Big(h_j(\varphi_j(v))-h_j(\psi_j(v))\Big)\,dv\,du\right|^p\right]\right\}\\
&\quad+4^{p-1}\sup_{t\geq 0}\left\{\mathbb{E}\left[\sum_{i=1}^{n}\sup_{t-\tau\leq s\leq t}\left|\int_0^s e^{-c_i(s-u)}\right.\right.\right.\\
&\qquad\left.\left.\left.\times\sum_{j=1}^{n}[\sigma_{ij}(u,\varphi_j(u),\varphi_j(u-\tau(u)))-\sigma_{ij}(u,\psi_j(u),\psi_j(u-\tau(u)))]\,dw_j(u)\right|^p\right]\right\}\\
&\leq 4^{p-1}\Bigg\{e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\
&\quad+\tau^p e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}+K_p n^p e^{pc\tau}q^p c^{1-p/2}(2c)^{-1}\Big(\mu^{p/2}+\nu^{p/2}\Big)\Bigg\}\\
&\qquad\times\sup_{t\geq 0}\sum_{j=1}^{n}\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|\varphi_j(s)-\psi_j(s)|^p\right]=\alpha\sup_{t\geq 0}\sum_{j=1}^{n}\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|\varphi_j(s)-\psi_j(s)|^p\right].
\end{aligned}$$

From (6.6), we obtain that $P:\mathcal{C}_\phi\to\mathcal{C}_\phi$ is a contraction mapping. □

We are now ready to prove Theorem 6.1.5

Proof. From Lemma 6.1.21, by a contraction mapping principle, we obtain that P has a unique fixed point $x(t)$, which is a solution of (6.1) with $x(t)=\phi(t)$ as $t\in[\vartheta,0]$ and $\sum_{i=1}^{n}\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|x_i(s)|^p\right]\to 0$ as $t\to\infty$.

We prove that the trivial solution of (6.1) is *pth* moment stable. Let $\varepsilon>0$ be given, we suppose that there exists $t^*>0$ such that

$$\begin{aligned}
\sum_{i=1}^{n}\mathbb{E}\left[\sup_{t^*-\tau\leq s\leq t^*}|x_i(s)|^p\right] &= \varepsilon,\\
\sum_{i=1}^{n}\mathbb{E}\left[\sup_{t-\tau\leq s\leq t}|x_i(s)|^p\right] &< \varepsilon \quad \text{for} \quad \vartheta\leq \mathrm{t}<\mathrm{t}^*,
\end{aligned}$$

choose $\delta>0$ ($\delta<\varepsilon$) satisfying

$$5^{p-1}e^{-pct^*}\delta<(1-\alpha)\varepsilon. \tag{6.27}$$

If $x(t) = (x_1(t), x_2(t), \cdots x_n(t))^T$ is a solution of (6.1) with the initial condtion satisfying $\|\phi\|^p < \delta$, then $x(t) = (Px)(t)$ defined in (6.15). We claim that $\|x\|^p < \varepsilon$ for all $t \geq 0$. It follows from (6.4) and (6.27), we obtain that

$$\begin{aligned}
&\sum_{i=1}^{n} \mathbb{E}\left[\sup_{t^*-\tau\leq s\leq t^*} |x_i(s)|^p\right] \\
&\leq 5^{p-1}\sum_{j=1}^{5}\sum_{i=1}^{n} \mathbb{E}\left[\sup_{t^*-\tau\leq s\leq t^*} |J_{ji}(s)|^p\right] \\
&\leq 5^{p-1}e^{-pct^*}\delta + 5^{p-1}\Bigg\{e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q} \\
&\qquad +\tau^p e^{c\tau}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q} + K_p n^p e^{pc\tau} q^p c^{1-p/2}(2c)^{-1}\left(\mu^{p/2}+\nu^{p/2}\right)\Bigg\}\varepsilon \\
&< (1-\alpha)\varepsilon + \alpha\varepsilon = \varepsilon,
\end{aligned}$$

which is a contradiction. Thus, the proof follows. □

6.1.4 Proof of Theorem 6.1.7

In this subsection, we prove Theorem 6.1.7. We start with a lemma presenting an integral inequality lemma.

Lemma 6.1.22. *Consider $c, \tau > 0$, positive constants $\lambda_1, \lambda_2, \lambda_3$ and a function $y : [-\tau, \infty) \to [0, \infty)$. If $\lambda_1 + \lambda_2 + \tau\lambda_3 < c$ and the following inequality holds,*

$$y(t) \leq \begin{cases} y_0 e^{-ct} + \lambda_1 \int_0^t e^{-c(t-s)} y(s)\, ds + \lambda_2 \int_0^t e^{-c(t-s)} y(s-\tau(s))\, ds \\ \qquad +\lambda_3 \int_0^t e^{-c(t-s)} \int_{s-r(s)}^{s} y(u)\, du\, ds \qquad t \geq 0, \\ y_0 e^{-ct}, \quad t \in [-\tau, 0], \end{cases} \tag{6.28}$$

then we have $y(t) \leq y_0 e^{-\gamma t} (t \geq -\tau)$, where γ is a positive root of the transcendental equation $\frac{1}{c-\gamma}\left(\lambda_1 + e^{\gamma\tau}\lambda_2 + \frac{e^{\gamma\tau}-1}{\gamma}\lambda_3\right) = 1$.

Proof. Let $F(\gamma) = \frac{1}{c-\gamma}\left(\lambda_1 + e^{\gamma\tau}\lambda_2 + \frac{e^{\gamma\tau}-1}{\gamma}\lambda_3\right) - 1$. We have $F(0)F(c^-) < 0$, that is, there exists a positive constant $\gamma \in (0, c)$ such that $F(\gamma) = 0$. For any $\varepsilon > 0$, let

$$C_\varepsilon = \varepsilon + y_0.$$

To prove the lemma, we claim that (6.28) implies

$$y(t) \leq C_\varepsilon e^{-\gamma t}, \quad t \geq -\tau. \tag{6.29}$$

It is easily shown that (6.29) holds for $t \in [-\tau, 0]$. Assume that there exists $t_1^* > 0$ such that

$$y(t) < C_\varepsilon e^{-\gamma t}, \quad t \in [-\tau, t_1^*), \qquad y(t_1^*) = C_\varepsilon e^{-\gamma t_1^*}. \tag{6.30}$$

Combining with (6.28), we have

$$\begin{aligned} y(t_1^*) &\leq y_0 e^{-ct_1^*} + \lambda_1 \int_0^{t_1^*} e^{-c(t_1^*-s)} y(s)\,ds + \lambda_2 \int_0^{t_1^*} e^{-c(t_1^*-s)} y(s-\tau(s))\,ds \\ &\quad + \lambda_3 \int_0^{t_1^*} e^{-c(t_1^*-s)} \int_{s-r(s)}^{s} y(u)\,du\,ds \\ &< y_0 e^{-ct_1^*} + C_\varepsilon \lambda_1 \int_0^{t_1^*} e^{-c(t_1^*-s)} e^{-\gamma s}\,ds + C_\varepsilon \lambda_2 \int_0^{t_1^*} e^{-c(t_1^*-s)} e^{-\gamma(s-\tau(s))}\,ds \\ &\quad + C_\varepsilon \lambda_3 \int_0^{t_1^*} e^{-c(t_1^*-s)} \int_{s-r(s)}^{s} e^{-\gamma u}\,du\,ds \\ &= \left[y_0 - \frac{C_\varepsilon}{c-\gamma} \left(\lambda_1 + e^{\gamma\tau}\lambda_2 + \frac{e^{\gamma\tau}-1}{\gamma}\lambda_3 \right) \right] e^{-ct_1^*} \\ &\quad + \frac{C_\varepsilon}{c-\gamma} \left(\lambda_1 + e^{\gamma\tau}\lambda_2 + \frac{e^{\gamma\tau}-1}{\gamma}\lambda_3 \right) e^{-\gamma t_1^*}. \end{aligned}$$

From the definition of C_ε, we have

$$y_0 - \frac{C_\varepsilon}{c-\gamma} \left(\lambda_1 + e^{\gamma\tau}\lambda_2 + \frac{e^{\gamma\tau}-1}{\gamma}\lambda_3 \right) = y_0 - C_\varepsilon < 0.$$

Then, together with the definition of γ, we obtain that $y(t_1^*) < C_\varepsilon e^{-\gamma t_1^*}$, which contradicts (6.30), so (6.29) holds. As $\varepsilon > 0$ is arbitrarily small, in view of (6.29), it follows that $y(t) \leq y_0 e^{-\gamma t}$ for $t \geq -\tau$. □

Proof. For the representation (6.14), using (6.17)-(6.19), we obtain that

$$\begin{aligned} &E \sum_{i=1}^{n} |x_i(t)|^p \\ &\leq 5^{p-1} e^{-ct} \sum_{i=1}^{n} \mathbb{E}|\phi_i(0)|^p \\ &\quad + 5^{p-1} c^{-p/q} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |a_{ij}|^q |\alpha_j|^q \right)^{p/q} \int_0^t e^{-c(t-s)} \mathbb{E} \left(\sum_{j=1}^{n} |x_j(s)|^p \right) ds \\ &\quad + 5^{p-1} c^{-p/q} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |b_{ij}|^q |\beta_j|^q \right)^{p/q} \int_0^t e^{-c(t-s)} \mathbb{E} \left[\sum_{j=1}^{n} |x_j(s-\tau(s)))|^p \right] ds \\ &\quad + 5^{p-1} \left(\frac{\tau}{c} \right)^{p/q} \sum_{i=1}^{n} \left(\sum_{j=1}^{n} |l_{ij}|^q |\gamma_j|^q \right)^{p/q} \int_0^t e^{-c(t-s)} \int_{s-r(s)}^{s} \mathbb{E} \left[\sum_{j=1}^{n} |x_j(u)|^p \right] du\,ds \\ &\quad + 5^{p-1} n^p c^{1-p/2} \Bigg\{ \mu^{p/2} \int_0^t e^{-c(t-s)} \mathbb{E} \left[\sum_{j=1}^{n} |x_j(s)|^p \right] ds \\ &\qquad + \nu^{p/2} \int_0^t e^{-c(t-s)} \mathbb{E} \left[\sum_{j=1}^{n} |x_j(s-\tau(s))|^p \right] ds \Bigg\}. \end{aligned}$$

Hence, by using Lemma 6.1.22 and (6.7), we obtain that the trivial solution of (6.1) is exponentially stable in *pth* moment. □

Corollary 6.1.23. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ ($\tau > 0$),*

(ii) *and such that*

$$5c^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}^2\alpha_j^2 + 5c^{-2}\sum_{i=1}^{n}\sum_{j=1}^{n} b_{ij}^2\beta_j^2 + 5c^{-2}\tau^2\sum_{i=1}^{n}\sum_{j=1}^{n} l_{ij}^2\gamma_j^2 + 20n^2c^{-1}(\mu+\nu) < 1,$$

where c, μ, ν are defined as in Theorem 6.1.4,

then the trivial solution of (6.1) *is exponentially stable in mean square.*

Corollary 6.1.24. *Let $p \geq 2$. Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ ($\tau > 0$),*

(ii) *and such that*

$$\begin{aligned} &4^{p-1}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 4^{p-1}c^{-p}\sum_{i=1}^{n}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q} \\ &+4^{p-1}n^pc^{-p/2}(\mu^{p/2}+\nu^{p/2}) < 1, \end{aligned}$$

where c, μ, ν are defined as in Theorem 6.1.4,

then the trivial solution of (6.20) *is exponentially stable in pth moment.*

6.1.5 Proof of Theorem 6.1.11

In this subsection, we prove Theorem 6.1.11. We start with some preparations.

Multiply both sides of (6.9) by $e^{c_i t}$ and integrate from 0 to t, we obtain that for $t \geq 0$,

$$\begin{aligned} x_i(t) &= e^{-c_i t}x_i(0) + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} a_{ij}g_j(x_j(s))\,ds + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} b_{ij}g_j(x_j(s-\tau(s)))\,ds \\ &\quad + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} d_{ij}\int_{s-r(s)}^{s} g_j(x_j(u))\,du\,ds, \quad i = 1,2,3,\cdots,n. \end{aligned} \tag{6.31}$$

Lemma 6.1.25. *Define an operator by $(Px)(\theta) = \phi(\theta)$ for $\vartheta \leq \theta \leq 0$, and for $t \geq 0$,*

$$\begin{aligned} (Px)_i(t) &= e^{-c_i t}x_i(0) + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} a_{ij}g_j(x_j(s))\,ds \\ &\quad + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} b_{ij}g_j(x_j(s-\tau(s)))\,ds \\ &\quad + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} d_{ij}\int_{s-r(s)}^{s} g_j(x_j(u))\,du\,ds := \sum_{i=1}^{4} I_i(t). \end{aligned} \tag{6.32}$$

If the conditions (i) *and* (i) *in Theorem 6.1.11 are satisfied, then $P : \mathcal{H}_\phi \to \mathcal{H}_\phi$ and P is a contraction mapping.*

Proof. First, we prove that $P\mathcal{H}_\phi \subseteq \mathcal{H}_\phi$. In view of (6.32), we have that, for fixed time $t_1 \geq 0$, it is easy to check that $\lim_{r\to 0}[(Px)_i(t_1+r)-(Px)_i(t_1)] = 0$. Thus, P is continuous on $[0,\infty)$. Note that $(Px)_i(\theta) = \phi(\theta)$ for $\vartheta \leq \theta \leq 0$, we obtain that P is indeed continuous on $[\vartheta,\infty)$.

Next, we prove that $\lim_{t\to\infty}(Px)_i(t) = 0$ for $x_i(t) \in \mathcal{H}_{i\phi}$. Since $x_i(t) \in \mathcal{H}_{i\phi}$, we have that $\lim_{t\to\infty} x_i(t) = 0$. Then for any $\varepsilon > 0$, there exists $T_i > 0$ such that $s \geq T_i$ implies $|x_i(s)| < \varepsilon$. Choose $T = \max_{i=1,2,\cdots,n}\{T_i\}$, combining with condition (A2),

$$\begin{aligned}
|I_2(t)| &= \left| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n a_{ij} f_j(x_j(s))\, ds \right| \\
&\leq \int_0^T e^{-c_i(t-s)} \sum_{j=1}^n |a_{ij}k_j||x_j(s)|\, ds + \int_T^t e^{-c_i(t-s)} \sum_{j=1}^n |a_{ij}\alpha_j||x_j(s)|\, ds \\
&\leq \sum_{j=1}^n |a_{ij}\alpha_j| \sup_{0\leq s\leq T} |x_j(s)| \int_0^T e^{-c_i(t-s)}\, ds + \varepsilon \sum_{j=1}^n |a_{ij}\alpha_j| \int_T^t e^{-c_i(t-s)}\, ds \\
&\leq e^{-c_i t} \sum_{j=1}^n |a_{ij}\alpha_j| \sup_{0\leq s\leq T} |x_j(s)| \int_0^T e^{-c_i s}\, ds + \frac{\varepsilon}{c_i} \sum_{j=1}^n |a_{ij}\alpha_j|. \qquad (6.33)
\end{aligned}$$

From the fact that $c_i > 0$ $(i = 1,2,\cdots,n)$ and estimate (6.33), we have that $I_2(t) \to 0$ as $t \to \infty$.

