Author: Fortes, Wagner Rodrigues
Title: Error bounds for discrete tomography
Issue Date: 2013-09-18

Chapter 4

Bounds on the difference between solutions in noisy binary tomography

4.1. Introduction

In Computed tomography one wants to find an image which satisfies a given set of projections. When this set of projections is obtained from a real scanning device, the projections are usually corrupted with noise and other artefacts. Due to this data perturbation, it is highly likely that there is no image satisfying the projection set. In any attempt to reconstruct an image which approximately satisfies the acquired measurements, the data perturbation is propagated into the reconstructed image which may not truly represent the original scanned object.

In Chapter 2 we studied error bounds for binary tomography with noiseless data, which are not applicable for noisy projection data. In this chapter, we aim to develop error bounds on the difference between binary solutions of the noiseless reconstruction problem even though the available projection set is perturbed. Also, following the bounds of Chapter 2, we bound the number of wrong pixels of a certain binary reconstruction with respect to the binary solutions of the noiseless problem.

A key idea of this chapter also relies on the fact that all binary solutions of the noiseless reconstruction problem must lie on a certain hypersphere. However, the center and radius of this hypersphere *cannot* be computed. Nevertheless, we compute the center and radius of a hypersphere which contains the set of binary solutions of the noiseless reconstruction problem. The center of this hypersphere is the least squares solution of the noisy tomography problem. Its radius is computed taking into account a bound on the distance between the centers of the two mentioned hyperspheres.

This chapter is structured as follows. Our notation is introduced in Section 4.2, after which a general class of reconstruction problems is introduced. In Section 4.3 we bound the difference between any two binary solutions of the noiseless reconstruction problem by bounding the Euclidean norm of the binary solutions of the noisy problem and the minimum norm solution of the noiseless reconstruction problem. Section 4.4 deals with bounds that are based on properties of the binary images that are obtained by rounding the central reconstruction. Section 4.5 presents the computational challenges and Section 4.6 concludes the chapter.

4.2. Notation and model

Throughout the discrete tomography literature, several imaging models have been considered: the grid model, the strip model, the line model, etc. [33, section 7.4.1]. We now define some general notation. An *image* is represented by a vector $\boldsymbol{x} = (x_i) \in \mathbb{R}^n$. A *binary image* corresponds with a vector $\bar{\boldsymbol{x}} \in \{0,1\}^n$. For a given set of k projection directions, the *projection map* maps an image $\boldsymbol{x}$ to a vector $\boldsymbol{p} \in \mathbb{R}^m$ of *projection data*, where m denotes the total number of line measurements. As the projection map is a linear transformation, it can be represented by a matrix $\boldsymbol{W} = (w_{ij}) \in \mathbb{R}^{m \times n}$, called the *projection matrix*. Entry w_{ij} represents the weight of the contribution of x_j to projected line i. From this point on, we assume that the projection matrix has the property that $\sum_{i=1}^{m} w_{ij} = k$ for all $j = 1, \ldots, n$, i.e., constant column sums.

The *general reconstruction problem* consists of finding a solution of the system $\boldsymbol{Wx} = \boldsymbol{p}$ for given projection data $\boldsymbol{p}$, i.e., to find an image that has the given projections. In *binary tomography*, one seeks a binary solution of the system. For a given projection matrix $\boldsymbol{W}$ and given projection data $\boldsymbol{p}$, let $S_{\boldsymbol{W}}(\boldsymbol{p}) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{Wx} = \boldsymbol{p}\}$, the set of all real-valued solutions corresponding with the projection data, and let $\bar{S}_{\boldsymbol{W}}(\boldsymbol{p}) = S_{\boldsymbol{W}}(\boldsymbol{p}) \cap \{0,1\}^n$, the set of *binary solutions* of the system. As the main goal of incorporating prior knowledge of the binary grey levels in the reconstruction is to reduce the number of required projections, we focus on the case where m is small with respect to n, such that the real-valued reconstruction problem is severely underdetermined.

