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Broken symmetries in field theory

Kok, M.O. de

Citation

Kok, M. O. de. (2008, June 26). *Broken symmetries in field theory*. Instituut-Lorentz for Theoretical Physics (Leiden) + NIKHEF, Faculty of Science, Leiden University. Retrieved from <https://hdl.handle.net/1887/12983>

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Broken Symmetries in Field Theory

M.O. de Kok

Broken Symmetries in Field Theory

P R O E F S C H R I F T

TER VERKRIJGING VAN
DE GRAAD VAN DOCTOR AAN DE UNIVERSITEIT LEIDEN,
OP GEZAG VAN RECTOR MAGNIFICUS
PROF. MR. P.F. VAN DER HEIJDEN,
VOLGENS BESLUIT VAN HET COLLEGE VOOR PROMOTIES
TE VERDEDIGEN OP DONDERDAG 26 JUNI 2008
KLOKKE 15.00 UUR

DOOR

Mark Okker de Kok

GEBOREN TE HAARLEM IN 1978

Promotiecommissie:

Promotores: Prof. dr. P.J. van Baal
Prof. dr. J.W. van Holten (NIKHEF, Vrije Universiteit A'dam)
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Overige leden: Prof. dr. J.M. van Ruitenbeek
Dr. K.E. Schalm
Prof. dr. K. Schoutens (Universiteit van Amsterdam)

ISBN 978-90-9023137-2

The work described in this thesis is part of the research programme of the Dutch Foundation for Fundamental Research on Matter (FOM), which is supported financially by the Netherlands Organisation for Scientific Research (NWO).

Et si volueris attendere, magna pars vitae elabitur male agentibus,
maxima nihil agentibus, tota vita aliud agentibus.

Lucius Annaeus Seneca
Epistulae Morales ad Lucilium
Liber I.I.I

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INTRODUCTION

The description of quantum mechanical systems with an arbitrary number of particles is most straightforwardly done using *quantum field theory*. In relativistic theories of physics, relevant for describing physics at high energies, there is no conservation of particle number. This is because relativistically energy can be converted into particles and vice versa via Einstein's famous mass-energy equivalence relation. Relativistic quantum field theory is thus the natural playground for describing the high energy physics of elementary particles. Quantum field theory has found its use not only in the realm of high energy physics, but has also proven to be of crucial importance in describing phenomena in condensed matter physics such as superconductivity.

One of the key concepts of modern field theory is *symmetry* [70, 94, 110, 111]. Symmetry in physics means invariance under some kind of transformation. Different types of symmetries must be distinguished. One type of symmetry is *internal symmetry*. For example when dealing with some kind of elementary particle that comes in two different 'flavours', the quantum mechanical description of this particle allows it to be in a mixture or superposition of the two flavours. Mathematically this flavour physics is described using a two dimensional vector space, where each flavour corresponds to one of the two vectors of an orthogonal basis. Since a choice of basis in this space is physically arbitrary, the model is invariant under rotations in this vector space. This symmetry does not affect the space-time the particles live in, but only the internal space that is chosen to describe the physics of the two flavours the particle comes in. It is therefore called an internal symmetry.

In this description of the flavour symmetry, the basis to describe the flavour physics is chosen the same for each space-time point. It is thus a *global symmetry*. A generalization of this symmetry concept is given by choosing the flavour basis for each space-time point independently.

The symmetry thus becomes a *local symmetry* or *gauge symmetry*. Doing this consistently requires the introduction of additional gauge particles. For example, in the *Standard Model* of elementary particle physics the quarks come in three different *colours*, and their physics is described by an $SU(3)$ gauge invariance, where $SU(3)$ is the group describing the internal rotational symmetry. The gauge particles are the gluons that mediate the *strong force* between the quarks. The model that describes quarks and gluons in this way is called *Quantum Chromodynamics* or *QCD*.

In a similar way the *electroweak force* describes the unification of the *electromagnetic force* and the *weak force* that is responsible for radioactivity using an $SU(2) \times U(1)$ gauge symmetry. This gauge symmetry describes the physics of the leptonic particles, such as the electron, muon, tauon and the neutrinos. Furthermore, the quarks transform under $SU(2) \times U(1)$ as well. The electroweak force is mediated via the photon and the W - and Z -bosons.