Since $x_i(t) \to 0$ and $t - \tau(t) \to \infty$ as $t \to \infty$, for each $\varepsilon > 0$, there exists $T_i' > 0$ such that $s \geq T_i'$ implies $|x_i(s-\tau(s))| < \varepsilon$ for $i = 1,2,\cdots,n$. Choose $T' = \max_{i=1,2,\cdots n}\{T_i'\}$, we obtain

$$\begin{aligned}
|I_3(t)| &= \left| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n b_{ij} g_j(x_j(s-\tau(s)))\, ds \right| \\
&\leq \int_0^{T'} e^{-c_i(t-s)} \sum_{j=1}^n |b_{ij}\beta_j||x_j(s-\tau(s))|\, ds + \int_{T'}^t e^{-c_i(t-s)} \sum_{j=1}^n |b_{ij}k_j||x_j(s-\tau(s))|\, ds \\
&\leq e^{-c_i t} \sum_{j=1}^n |b_{ij}\beta_j| \sup_{\vartheta\leq s\leq T'} |x_j(s)| \int_0^{T'} e^{c_i s}\, ds + \frac{\varepsilon}{c_i} \sum_{j=1}^n |b_{ij}\beta_j|. \qquad (6.34)
\end{aligned}$$

From the fact that $c_i > 0$ $(i = 1,2,\cdots,n)$ and estimate (6.34), we have that $I_3(t) \to 0$ as $t \to \infty$.

Since $x_i(t) \to 0$ and $t - r(t) \to \infty$ as $t \to \infty$, for each $\varepsilon > 0$, there exists $T_i^* > 0$ such that $s \geq T_i^*$ implies $|x_i(s-r(s))| < \varepsilon$ for $i = 1,2,\cdots,n$. Choose $T^* = \max_{i=1,2,\cdots n}\{T_i^*\}$, we obtain

$$\begin{aligned}
|I_4(t)| &= \left| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n d_{ij} \int_{s-r(s)}^s h_j(x_j(u))\, du\, ds \right| \\
&\leq \int_0^{T^*} e^{-c_i(t-s)} \sum_{j=1}^n |d_{ij}\gamma_j| \int_{s-r(s)}^s |x_j(u)|\, du\, ds + \varepsilon r \int_{T^*}^t e^{-c_i(t-s)} \sum_{j=1}^n |d_{ij}\gamma_j|\, ds \\
&\leq r \sum_{j=1}^n |d_{ij}\gamma_j| \sup_{\vartheta\leq u\leq T^*} |x_j(u)| \int_0^{T^*} e^{-c_i(t-s)}\, ds + \frac{\varepsilon r}{c_i} \sum_{j=1}^n |d_{ij}\gamma_j|. \qquad (6.35)
\end{aligned}$$

From the fact that $c_i > 0 (i = 1, 2, \cdots, n)$ and estimate (6.35), we have that $I_4(t) \to 0$ as $t \to \infty$. From the above estimate, we conclude that $\lim_{t\to\infty}(Px)_i)(t) = 0$ for $x_i(t) \in \mathcal{H}_{i\phi}$. Therefore, $P : \mathcal{H}_\phi \to \mathcal{H}_\phi$.

Now, we prove that P is a contraction mapping. For any $x, y \in \mathcal{H}_\phi$, from (6.33) and (6.35), we obtain that

$$\begin{aligned}
&\sum_{i=1}^{n} |(Px)_i(t) - (Py)_i(t)| \\
&\quad \le \sum_{i=1}^{n} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} |x_j(s) - y_j(s)|\, ds \\
&\qquad + \sum_{i=1}^{n} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} |x_j(s-\tau(s)) - y_j(s-\tau(s))|\, ds \\
&\qquad + \sum_{i=1}^{n} \max_{j=1,2,\cdots,n} |d_{ij}\gamma_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} \int_{s-r(s)}^{s} |x_j(u) - y_j(u)|\, du\, ds \\
&\quad \le \sum_{i=1}^{n} \left\{ \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \frac{r}{c_i} \max_{j=1,2,\cdots,n} |d_{ij}\gamma_j| \right\} \\
&\qquad \times \sup_{\vartheta \le s \le t} \sum_{j=1}^{n} |x_j(s) - y_j(s)| = \alpha \sup_{\vartheta \le s \le t} \sum_{j=1}^{n} |x_j(s) - y_j(s)|.
\end{aligned}$$

Hence, we obtain that P is a contraction mapping. □

We are now ready to prove Theorem 6.1.11.

Proof. Let P be defined as in Lemma 6.1.25, by a contraction mapping principle, P has a unique fixed point $x \in \mathcal{H}_\phi$ with $x(\theta) = \phi(\theta)$ on $\vartheta \le \theta \le 0$ and $x(t) \to 0$ as $t \to \infty$.

To obtain asymptotic stability, it remains to prove that the trivial solution $x = 0$ of (6.9) is stable. For any $\varepsilon > 0$, choose $\sigma > 0$ and $\sigma < \varepsilon$ satisfying the condition $\sigma + \varepsilon\alpha < \varepsilon$.

If $x(t, s, \phi) = (x_1(t, s, \phi), x_2(t, s, \phi), \cdots, x_n(t, s, \phi))$ is the solution of (6.9) with the initial condition $\|\phi\| < \sigma$, then we claim that $\|x(t, s, \phi)\| < \varepsilon$ for all $t \ge 0$. Indeed, we suppose that there exists $t^* > 0$ such that

$$\sum_{i=1}^{n} |x_i(t^*; s, \phi)| = \varepsilon, \qquad \text{and} \qquad \sum_{i=1}^{n} |x_i(t; s, \phi)| < \varepsilon \quad \text{for} \quad 0 \le t < t^*. \tag{6.36}$$

From (6.12) and (6.31), we obtain

$$\begin{aligned}
\sum_{i=1}^{n} |x_i(t^*; s, \phi)| \quad \le \quad & \sum_{i=1}^{n} \Bigg[|e^{-c_i t^*} x_i(0)| + \int_0^{t^*} e^{-c_i(t^*-s)} \sum_{j=1}^{n} |a_{ij} f_j(x_j(s))|\, ds \\
& \quad + \int_0^{t^*} e^{-c_i(t^*-s)} \sum_{j=1}^{n} |b_{ij} g_j(x_j(s-\tau(s)))|\, ds
\end{aligned}$$

$$
\begin{aligned}
& + \int_0^{t^*} e^{-c_i(t^*-s)} \sum_{j=1}^n |d_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))|\, du\, ds \Big] \\
< \ & \sigma + \varepsilon \sum_{i=1}^n \left(\frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \frac{r}{c_i} \max_{j=1,2,\cdots,n} |d_{ij}\gamma_j| \right) \\
\leq \ & \sigma + \varepsilon\alpha < \varepsilon,
\end{aligned}
$$

which contradicts (6.36). Therefore, $\|x(t,s,\phi)\| < \varepsilon$ for all $t \geq 0$. This completes the proof. □

Let $d_{ij} \equiv 0$ for $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$. The system (6.9) is then reduced to

$$
\frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} g_j(x_j(t - \tau(t))), \tag{6.37}
$$

which is the description of a cellular neural network with time-varying delays. Following the result of Theorem 6.1.11, we have the following corollary.

Corollary 6.1.26. *Suppose that the assumptions* (A1)-(A3) *hold. If the following condition is satisfied,*

$$
\sum_{i=1}^n \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^n \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| < 1, \tag{6.38}
$$

then the trivial solution of (6.37) *is asymptotically stable.*

Remark 6.1.27. *Note that the delay in Corollary* 6.1.26 *can be unbounded. Lai and Zhang* [74] *studied the asymptotic stability* (6.37) *as well. However, the additional condition*

$$
\max_{i=1,2,\cdots n} \left[\frac{1}{c_i} \sum_{j=1}^n |a_{ij}k_j| + \frac{1}{c_i} \sum_{j=1}^n |b_{ij}k_j| \right] < \frac{1}{\sqrt{n}} \tag{6.39}
$$

is needed in Theorem 4.1 *of* [74]*. It is clear that Corollary* 6.1.26 *is an improvement of the result in* [74].

6.1.6 Proof of Theorem 6.1.13

Proof. From the represention (6.31), we obtain that

$$
\begin{aligned}
\sum_{i=1}^n |x_i(t)| \ \leq \ & e^{-ct} \sum_{i=1}^n |x_i(0)| + \sum_{i=1}^n \max_{j=1,2,\cdots n} \{|a_{ij}k_j|\} \int_0^t e^{-c(t-s)} \sum_{j=1}^n |x_j(s)|\, ds \\
& + \sum_{i=1}^n \max_{j=1,2,\cdots n} \{|b_{ij}k_j|\} \int_0^t e^{-c(t-s)} \sum_{j=1}^n |x_j(s - \tau(s))|\, ds \\
& + \sum_{i=1}^n \max_{j=1,2,\cdots n} \{|d_{ij}k_j|\} \int_0^t e^{-c(t-s)} \sum_{j=1}^n \int_{s-r(s)}^{s} |x_j(u)|\, du\, ds.
\end{aligned}
$$

Combining with Lemma 6.1.22, we obtain that the trivial solution of (6.9) with initial condition (6.11) is exponentially stable. □

For the cellular neural network (6.37), we have the following result.

Corollary 6.1.28. *Suppose that the assumptions* (A1)-(A3) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ,*

(ii) *and such that*

$$\sum_{i=1}^{n} \max_{j=1,2,\cdots,n} |a_{ij}k_j| + \sum_{i=1}^{n} \max_{j=1,2,\cdots,n} |b_{ij}k_j| < c, \quad c = \min\{c_1, c_2, \cdots c_1\},$$

then the trivial solution of (6.37) *with initial condition* (6.11) *is exponentially stable.*

6.1.7 Examples

Example 6.1.29. *Consider the following two-dimensional cellular neural network*

$$\frac{dx(t)}{dt} = -Cx(t) + Ag(x(t)) + Bg(x - \tau(t)),$$

where

$$C = \begin{pmatrix} c_1 & 0 \\ 0 & c_2 \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}, \quad A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 6/7 & 3/7 \\ -1/7 & -1/7 \end{pmatrix}$$

$$B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 6/7 & 2/7 \\ 3/7 & 1/7 \end{pmatrix}.$$

The activation function is described by $g_i(x) = \frac{|x+1|-|x-1|}{2}$ for $i = 1, 2$. The time-varying delay $\tau(t)$ is continuous and $|\tau(t)| \leq \tau$, where τ is a constant.

It is clear that $\alpha_i = \beta_i = 1$ for $i = 1, 2$. We check the condition (6.38) in Corollary 6.1.26,

$$\sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |a_{ij}\alpha_j| + \sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |b_{ij}\beta_j| \leq \frac{1}{3} \times \left(\frac{6}{7} + \frac{1}{7} + \frac{6}{7} + \frac{3}{7}\right) = \frac{16}{21} < 1.$$

Hence, by Corollary 6.1.26, the trivial solution $x = 0$ of this cellular neural network is asymptotically stable.

However, the condition (6.39) becomes

$$\max_{i=1,2} \left\{ \frac{1}{c_i} \sum_{j=1}^{2} |a_{ij}\alpha_j| + \frac{1}{c_i} \sum_{j=1}^{2} |b_{ij}\beta_j| \right\} = \frac{17}{21} > \frac{1}{\sqrt{2}}.$$

Hence, Theorem 4.1 of [74] is not applicable.

Example 6.1.30. *Consider a two-dimensional stochastic recurrent neural network with time-varying delays*

$$\begin{aligned} dx(t) &= -\begin{pmatrix} 6 & 0 \\ 0 & 5 \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} dt + \begin{pmatrix} 2 & 0.4 \\ 0.6 & 1 \end{pmatrix}\begin{pmatrix} 0.2\tanh(x_1(t)) \\ 0.2\tanh(x_2(t)) \end{pmatrix} dt \\ &\quad + \begin{pmatrix} -0.8 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0.2\tanh(x_1(t-\tau_1(t))) \\ 0.2\tanh(x_2(t-\tau_2(t))) \end{pmatrix} dt \\ &\quad + \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} \int_{t-r(t)}^{t} 0.2\tanh(x_1(s))\,ds \\ \int_{t-r(t)}^{t} 0.2\tanh(x_2(s))\,ds) \end{pmatrix} dt \\ &\quad + \sigma(t, x(t), x(t-\tau(t)))\,dw(t), \end{aligned} \tag{6.40}$$

where $\tau_1(t), \tau_2(t), r(t)$ are continuous functions such that $t-\tau(t) \to \infty$ as $t \to \infty$ and $|r(t)| \le 1$, $\sigma : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ satisfies

$$trace\left[\sigma^T(t,x,y)\sigma(t,x,y)\right] \le 0.003(x_1^2 + x_2^2 + y_1^2 + y_2^2),$$

and $w(t)$ is a two dimensional Brownian motion.

We suppose $p = 2$, and take $\mu_i = \nu_i = 0.003$ for $i = 1, 2$, by simple computation, we have $\alpha_i = 0.2$ for $i = 1, 2$, $c = \min\{c_1, c_2\} = 5$, $\mu = \nu = 0.003$. From Corollary 6.1.18, we have that

$$\begin{aligned} &5\sum_{i=1}^{2} c_i^{-2}\left(\sum_{j=1}^{2} a_{ij}^2\alpha_j^2\right) + 5\sum_{i=1}^{2} c_i^{-2}\left(\sum_{j=1}^{2} b_{ij}^2\alpha_j^2\right) + 5\sum_{i=1}^{2}\left(\frac{\tau}{c_i}\right)^2\left(\sum_{j=1}^{2} l_{ij}^2\alpha_j^2\right) \\ &+20 \times 2 \times \sum_{i=1}^{2} c_i^{-1}(\mu+\nu) < 0.256 < 1. \end{aligned}$$

Then the trivial solution of (6.40) is mean square asymptotically stable.