For this chapter, the given projection vector $\hat{\boldsymbol{q}}$ is perturbed by noise with $\hat{\boldsymbol{q}} = \boldsymbol{p} + \boldsymbol{\eta}$ where $\boldsymbol{p}$ is the unknown noiseless projection data and $\boldsymbol{\eta}$ is the unkown vector of noise. As the system $\boldsymbol{Wx} = \hat{\boldsymbol{q}}$ is typically inconsistent, one may try to solve the it approximately. Alternatively, it can be useful to consider a *consistent* linear system with a righthand side that is different, but close to $\boldsymbol{p}$. We therefore study the consistent linear system $\boldsymbol{Wx} = \boldsymbol{q}$, where $\boldsymbol{q}$ is the orthogonal projection of $\hat{\boldsymbol{q}}$ onto the range of $\boldsymbol{W}$. As $\boldsymbol{p}$ is not available, we consider the use of an estimated upper bound for the distance between $\boldsymbol{p}$ and $\boldsymbol{q}$ measured by some norm, i.e., $\delta \geq \|\boldsymbol{p} - \boldsymbol{q}\|$ is given for a defined norm.

4.3. A bound on the difference between all binary solutions

Lemma 3 from Chapter 2 allows us to compute the radius of the hypersphere which contains all binary solutions of the noiseless problem. The computation of the radius depends on $\|\boldsymbol{x}^*\|_2$ and $\|\bar{\boldsymbol{x}}\|_2$, the Euclidean norm of the minimum norm solution and the Euclidean norm of any binary solution of $\boldsymbol{Wx} = \boldsymbol{p}$, respectively. Recall that all binary solutions of $\boldsymbol{Wx} = \boldsymbol{p}$

have the same length, given by $||\bar{\boldsymbol{x}}||_2^2 = \frac{||\boldsymbol{p}||_1}{k}$ as proved in Lemma 1.

This section starts by showing that the Euclidean norm of any binary solution of $\boldsymbol{W}\boldsymbol{x} = \boldsymbol{p}$ can be bounded even if $\boldsymbol{p}$ itself is not available, but only a perturbed version of $\boldsymbol{p}$.

Lemma 6. *Let $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^m$ and $\bar{\boldsymbol{x}} \in \bar{S}_W(\boldsymbol{p})$. Assume that $\delta_1 \geq ||\boldsymbol{p} - \boldsymbol{q}||_1$ and define $U_{\bar{\boldsymbol{x}}} = \sqrt{\frac{||\boldsymbol{q}||_1 + \delta_1}{k}}$. Then $||\bar{\boldsymbol{x}}||_2 \leq U_{\bar{\boldsymbol{x}}}$.*

Proof. Write $\boldsymbol{p} = \boldsymbol{q} + \boldsymbol{p} - \boldsymbol{q}$ and use the triangle inequality to obtain

$$||\boldsymbol{p}||_1 \leq ||\boldsymbol{q}||_1 + ||\boldsymbol{p} - \boldsymbol{q}||_1, \tag{4.1}$$

and therefore

$$||\boldsymbol{p}||_1 \leq ||\boldsymbol{q}||_1 + \delta_1. \tag{4.2}$$

Using Eq. (4.2) with $||\bar{\boldsymbol{x}}||_2^2 = \frac{||\boldsymbol{p}||_1}{k}$ and applying Lemma 1 of Chapter 2, we find that

$$||\bar{\boldsymbol{x}}||_2^2 \leq \frac{||\boldsymbol{q}||_1 + \delta_1}{k}, \tag{4.3}$$

as required. □

The next two lemmas are meant to develop a lower bound for $||\boldsymbol{x}^*||_2$.

Lemma 7. *Let $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^m$, $\boldsymbol{x}^* = \boldsymbol{W}^\dagger \boldsymbol{p}$ and $\boldsymbol{z}^* = \boldsymbol{W}^\dagger \boldsymbol{q}$, where $\boldsymbol{W}^\dagger$ denotes the Moore-Penrose inverse of $\boldsymbol{W}$ [12].. Assume that $||\boldsymbol{p} - \boldsymbol{q}||_2 \leq \delta_2$. Then $||\boldsymbol{x}^* - \boldsymbol{z}^*||_2 \leq ||\boldsymbol{W}^\dagger||_2 \delta_2$.*

Proof. As the ℓ_2-norm is a subordinate norm, the following inequality property holds

$$||\boldsymbol{W}^\dagger(\boldsymbol{p} - \boldsymbol{q})||_2 \leq ||\boldsymbol{W}^\dagger||_2 ||\boldsymbol{p} - \boldsymbol{q}||_2 \leq ||\boldsymbol{W}^\dagger||_2 \delta_2. \tag{4.4}$$

