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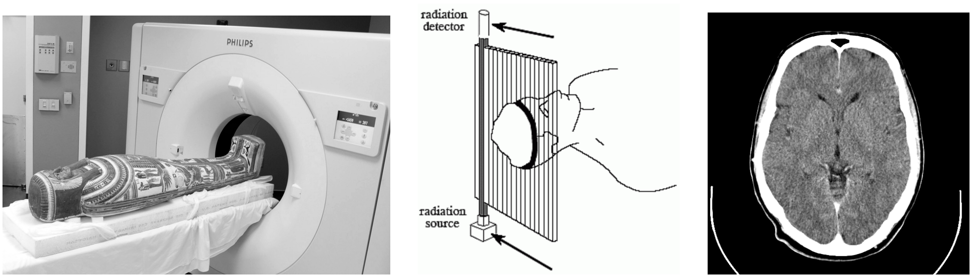
Summary

The research in this thesis is devoted to the development of quantitative guarantees on the quality of reconstructions in discrete tomography.

Tomography refers to imaging by sections through the use of any kind of penetrating beam. The word tomography is derived from the Greek tomē (“cut") or tomos (“part" or “section") and graphein (“to write"). The goal in tomography is to obtain information from the interior of an object without having to open or destroy it. An example is the Computerized Tomography (CT) scan which is often used to scan medical patients in order to detect tumors or other irregularities in the tissues of internal organs; see Fig. 8.6(a). The scanning device used in tomography is called a *tomograph*. A source, emitting a penetrating beam, and a detector are positioned on opposite sides of the object. The tomograph records the intensity profile of the beam after it has passed through the object (or patient), acquiring a *projection* of the object at a particular angle, as in Fig. 8.6(b). From projection data obtained from different angles, a mathematical method produces a *tomogram* (also called *reconstruction*): an image of the interior of the object. A 2-dimensional reconstructed image resembles a cut of the object, as in Fig. 8.6(c).



(a) A medical CT scanner[1] (b) Projection acquisition[2] (c) A brain tomogram[3]

Figure 8.6: *After the scanning process, a reconstruction algorithm generates a tomogram*

[1] Thanks to Dan Dry and the University of Chicago Medicine
[2] Image from the book “The Scientist and Engineer’s Guide to Digital Signal Processing" by Steven W. Smith
[3] By Mikael HÃď ggstrÃűm [Public domain], via Wikimedia Commons

In some cases, the judgment of a medical doctor relies on the quality of the reconstructed images (the tomograms). Tomographs often make use of harmful radiation, such as X-rays, and the dosage must be controlled in order not to harm the patient. Dose restriction is one of the limitations that affects the quality of reconstructions. Unfortunately, with the current technology and techniques available, reconstructed images are far from perfect. If doctors encounter a damaged organ or a strange object inside a patient by observing the tomogram, they would be glad to know how much they can rely on that image before sending the patient to emergency surgery. Tomography is not restricted to medical applications and can be found, for example, in industrial production lines to identify defects inside the produced products for quality control.

The seek for guarantees on the quality of reconstructions is inspired by the need of accuracy in tomography. Providing theoretical quality bounds for tomographic reconstructions is in general an inherently very difficult problem due to the ill-posedness of the mathematical problem. Firstly, the modeled problem of reconstructing images from a set of projections is typically underdetermined, having an infinite number of solutions. Finding one solution means reconstructing an image that may – or may not – be equal to the original object. Obtaining a solution is not a trivial mathematical procedure. Obtaining a solution among an infinite number of other solutions, such that it is equal to the original object, is in practice often impossible. Secondly, image reconstruction algorithms usually do not find an exact solution, but rather an approximate solution, i.e, an image with projections that approximate the given set of projections. As a consequence, one should not expect to "know" the inside of the scanned object but rather something that resembles it. Thirdly, if the projection set is acquired by a real scanning device, the projection data is corrupted by noise. The problem of finding an image which corresponds perfectly to the noisy projection data can typically not be solved exactly, because the noise perturbs the data in such a way that the reconstruction problem has contradicting information. An attempt to reconstruct an image from noisy projection data propagates the noise into the reconstructed image. Furthermore, in several applications, there are limitations on the number of projections or on the angular range for which projections can be acquired, which directly affects the quality of reconstructions. In summary, in tomography the reconstructed image might substantially differ from the true object that has been scanned.

In the process of modelling the reconstruction problem, the interaction between the beam and the object must be defined mathematically in a *projection model*. In this thesis we model the unknown object by representing it on a grid of square pixels. Some projection models have the property that the sum of the intensities of the pixels is equal to the sum of the intensities of the projection for each projection angle. For these cases, we have developed computable error bounds for *binary tomography* (where the unkown image is known to be black and white), which provide guarantees on the number of pixel errors between a reconstructed image of the object and the true object. Although it is not possible to measure the error of the reconstructed image without having access to all the information of the original object, we can guarantee a bound, a measure that is equal or larger than the actual error. With this information, one may be able to evaluate whether the reconstruction is good enough or not. A key idea in deriving these bounds is an observation first made by Hajdu and Tijdeman in [22], concerning the fact that all binary solutions have the same number of white pixels and lie on a hypersphere of which the center and the radius can be computed.

With this information, we compute bounds on the difference between binary solutions of the reconstruction problem, which also bounds the difference between any binary solution and the original binary object. In addition, we have presented bounds on the difference between any given binary image and all binary solutions.

We have developed similar bounds for the problem of high resolution binary image reconstruction from lower resolution scans. In this case, a scanning device (a video camera, for example) scans an image but due to the low resolution of the device, certain characteristics of the image cannot be identified or differentiated. A superresolution technique is used to obtain a higher resolution image from which those characteristics are now better defined. However, this new image is not guaranteed to provide an accurate representation of the true object that was scanned. This problem calls for a technique which can provide error guarantees.

When studying the problem of generating error bounds for binary tomography, we obtained a sufficient condition for the existence of binary solutions for the reconstruction problem. If one searches for a binary solution with certain specific characteristics, these characteristics can be enforced into the modeled reconstruction problem, generating a modified problem. Checking whether the modified reconstruction problem satisfies the condition of existence of binary solutions, we can determine if the feature of interest may be part of a binary solution or if it can certainly not be part of a binary solution.

Expanding the research on bounding errors in binary tomography, we developed another approach with the goal of bounding the difference between binary solutions of any underdetermined algebraic linear system of equations. It includes the case of fan beam tomography and the use of any projection model. Also, we have studied the case of binary tomography with noisy projection data. In this case, it is likely that no binary image has projections that equal the given set of projections. We assume that a noiseless projection set exists (yet not known) and take the noise level into account to compute approximate error bounds on binary solutions of the noiseless reconstruction problem (the problem for which has the true scanned object as a solution).

Discrete tomography does not only deal with black and white images, but also with more than two grey levels (say, 3 or 4). In some cases in discrete tomography, we may assume that the grey levels of the image to be reconstructed are known in advance. This prior knowledge can help to improve the quality of reconstructions. However, reconstructing discrete images from few projections is still a highly challenging problem. Many discrete reconstruction algorithms obtain accurate reconstructions in practice but do not offer mathematical guarantees. We have developed a discrete reconstruction algorithm which guarantees that the projections of the reconstructed discrete image are close to the given set of projections.