If $\tau(t)$ is bounded, from Corollary 6.1.23, we obtain that

$$5c^{-2}\sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}^2\alpha_j^2 + 5c^{-2}\sum_{i=1}^{2}\sum_{j=1}^{n} b_{ij}^2\alpha_j^2 + 5c^{-2}\tau^2\sum_{i=1}^{2}\sum_{j=1}^{2} l_{ij}^2\alpha_j^2 + 20 \times 4c^{-1}(\mu+\nu) < 0.298.$$

Hence, the trivial solution of (6.40) is mean square exponentially stable.

Example 6.1.31. *Consider a two-dimensional stochastically perturbed HNN with time-varying delays,*

$$dx(t) = [-Cx(t) + Af(x(t)) + Bg(x_\tau(t))]\,dt + \sigma(t, x(t), x_\tau(t))\,dw(t), \tag{6.41}$$

where $f_i(x) = \frac{1}{5}\arctan x$, $g_i(x) = \frac{1}{5}\tanh x = \frac{1}{5}(e^x - e^{-x})/(e^{x_i} + e^{-x})$, $i = 1, 2$, $\tau(t) = \frac{1}{2}\sin t + \frac{1}{2}$,

$$C = \begin{pmatrix} 5 & 0 \\ 0 & 4.5 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 0.4 \\ 0.6 & 1 \end{pmatrix} \quad and \quad B = \begin{pmatrix} -0.8 & 2 \\ 1 & 4 \end{pmatrix}.$$

In this example, let $p = 3$, take $\alpha_j = 0.2$, $\beta_j = 0.2$, $j = 1, 2$, $\sigma : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ satisfies

$$\sigma_{i1}(t,x,y)^2 \le 0.01(x_1^2 + y_1^2) \quad \text{and} \quad \sigma_{i2}(t,x,y)^2 \le 0.01(x_2^2 + y_2^2), \quad i = 1, 2,$$

and $w(t)$ is a two dimensional Brownian motion.

Note that the exponential stability of (6.41) has been studied in Sun and Cao [120] by employing the method of variation of parameter, inequality technique and stochastic analysis.

Now, we check the condition in Corollary 6.1.24,

$$4^{p-1}c^{-(1+p/q)}\sum_{i=1}^{2}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+4^{p-1}c^{-(1+p/q)}\sum_{i=1}^{2}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}$$
$$+4^{p-1}2^pc^{-p/2}(\mu^{p/2}+\nu^{p/2})<0.18<1.$$

From Corollary 6.1.24, the trivial solution of (6.41) is exponentially stable.

6.2 Stability of stochastic delayed neural networks with impulses

6.2.1 Introduction and main results

Besides delay and stochastic effects, impulsive effects are also likely to exist in the neural networks systems, which could stabilize or destabilize the systems. Therefore, it is of interest to take delay effects, stochastic effects and impulsive effects into account in investigations of the dynamical behavior of neural networks.

In this section, we apply fixed point methods to study asymptotic stability and exponential stability of a class of stochastic delayed neural networks with impulsive effects, which is described by

$$\begin{cases} dx_i(t)=\Big[-c_ix_i(t)+\sum_{j=1}^n a_{ij}f_j(x_j(t))+\sum_{j=1}^n b_{ij}g_j(x_j(t-\tau(t))) \\ \qquad +\sum_{j=1}^n l_{ij}\int_{t-r(t)}^t h_j(x_j(s))\,ds\Big]\,dt \\ \qquad +\sum_{j=1}^n \sigma_{ij}(t,x_j(t),x_j(t-\tau(t)))\,dw_j(t), \quad t\neq t_k \\ \\ \Delta x_i(t_k)=I_{ik}(x_i(t_k)), \quad t=t_k, \quad k=1,2,3,\cdots \end{cases} \tag{6.42}$$

or

$$\begin{cases} dx(t)=\Big[-Cx(t)+Af(x(t))+Bg(x(t-\tau(t)))+W\int_{t-r(t)}^t h(x(s))\,ds\Big]\,dt \\ \qquad +\sigma(t,x(t),x(t-\tau(t)))\,dw(t), \quad t\neq t_k \\ \\ \Delta x(t_k)=I_k(x(t_k)), \quad t=t_k, \quad k=1,2,3,\cdots \end{cases}$$

$i=1,2,3,\cdots,n$, where $x(t)=(x_1(t),x_2(t),\cdots x_n(t))^T\in\mathbb{R}^n$ is the state vector associated with the neurons; $C=\text{diag}(c_1,c_2,\cdots,c_n)>0$ where $c_i>0$ represents the rate with which the ith unit will reset its ponential to the resting state in isolation when disconnected from the network and the external stochastic perturbations; $A=(a_{ij})_{n\times n}$, $B=(b_{ij})_{n\times n}$ and $W=(l_{ij})_{n\times n}$ represent the connection weight matrix, delayed connection weight matrix and distributed delayed connection weight matrix, respectively; f_j, g_j, h_j are activation functions, $f(x(t))=(f_1(x(t)),f_2(x(t)),\cdots f_n(x(t)))^T\in\mathbb{R}^n$, $g(x(t))=(g_1(x(t)),g_2(x(t)),\cdots,g_n(x(t)))^T\in\mathbb{R}^n$, $h(x(t))=(h_1(x(t)),h_2(x(t)),\cdots,h_n(x(t)))^T\in\mathbb{R}^n$, $w(t)=(w_1(t),w_2(t),\cdots,w_n(t))^T\in\mathbb{R}^n$ is an n-dimensional Brownion motion defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ with natural complete filtration $\{\mathcal{F}_t\}_{t\geq 0}$ (i.e. $\mathcal{F}_t$ = completion of $\sigma\{w(s) : 0 \leq s \leq t\}$) and $\sigma : \mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $\sigma = (\sigma_{ij})_{n\times n}$ is the diffusion coefficient matrix. $\Delta x_i(t_k) = I_{ik}(x_i(t_k)) = x_i(t_k^+) - x_i(t_k^-)$ is the impulse at moment t_k, and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to\infty} t_k = +\infty$, $x_i(t_k^+)$ and $x_i(t_k^-)$ stand for the right-hand and left-hand limit of $x_i(t)$ at $t = t_k$, respectively. $I_{ik}(x_i(t_k))$ shows the abrupt change of $x_i(t)$ at the impulsive moment t_k and $I_{ik}(\cdot) \in C\left(L^p_{\mathcal{F}_t}(\Omega;\mathbb{R}^n), L^p_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)\right)$. $\tau(t)$ and $r(t)$ denote a discrete time varying delay and the bound of a distributed time varying delay, respectively. Denote $\vartheta = \inf_{t\geq 0}\{t-\tau(t), t-r(t)\}$.

The initial condition for the system (6.42) is given by

$$x(t) = \phi(t), \qquad t \in [\vartheta, 0], \tag{6.43}$$

where $t \mapsto \phi(t) = (\phi_1(t), \phi_2(t), \cdots, \phi_n(t))^T \in C\left([\vartheta,0], L^p_{\mathcal{F}_0}(\Omega;\mathbb{R}^n)\right)$ with the norm is defined as

$$\|\phi\|^p = \sup_{\vartheta\leq s\leq 0}\left(\mathbb{E}\sum_{i=1}^n |\phi_i(s)|^p\right),$$

where $\mathbb{E}$ denotes expectation with respect to the probability measure $\mathbb{P}$ and $p \geq 2$.

To obtain our main results, we suppose the following conditions are satisfied:

(A1) the delays $\tau(t), r(t)$ are continuous functions such that $t-\tau(t)\to\infty$ and $t - r(t) \to \infty$ as $t \to \infty$;

(A2) $f_i(x)$, $g_i(x)$, and $h_i(x)$ satisfy Lipschitz condition. That is, for each $i = 1, 2, 3, \cdots, n$, there exist constants α_i, β_i, γ_i such that for every $x, y \in \mathbb{R}^n$,

$$|f_j(x) - f_i(y)| \leq \alpha_i|x-y|, \quad |g_i(x) - g_i(y)| \leq \beta_i|x-y|, \quad |h_j(x) - h_i(y)| \leq \gamma_i|x-y|;$$

(A3) there exists nonegative constants p_{ik} such that for any $x, y \in \mathbb{R}^n$,

$$|I_{ik}(x) - I_{ik}(y)| \leq p_{ik}|x-y|,\ i = 1,2,\cdots,n,\ k = 1,2,3,\cdots;$$

(A4) assume that $f(0)\equiv 0$, $g(0)\equiv 0$, $h(0)\equiv 0$, $\sigma(t,0,0)\equiv 0$, $I_{ik}(0)\equiv 0$, $i = 1,2,\cdots,n$, $k = 1,2,3,\cdots$;

(A5) $\sigma(t,x,y)$ satisfies a Lipschitz condition. That is, there are nonnegative constants μ_i and ν_i such that $\forall\ i,j$,

$$(\sigma_{ij}(t,x,y) - \sigma_{ij}(t,u,v))^2 \leq \mu_j(x_j-u_j)^2 + \nu_j(y_j-v_j)^2.$$

The solution $x(t) := x(t,\phi)$ of the system (6.42) is, for the time t, a piecewise continuous vector-valued function with the first kind discontinuity at the points t_k $(k = 1, 2, \cdots)$, where it is left continuous, i.e.,

$$x_i(t_k^-) = x_i(t_k), \qquad x_i(t_k^+) = x_i(t_k) + I_{ik}(x_i(t_k)), \qquad i = 1,2,\cdots n, \qquad k = 1,2,\cdots.$$

Define $\mathcal{S}_\phi$ the space of all $\mathcal{F}_t$-adapted processes $\varphi(t,w) : [\vartheta,\infty)\times\Omega\to\mathbb{R}^n$ such that $\varphi : [\vartheta,\infty)\mapsto L^p_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ is continuous on $t\neq t_k$ $(k = 1,2,\cdots)$, $\lim_{t\to t_k^-}\varphi(t,\cdot)$ and $\lim_{t\to t_k^+}\varphi(t,\cdot)$ exist, and

$\lim_{t\to t_k^-}\varphi(t,\cdot)=\varphi(t_k,\cdot)$ for $k=1,2,\cdots$. Moreover, we set $\varphi(t,\cdot)=\phi(t)$ for $t\in[\vartheta,0]$ and $\mathbb{E}\left(\sum_{i=1}^n|\varphi_i(t)|^p\right)\to 0$ as $t\to\infty$, $i=1,2,\cdots,n$. If we define the metric as the form

$$\|\varphi\|^p := \sup_{t\geq\vartheta}\left(\mathbb{E}\sum_{i=1}^n|\varphi_i(t)|^p\right), \tag{6.44}$$

then $\mathcal{S}_\phi$ is a complete metric space with respect to the norm (6.44). Using the contraction mapping defined on the space $\mathcal{S}_\phi$ and applying a contraction mapping principle, we obtain our first result, which is proved in Subsection 6.2.2.

Theorem 6.2.1. *Suppose that the assumptions* (A1)-(A5) *hold. If the following conditions are satisfied,*

(i) *the function $r(t)$ is bounded by a constant r $(r>0)$;*

(ii) *there exist constants p_i $(i=1,2,\cdots,n)$ such that $p_{ik}\leq p_i(t_k-t_{k-1})$, $k=1,2,\cdots$;*

(iii) *and such that*

$$\begin{aligned}\alpha \;\triangleq\;& 6^{p-1}\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+6^{p-1}\sum_{i=1}^n c_i^{-p}\left(\sum_{j=1}^n|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\ &+6^{p-1}\sum_{i=1}^n\left(\frac{r}{c_i}\right)^p\left(\sum_{j=1}^n|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\\ &+6^{p-1}n^{p-1}\sum_{i=1}^n c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)+6^{p-1}c^{-1}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}<1,\end{aligned} \tag{6.45}$$

where $c=\min\{c_1,c_2,\cdots,c_n\}$, $\mu=\max\{\mu_1,\mu_2,\cdots,\mu_n\}$, $\nu=\max\{\nu_1,\nu_2,\cdots,\nu_n\}$,

then the trivial solution of (6.42) *is pth moment asymptotically stable.*

Define $\mathcal{C}_\phi$ the space of all $\mathcal{F}_t$-adapted processes $\varphi(t,\omega):[\vartheta,\infty)\times\Omega\to\mathbb{R}^n$ such that $\varphi:[\vartheta,\infty)\mapsto L^p_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ is continuous on $t\neq t_k$ $(k=1,2,\cdots)$, $\lim_{t\to t_k^-}\varphi(t,\cdot)$ and $\lim_{t\to t_k^+}\varphi(t,\cdot)$ exist, and $\lim_{t\to t_k^-}\varphi(t,\cdot)=\varphi(t_k,\cdot)$ for $k=1,2,\cdots$. Moreover, we set $\varphi(t,\cdot)=\phi(t)$ for $t\in[\vartheta,0]$ and $e^{\lambda t}\mathbb{E}\left(\sum_{i=1}^n|\varphi_i(t)|^p\right)\to 0$ as $t\to\infty$, $\lambda<\min\{c_1,c_2,\cdots,c_n\}$, $i=1,2,\cdots,n$. Then $\mathcal{C}_\phi$ is a complete metric space with respect to the norm (6.44). Using a contraction mapping defined on the space $\mathcal{C}_\phi$ and applying a contraction mapping principle, we obtain our second result. For its proof, see Subsection 6.2.3.