□

Lemma 8. *Let $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^m$, $\boldsymbol{x}^* = \boldsymbol{W}^\dagger \boldsymbol{p}$ and $\boldsymbol{z}^* = \boldsymbol{W}^\dagger \boldsymbol{q}$. Assume $||\boldsymbol{p} - \boldsymbol{q}||_1 \leq \delta_1$ and $||\boldsymbol{p} - \boldsymbol{q}||_2 \leq \delta_2$. Define*

$$L_{\boldsymbol{x}^*} = \max\left\{||\boldsymbol{z}^*||_2 - ||\boldsymbol{W}^\dagger||_2 \delta_2, 0\right\}.$$

Then $L_{\boldsymbol{x}^*} \leq ||\boldsymbol{x}^*||_2$.

Proof. A reverse triangle inequality is used to give

$$\Big| ||\boldsymbol{z}^*||_2 - ||\boldsymbol{x}^* - \boldsymbol{z}^*||_2 \Big| \leq ||\boldsymbol{x}^*||_2. \tag{4.5}$$

From Lemma 7 we have $||\boldsymbol{x}^* - \boldsymbol{z}^*||_2 \leq ||\boldsymbol{W}^\dagger||_2 \delta_2$ which in combination with Eq. (4.5) gives the bound $||\boldsymbol{x}^*||_2 \geq ||\boldsymbol{z}^*||_2 - ||\boldsymbol{W}^\dagger||_2 \delta_2$. Hence

$$\max\left\{||\boldsymbol{z}^*||_2 - ||\boldsymbol{W}^\dagger||_2 \delta_2, 0\right\} \leq ||\boldsymbol{x}^*||_2. \tag{4.6}$$

□

Supposing the existence of at least two different binary solutions of the reconstruction problem $\boldsymbol{W}\boldsymbol{x} = \boldsymbol{p}$, Lemmas 6 and 8 allow us to derive an upper bound for the Euclidean distance between those solutions.

Theorem 7. *Let $\boldsymbol{p},\boldsymbol{q} \in \mathbb{R}^m$, $\bar{\boldsymbol{x}},\bar{\boldsymbol{y}} \in \bar{S}_{\boldsymbol{W}}(\boldsymbol{p})$ and $\boldsymbol{x}^* = \boldsymbol{W}^\dagger\boldsymbol{p}$. Define $U_{\bar{\boldsymbol{x}}}$ as in Lemma 6, and $L_{\boldsymbol{x}^*}$ as in Lemma 8. Define $R_{\boldsymbol{p}}^2 = U_{\bar{\boldsymbol{x}}}^2 - L_{\boldsymbol{x}^*}^2$. Then $||\bar{\boldsymbol{y}} - \bar{\boldsymbol{x}}||_2 \le 2R_{\boldsymbol{p}}$.*

Proof. From Theorem 3 of Chapter 2, we have $||\bar{\boldsymbol{x}} - \boldsymbol{x}^*||_2^2 = ||\bar{\boldsymbol{x}}||_2^2 - ||\boldsymbol{x}^*||_2^2$. Using the bound $U_{\bar{\boldsymbol{x}}}$ for $||\bar{\boldsymbol{x}}||_2$ of Lemma 6 and the bound $L_{\boldsymbol{x}^*}$ for $||\boldsymbol{x}^*||_2$ of Lemma 8, both depending on $\boldsymbol{q}$, we have

$$||\bar{\boldsymbol{x}} - \boldsymbol{x}^*||_2^2 \le R_{\boldsymbol{p}}^2. \tag{4.7}$$

Equation 4.7 says that any $\bar{\boldsymbol{x}} \in \bar{S}_{\boldsymbol{W}}(\boldsymbol{p})$ is *not outside* of the n-dimensional hypersphere with radius $R_{\boldsymbol{p}}$ and center $\boldsymbol{x}^*$. Therefore,

$$||\bar{\boldsymbol{x}} - \bar{\boldsymbol{y}}||_2 \le ||\bar{\boldsymbol{x}} - \boldsymbol{x}^*||_2 + ||\bar{\boldsymbol{y}} - \boldsymbol{x}^*||_2 = 2R_{\boldsymbol{p}}. \tag{4.8}$$

□

Corollary 9. *Let $\bar{\boldsymbol{x}},\bar{\boldsymbol{y}} \in \bar{S}_{\boldsymbol{W}}(\boldsymbol{p})$ and define $R_{\boldsymbol{p}}$ as in Theorem 7. Then* $\mathrm{d}(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}) \le 4R_{\boldsymbol{p}}^2$.