At energies below the electroweak breaking scale however, the internal $SU(2) \times U(1)$ gauge symmetry is broken to a $U(1)$ gauge symmetry via the *Higgs mechanism*. This remaining $U(1)$ symmetry describes electromagnetism. The breaking is done by adding a new field to the Standard Model, the *Higgs field*, that has a potential energy term in such a way that below the energy of the electroweak breaking scale a preferred direction in the internal vector space of the $SU(2) \times U(1)$ gauge symmetry needs to be taken. This breaks the $SU(2) \times U(1)$ symmetry down to a $U(1)$ symmetry. In this mechanism of *spontaneous symmetry breaking* the W - and Z -bosons describing the weak force acquire a mass. This mechanism furthermore leads to the *Higgs particle*, which is the only particle in the Standard Model that remains to be found. Spontaneous symmetry breaking is in contrast with *dynamical symmetry breaking* to be discussed later.

Another type of symmetry is *space-time symmetry*. These combine transformations of the coordinates of space-time with transformations of the fields in the model in such a way that the model remains invariant. For example, physical theories are usually required to be invariant under translations in both time and space and under rotations of space-time. Invariance under time- and space-translations signifies that physics is assumed to be independent of when and where an experiment is performed. When a model is rotationally invariant this means there is assumed to be no preferred direction in space.

Furthermore, relativistic theories are required to be invariant under Lorentz transformations of the space-time coordinates, which is a manifestation of Einstein's principle that all inertial frames are physically equivalent. Along the same lines non-relativistic theories are required to be invariant under Galilei transformations.

An interesting class of space-time symmetries is given by the *conformal symmetries*. Conformal symmetries include the *scale symmetry* of a model. This symmetry means that under a rescaling of the space and time coordinates the model remains invariant. Since zooming into space-time is equivalent to studying a model at a higher energy scale, this means that the physics of a scale invariant model is independent of the energy scale one is looking at. Only very few models are known to be exactly scale invariant, except in two dimensions [7]. In models which are classically scale invariant, such as QCD with massless quarks, this symmetry is generally broken because quantum mechanical effects in perturbation theory generate an energy scale dependence in the model. This happens via *renormalization* effects. The infinities that generically arise when calculating quantum mechanical corrections to the fields, masses and coupling constants of a model in a perturbative expansion are dealt with in the following way: it is assumed that the original fields, masses and coupling constants are actually the ones that are (possibly) infinite, in such a way that the corrected or renormalized ones are finite. When this procedure is possible, the model is called *renormalizable*. Doing this consistently, the *renormalization group equations* dictate that the fields, masses and coupling constants of the model become a function of the energy scale [16, 94, 105, 120]. The breaking of the scale invariance is thus seen to dictate the range of applicability of the model in perturbation theory, since this requires the coupling constants to be relatively small. In this thesis precisely such an effect is observed in the non-relativistic Schrödinger model, as published in [38] and discussed in chapter 2, and in the Jackiw-Pi model, as discussed in chapter 3.

This breaking of the scale invariance is an example of a dynamical breaking of a symmetry by quantum effects. When a symmetry of a classical model is broken after quantizing the model, this symmetry is called *anomalous*.

A third type of symmetry is *supersymmetry*. Supersymmetry is both an internal and a space-time symmetry. It is an internal symmetry since su-

persymmetry in essence is a symmetry between bosonic particles (having integer spin like the photon) and fermionic particles (having half-integer spin like the electron), such that each bosonic particle gets a fermionic partner of the same mass and vice versa. At the same time supersymmetry can be understood as the only possible non-trivial extension of the translation, rotation plus Lorentz or Galilei space-time symmetries.

Supersymmetry is a symmetry in the theory of elementary particles that was developed in the early 1970's [55,60,89,90,97,109,114,115]. Since supersymmetry is not observed in nature at the energy scales investigated so far, it is clear that it is not a true symmetry of nature below energies of a few hundred GeV. However, supersymmetry might exist above this energy scale. If this is the case, it would solve many theoretical problems in high energy physics [10,70,112,113,116]. Supersymmetric theories suffer much less from the quantum mechanical effects that are seen to break for example the scale symmetry in many models. This effect is due to cancellations between fermionic quantum loops (carrying a minus sign) and bosonic quantum loops (carrying a plus sign) in perturbation theory, because of the symmetry between bosons and fermions; this is manifested by the *nonrenormalization theorems*. For example, in four dimensions the $N = 4$ supersymmetric Yang-Mills model is known to be scale invariant to all orders in perturbation theory [83].

Because of this reason supersymmetry is very succesful in addressing for example the problem of the smallness of the cosmological constant, the hierarchy problem of the huge energy gap between the scale at which grand unification is expected and the scale at which electromagnetism and the weak force unify, or the issue of renormalisation of quantum gravity. However, considering for example the hierarchy problem, there might be a lot of new and unexpected physics in between the two distant energy scales, obscuring the role that supersymmetry should play.