Theorem 6.2.2. *Suppose that the assumptions* (A1)-(A5) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ $(\tau>0)$;*

(ii) *there exist constants p_i $(i=1,2,\cdots,n)$ such that $p_{ik}\leq p_i(t_k-t_{k-1})$, $k=1,2,\cdots$;*

(iii) *and such that*

$$\begin{aligned}\alpha \triangleq\ & 6^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 6^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\ & +6^{p-1}\sum_{i=1}^{n}\left(\frac{\tau}{c_i}\right)^p\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\\ & +6^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)+6^{p-1}c^{-1}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}<1, \quad (6.46)\end{aligned}$$

where $c=\min\{c_1,c_2,\cdots,c_n\}$, $\mu=\max\{\mu_1,\mu_2,\cdots,\mu_n\}$, $\nu=\max\{\nu_1,\nu_2,\cdots,\nu_n\}$;

then the trivial solution of (6.42) *is pth moment exponentially stable.*

Remark 6.2.3. *In Theorem 6.2.2, both the discrete delay $\tau(t)$ and distributed delay $r(t)$ are required to be bounded, while the discrete delay $\tau(t)$ in Theorem 6.2.1 can be unbounded. It is clear that the conditions in Theorem 6.2.1 and Theorem 6.2.2 do not require the differentiability of delays. In addition, condition* (A2) *implies that the activation functions discussed in this section may be unbounded, non-monotonic and non-differentiable.*

Remark 6.2.4. *The system* (6.42) *is quite general and it includes several well-known neural network models as its special cases, see, for example, the models in* [54, 74, 78, 83, 116, 120, 129, 142]. *Sakthivel et al.* [116] *has considered asymptotic stability in mean square of the system* (6.42) *with linear impulsive effects, by employing Liapunov functional method and using linear matrix inequality optimization approach. However, the time varying delays in* [116] *should satisfy*

$$(\mathrm{H}_1)\quad 0\le h_1\le\tau(t)\le h_2,\qquad\qquad \tau'(t)\le\mu,$$

where h_1, h_2 are constants, the distributed delay $r(t)$ is bounded, $0\le r(t)\le\overline{r}$, $\overline{r}$ is a constant. In our results, the condition (H_1) *is replaced by other assumptions, and the assumptions in Theorem 6.2.1 and Theorem 6.2.2 may be satisfied if* (H_1) *is not.*

Remark 6.2.5. *In this section, our approach is based on fixed point methods, and in one step, a fixed point argument can yield the existence and stability criteria of the considered system. However, when using Liapunov's direct method, one must independently verify that a solution exists. The stability criteria we provided in our main results are only in terms of the system parameters c_i, a_{ij}, b_{ij}, l_{ij}, p_i etc. Hence, these criteria can be verified easily in applications.*

Consider the a when there are no stochastic perturbations on the system (6.42), the stochastic neural networks become usual neural network which can be described as

$$\begin{cases}\frac{dx_i(t)}{dt}=-c_ix_i(t)+\sum_{j=1}^n a_{ij}f_j(x_j(t))+\sum_{j=1}^n b_{ij}g_j(x_j(t-\tau(t)))\\ \qquad +\sum_{j=1}^n l_{ij}\int_{t-r(t)}^t h_j(x_j(s))\,ds, \quad t\ne t_k\\ \\ \Delta x_i(t_k)=I_{ik}(x_i(t_k)),\quad t=t_k,\quad k=1,2,3,\cdots\end{cases}\qquad (6.47)$$

or

$$\begin{cases}\frac{dx(t)}{dt}=-Cx(t)+Af(x(t))+Bg(x-\tau(t))+D\int_{t-r(t)}^t h(x(s))\,ds,\quad t\ne t_k\\ \\ \Delta x(t_k)=I_k(x(t_k)),\quad t=t_k,\quad k=1,2,3,\cdots\end{cases}$$

for $i = 1, 2, 3, \cdots, n$, where $x(\cdot) = (x_1(\cdot), x_2(\cdot), \cdots, x_n(\cdot))^T$ is the neuron state vector of the transformed system (6.47).

The initial condition for the system (6.47) is

$$x(t) = \phi(t), \qquad t \in [\vartheta, 0], \tag{6.48}$$

where ϕ is a continuous function with the norm defined by $\|\phi\| = \sum_{i=1}^{n} \sup_{\vartheta \le s \le 0} |\phi_i(s)|$. Define $\mathcal{H}_\phi = \mathcal{H}_{1\phi} \times \mathcal{H}_{2\phi} \times \cdots \mathcal{H}_{n\phi}$, where $\mathcal{H}_{i\phi}$ is the space consisting of continuous functions $\varphi_i(t) : \mathbb{R}^+ \to \mathbb{R}$ such that $\varphi_i(t)$ is continuous on $t \neq t_k$ $(k = 1, 2, \cdots)$, $\lim_{t \to t_k^-} \varphi_i(t)$ and $\lim_{t \to t_k^+} \varphi_i(t)$ exist, and $\lim_{t \to t_k^-} \varphi_i(t) = \varphi_i(t_k)$. Moreover, we set $\varphi_i(\theta) = \phi(\theta)$ for $\vartheta \le \theta \le 0$ and $\varphi_i(t) \to 0$ as $t \to \infty$, $i = 1, 2 \cdots n$. For any $\varphi(t), \eta(t) \in \mathcal{H}_\phi$, if we define the metric as

$$d(\varphi, \eta) = \sum_{i=1}^{n} \sup_{t \ge \vartheta} |\varphi_i(t) - \eta_i(t)|, \tag{6.49}$$

then $\mathcal{H}_\phi$ is a complete metric space with respect to the norm (6.49). Using a contraction mapping defined on the space H_ϕ and applying a contraction mapping principle, we obtain our third result, which is proved in Subsection 6.2.4.

Theorem 6.2.6. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the function $r(t)$ is bounded by a constant r $(r > 0)$;*

(ii) *there exist constants p_i $(i = 1, 2, \cdots, n)$ such that $p_{ik} \le p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;*

(iii) *and such that*

$$\begin{aligned} \alpha \triangleq & \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| \\ & + \sum_{i=1}^{n} \frac{r}{c_i} \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| + \max_{i=1,2,\cdots,n} \left\{ \frac{p_i}{c_i} \right\} < 1; \end{aligned} \tag{6.50}$$

then the trivial solution of (6.47) *is asymptotically stable.*

Define $\mathcal{B}_\phi = \mathcal{B}_{1\phi} \times \mathcal{B}_{2\phi} \times \cdots \mathcal{B}_{n\phi}$, where $\mathcal{B}_{i\phi}$ is the space consisting of continuous functions $\varphi_i(t) : \mathbb{R}^+ \to \mathbb{R}$ such that $\varphi_i(t)$ is continuous on $t \neq t_k$ $(k = 1, 2, \cdots)$, $\lim_{t \to t_k^-} \varphi_i(t)$ and $\lim_{t \to t_k^+} \varphi_i(t)$ exist, and $\lim_{t \to t_k^-} \varphi_i(t) = \varphi_i(t_k)$. Moreover, we set $\varphi_i(\theta) = \phi(\theta)$ for $\vartheta \le \theta \le 0$ and $e^{\lambda t}\varphi_i(t) \to 0$ as $t \to \infty$, where $\lambda < \min\{c_1, c_2, \cdots, c_n\}$, $i = 1, 2 \cdots n$. Then $\mathcal{B}_\phi$ is a complete metric space with respect to the metric (6.49). Using a contraction mapping defined on the space $\mathcal{B}_\phi$ and applying a contraction mapping principle, we obtain our fourth result, which is proved in Subsection 6.2.5.

Theorem 6.2.7. *Suppose that the assumptions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the discrete delay $\tau(t)$ and $r(t)$ in the distributed delay are bounded by a constant τ $(\tau > 0)$;*

(ii) *there exist constants p_i $(i = 1, 2, \cdots n)$ such that $p_{ik} \le p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;*

(iii) *and such that*

$$\begin{aligned}\alpha \ &\triangleq\ \sum_{i=1}^{n}\frac{1}{c_i}\max_{j=1,2,\cdots,n}|a_{ij}\alpha_j|+\sum_{i=1}^{n}\frac{1}{c_i}\max_{j=1,2,\cdots,n}|b_{ij}\beta_j| \\ &\quad+\sum_{i=1}^{n}\frac{\tau}{c_i}\max_{j=1,2,\cdots,n}|l_{ij}\gamma_j|+\max_{i=1,2,\cdots,n}\left\{\frac{p_i}{c_i}\right\}<1; \qquad (6.51)\end{aligned}$$

then the trivial solution of (6.47) *is exponentially stable.*

Remark 6.2.8. *Zhang et al.* [142, 143] *have investigated exponential stability and asymptotic stability of a class of impulsive cellular neural networks by using fixed point methods, which is a special case of the system* (6.47)*. Our results in Theorem* 6.2.6 *and Theorem* 6.2.7 *improve and extend the results in* [142, 143] (*see Remark* 6.2.15 *and Remark* 6.2.17 *for more information*)*.*

The rest of this section is organized as follows. The proofs of Theorem 6.2.1 and Theorem 6.2.2 are presented in Subsection 6.2.2 and Subsection 6.2.3, respectively. The proofs of Theorem 6.2.6 and Theorem 6.2.7 are provided in Subsection 6.2.4 and Subsection 6.2.5, respectively. Some examples are given to illustrate our main results in Subsection 6.2.6.

6.2.2 Proof of Theorem 6.2.1

In this subsection, we prove Theorem 6.2.1. We start with some preparations.

Multiply both sides of (6.42) by $e^{c_i t}$, we obtain that for $t \neq t_k$, $i = 1, 2, 3, \cdots, n$,

$$\begin{aligned} d(e^{c_i t}x_i(t)) \ &=\ e^{c_i t}\left[\sum_{j=1}^{n}a_{ij}f_j(x_j(t))+\sum_{j=1}^{n}b_{ij}g_j(x_j(t-\tau(t)))+\sum_{j=1}^{n}l_{ij}\int_{t-r(t)}^{t}h_j(x_j(u))\,du\right]dt \\ &\quad+e^{c_i t}\sum_{j=1}^{n}\sigma_{ij}(t,x_j(t),x_j(t-\tau(t)))\,dw_j(t). \qquad (6.52)\end{aligned}$$

Integrate (6.52) from $t_{k-1}+\varepsilon$ ($\varepsilon > 0$) to $t \in (t_{k-1}, t_k)$ ($k = 1, 2, \cdots$), we obtain that

$$\begin{aligned} e^{c_i t}x_i(t) \ &=\ e^{c_i(t_{k-1}+\varepsilon)}x_i(t_{k-1}+\varepsilon)+\int_{t_{k-1}+\varepsilon}^{t}e^{c_i s}\sum_{j=1}^{n}\sigma_{ij}(s,x_j(s),x_j(s-\tau(s)))\,dw_j(s) \\ &\quad+\int_{t_{k-1}+\varepsilon}^{t}e^{c_i s}\left[\sum_{j=1}^{n}a_{ij}f_j(x_j(s))\right. \\ &\quad\left.+\sum_{j=1}^{n}b_{ij}g_j(x_j(s-\tau(s)))+\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}h_j(x_j(u))\,du\right]ds. \qquad (6.53)\end{aligned}$$

Let $\varepsilon \to 0$ in (6.53), for $t \in (t_{k-1}, t_k)$ $(k = 1, 2, \cdots)$, we obtain that

$$\begin{aligned} e^{c_i t} x_i(t) &= e^{c_i t_{k-1}} x_i(t_{k-1}^+) + \int_{t_{k-1}}^{t} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, dw_j(s) \\ &\quad + \int_{t_{k-1}}^{t} e^{c_i s} \Bigg[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) \\ &\quad + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\, du \Bigg] ds. \end{aligned} \tag{6.54}$$

Set $t = t_k - \varepsilon$ $(\varepsilon > 0)$ in (6.54), we obtain that

$$\begin{aligned} e^{c_i (t_k - \varepsilon)} x_i(t_k - \varepsilon) &= e^{c_i t_{k-1}} x_i(t_{k-1}^+) + \int_{t_{k-1}}^{t_k - \varepsilon} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, dw_j(s) \\ &\quad + \int_{t_{k-1}}^{t_k - \varepsilon} e^{c_i s} \Bigg[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) \\ &\quad + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\, du \Bigg] ds. \end{aligned} \tag{6.55}$$

Let $\varepsilon \to 0$ in (6.55), we obtain that

$$\begin{aligned} e^{c_i t_k} x_i(t_k^-) &= e^{c_i t_{k-1}} x_i(t_{k-1}^+) + \int_{t_{k-1}}^{t_k} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, dw_j(s) \\ &\quad + \int_{t_{k-1}}^{t_k} e^{c_i s} \Bigg[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) \\ &\quad + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\, du \Bigg] ds. \end{aligned} \tag{6.56}$$

Note that $x_i(t_k) = x_i(t_k^-)$, from (6.54) and (6.56), we obtain that for $t \in (t_{k-1}, t_k]$ $(k = 1, 2, \cdots)$,

$$\begin{aligned} e^{c_i t} x_i(t) &= e^{c_i t_{k-1}} x_i(t_{k-1}^+) + \int_{t_{k-1}}^{t} e^{c_i s} \Bigg[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) \\ &\quad + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\, du \Bigg] ds \\ &\quad + \int_{t_{k-1}}^{t} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, dw_j(s) \\ &= e^{c_i t_{k-1}} x_i(t_{k-1}) + \int_{t_{k-1}}^{t} e^{c_i s} \Bigg[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) \\ &\quad + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\, du \Bigg] ds \\ &\quad + \int_{t_{k-1}}^{t} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\, dw_j(s) + e^{c_i t_{k-1}} I_{i(k-1)}(x_i(t_{k-1})). \end{aligned}$$

Hence, we obtain that

$$
\begin{aligned}
e^{c_i t_{k-1}} x_i(t_{k-1}) &= e^{c_i t_{k-2}} x_i(t_{k-1}) + \int_{t_{k-2}}^{t_{k-1}} e^{c_i s} \left[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) \right. \\
&\quad \left. + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\,du \right] ds \\
&\quad + \int_{t_{k-2}}^{t_{k-1}} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\,dw_j(s) + e^{c_i t_{k-2}} I_{i(k-2)}(x_i(t_{k-2})) \\
&\vdots \\
&\vdots \\
e^{c_i t_2} x_i(t_2) &= e^{c_i t_1} x_i(t_1) + \int_{t_1}^{t_2} e^{c_i s} \left[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) \right. \\
&\quad \left. + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\,du \right] ds \\
&\quad + \int_{t_1}^{t_2} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\,dw_j(s) + e^{c_i t_1} I_{i1}(x_i(t_1)) \\
e^{c_i t_1} x_i(t_1) &= \phi_i(0) + \int_{0}^{t_1} e^{c_i s} \left[\sum_{j=1}^{n} a_{ij} f_j(x_j(s)) \right. \\
&\quad \left. + \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s))) + \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\,du \right] ds \\
&\quad + \int_{0}^{t_1} e^{c_i s} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\,dw_j(s),
\end{aligned}
$$

which yields that for $t > 0$,

$$
\begin{aligned}
x_i(t) &= e^{-c_i t}\phi_i(0) + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} a_{ij} f_j(x_j(s))\,ds \\
&\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} b_{ij} g_j(x_j(s-\tau(s)))\,ds \\
&\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))\,du\,ds \\
&\quad + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^{n} \sigma_{ij}(s, x_j(s), x_j(s-\tau(s)))\,dw_j(s) + \sum_{0<t_k<t} e^{-c_i(t-t_k)} I_{ik}(x_i(t_k)).
\end{aligned}
$$

Lemma 6.2.9. *Define an operator by* $(Q\varphi)(t) = \phi(t)$ *for* $t \in [\vartheta, 0]$, *and for* $t \geq 0$,

$i = 1, 2, 3, \cdots, n,$

$$
\begin{aligned}
&(Q\varphi)_i(t) \qquad (6.57)\\
&= e^{-c_i t}\phi_i(0) + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n a_{ij} f_j(\varphi_j(s))\, ds + \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n b_{ij} g_j(\varphi_j(s-\tau(s)))\, ds \\
&+ \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(\varphi_j(u))\, du\, ds \\
&+ \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n \sigma_{ij}(s, \varphi_j(s), \varphi_j(s-\tau(s)))\, dw_j(s) + \sum_{0<t_k<t} e^{-c_i(t-t_k)} I_{ik}(\varphi_i(t_k)).
\end{aligned}
$$

Suppose that the assumptions (A1)-(A5) *hold. If the conditions* (i)-(iii) *in Theorem* 6.2.1 *are satisfied, then $Q : \mathcal{S}_\phi \to \mathcal{S}_\phi$ and Q is a contraction mapping.*

Proof. Denote $(Q\varphi)_i(t) := J_{1i}(t) + J_{2i}(t) + J_{3i}(t) + J_{4i}(t) + J_{5i}(t) + J_{6i}(t)$, where

$$
\begin{aligned}
J_{1i}(t) &= e^{-c_i t}\varphi_i(0), \qquad J_{2i}(t) = \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n a_{ij} f_j(\varphi_j(s))\, ds, \\
J_{3i}(t) &= \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n b_{ij} g_j(\varphi_j(s-\tau(s)))\, ds, \\
J_{4i}(t) &= \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n l_{ij} \int_{s-r(s)}^s h_j(\varphi_j(u))\, du\, ds, \\
J_{5i}(t) &= \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n \sigma_{ij}(s, \varphi_j(s), \varphi_j(s-\tau(s)))\, dw_j(s), \\
J_{6i}(t) &= \sum_{0<t_k<t} e^{-c_i(t-t_k)} P_{ik}(x_i(t_k)).
\end{aligned}
$$

Step1. From the definition of the metric space $\mathcal{S}_\phi$, we have that $\mathbb{E}\sum_{i=1}^n |\varphi_i(t)|^p < \infty$ for all $t \geq 0$, $\varphi \in \mathcal{S}_\phi$.