Proof. As $\bar{\boldsymbol{x}}$ and $\bar{\boldsymbol{y}}$ are binary, we have $\mathrm{d}(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}) = ||\bar{\boldsymbol{x}} - \bar{\boldsymbol{y}}||_2^2 \le 4R_{\boldsymbol{p}}^2$. □

4.4. Bounds based on rounding

The first result of this section shows that any binary solution of $\boldsymbol{W}\boldsymbol{x} = \boldsymbol{p}$ is *not outside* of a hypersphere centered in $\boldsymbol{z}^*$, the minimal Euclidean norm solution of $\boldsymbol{W}\boldsymbol{x} = \boldsymbol{q}$, of which the radius can be computed based on the given perturbed projection data.

Theorem 8. *Let $\boldsymbol{p},\boldsymbol{q} \in \mathbb{R}^m$, $\boldsymbol{x}^* = \boldsymbol{W}^\dagger\boldsymbol{p}$ and $\boldsymbol{z}^* = \boldsymbol{W}^\dagger\boldsymbol{q}$. Suppose that $\bar{\boldsymbol{x}} \in \bar{S}_{\boldsymbol{W}}(\boldsymbol{p})$. Put $R = \sqrt{||\boldsymbol{x}^* - \boldsymbol{z}^*||_2^2 + ||\bar{\boldsymbol{x}} - \boldsymbol{x}^*||_2^2}$. Then $||\bar{\boldsymbol{x}} - \boldsymbol{z}^*||_2 = R$.*

Proof. As $\boldsymbol{W}\boldsymbol{x}^*, \boldsymbol{W}\boldsymbol{z}^* \in \mathcal{R}(\boldsymbol{W}) = \{\boldsymbol{W}\boldsymbol{x} : \boldsymbol{x} \in \mathbb{R}^n\}$, we also have $(\boldsymbol{W}\boldsymbol{x}^* - \boldsymbol{W}\boldsymbol{z}^*) \in \mathcal{R}(\boldsymbol{W})$. Therefore $\boldsymbol{x}^* - \boldsymbol{z}^* = \boldsymbol{W}^\dagger(\boldsymbol{p} - \boldsymbol{q})$ is the minimal Euclidean norm solution of $\boldsymbol{W}\boldsymbol{x} = (\boldsymbol{p} - \boldsymbol{q})$. Consequently, $(\boldsymbol{x}^* - \boldsymbol{z}^*) \perp (\bar{\boldsymbol{x}} - \boldsymbol{x}^*) \in \mathcal{N}(\boldsymbol{W})$.

Using Pythagoras' theorem we have

$$||\bar{\boldsymbol{x}} - \boldsymbol{z}^*||_2^2 = ||\boldsymbol{x}^* - \boldsymbol{z}^*||_2^2 + ||\bar{\boldsymbol{x}} - \boldsymbol{x}^*||_2^2. \tag{4.9}$$

□

Although we cannot compute the radius $R = \sqrt{||\boldsymbol{x}^* - \boldsymbol{z}^*||_2^2 + ||\bar{\boldsymbol{x}} - \boldsymbol{x}^*||_2^2}$ from Theorem 8, we are going to bound it from above. In order to do that, we need to find a way to bound the term $||\boldsymbol{x}^* - \boldsymbol{z}^*||_2$. Notice that an upper bound for the term $||\bar{\boldsymbol{x}} - \boldsymbol{x}^*||_2^2$ is given in the proof of Theorem 7.

Corollary 10. *Let $\bar{\boldsymbol{x}} \in \bar{S}_{\boldsymbol{W}}(\boldsymbol{p})$, $\boldsymbol{x}^* = \boldsymbol{W}^\dagger\boldsymbol{p}$, $\boldsymbol{z}^* = \boldsymbol{W}^\dagger\boldsymbol{q}$, and $R_{\boldsymbol{p}}$ defined as in Theorem 7. Then $||\bar{\boldsymbol{x}} - \boldsymbol{z}^*||_2 \le R_{\boldsymbol{q}}$, where $R_{\boldsymbol{q}} = \sqrt{\left[\min\left\{||\boldsymbol{W}^\dagger||_2\delta_2,\ \sqrt{\frac{||\boldsymbol{q}||_1+\delta_1}{k}} + ||\boldsymbol{z}^*||_2\right\}\right]^2 + R_{\boldsymbol{p}}^2}$.*

