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Error bounds for discrete tomography

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Chapter 1

Introduction

In this chapter we introduce the concept of discrete tomography and explain the basic concepts involved in this thesis. Furthermore, we describe the key problems considered in this thesis and outline the main results.

1.1. Discrete Tomography

Tomography is a technique for reconstructing an image from a series of projections of it. Such projections are acquired from a range of viewing angles. Images can be continuous or discrete in space. Space continuous 2-D images are often discretized in small squares, named *pixels*, which are assumed to have only one intensity value associated with it. Space discrete 2-D images are defined on a subset of $\mathbb{Z}^2$ and its points (or pixels) are assigned an intensity value. In both cases, reconstructed images are represented as a finite vector with each entry representing one pixel of the reconstruction region. All pixels outside of the region of interest are assumed to have zero intensity.

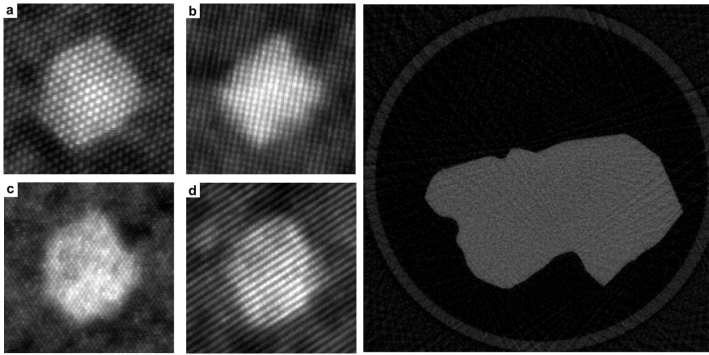
In this thesis, the term *discrete tomography* refers to the reconstruction of images with intensity values belonging to a small discrete subset of $\mathbb{R}$ and either continuous or discrete in space. Images with intensity values restricted to $\{0, 1\}$ are called *binary images*. We refer to the corresponding reconstruction techniques as *binary tomography*.

The projection process is defined by the projection model and the set of angles and can be represented by a linear transformation. Given a set of projections, finding an image satisfying these projections is an inverse problem for which the existence and uniqueness of the solution is, in general, not guaranteed. The problem of finding an image from a set of projections is known as the *reconstruction problem*. Methods to find a solution to the reconstruction problem often yield an image that approximately satisfies the given set of projections. A solution or approximate solution of the reconstruction problem is called a *reconstruction*. If only a few projection angles are available, there may be large differences between the solutions of the *discrete reconstruction problem*, the reconstruction problem in discrete tomography.

An application of discrete tomography is the reconstruction of nano-crystals at atomic

resolution. In this problem, discrete atoms are positioned on a regular grid (in this case, it is a 3-D image). By using an electron microscope, 2-D projections are acquired from various angles, as displayed in Fig. 1.1(a) [1, 31].

Another application of binary tomography can be found in the imaging of diamonds. Diamonds are structures composed of only one element: carbon. The problem of reconstructing a diamond is a binary tomography problem where the carbon corresponds with the foreground and its absence corresponds with the background. Fibrous diamonds, however, may have cracks containing material with a small number of different densities [51]. In Fig. 1.1(b) we show a slice of a 3-D reconstruction of a diamond on a cylindrical holder. In Chapter 5 we present experiments with projections of a raw diamond.



(a) 2-D projections of a gold nano- (b) Tomographic reconstruction of a dia-
particle, reproduced with permission from mond from X-ray micro-CT data; cour-
[1]. tesy of DiamCAD, Antwerp.

Figure 1.1: *Discrete tomography examples.*

There are also applications of discrete tomography in medical imaging, despite of its restricted use. Discrete tomography can be applied when some physical property of the organ of interest can be enhanced as compared to the surrounding tissue. As an example, see [29] for angiographic applications.

1.2. Difference between reconstructions

Despite the strong constraint imposed on the grey values in discrete tomography, many solutions of the reconstruction problem can exist, all corresponding to the same set of projections. If the projections are obtained by performing measurements on some unknown ground truth image, the reconstruction can then deviate substantially from the original image, as exemplified in Fig. 1.2. As a consequence, there is a need for a measurement on the difference between solutions of the reconstruction problem. As the ground truth is a solution by itself, this would also yield a bound on the reconstruction error with respect to the ground truth.