Step2. We prove the continuity in *pth* moment of Qx on $[0, \infty) \setminus \{t_1, t_2, \cdots\}$ for $x \in \mathcal{S}_\phi$ and left continuity and existence of a right limit at each t_k $(k = 1, 2, \cdots)$. It is clear that $(Q\varphi)_i(t)$ is continuous on $[\vartheta, 0]$. For a fixed time $t > 0$, it is easy to check that $J_{1i}(t), J_{2i}(t), J_{3i}(t), J_{4i}(t), J_{5i}(t)$, $J_{6i}(t)$ are continuous in *pth* moment on the fixed time $t \neq t_k$ $(k = 1, 2, \cdots)$. Hence, $(Q\varphi_i)(t)$ is continuous in *pth* moment on the fixed time $t \neq t_k$ $(k = 1, 2, \cdots)$. On the other hand, as $t = t_k$, it is easy to check that $J_{1i}(t), J_{2i}(t), J_{3i}(t), J_{4i}(t), J_{5i}(t)$ are continuous in *pth* moment on the fixed time $t = t_k$ $(k = 1, 2, \cdots)$. In the following, we check *pth* moment left continuity of $J_{6i}(t)$ on $t = t_k$ $(k = 1, 2, \cdots)$. Let $r < 0$ be small enough,

$$
\begin{aligned}
&\mathbb{E}\sum_{i=1}^n |J_{6i}(t_k + r) - J_{6i}(t_k)|^p \\
&= \mathbb{E}\sum_{i=1}^n \left| \sum_{0<t_m<t_k+r} e^{-c_i(t_k+r-t_m)} I_{im}(\varphi_i(t_m)) - \sum_{0<t_m<t_k} e^{-c_i(t_k-t_m)} I_{im}(\varphi_i(t_m)) \right|^p \\
&\leq \mathbb{E}\sum_{i=1}^n \left| \left(e^{-c_i(t_k+r)} - e^{-c_i t_k}\right) \sum_{0<t_m<t_k} e^{c_i t_m} I_{im}(\varphi_i(t_m)) \right|^p,
\end{aligned}
$$

which implies that $\lim_{r\to 0^-}\mathbb{E}\sum_{i=1}^n|J_{6i}(t_k+r)-J_{6i}(t_k)|^p=0$. Let $r>0$ be small enough,

$$\begin{aligned}
&\mathbb{E}\sum_{i=1}^n|J_{6i}(t_k+r)-J_{6i}(t_k)|^p\\
&=\mathbb{E}\sum_{i=1}^n\left|\sum_{0<t_m<t_k+r}e^{-c_i(t_k+r-t_m)}I_{im}(\varphi_i(t_m))-\sum_{0<t_m<t_k}e^{-c_i(t_k-t_m)}I_{im}(\varphi_i(t_m))\right|^p\\
&=\mathbb{E}\sum_{i=1}^n\left|e^{-c_i(t_k+r)}\left[\sum_{0<t_m<t_k}e^{c_it_m}I_{im}(\varphi_i(t_m))+e^{c_it_k}I_{ik}(\varphi_i(t_k))\right]\right.\\
&\qquad\left.-e^{-c_it_k}\sum_{0<t_m<t_k}e^{c_it_m}I_{im}(\varphi_i(t_m))\right|^p\\
&=\mathbb{E}\sum_{i=1}^n\left|\left(e^{-c_i(t_k+r)}-e^{-c_it_k}\right)\sum_{0<t_m<t_k}e^{c_it_m}I_{im}(\varphi_i(t_m))+e^{-c_ir}I_{ik}(\varphi_i(t_k))\right|^p,
\end{aligned}$$

which implies that $\lim_{r\to 0^+}\mathbb{E}\sum_{i=1}^n|J_{6i}(t_k+r)-J_{6i}(t_k)|^p=\mathbb{E}\sum_{i=1}^n|I_{ik}(\varphi_i(t_k))|^p$.

Based on the above discussion, we obtain that $(Q\varphi)_i(t):[\vartheta,\infty)\to L^p_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$ is continuous in pth moment on $t\neq t_k$ $(k=1,2,\cdots)$, and for $t=t_k$ $(k=1,2,\cdots)$, $\lim_{t\to t_k^+}(Q\varphi)_i(t)$ and $\lim_{t\to t_k^-}(Q\varphi)_i(t)$ exist. Furthermore, we also obtain that $\lim_{t\to t_k^-}(Q\varphi)_i(t)=(Q\varphi)_i(t_k)\neq\lim_{t\to t_k^+}(Q\varphi)_i(t)$.

Step3. We prove that $Q(\mathcal{S}_\phi)\subseteq\mathcal{S}_\phi$. From (6.57),

$$\mathbb{E}\sum_{i=1}^n|Q\varphi_i(t)|^p=\mathbb{E}\sum_{i=1}^n\left|\sum_{j=1}^6J_{ji}(t)\right|^p\le 6^{p-1}\sum_{j=1}^6\mathbb{E}\left(\sum_{i=1}^n|J_{ji}(t)|^p\right).\tag{6.58}$$

Now, we estimate the right-hand terms of (6.58). From (A3), we know that $|I_{ik}(x_i(t_k))|\le p_{ik}|x_i(t_k)|$, combining with the condition (ii), we obtain that

$$\begin{aligned}
\mathbb{E}\sum_{i=1}^n|J_{6i}(t)|^p&\le\mathbb{E}\sum_{i=1}^n\left[\sum_{0<t_k<t}e^{-c_i(t-t_k)}p_{ik}|\varphi_i(t_k)|\right]^p\\
&\le\mathbb{E}\sum_{i=1}^n\left[p_i\sum_{0<t_k<t}e^{-c_i(t-t_k)}|\varphi_i(t_k)|(t_k-t_{k-1})\right]^p\\
&\le\mathbb{E}\sum_{i=1}^n\left[p_i\int_0^te^{-c_i(t-s)}|\varphi_i(s)|\,ds\right]^p\\
&\le\mathbb{E}\sum_{i=1}^np_i^p\left(\int_0^te^{-c_i(t-s)}\,ds\right)^{p/q}\int_0^te^{-c_i(t-s)}|\varphi_i(s)|^p\,ds\\
&\le\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}\int_0^te^{-c(t-s)}\mathbb{E}\left(\sum_{i=1}^n|\varphi_i(s)|^p\right)ds.
\end{aligned}\tag{6.59}$$

Since $\mathbb{E}\sum_{i=1}^n|\varphi_i(t)|\to 0$, $t-\tau(t)\to\infty$ and $t-r(t)\to\infty$ as $t\to\infty$. Thus, from (6.58), (6.59) and combining with (6.17), (6.18) and (6.19), we obtain that $\mathbb{E}\sum_{i=1}^n|Q\varphi_i(t)|^p\to 0$ as

$\mathbb{E}\sum_{i=1}^{n}|\varphi_i(t)|^p \to 0$. Therefore, $Q:\mathcal{S}_\phi \to \mathcal{S}_\phi$.

Step4. We prove that Q is a contraction mapping. For any $\varphi,\psi \in \mathcal{S}_\phi$, from (6.17), (6.18), (6.19), (6.58) and (6.59), we obtain

$$\begin{aligned}
&\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}|Q\varphi_i(s)-Q\psi_i(s)|^p\right\}\\
&\leq 5^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}a_{ij}\Big(f_j(x_j(u))-f_j(y_j(u))\Big)\,du\right|^p\right\}\\
&\quad+5^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}b_{ij}\Big(g_j(x_j(u-\tau(u)))-g_j(y_j(u-\tau(u)))\Big)\,du\right|^p\right\}\\
&\quad+5^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}l_{ij}\int_{s-r(s)}^{s}\Big(h_j(\varphi_j(v))-h_j(\psi_j(v))\Big)\,dv\,du\right|^p\right\}\\
&\quad+5^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\int_0^s e^{-c_i(s-u)}\sum_{j=1}^{n}\Big(\sigma_{ij}(s,x_j(s),x_j(u-\tau(u)))\right.\right.\\
&\qquad\left.\left.-\sigma_{ij}(s,y_j(s),y_j(s-\tau(u)))\Big)\,dw_j(u)\right|^p\right\}\\
&\quad+5^{p-1}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{i=1}^{n}\left|\sum_{0<t_k<t}e^{-c_i(t-t_k)}\left(I_{ik}(\varphi_i(t_k))-I_{ik}(\psi_i(t_k))\right)\right|^p\right\}\\
&\leq 5^{p-1}\left\{\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\right.\\
&\quad+\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^p\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}+n^{p-1}\sum_{i=1}^{n}c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)\\
&\quad\left.+\frac{1}{c}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}\right\}\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{j=1}^{n}|\varphi_j(s)-\psi_j(s)|^p\right\}=\alpha\sup_{s\geq\vartheta}\left\{\mathbb{E}\sum_{j=1}^{n}|\varphi_j(s)-\psi_j(s)|^p\right\}.
\end{aligned}$$

From (6.45), we obtain that $Q:\mathcal{S}_\phi\to\mathcal{S}_\phi$ is a contraction mapping. □

We are now ready to prove Theorem 6.2.1.

Proof. From Lemma 6.2.9, by a contraction mapping principle, we obtain that Q has a unique fixed point $x(t)$, which is a solution of (6.42) with $x(t)=\phi(t)$ as $t\in[\vartheta,0]$ and $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p\to 0$ as $t\to\infty$.

Now, we prove that the trivial solution of (6.42) is *pth* moment stable. From (6.45), For any $\varepsilon>0$, we choose $\delta>0$ $(\delta<\varepsilon)$ such that $6^{p-1}\delta<(1-\alpha)\varepsilon$.

If $x(t)=(x_1(t),x_2(t),\cdots,x_n(t))^T$ is a solution of (6.42) with the initial condtion satisfying $\mathbb{E}\sum_{i=1}^{n}|\phi_i(t)|^p<\delta$, then $x(t)=(Qx)(t)$ defined in (6.57). We claim that $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p<\varepsilon$ for all $t\geq 0$. Notice that $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p<\varepsilon$ for $t\in[\vartheta,0]$, we suppose that there exists $t^*>0$ such that $\mathbb{E}\sum_{i=1}^{n}|x_i(t^*)|^p=\varepsilon$ and $\mathbb{E}\sum_{i=1}^{n}|x_i(t)|^p<\varepsilon$ for $-\tau\le t<t^*$, then it follows from (6.45), we obtain that

$$
\begin{aligned}
&\mathbb{E}\sum_{i=1}^{n}|x_i(t^*)|^p\\
&\le 6^{p-1}\mathbb{E}\sum_{i=1}^{n}e^{-pc_it^*}|x_i(0)|^p\\
&\quad +6^{p-1}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^{t^*}e^{-c_i(t^*-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s)|^p\right)ds\\
&\quad +6^{p-1}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^{t^*}e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s-\tau(s)))|^p\right)ds\\
&\quad +6^{p-1}\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^{t^*}e^{-c_i(t^*-s)}\int_{s-r(s)}^{s}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(u)|^p\right)du\,ds\\
&\quad +6^{p-1}n^{p-1}\sum_{i=1}^{n}c_i^{1-p/2}\Bigg[\mu^{p/2}\int_0^{t^*}e^{-c_i(t^*-s)}\mathbb{E}\left(\sum_{j=1}^{n}|x_j(s)|^p\right)ds\\
&\qquad +\nu^{p/2}\int_0^{t^*}e^{-c_i(t-s)}E\left(\sum_{j=1}^{n}|x_j(s-\tau(s))|^p\right)ds\Bigg]\\
&\quad +6^{p-1}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}\int_0^{t^*}e^{-c(t^*-s)}\mathbb{E}\left(\sum_{i=1}^{n}|x_i(s)|^p\right)ds\\
&\le 6^{p-1}\delta+\Bigg[6^{p-1}\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}+6^{p-1}\sum_{i=1}^{n}c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\\
&\quad +6^{p-1}\sum_{i=1}^{n}\left(\frac{r}{c_i}\right)^{p}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}+6^{p-1}n^{p-1}\sum_{i=1}^{n}c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right)\\
&\quad +\frac{1}{c}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\}\Bigg]\varepsilon<(1-\alpha)\varepsilon+\alpha\varepsilon=\varepsilon,
\end{aligned}
$$

which is a contradiction. Therefore, the trivial solution of (6.42) is asymptotically stable in *pth* moment. □

Let $l_{ij}\equiv 0$, the system (6.42) is reduced to

$$
\begin{cases}
dx(t)=[-Cx(t)+Af(x(t))+Bg(x(t-\tau(t)))]\,dt\\
\qquad\qquad +\sigma(t,x(t),x(t-\tau(t)))\,dw(t), \quad t\neq t_k\\
\Delta x(t_k)=I_k(x(t_k)), \quad t=t_k, \quad k=1,2,3,\cdots.
\end{cases}
\tag{6.60}
$$

which is a description of a stochastically perturbed Hopfield neural networks with time-varying delays.