Proof. In Theorem 8 we have shown that

$$\|\bar{\boldsymbol{x}} - \boldsymbol{z}^*\|_2^2 = \|\boldsymbol{x}^* - \boldsymbol{z}^*\|_2^2 + \|\bar{\boldsymbol{x}} - \boldsymbol{x}^*\|_2^2. \tag{4.10}$$

Using the upper bounds from Lemma 7 and the proof of Theorem 7,

$$\|\bar{\boldsymbol{x}} - \boldsymbol{z}^*\|_2^2 \leq \left[\min\left\{\|\boldsymbol{W}^\dagger\|_2\delta_2,\ \sqrt{\frac{\|\boldsymbol{q}\|_1 + \delta_1}{k}} + \|\boldsymbol{z}^*\|_2\right\}\right]^2 + R_{\boldsymbol{p}}^2, \tag{4.11}$$

completing the proof. □

Let $\langle\alpha\rangle = \min(|\alpha|, |\alpha - 1|)$ for $\alpha \in \mathbb{R}$, and put $T_{\boldsymbol{q}} = \sqrt{\sum_{i=1}^n \langle z_i^*\rangle^2}$, i.e., the Euclidean distance from $\boldsymbol{z}^*$ to the nearest binary vector. Let $\mathcal{T} = \{\bar{\boldsymbol{v}} \in \{0,1\}^n : \|\bar{\boldsymbol{v}} - \boldsymbol{z}^*\|_2 = T_{\boldsymbol{q}}\}$ and let $\bar{\boldsymbol{r}} \in \mathcal{T}$, i.e., $\bar{\boldsymbol{r}}$ is among the binary vectors that are nearest to $\boldsymbol{z}^*$ in the Euclidean sense.

If $T_{\boldsymbol{q}} < R_{\boldsymbol{q}}$ and $R_{\boldsymbol{q}} - T_{\boldsymbol{q}}$ is small, it is possible to say that a fraction of the rounded values are correct, i.e., to provide an upper bound on the *number* of pixel differences between any solution in $\bar{S}_{\boldsymbol{W}}(\boldsymbol{p})$ and $\bar{\boldsymbol{r}}$. In most cases we can not say *which* rounded values are correct.

For any two vectors $\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}} \in \{0,1\}^n$, define the *difference set* $\mathrm{D}(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}) = \{i : \bar{u}_i \neq \bar{v}_i\}$ and the *number of differences* $\mathrm{d}(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}) = \#\mathrm{D}(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}})$, where the symbol # denotes the cardinality operator for a finite set. Note that $\mathrm{d}(\bar{\boldsymbol{u}}, \bar{\boldsymbol{v}}) = \|\bar{\boldsymbol{u}} - \bar{\boldsymbol{v}}\|_1$.

Lemma 9. *Let $\bar{\boldsymbol{r}} \in \mathcal{T}$ and let $\bar{\boldsymbol{v}} \in \{0,1\}^n$ be any binary vector.*
Then $\|\bar{\boldsymbol{v}} - \boldsymbol{z}^*\|_2^2 = T_{\boldsymbol{q}}^2 + \sum_{i \in \mathrm{D}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{r}})} |2z_i^* - 1|$.

Proof. We have the following identities:

$$\begin{aligned}
\|\bar{\boldsymbol{v}} - \boldsymbol{z}^*\|_2^2 &= \|\bar{\boldsymbol{v}} - \bar{\boldsymbol{r}} + \bar{\boldsymbol{r}} - \boldsymbol{z}^*\|_2^2 \\
&= \|\bar{\boldsymbol{r}} - \boldsymbol{z}^*\|_2^2 + 2\langle\bar{\boldsymbol{r}} - \boldsymbol{z}^*, \bar{\boldsymbol{v}} - \bar{\boldsymbol{r}}\rangle + \langle\bar{\boldsymbol{v}} - \bar{\boldsymbol{r}}, \bar{\boldsymbol{v}} - \bar{\boldsymbol{r}}\rangle \\
&= T_{\boldsymbol{q}}^2 + 2\langle\bar{\boldsymbol{r}} - \bar{\boldsymbol{v}}, \boldsymbol{z}^*\rangle + \|\bar{\boldsymbol{v}}\|_2^2 - \|\bar{\boldsymbol{r}}\|_2^2 \\
&= T_{\boldsymbol{q}}^2 + 2\sum_{i=1}^n (\bar{r}_i - \bar{v}_i) z_i^* + \sum_{i=1}^n \bar{v}_i - \sum_{i=1}^n \bar{r}_i \\
&= T_{\boldsymbol{q}}^2 + \sum_{i=1}^n (\bar{r}_i - \bar{v}_i)(2z_i^* - 1) \\
&= T_{\boldsymbol{q}}^2 + \sum_{i \in \mathrm{D}(\bar{\boldsymbol{v}}, \bar{\boldsymbol{r}})} |2z_i^* - 1|.
\end{aligned}$$