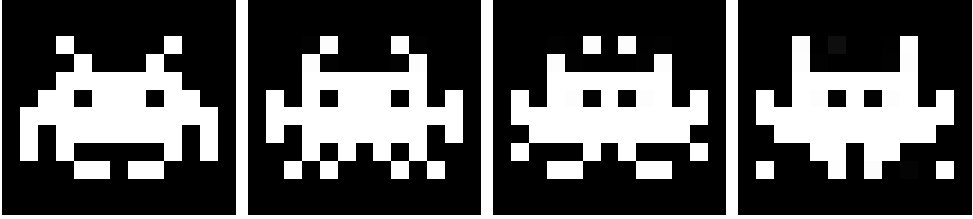


Figure 1.2: All images have equal line sums in the vertical and horizontal directions.

For images represented on a discrete grid, both lower and upper bounds have been obtained for the magnitude of changes in discrete solutions when the projections are slightly perturbed [2–4, 46]. For the case of binary image reconstruction from just two projections, horizontal and vertical, bounds on the difference between binary images having the same projections have been obtained by Van Dalen [44, 45].

In this thesis, we develop a series of computable upper bounds on the difference between reconstructions of binary images. Our approach is based on ideas initially proposed by Hajdu and Tjeldeman [22]. When computing the absolute difference between corresponding pixels of two known binary images, one can identify whether such pixels differ or not, which allows the classification of a pixel of an image as correct or wrong with respect to the other image. Despite the fact that our methodology can be adapted to compute the cited bounds for non-binary images with small number of different grey values, we have restricted its development to binary images only.

In Chapter 2 we demonstrate that for parallel beam tomography, some projection models yield the property that the total intensity of all binary solutions of the reconstruction problem must be the same, and it can be computed directly from the projection data. Furthermore, all binary solutions lie on a hypersphere of which the center and radius are known. Based on these observations, a methodology is derived to compute an upper bound on the difference between any two binary solutions. As reconstruction algorithms may find an approximate solution, we also derive a bound on the difference between a given binary reconstruction and any binary solution.

Similar to the tomography problem, we investigate the problem of image reconstruction from low resolution scans. Video recordings, picture cameras and other devices record images subject to resolution restrictions. A sequence of recordings may be used to acquire an image with higher resolution than each recording individually. For such settings, Chapter 3 presents error bounds on the higher resolution reconstructed images.

Chapter 4 extends the previously developed bounds to deal with noise in the projection data. When perturbing the projection data, the reconstruction problem is likely to be inconsistent, having no solution. When solving inconsistent reconstruction problems, one finds solutions of an approximate problem. In this case, the set of solutions of the reconstruction problem may change significantly, yielding big differences between reconstructions due to the ill-posed nature of these inverse problems. Furthermore, the set of solutions may not contain the original image which generated the projection data. The error bounds developed

in this chapter provide bounds on the image error with respect to the solutions of the noiseless reconstruction problem, for which the projections are not known. Intending to make the error bounds useful in practice, we introduce parametrized approximations in these bounds in Chapter 5. The parameters are based on experiments with images in a controlled environment and then applied to a similar problem with real projection data. The resulting approximate bounds are no longer mathematical bounds, but still follow the same general behaviour as the true errors measured for phantom images.

So far, the error bounds were restricted to projections models with the property of conservation of total intensity of binary solutions. As a consequence, all binary solutions have the same total intensity. Chapter 6 studies the general case where any projection model can be used to compute reconstruction error bounds. In fact, these techniques can be used for any application modelled as an algebraic linear system of equations with binary solutions.

1.3. Discrete reconstruction algorithms

Discrete reconstruction algorithms exploit prior knowledge about the discreteness of the image that is being reconstructed. This can improve the reconstruction quality, or reduce the number of projections required while keeping the reconstruction quality the same. Another advantage of discrete tomography reconstruction algorithms is that the resulting reconstruction is an image that is already segmented.

A range of reconstruction algorithms for discrete tomography have been proposed in the literature, [6, 9, 28, 40], however none of them comes with a guarantee that an exact solution of the discrete tomography problem is found for the general problem, from more than two directions. At the same time, the results of computational experiments show many cases of a near-optimal approximate solution, or even a reconstruction that is completely identical to the original image from which the projections were taken.

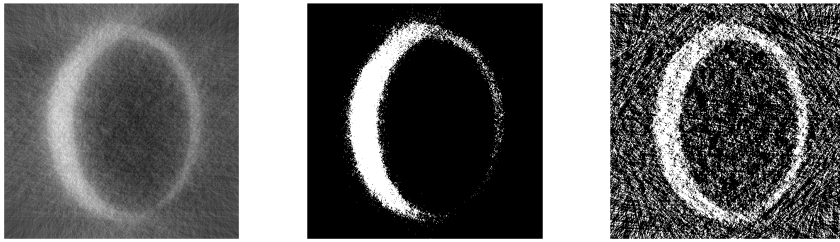
A major problem with these algorithms is the fact that the error - both in the reconstructed image (with respect to the ground truth image) and in the projected reconstructed image (with respect to the given projections) - depends on the particular problem instance and cannot be bounded sharply. We are not aware of any algorithm for which non-trivial bounds have been described for the difference between the projections of the reconstructed (discrete) image and the given projections.