Corollary 6.2.10. *Suppose that the assumptions* (A1)-(A5) *hold. If the following conditions are satisfied,*

(i) *there exist constants* p_i *(*$i = 1, 2, \cdots, n$*) such that* $p_{ik} \le p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;

(ii) *and such that*

$$5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}$$
$$+5^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right) + 5^{p-1}c^{-1}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\} < 1,$$

where $c = \min\{c_1, c_2, \cdots, c_n\}$, $\mu = \max\{\mu_1, \mu_2, \cdots, \mu_n\}$, $\nu = \max\{\nu_1, \nu_2, \cdots, \nu_n\}$;

then the trivial solution of (6.60) *is pth moment asymptotically stable.*

Remark 6.2.11. *Note that the delay* $\tau(t)$ *in Corollary* 6.2.10 *can be unbounded.*

6.2.3 Proof of Theorem 6.2.2

Define an operator $(Q\varphi)(t) = \phi(t)$ for $t \in [\vartheta, 0]$ and for $t \ge 0$, $(Q\varphi)(t)$ is defined as the right hand side of (6.57). Following the proof of Theorem 6.2.1, we find that to show Theorem 6.2.2, we only need to prove that $e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|(Q\varphi)_i(t)|^p \to 0$ as $t \to \infty$. It follows from (6.57) that

$$e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|(Q\varphi)_i(t)|^p = e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}\left|\sum_{j=1}^{6}J_{ji}(t)\right|^p \le 6^{p-1}e^{\lambda t}\sum_{j=1}^{6}\mathbb{E}\left(\sum_{i=1}^{n}|J_{ji}(t)|^p\right). \tag{6.61}$$

Now, we estimate the right-hand terms of (6.61). First, by using Hölder's inequality,

$$\begin{aligned}
e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{2i}(t)|^p &= e^{\lambda t}\sum_{i=1}^{n}\mathbb{E}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}a_{ij}f_j(\varphi_j(s))\,ds\right|^p \qquad (6.62)\\
&\le e^{\lambda t}e^{\lambda t}\sum_{i=1}^{n}\mathbb{E}\left[\int_0^t e^{-\frac{c_i(t-s)}{q}}e^{-\frac{c_i(t-s)}{p}}\sum_{j=1}^{n}|a_{ij}||f_j(\varphi_j(s))|\,ds\right]^p\\
&\le e^{\lambda t}\sum_{i=1}^{n}\mathbb{E}\left\{\left[\int_0^t e^{-c_i(t-s)}\,ds\right]^{p/q}\int_0^t e^{-c_i(t-s)}\left[\sum_{j=1}^{n}|a_{ij}||f_j(\varphi_j(s))|\right]^p ds\right\}\\
&\le e^{\lambda t}\sum_{i=1}^{n}c_i^{-p/q}\mathbb{E}\left\{\int_0^t e^{-c_i(t-s)}\left[\sum_{j=1}^{n}|a_{ij}||\alpha_j||\varphi_j(s)|\right]^p ds\right\}\\
&\le e^{\lambda t}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds\\
&= \sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q}\int_0^t e^{(\lambda-c_i)(t-s)}e^{\lambda s}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(s)|^p\right)ds.
\end{aligned}$$

With a similar computation to (6.62), we obtain that

$$
\begin{aligned}
& e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{3i}(t)|^p \qquad (6.63)\\
&\leq e^{\lambda t}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\right]ds\\
&\leq e^{\lambda \tau}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda(s-\tau(s))}\mathbb{E}\left[\sum_{j=1}^{n}|\varphi_j(s-\tau(s)))|^p\right]ds.
\end{aligned}
$$

$$
\begin{aligned}
& e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{4i}(t)|^p \qquad (6.64)\\
&\leq e^{\lambda t}\sum_{i=1}^{n}c_i^{-p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\mathbb{E}\left[\sum_{j=1}^{n}\left|\int_{s-r(s)}^{s}\varphi_j(u)\,du\right|^p\right]ds\\
&\leq e^{\lambda t}\sum_{i=1}^{n}\left(\frac{\tau}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^t e^{-c_i(t-s)}\int_{s-r(s)}^{s}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)|^p\right)du\,ds\\
&\leq e^{\lambda \tau}\sum_{i=1}^{n}\left(\frac{\tau}{c_i}\right)^{p/q}\left(\sum_{j=1}^{n}|l_{ij}|^q|\gamma_j|^q\right)^{p/q}\int_0^t e^{-(c_i-\lambda)(t-s)}\int_{s-r(s)}^{s}e^{\lambda u}\mathbb{E}\left(\sum_{j=1}^{n}|\varphi_j(u)|^p\right)du\,ds.
\end{aligned}
$$

Using Lemma 6.1.16 and Hölder's inequality, we obtain that

$$
\begin{aligned}
& e^{\lambda t}\mathbb{E}\sum_{i=1}^{n}|J_{5i}(t)|^p \qquad (6.65)\\
&= e^{\lambda t}\sum_{i=1}^{n}\mathbb{E}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n}\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,dw_j(s)\right|^p\\
&\leq e^{\lambda t}n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left\{\left[\int_0^t e^{-c_i(t-s)}|\sigma_{ij}(s,\varphi_j(s),\varphi_j(s-\tau(s)))|\,dw_j(s)\right]^2\right\}^{p/2}\\
&= e^{\lambda t}n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\int_0^t e^{-2c_i(t-s)}\sigma_{ij}^2(s,\varphi_j(s),\varphi_j(s-\tau(s)))\,ds\right]^{p/2}\\
&\leq e^{\lambda t}n^{p-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\left[\int_0^t e^{-2c_i(t-s)}\left(\mu_j\varphi_j^2(s)+\nu_j\varphi_j^2(s-\tau(s))\right)ds\right]^{p/2}\\
&\leq e^{\lambda t}n^{p-1}2^{p/2-1}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbb{E}\Bigg[\left(\int_0^t e^{-2c_i(t-s)}\mu_j\varphi_j^2(s)\,ds\right)^{p/2}\\
&\qquad +\left(\int_0^t e^{-2c_i(t-s)}\nu_j\varphi_j^2(s-\tau(s))\,ds\right)^{p/2}\Bigg]
\end{aligned}
$$

$$
\begin{aligned}
&\leq e^{\lambda t} n^{p-1} 2^{p/2-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E}\left[\left(\int_0^t e^{-2c_i(t-s)}\, ds\right)^{p/2-1} \int_0^t e^{-2c_i(t-s)} \mu_j^{p/2} |\varphi_j(s)|^p\, ds\right] \\
&\quad + e^{\lambda t} n^{p-1} 2^{p/2-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbb{E}\left[\left(\int_0^t e^{-2c_i(t-s)}\, ds\right)^{p/2-1} \int_0^t e^{-2c_i(t-s)} \nu_j^{p/2} |\varphi_j(s-\tau(s))|^p\, ds\right] \\
&\leq e^{\lambda t} n^{p-1} \sum_{i=1}^{n} c_i^{1-p/2} \left[\mu^{p/2} \int_0^t e^{-c_i(t-s)} \mathbb{E}\left(\sum_{j=1}^{n} |\varphi_j(s)|^p\right) ds\right] \\
&\quad + e^{\lambda t} n^{p-1} \sum_{i=1}^{n} c_i^{1-p/2} \left[\nu^{p/2} \int_0^t e^{-c_i(t-s)} \mathbb{E}\left(\sum_{j=1}^{n} |\varphi_j(s-\tau(s))|^p\right) ds\right] \\
&\leq n^{p-1} \sum_{i=1}^{n} c_i^{1-p/2} \left[\mu^{p/2} \int_0^t e^{-(c_i-\lambda)(t-s)} e^{\lambda s} \mathbb{E}\left(\sum_{j=1}^{n} |\varphi_j(s)|^p\right) ds\right] \\
&\quad + e^{\lambda \tau} n^{p-1} \sum_{i=1}^{n} c_i^{1-p/2} \left[\nu^{p/2} \int_0^t e^{-(c_i-\lambda)(t-s)} e^{\lambda(s-\tau(s))} \mathbb{E}\left(\sum_{j=1}^{n} |\varphi_j(s-\tau(s))|^p\right) ds\right].
\end{aligned}
$$

Further, from (A3), we know that $|I_{ik}(x_i(t_k))| \leq p_{ik}|x_i(t_k)|$ for $i = 1, 2, \cdots, n$, $k = 1, 2, \cdots$. Combining with the condition that $p_{ik} \leq p_i(t_k - t_{k-1})$, we obtain that

$$
\begin{aligned}
e^{\lambda t} \mathbb{E} \sum_{i=1}^{n} |J_{6i}(t)|^p &\leq e^{\lambda t} \mathbb{E} \sum_{i=1}^{n} \left[\sum_{0<t_k<t} e^{-c_i(t-t_k)} p_{ik} |\varphi_i(t_k)|\right]^p \\
&\leq e^{\lambda t} \mathbb{E} \sum_{i=1}^{n} \left[p_i \sum_{0<t_k<t} e^{-c_i(t-t_k)} |\varphi_i(t_k)| (t_k - t_{k-1})\right]^p \\
&\leq e^{\lambda t} \mathbb{E} \sum_{i=1}^{n} \left[p_i \int_0^t e^{-c_i(t-s)} |\varphi_i(s)|\, ds\right]^p \\
&\leq e^{\lambda t} \mathbb{E} \sum_{i=1}^{n} p_i^p \left(\int_0^t e^{-c_i(t-s)}\, ds\right)^{p/q} \int_0^t e^{-c_i(t-s)} |\varphi_i(s)|^p\, ds \\
&\leq \max_{i=1,2,\cdots,n} \left\{\frac{p_i^p}{c_i^{p-1}}\right\} \int_0^t e^{-(c-\lambda)(t-s)} e^{\lambda s} \mathbb{E}\left(\sum_{i=1}^{n} |\varphi_i(s)|^p\right) ds. \qquad (6.66)
\end{aligned}
$$

Since $\mathbb{E}\sum_{i=1}^{n} |\varphi_i(t)| \to 0$, $t - \tau(t) \to \infty$ and $t - r(t) \to \infty$ as $t \to \infty$. Thus, from (6.61) to (6.66), we obtain that $e^{\lambda t}\mathbb{E}\sum_{i=1}^{n} |(Q\varphi)_i(t)|^p \to 0$ as $e^{\lambda t}\mathbb{E}\sum_{i=1}^{n} |\varphi_i(t)|^p \to 0$. Hence, combining the proof of Theorem 6.2.1, there exists a unique fixed point $\varphi(\cdot)$ of Q in $\mathcal{C}_\phi$, which is a solution of the system (6.42) such that $e^{\lambda t}\mathbb{E}\sum_{i=1}^{n} |\varphi_i(t)|^p \to 0$ as $t \to \infty$. This completes the proof.

Corollary 6.2.12. *Suppose that the assumptions* (A1)-(A5) *hold. Assume that*

(i) *the discrete delay $\tau(t)$ is bounded by a constant τ ($\tau > 0$);*

(ii) *there exist constants p_i ($i = 1, 2, \cdots, n$) such that $p_{ik} \leq p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;*

(iii) *and such that*

$$5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}$$
$$+5^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{-p/2}\left(\mu^{p/2}+\nu^{p/2}\right) + 5^{p-1}c^{-1}\max_{i=1,2,\cdots,n}\left\{\frac{p_i^p}{c_i^{p-1}}\right\} < 1,$$

where $c=\min\{c_1,c_2,\cdots,c_n\}$, $\mu=\max\{\mu_1,\mu_2,\cdots,\mu_n\}$, $\nu=\max\{\nu_1,\nu_2,\cdots,\nu_n\}$, *then the trivial solution of* (6.60) *is pth moment exponentially stable.*

6.2.4 Proof of Theorem 6.2.6

In this subsection, we prove Theorem 6.2.6. We start with some preparations.

Using similar computations as in Subsection 6.2.2, we obtain that for $t \geq 0$, the system (6.47) is equivalent to

$$\begin{aligned} x_i(t) &= e^{-c_i t}x_i(0) + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} a_{ij}g_j(x_j(s))\,ds + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} b_{ij}g_j(x_j(s-\tau(s)))\,ds \\ &\quad + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} l_{ij}\int_{s-r(s)}^{s} g_j(x_j(u))\,du\,ds + \sum_{0<t_k<t} e^{-c_i(t-t_k)}I_{ik}(x_i(t_k)), \end{aligned}$$

$i=1,2,3,\cdots,n$, $k=1,2,\cdots$.