□

Define $s_i = |2z_i^* - 1|$ and let π be a permutation of $\{1, \ldots, n\}$ such that $s_{\pi(1)} \leq s_{\pi(2)} \leq \ldots \leq s_{\pi(n)}$, which can be obtained by sorting the entries s_i in increasing order.

Theorem 9. *Let $\bar{\boldsymbol{r}} \in \mathcal{T}$, $\bar{\boldsymbol{x}} \in \bar{S}_{\boldsymbol{W}}(\boldsymbol{p})$. Put*

$$U = \max\left\{0 \leq \ell \leq n : \sum_{i=1}^{\ell} s_{\pi(i)} \leq R_{\boldsymbol{q}}^2 - T_{\boldsymbol{q}}^2\right\}.$$

Then $\mathrm{d}(\bar{\boldsymbol{x}}, \bar{\boldsymbol{r}}) \leq U$.

Proof. As $\bar{\boldsymbol{x}} \in \bar{S}_{\boldsymbol{W}}(\boldsymbol{p})$, we have $\|\bar{\boldsymbol{x}} - \boldsymbol{z}^*\|_2^2 \le R_{\boldsymbol{q}}^2$. Applying Lemma 9, we find that

$$R_{\boldsymbol{q}}^2 - T_{\boldsymbol{q}}^2 \ge \sum_{i \in D(\bar{\boldsymbol{x}},\bar{\boldsymbol{r}})} s_i \ge \sum_{i=1}^{\mathrm{d}(\bar{\boldsymbol{x}},\bar{\boldsymbol{r}})} s_{\pi(i)},$$

which implies that $\mathrm{d}(\bar{\boldsymbol{x}},\bar{\boldsymbol{r}}) \le U$. □

Similarly to Theorem 9, another bound on the difference between any two binary solutions of $\boldsymbol{W}\boldsymbol{x} = \boldsymbol{q}$ can be derived.

Theorem 10. *Let $\bar{\boldsymbol{r}} \in \mathcal{T}$, $\bar{\boldsymbol{x}},\bar{\boldsymbol{y}} \in \bar{S}_{\boldsymbol{W}}(\boldsymbol{p})$. Put*

$$V = \max\left\{0 \le \ell \le n : \sum_{i=1}^{\ell} s_{\pi(i)} \le 2(R_{\boldsymbol{q}}^2 - T_{\boldsymbol{q}}^2)\right\}.$$

Then $\mathrm{d}(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}) \le V$.

Proof. Define $\hat{\boldsymbol{x}}$ by $\hat{x}_i = \bar{r}_i$ if $\bar{x}_i = \bar{y}_i$, and $\hat{x}_i = \bar{x}_i$ otherwise. Define $\hat{\boldsymbol{y}}$ analogously. Then $\mathrm{d}(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}}) = \mathrm{d}(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}})$, $\|\hat{\boldsymbol{x}} - \boldsymbol{z}^*\|_2^2 \le R_{\boldsymbol{q}}^2$, and $\|\hat{\boldsymbol{y}} - \boldsymbol{z}^*\|_2^2 \le R_{\boldsymbol{q}}^2$. Hence,

$$2R_{\boldsymbol{q}}^2 \ge \|\hat{\boldsymbol{x}} - \boldsymbol{z}^*\|_2^2 + \|\hat{\boldsymbol{y}} - \boldsymbol{z}^*\|_2^2 = 2T_{\boldsymbol{q}}^2 + \sum_{i \in D(\bar{\boldsymbol{r}},\hat{\boldsymbol{x}})} s_i + \sum_{i \in D(\bar{\boldsymbol{r}},\hat{\boldsymbol{y}})} s_i.$$