In Chapter 7, we present a discrete approximate reconstruction algorithm that comes with such guarantees. The algorithm computes an image that has only grey values belonging to the given finite set (prior knowledge). It also guarantees that the difference between the given projections and the projections of the reconstructed discrete image is bounded. The bound is independent of the image size and proportional to the number of projection angles.

1.4. Feature detection

As mentioned in Section 1.2, the reconstruction problem may allow the existence of binary solutions that are substantially different from each other. Furthermore, Section 1.3 indicates how discrete reconstruction algorithms may not find an exact solution of the reconstruction

problem, but an approximation of it. When one is interested in finding a specific feature in binary tomography images, a specific reconstruction (either a solution or an approximate one) may not truly represent the original ground truth image. As an example, Figure 1.3 presents three different reconstructions of a ring-like shaped original image in which it is very difficult to determine, based on these reconstructions, whether the bottom right part of the ring is open.



(a) A continuous reconstruction (b) A binary reconstruction (c) Another binary reconstruction

Figure 1.3: Reconstructions of a ring-like shaped object using three different reconstruction methods. Is the “ring” open?

Even in cases when insufficient information is available to compute an accurate reconstruction of the complete image, it may still be possible to answer certain questions about the original image, or to determine certain features of it. Although finding a binary solution of the reconstruction problem is typically hard, it is often easier to prove that a solution with a specific feature *cannot* exist. For example, if the projections do not satisfy certain consistency conditions, a solution will certainly not exist.

When developing the error bounds for binary tomography, an existence condition for binary solutions of the reconstruction problem was found. In Chapter 8 we study the case where a pre-defined binary structure is enforced in the reconstruction problem and then a *consistency* condition for binary solutions is checked. By applying this methodology, it can be determined whether such a substructure can possibly occur, or whether it can certainly not occur in any binary image of the solution set.

1.5. Overview

As a conclusion of this introduction, we now provide a brief overview of the material contained in each of the next chapters.

In **Chapter 2**, we derive a series of upper bounds that can be used to guarantee the quality of a reconstructed binary image. The bounds limit the number of pixels that can be incorrect in the reconstructed image, in binary tomography, with respect to the original image. We provide several versions of these bounds, ranging from bounds on the difference between any two binary solutions of a tomography problem to bounds on the difference between

approximate solutions and the original object.

In **Chapter 3**, we consider the problem of reconstructing a high-resolution binary image from several low-resolution scans. Each of the pixels in a low-resolution scan yields the value of the sum of the pixels in a rectangular region of the high-resolution image. For any given set of such pixel sums, we derive an upper bound on the difference between a certain binary image which can be computed efficiently and *any* binary image that corresponds with the given measurements. We also derive a bound on the difference between any two binary images having these pixel sums.

In **Chapter 4** we expand the theory of error bounds for binary tomography of Chapter 2 to the case of noisy projection data. Despite the fact that the noiseless projection data is not available, we develop error bounds with respect to the solution set of the noiseless reconstruction problem.

In **Chapter 5** we show how the error bounds of Chapter 2 can be adapted to be useful for bounding the quality of experimental images. Our experimental results suggest that even though approximations have to be made due to noise and other errors in the data, the resulting bounds can still provide guidance on estimating the reconstruction quality in practice.

In **Chapter 6**, we present a series of computable bounds that can be used with any projection model. The approach developed in Chapter 2 is restricted to projection models where the corresponding matrix has constant column sums. We generalize these results and thereby broaden their applicability to include fan beam and cone beam projection models. In fact, the study presented here is not restricted to tomography and works for more general linear systems. We report the results of computational experiments for several phantom images, focused on parallel and fan beam projection models.

In **Chapter 7**, we develop a discrete approximate reconstruction algorithm. Our algorithm computes an image that has only grey values belonging to a given finite set. It also guarantees that the difference between the given projections and the projections of the reconstructed discrete image is bounded. The bound, which is computable, is independent of the image size.

In **Chapter 8**, we present a computational technique for discovering the possible presence of *features* (such as straight boundaries or homogeneous regions) in the unknown original image from its projections. We show that it is often possible to accurately identify the presence of certain features, even when insufficient information is available to compute an accurate reconstruction of the complete image.