Lemma 6.2.13. *Define an operator by* $(P\varphi)(t)=\phi(t)$ *for* $-\tau \leq t \leq 0$, *and for* $t \geq 0$,

$$\begin{aligned} (P\varphi)_i(t) &= e^{-c_i t}\varphi_i(0) + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} a_{ij}g_j(\varphi_j(s))\,ds \\ &\quad + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} b_{ij}g_j(\varphi_j(s-\tau(s)))\,ds \\ &\quad + \int_0^t e^{-c_i(t-s)}\sum_{j=1}^{n} l_{ij}\int_{s-r(s)}^{s} g_j(\varphi_j(u))\,du\,ds + \sum_{0<t_k<t} e^{-c_i(t-t_k)}I_{ik}(x_i(t_k)) \\ &:= I_1(t)+I_2(t)+I_3(t)+I_4(t)+I_5(t). \end{aligned} \tag{6.67}$$

If the conditions (i)-(iii) *in Theorem* 6.2.6 *are satisfied, then* $P:\mathcal{S}_\phi \to \mathcal{S}_\phi$ *and* P *is a contraction mapping.*

Proof. First, we prove that $P\mathcal{S}_\phi \subseteq \mathcal{S}_\phi$. In view of (6.67), it is easy to check that $(Px_i)(t)$ is continuous on fixed time $t \neq t_k$ $(k=1,2,\cdots)$. On the other hand, as $t=t_k$ $(k=1,2,\cdots)$, it is not difficult to show that $I_1(t), I_2(t), I_3(t), I_4(t)$ is continuous on fixed time $t=t_k$ $(k=1,2,\cdots)$. Let $r<0$ be small enough, we obtain that

$$\begin{aligned} |J_{5i}(t_k+r)-J_{5i}(t_k)| &= \left|\sum_{0<t_m<t_k+r} e^{-c_i(t_k+r-t_m)}I_{im}(\varphi_i(t_m)) - \sum_{0<t_m<t_k} e^{-c_i(t_k-t_m)}I_{im}(\varphi_i(t_m))\right| \\ &\leq \left|\left(e^{-c_i(t_k+r)}-e^{-c_i t_k}\right)\sum_{0<t_m<t_k} e^{c_i t_m}I_{im}(\varphi_i(t_m))\right|, \end{aligned}$$

which implies that $\lim_{r\to 0^-} |J_{5i}(t_k + r) - J_{5i}(t_k)| = 0$. Let $r > 0$ be small enough, we obtain that

$$\begin{aligned}
|J_{5i}(t_k + r) - J_{5i}(t_k)| &= \left| \sum_{0<t_m<t_k+r} e^{-c_i(t_k+r-t_m)} I_{im}(\varphi_i(t_m)) - \sum_{0<t_m<t_k} e^{-c_i(t_k-t_m)} I_{im}(\varphi_i(t_m)) \right| \\
&= \Bigg| e^{-c_i(t_k+r)} \left[\sum_{0<t_m<t_k} e^{c_i t_m} I_{im}(\varphi_i(t_m)) + e^{c_i t_k} I_{ik}(\varphi_i(t_k)) \right] \\
&\qquad - e^{-c_i t_k} \sum_{0<t_m<t_k} e^{c_i t_m} I_{im}(\varphi_i(t_m)) \Bigg| \\
&= \left| \left(e^{-c_i(t_k+r)} - e^{-c_i t_k}\right) \sum_{0<t_m<t_k} e^{c_i t_m} I_{im}(\varphi_i(t_m)) + e^{-c_i r} I_{ik}(\varphi_i(t_k)) \right|,
\end{aligned}$$

which implies that $\lim_{r\to 0^+} |J_{5i}(t_k + r) - J_{5i}(t_k)| = |I_{ik}(\varphi_i(t_k))|$.

Based on the above discussion, we obtain that $(P\varphi)_i(t) : [\vartheta, \infty) \to \mathbb{R}^n$ is continuous on $t \neq t_k$ $(k = 1, 2, \cdots)$, and for $t = t_k$ $(k = 1, 2, \cdots)$, $\lim_{t\to t_k^+}(P\varphi)_i(t)$ and $\lim_{t\to t_k^-}(P\varphi)_i(t)$ exist. Furthermore, we also obtain that $\lim_{t\to t_k^-}(P\varphi)_i(t) = (P\varphi)_i(t_k) \neq \lim_{t\to t_k^+}(P\varphi)_i(t)$.

Next, we prove that $\lim_{t\to\infty}(P\varphi)_i(t) = 0$ for $\varphi_i(t) \in \mathcal{S}_{i\phi}$.

$$\begin{aligned}
|I_5(t)| = \left| \sum_{0<t_k<t} e^{-c_i(t-t_k)} I_{ik}(x_i(t_k)) \right| &\leq \left| \sum_{0<t_k<t} e^{-c_i(t-t_k)} p_i(t_k - t_{k-1}) x_i(t_k) \right| \\
&\leq p_i \int_0^t e^{-c_i(t-s)} |x_i(s)|\, ds. \qquad (6.68)
\end{aligned}$$

From the fact that $c_i > 0$ $(i = 1, 2, \cdots, n)$ and the estimate (6.33), (6.34), (6.35) and (6.68), we conclude that $\lim_{t\to\infty}(Px_i)(t) = 0$ for $x_i(t) \in \mathcal{S}_{i\phi}$. Therefore, $P : \mathcal{S}_\phi \to \mathcal{S}_\phi$.

Now, we prove that P is a contraction mapping. For any $x(t), y(t) \in \mathcal{S}_\phi$, we obtain that

$$\begin{aligned}
&\sum_{i=1}^n \sup_{\vartheta\leq s\leq t} |(Px)_i(t) - (Py)_i(t)| \\
&\leq \sum_{i=1}^n \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n |x_j(s) - y_j(s)|\, ds \\
&\quad + \sum_{i=1}^n \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n |x_j(s - \tau(s)) - y_j(s - \tau(s))|\, ds \\
&\quad + \sum_{i=1}^n \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| \int_0^t e^{-c_i(t-s)} \sum_{j=1}^n \int_{s-r(s)}^s |x_j(u) - y_j(u)|\, du\, ds \\
&\quad + \sum_{i=1}^n p_i \int_0^t e^{-c_i(t-s)} |x_i(s) - y_i(s)|\, ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \left[\sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| \right.\\
&\quad \left. + \sum_{i=1}^{n} \frac{\tau}{c_i} \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| + \max_{j=1,2,\cdots,n} \left\{\frac{p_i}{c_i}\right\}\right] \sum_{j=1}^{n} \left[\sup_{\vartheta\leq s\leq t} |x_j(s) - y_j(s)|\right]\\
&= \alpha \sum_{j=1}^{n} \left[\sup_{\vartheta\leq s\leq t} |x_j(s) - y_j(s)|\right].
\end{aligned}
$$

From (6.50), we obtain that P is a contraction mapping. □

We are now ready to prove Theorem 6.2.6.

Proof. Let P be defined as in Lemma 6.2.13, by a contraction mapping principle, P has a unique fixed point $x \in \mathcal{S}_\phi$ with $x(\theta) = \phi(\theta)$ on $-\tau \leq \theta \leq 0$ and $x(t) \to 0$ as $t \to \infty$.

To obtain asymptotically stable, we need to prove that the trivial equilibrium $x = 0$ of (6.47) is stable. From (6.50), For any $\varepsilon > 0$, choose $\sigma > 0$ and $\sigma < \varepsilon$ satisfying the condition $\sigma + \varepsilon\alpha < \varepsilon$.

If $x(t,\phi) = (x_1(t,\phi), x_2(t,\phi), \cdots, x_n(t,\phi))$ is the solution of (6.47) with the initial condition $\|\phi\| < \sigma$, the we claim that $\|x(t,\phi)\| < \varepsilon$ for all $t \geq 0$. Indeed, we suppose that there exists $t^* > 0$ such that

$$
\sum_{i=1}^{n} |x_i(t^*,\phi)| = \varepsilon, \qquad \text{and} \qquad \sum_{i=1}^{n} |x_i(t,\phi)| < \varepsilon \quad \text{for} \quad 0 \leq t < t^*. \tag{6.69}
$$

From (6.50), we obtain

$$
\begin{aligned}
&\sum_{i=1}^{n} |x_i(t^*,\phi)|\\
&\leq \sum_{i=1}^{n} \left[|e^{-c_i t^*} x_i(0)| + \int_0^{t^*} e^{-c_i(t^*-s)} \sum_{j=1}^{n} |a_{ij} f_j(x_j(s))|\, ds \right.\\
&\quad + \int_0^{t^*} e^{-c_i(t^*-s)} \sum_{j=1}^{n} |b_{ij} g_j(x_j(s-\tau(s)))|\, ds\\
&\quad \left. + \int_0^{t^*} e^{-c_i(t^*-s)} \sum_{j=1}^{n} |l_{ij} \int_{s-r(s)}^{s} h_j(x_j(u))|\, du\, ds + p_i \int_0^{t^*} e^{-c_i(t^*-s)} |x_i(s)|\, ds \right]\\
&< \sigma + \varepsilon \left[\sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| \right.\\
&\quad \left. + \sum_{i=1}^{n} \frac{\tau}{c_i} \max_{j=1,2,\cdots,n} |l_{ij}\gamma_j| + \max_{j=1,2,\cdots,n} \left\{\frac{p_i}{c_i}\right\}\right]\\
&\leq \sigma + \varepsilon\alpha < \varepsilon.
\end{aligned}
$$

which contradicts (6.69). Therefore, $\|x(t,\phi)\| < \varepsilon$ for all $t \geq 0$. This completes the proof. □

Let $l_{ij} \equiv 0$ for $i = 1, 2, \cdots, n$, $j = 1, 2, \cdots, n$, the system (6.47) is reduced to

$$\begin{cases} \frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^n a_{ij} f_j(x_j(t)) + \sum_{j=1}^n b_{ij} g_j(x_j(t-\tau(t))), \quad t \neq t_k \\ \\ \Delta x_i(t_k) = I_{ik} x_i(t_k), \quad t = t_k, \quad k = 1, 2, 3, \cdots, \end{cases} \tag{6.70}$$

which is the description of cellular neural network with time-varying delays. Following the result of Theorem 6.2.6, we have the following corollary. Note that the delay in Corollary 6.2.14 can be unbounded.

Corollary 6.2.14. *Suppose that the conditions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *there exist constants p_i $(i = 1, 2, \cdots, n)$ such that $p_{ik} \leq p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;*

(ii) *and such that*

$$\alpha \triangleq \sum_{i=1}^n \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^n \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \max_{i=1,2,\cdots,n} \left\{ \frac{p_i}{c_i} \right\} < 1, \tag{6.71}$$

then the trivial solution of (6.70) *is asymptotically stable.*

Remark 6.2.15. *Zhang and Guan* [143] *has studied asymptotic stability of* (6.70) *by using fixed point theory. The conditions in* [143] *are as follows*

(i) *there exists a constant μ such that $\inf_{k=1,2,\cdots}\{t_k - t_{k-1}\} \geq \mu$;*

(ii) *there exist constants p_i $(i = 1, 2, \cdots, n)$ such that $p_{ik} \leq p_i\mu$, $k = 1, 2, \cdots$;*

(iii) *and such that*

$$\lambda^* \triangleq \sum_{i=1}^n \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^n \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \max_{i=1,2,\cdots,n} \left\{ \frac{p_i}{c_i} + p_i\mu \right\} < 1;$$

(iv)

$$\max_{i=1,2,\cdots,n} \{\lambda_i\} < \frac{1}{\sqrt{n}}, \quad \text{where } \lambda_i = \frac{1}{c_i} \sum_{j=1}^n |a_{ij}\alpha_j| + \frac{1}{c_i} \sum_{j=1}^n |b_{ij}\beta_j| + \left(\frac{p_i}{c_i} + p_i\mu \right).$$

It is clear that Corollary 6.2.16 is an improvement of the result in [143].

6.2.5 Proof of Theorem 6.2.7

Define the operator P as in Subsection 6.2.4. Following the proof of Theorem 6.2.6, we only need to prove that $e^{\lambda t}(P\varphi)_i(t) \to 0$ as $t \to \infty$. We estimate the right-hand terms of (6.67), we obtain that

$$\begin{aligned} e^{\lambda t}|I_2(t)| &= \left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n a_{ij}f_j(\varphi_j(s))\,ds\right| \\ &\le e^{\lambda t}\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n |a_{ij}\alpha_j||\varphi_j(s)|\,ds \\ &\le \max_{j=1,2,\cdots,n}|a_{ij}\alpha_j|\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda s}\sum_{j=1}^n|\varphi_j(s)|\,ds, \end{aligned} \tag{6.72}$$

$$\begin{aligned} e^{\lambda t}|I_3(t)| &= e^{\lambda t}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n b_{ij}g_j(\varphi_j(s-\tau(s)))\,ds\right| \\ &\le e^{\lambda t}\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n |b_{ij}\beta_j||\varphi_j(s-\tau(s))|\,ds \\ &\le e^{\lambda\tau}\max_{j=1,2,\cdots,n}|b_{ij}\beta_j|\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda(s-\tau(s))}\sum_{j=1}^n|\varphi_j(s-\tau(s))|\,ds, \end{aligned} \tag{6.73}$$

$$\begin{aligned} e^{\lambda t}|I_4(t)| &= e^{\lambda t}\left|\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n l_{ij}\int_{s-r(s)}^s h_j(\varphi_j(u))\,du\,ds\right| \\ &\le e^{\lambda t}\int_0^t e^{-c_i(t-s)}\sum_{j=1}^n |l_{ij}\gamma_j|\int_{s-r(s)}^s|\varphi_j(u)|\,du\,ds \\ &\le e^{\lambda\tau}\max_{j=1,2,\cdots,n}|l_{ij}\gamma_j|\int_0^t e^{-(c_i-\lambda)(t-s)}\int_{s-r(s)}^s e^{\lambda u}\sum_{j=1}^n|\varphi_j(u)|\,du\,ds, \end{aligned} \tag{6.74}$$

$$\begin{aligned} e^{\lambda t}|I_5(t)| = e^{\lambda t}\left|\sum_{0<t_k<t}e^{-c_i(t-t_k)}I_{ik}(x_i(t_k))\right| &\le e^{\lambda t}\left|\sum_{0<t_k<t}e^{-c_i(t-t_k)}p_i(t_k-t_{k-1})x_i(t_k)\right| \\ &\le p_i\int_0^t e^{-(c_i-\lambda)(t-s)}e^{\lambda s}|x_i(s)|\,ds. \end{aligned} \tag{6.75}$$

From the fact that $\lambda < \min\{c_1, c_2, \cdots, c_n\}$, $c_i > 0$ $(i = 1, 2, \cdots, n)$ and the above estimate, we obtain that $e^{\lambda t}(P\varphi)_i(t) \to 0$ as $t \to \infty$.

Corollary 6.2.16. *Suppose that the conditions* (A1)-(A4) *hold. If the following conditions are satisfied,*

(i) *the delay $\tau(t)$ is bounded by a constant τ $(\tau > 0)$;*

(ii) *there exist constants p_i $(i = 1, 2, \cdots, n)$ such that $p_{ik} \le p_i(t_k - t_{k-1})$, $k = 1, 2, \cdots$;*

(iii) *and such that*

$$\alpha \triangleq \sum_{i=1}^n \frac{1}{c_i}\max_{j=1,2,\cdots,n}|a_{ij}\alpha_j| + \sum_{i=1}^n \frac{1}{c_i}\max_{j=1,2,\cdots,n}|b_{ij}\beta_j| + \max_{j=1,2,\cdots,n}\left\{\frac{p_i}{c_i}\right\} < 1, \tag{6.76}$$

then the trivial solution of (6.70) is exponentially stable.