As $D(\bar{\boldsymbol{r}},\hat{\boldsymbol{x}})$ and $D(\bar{\boldsymbol{r}},\hat{\boldsymbol{y}})$ are disjoint, we have $2R_{\boldsymbol{q}}^2 - 2T_{\boldsymbol{q}}^2 \ge \sum_{i=1}^{d(\bar{\boldsymbol{r}},\hat{\boldsymbol{x}})+d(\bar{\boldsymbol{r}},\hat{\boldsymbol{y}})} s_{\pi(i)}$. This implies that $d(\bar{\boldsymbol{x}},\bar{\boldsymbol{y}}) = d(\hat{\boldsymbol{x}},\hat{\boldsymbol{y}}) \le d(\bar{\boldsymbol{r}},\hat{\boldsymbol{x}}) + d(\bar{\boldsymbol{r}},\hat{\boldsymbol{y}}) \le V$. □

4.5. Computations

In several intermediate bounds developed in this chapter, and in particular in Corollary 10, the computation of the ℓ_2-norm of $\boldsymbol{W}^\dagger$, the Moore-Penrose pseudo-inverse of the projection matrix, is required. Computing the pseudo-inverse $\boldsymbol{W}^\dagger$ is expensive in terms of number of operations and memory requirements. A computer equipped with 8 GB memory ran out of memory when computing the matrix $\boldsymbol{W}^\dagger$ for images larger than 64×64. There is, however, a way to compute $\|\boldsymbol{W}^\dagger\|_2$ without computing $\boldsymbol{W}^\dagger$ explicitly.

A matrix norm based on a vector norm satisfies certain properties that are analogous to properties of this vector norm [30]. As the ℓ_2- matrix norm is unitary invariant, using the SVD decomposition of $\boldsymbol{W}^\dagger$ we have

$$\|\boldsymbol{W}^\dagger\|_2 = \|\boldsymbol{V}\boldsymbol{\Sigma}^\dagger\boldsymbol{U}^T\|_2 = \|\boldsymbol{\Sigma}^\dagger\|_2 = \sigma_{\max}(\boldsymbol{W}^\dagger),$$

where $\boldsymbol{V} \in \mathbb{R}^{n\times n}$ is unitary, $\boldsymbol{\Sigma}^\dagger \in \mathbb{R}^{n\times m}$ is a rectangular diagonal matrix with non-negative entries, $\boldsymbol{U}^T \in \mathbb{R}^{m\times m}$ is unitary, and $\sigma_{\max}(\boldsymbol{W}^\dagger)$ is the largest singular value of $\boldsymbol{W}^\dagger$. The matrix $\boldsymbol{\Sigma}^\dagger$ is the pseudo-inverse of $\boldsymbol{\Sigma}$, which is formed by replacing every non-zero diagonal entry by its reciprocal and transposing the resulting matrix. The diagonal entries of $\boldsymbol{\Sigma}$ are the singular values of $\boldsymbol{W}$. Therefore,

$$\|\boldsymbol{W}^\dagger\|_2 = \frac{1}{\sigma_{\min}(\boldsymbol{W})}.$$

There are computational techniques for computing the smallest singular value of a matrix without a complete SVD decomposition, such as the *power iteration* [48] (also known as *Von Mises iteration*) applied to the shifted matrix $\boldsymbol{W}\boldsymbol{W}^{\boldsymbol{T}} - \lambda_{\max}\boldsymbol{I}$ with $\lambda_{\max}$ being the largest eigenvalue of $\boldsymbol{W}\boldsymbol{W}^{\boldsymbol{T}}$. The singular values of $\boldsymbol{W}$ are the square roots of the non-negative eigenvalues of $\boldsymbol{W}\boldsymbol{W}^{\boldsymbol{T}}$. However, due to the ill-posed nature of the reconstruction problem, the smallest singular value can often not be distinguished from small values originated from numerical errors.

4.6. Conclusions

The nature of the reconstruction problem and the numerical errors in the computation make the computed smallest singular value of $\boldsymbol{W}$ unsuitable for effective use in the bounds presented in this chapter. Very small computed values for $\sigma_{\min}$, which can possibly be attributed to numerical errors, result in computed upper bounds that are too big to be useful in practice. In an attempt to overcome this problem, we will demonstrate in the next chapter how approximations can be made that avoid the computation of $||\boldsymbol{W}^{\dagger}||_2$, resulting in non-guaranteed bounds that can still be useful in practice.