Remark 6.2.17. *Zhang and Luo [142] has studied exponential stability of (6.70) by using fixed point theory. The conditions in [142] are as follows*

(i) *the delay $\tau(t)$ is bounded by a constant τ $(\tau > 0)$;*

(ii) *there exists a constant μ such that $\inf_{k=1,2,\cdots}\{t_k - t_{k-1}\} \geq \mu$;*

(iii) *there exist constants p_i $(i = 1, 2, \cdots, n)$ such that $p_{ik} \leq p_i\mu$, $k = 1, 2, \cdots$;*

(iv) *and such that*

$$\alpha \triangleq \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |a_{ij}\alpha_j| + \sum_{i=1}^{n} \frac{1}{c_i} \max_{j=1,2,\cdots,n} |b_{ij}\beta_j| + \max_{i=1,2,\cdots,n} \left\{\frac{p_i}{c_i} + p_i\mu\right\} < 1.$$

It is clear that Corollary 6.2.16 is an improvement of the result in [142].

6.2.6 Examples

Example 6.2.18. *Consider the following two-dimensional cellular neural network*

$$\begin{cases} \frac{dx_i(t)}{dt} = -c_i x_i(t) + \sum_{j=1}^{2} a_{ij} g_j(x_j(t)) + \sum_{j=1}^{2} b_{ij} g_j(x_j(t - \tau(t))) \quad i = 1, 2, \quad t \neq t_k \\ \\ \Delta x_i(t_k) = I_{ik} x_i(t_k), \quad t = t_k, \quad k = 1, 2, 3, \cdots, \end{cases} \tag{6.77}$$

with the initial conditions $x_1(s) = \cos(s)$, $x_2(s) = \sin(s)$ on $-\frac{1}{2} \leq s \leq 0$, where $c_1 = c_2 = 3$, $a_{11} = 6/7$, $a_{12} = 3/7$, $a_{21} = -1/7$, $a_{22} = -1/7$, $b_{11} = 6/7$, $b_{12} = 2/7$, $b_{21} = 3/7$, $b_{22} = 1/7$, the activation function is described by $g_i(x) = \frac{|x+1|-|x-1|}{2}$, $\tau(t) = 0.4t + 1$. $I_{ik}(x_i(t_k)) = \arctan(0.4x_i(t_k))$, $t_k = t_{k-1} + 0.5k$, $i = 1, 2$ and $k = 1, 2, \cdots$.

It is clear that $\alpha_i = \beta_i = 1$, $p_{ik} = 0.4$ for $i = 1, 2$, $k = 1, 2, \cdots$, we select $p_i = 0.8$, then

$$\begin{aligned} \sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |a_{ij}\alpha_j| + \sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |b_{ij}\beta_j| + \max_{j=1,2,\cdots,n} \left\{\frac{p_i}{c_i}\right\} &\leq \frac{1}{3} \times \left(\frac{6}{7} + \frac{1}{7} + \frac{6}{7} + \frac{3}{7}\right) + \frac{4}{35} \\ &= \frac{16}{21} + \frac{4}{35} < 0.88 < 1. \end{aligned}$$

Hence, by Corollary 6.2.14, the trivial solution of (6.77) is asymptotically stable. However,

$$\sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |a_{ij}\alpha_j| + \sum_{i=1}^{2} \frac{1}{c_i} \max_{j=1,2} |b_{ij}\beta_j| + \max_{i=1,2,\cdots,n} \left\{\frac{p_i}{c_i} + p_i\mu\right\} > 1,$$

which implies that the result in [143] is not applicable.

Example 6.2.19. *Consider a two-dimensional stochastically perturbed Hopfield neural network with time-varying delays,*

$$\begin{cases} dx(t) = [-Cx(t) + Af(x(t)) + Bg(x_\tau(t))]\, dt + \sigma(t, x(t), x_\tau(t))\, dw(t), \quad t \neq t_k \\ \\ \Delta x(t_k) = I_k(x(t_k)), \quad t = t_k, \quad k = 1, 2, 3, \cdots, \end{cases} \tag{6.78}$$

where $f(x) = \frac{1}{5}\arctan x$, $g(x) = \frac{1}{5}\tanh x = \frac{1}{5}(e^x - e^{-x})/(e^x + e^{-x})$, $\tau(t) = \frac{1}{2}\sin t + \frac{1}{2}$,

$$C = \begin{pmatrix} 5 & 0 \\ 0 & 4.5 \end{pmatrix}, \quad A = \begin{pmatrix} 2 & 0.4 \\ 0.6 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -0.8 & 2 \\ 1 & 4 \end{pmatrix}.$$

In this example, let $p = 3$, *take* $\alpha_j = 0.2$, $\beta_j = 0.2$, $j = 1, 2$, $\sigma : \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2$ *satisfies*

$$\sigma_{i1}^2(t, x, y) \leq 0.01(x_1^2 + y_1^2) \quad \text{and} \quad \sigma_{i2}^2(t, x, y) \leq 0.01(x_2^2 + y_2^2), \quad i = 1, 2.$$

$I_{ik}(x_i(t_k)) = 0.1x_i(t_k)$, $t_k = t_{k-1} + 0.5$, $i = 1, 2$ *and* $k = 1, 2, \cdots$.

It is clear that $p_{ik} = 0.1$, we choose $p_i = 0.2$, let $p = 2$, we check the condition in Corollary 6.2.10,

$$5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|a_{ij}|^q|\alpha_j|^q\right)^{p/q} + 5^{p-1}\sum_{i=1}^{n} c_i^{-p}\left(\sum_{j=1}^{n}|b_{ij}|^q|\beta_j|^q\right)^{p/q}$$
$$+5^{p-1}n^{p-1}\sum_{i=1}^{n} c_i^{-p/2}\left(\mu^{p/2} + \nu^{p/2}\right) + 5^{p-1}c^{-1}\max_{i=1,2}\left\{\frac{p_i^p}{c_i^{p-1}}\right\} < 0.53 < 1.$$

From Corollary 6.2.10, the trivial solution of (6.78) is asymptotically stable. On the other hand, since $|\tau(t)| = \left|\frac{1}{2}\sin t + \frac{1}{2}\right| \leq 1$, from Corollary 6.2.12, the trivial solution of (6.78) is exponentially stable.

6.3 Notes and remarks

Neural networks have received an increasing interest in various areas [34, 119]. The stability of neural networks [38, 82, 139, 140] is critical for signal processing, especially in image processing and solving some classes of optimization problems. For the stochastic effects to the dynamical behaviors of neural networks, Liao and Mao [79, 80] initiated the study of stability and instability of stochastic neural networks.

Many articles [54, 55, 56, 120, 129] have considered a special case of the stochastic equation (6.1). Hu et al.[54] and Wan and Sun [129] studied a special case of (6.1) with the delays constant and discrete. The activation functions appearing in [54] are required to be bounded. Liao and Mao [81] investigated exponential stability of stochastic delay interval systems via Razumikihin-type theorems developed in [95], several exponential stability results were provided. However, the results are not only difficult to verify but also restrict to a case of the interval matrices $\tilde{A} = \tilde{B} = \tilde{C} = 0$. Sun and Cao [120] investigated the *pth* moment exponential stability of stochastic differential equations with discrete bounded delays by using the method of variation parameter, inequality technique and stochastic analysis. This method was firstly used in [129], which does not require the boundedness, monotonicity and differentiability of the activation functions. However, the stability criteria in [120] requires that the delay functions are bounded, differentiable and their derivatives are simultaneously required to be not greater than 1, this may impose a very strict constraint on model (see [138]). Huang et al. [55, 56] investigated the exponential stability of stochastic differential equations with discrete time-varying delays with the help of the Liapunov function and Dini derivative. However, the use of their criteria depends very much on the choice of positive numbers k_{ij} etc. and a positive diagonal matrix M (see Theorem 3.3 in [55] and Theorem 3.3 in [56]).

Based on the contents of this chapter, two papers [19, 20] have been submitted for possible publication.

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Notational conventions

Here we state some conventions regarding mathematical notation that we will use in this thesis.

- Denote by $\mathbb{R} = (-\infty, \infty)$, $\mathbb{R}^+ = [0, \infty)$, $\mathbb{Z}^+ = \{1, 2, 3, \cdots\}$.
- For any positive integer n, R^n denotes the set of all n-tuples of real numbers forms an n-dimensional vector space over R.
- $R^{m\times n}$ denotes the set of real $m \times n$ matrices.
- A^T denotes the transpose of matrix A.
- $Re(a)$ denotes the real part of the complex number a.
- $\lambda_d(A)$ denotes dominant eigenvalue of matrix A.
- $diag(d_1, d_2, \cdots d_n)$ denotes a diagonal matrix.
- $C([a, b]; \mathbb{R})$ denotes the space of continuous, real-valued functions on $[a, b]$.
- $C([a, b]; \mathbb{R}^n)$ denotes the space of continuous, $\mathbb{R}^n$-valued functions on $[a, b]$.
- $C([a, b]; X)$ denotes the space of continuous, X-valued functions on $[a, b]$.
- $BC([a, b]; X)$ denotes the space of bounded continuous, X-valued functions on $[a, b]$.
- $L_2[0, \infty)$ denotes the space of square-integrable functions on $[0, \infty)$.
- $(\Omega, \mathcal{F}, \mathbb{P})$ denotes a probability space, where Ω is the collection of all possible outcomes, and $\mathcal{F}$ is the set of all events A to which a probability $\mathbb{P}(A)$ can be attached. $\mathcal{F}$ is $\sigma-$algebra, and $\mathbb{P}$ a probability measure.
- $\mathbb{E}$ denotes expectation with respect to the probability measure $\mathbb{P}$.

Summary

Asymptotic behavior and stability of solutions of delay differential equations play an important role in the qualitative analysis of delay differential equations. This thesis studies asymptotic behavior and stability of determinsitic and stochastic delay differential equations.

The approach used in this thesis is based on fixed point theory, which does not resort to any Liapunov function or Liapunov functional. This approach relies mainly on three principles: an elementary variation of parameters formula, a complete metric space and a constraction mapping principle. The benefit of this approach is that the fixed point arguments can yield existence, uniqueness and stability of a system in one step. The main difficulty of this approach is to define a suitable complete metric space and a suitable mapping. Different choices of norms can be considered on defined spaces. The norms we choose should be such that the space under consideration is complete and the equation yields a contraction with respect to the norm.

The main contribution of this thesis is to study the approach using fixed point theory in a systematic way and to unify recent results in the literature by considering some general classes of equations. The equation we considered is a combination of time dependent delays, distributed delays, impulses and stochastic perturbations. In addition, an application to stochastic delayed neural networks is investigated. The results in this thesis extend and improve some exist results in the literature in some ways. Examples are discussed in each chapter to illustrate our main results. Chapter 2 presents three methods concerning asymptotic behavior of autonomous neutral delay differential equations. More specifically, we address the relations of the spectral method and the ODE method by considering a class of second order delay differential equations. For a case when there are no neutral terms to the considered equations, we illustrate a third method, fixed point method. Chapter 3 focuses on asymptotic behavior of a class of nonautonomous neutral delay differential equations. Chapter 4 addresses a fixed point approach to stability of deterministic delay differential equations and Chapter 5 discusses the stability of two classes of neutral stochastic delay differential equations with impulses. Chapter 6 studies stability properties of a class of stochastic delayed neural networks.

Samenvatting

Asymptotisch gedrag en stabiliteit van oplossingen spelen een belangrijke rol in de analyse van het kwalitatieve gedrag van delay differentiaalvergelijkingen. Dit proefschrift bestudeert asymptotisch gedrag en stabiliteit van deterministische en stochastische delay differentiaalvergelijkingen.

De gekozen aanpak is gebaseerd op theorie van vaste punten, zonder gebruik te maken van Lyapunov-functies of Lyapunov-functionalen. De aanpak berust hoofdzakelijk op drie uitgangspunten: een elementaire variatie-van-constantenformule, een volledige metrische ruimte en een contractieve afbeelding. Het voordeel van deze aanpak is dat de vaste-puntargumenten in één moeite door zowel existentie en eenduidigheid van oplossingen als stabiliteit van een vergelijking kunnen geven. Er kunnen verschillende metrieken gekozen worden. Er moet daarbij gezorgd worden dat de metrische ruimte volledig is en dat de vergelijking een contractieve afbeelding geeft.

De belangrijkste bijdrage van dit proefschrift is een systematisch onderzoek naar het gebruik van vaste-punttheorie voor stabiliteit van vergelijkingen en het verenigen van verschillende recente resultaten uit de vakliteratuur door enkele algemene klassen van vergelijkingen te beschouwen. De differentiaalvergelijkingen die we beschouwen kunnen een combinatie bevatten van tijdsafhankelijke delays, gespreide delays, neutral termen, impulsen en stochastische verstoringen. Verder wordt toepassing op stochastische neurale netwerken met delays onderzocht.

De resultaten in dit proefschrift verbeteren bestaande resultaten op verschillende punten en breiden die ook in verschillende richtingen uit. In ieder hoofdstuk worden voorbeelden besproken die de resultaten illustreren.

In Hoofdstuk 2 worden drie methoden besproken om het asymptotisch gedrag van autonome neutral delay differentiaalvergelijkingen te analyseren. Meer specifiek worden een spectrale methode en een methode gebaseerd op gewone differentiaalvergelijkingen vergeleken aan de hand van een klasse van tweede orde delay differentiaalvergelijkingen. Voor het geval de vergelijkingen geen neutral term hebben vergelijken we ook met een derde methode gebaseerd op vaste-punttheorie. Hoofdstuk 3 richt zich op het asymptotisch gedrag van een klasse van niet-autonome neutral delay differentiaalvergelijkingen. Hoofdstuk 4 gaat over een vaste-puntbenadering voor stabiliteit van deterministische delay differentiaalvergelijkingen en Hoofdstuk 5 bespreekt de stabiliteit van twee klassen van stochastische neutral delay differentiaalvergelijkingen met impulsen. Hoofdstuk 6, ten slotte, bestudeert stabiliteitseigenschappen van een klasse van stochastische neurale netwerken met delays.

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Curriculum Vitae

Guiling Chen was born on Feb. 27, 1983 in Shandong Province, China. In 2002, she completed her high middle school education at Yiyuan No.1 middle school in Shandong. At the same year, she started her combined studies at School of Mathematical Science, Qufu Normal University, Shandong, China. In 2006, she obtained her bachelor degree. After that, she continued her master programme at School of Mathematics in Shandong University, China. During her studies in Shandong University, she was a teaching assistant from 2007 to 2008. Under the supervison of Prof. Peichu Hu, she finished her master's thesis entitled *Uniqueness of meromorphic functions* and received her master degree in July, 2009. After she got a scholarship from Chinese Scholarship Council in Sep. 2009, she started her PhD study in Analysis and Dynamical System group under Mathematical Institute in Leiden University, the Netherlands. Her supervisors were Prof. dr. S.M.Verduyn Lunel and Dr. O.van Gaans, and her PhD research topic was *A fixed point approach towards stability of delay differential equations with application to neural networks.*