Furthermore, there is another, more philosophical, argument to study supersymmetry. As mentioned, supersymmetry is the only possible non-trivial extension [27] of ordinary symmetries of space-time. Since nature is often seen to use the maximal freedom it has according to mathematical physics, it is an interesting issue whether this is also the case for supersymmetry.

Postulating supersymmetry to exist in quantum field theories of elementary particles at higher energy scales makes it necessary to have some mechanism that breaks the supersymmetry at lower energy scales.

This breaking can be done by introducing explicit breaking terms in the model; however, this amounts to introducing all mass scales of the particles and their supersymmetric partners by hand rather than explaining them. One would like to have a dynamical quantum mechanism of supersymmetry breaking, like the breaking of the scale symmetry discussed before. In case of four dimensional supersymmetric theories there is a theorem stating that supersymmetry will not be broken by quantum effects in perturbation theory if it was not already broken directly upon quantization [54]. One is therefore led to investigate non-perturbative mechanisms for the breaking of supersymmetry. Instantons for example might play a role here [28, 99, 121]. Due to the presence of extra symmetries in supersymmetric field theory models, some of them have been fully solved non-perturbatively in an analytic way, whereas this is difficult if not impossible in non-supersymmetric theories [100, 101].

In general non-perturbative effects in field theories can systematically be investigated by doing computer simulations, as is done extensively in gauge theories to study for example the problem of quark confinement [119]. Since computers can only handle a finite amount of data, the model will have to be defined on a discrete spacetime, i.e. on a (usually square rectangular) lattice, where fields are either defined as living on the lattice sites or on the links between them. This explains the interest in supersymmetric lattice field theories. Unfortunately it is very difficult to get a lattice version of supersymmetry since the discretization naturally breaks the supersymmetry [43, 45, 49, 57, 59, 76].

In this thesis two approaches towards supersymmetry on the lattice are studied. One of them, the *noncommutativity approach* [32–34], is studied in the context of supersymmetric quantum mechanics [28, 29]. This model can be formulated as a field theory in one time and zero space dimensions, and is one of the simplest supersymmetric systems. With its minimal field content this system is the simplest toy model to test non-perturbative features of supersymmetric field theories.

The other approach studied is the *link approach* [35–37]. Unfortunately, both approaches are explained to be inconsistent [13–15], which is the main topic of chapter 1 on lattice supersymmetry.

CHAPTER 1

SUPERSYMMETRY ON THE LATTICE

1.1 Introduction

This chapter begins with a discussion of supersymmetry that puts emphasis on those aspects that are especially important when dealing with lattice supersymmetry. It continues with an overview of the main problem with and main approaches to defining supersymmetric field theories on a lattice. Two of those approaches are discussed in great detail. First the noncommutativity approach is studied, explaining both how it applies to supersymmetric quantum mechanics on the lattice and how this approach is actually seen to be inconsistent in general. Second the related link approach to supersymmetry on the lattice, which is especially suited for dealing with gauge fields, is studied and shown to be inconsistent as well.

1.2 Essentials of Supersymmetry

Supersymmetry in field theories essentially means a symmetry between the bosonic and fermionic fields. Under the supersymmetry transformations that mix these types of fields in a particular way, the action is unaltered. The supersymmetry transformations do not mix known particle fields like for instance electrons and photons, rather the concept of supersymmetry in field theory actually requires the introduction of new particle fields.

Each bosonic field gets a fermionic partner of the same mass and vice versa. Moreover, supersymmetry invariance generally fixes the coupling constants of the interactions in the theory with respect to each other.

A One Dimensional Example

One of the simplest examples of a supersymmetric field theory available is the following one dimensional theory with one real scalar field ϕ and one complex Dirac field ψ , a single component Grassmannian:

$$S = \int dt \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}m^2 \phi^2 + \psi^\dagger (i\partial_t - m)\psi, \quad (1.2.1)$$

where the dagger $\dagger$ denotes Hermitian conjugation. This is the action of *supersymmetric quantum mechanics* without interactions [28, 29, 121], formulated as a field theory. This theory is amongst the simplest supersymmetric field theories, but still has the essential supersymmetry structures.

The action can be seen to be invariant under two independent sets of transformations, T and $T^\dagger$ (see table 1.1), making it an example of $N = 2$ supersymmetry. The parameters ϵ and $\epsilon^\dagger$ are arbitrary Grassmannians. Dealing with global supersymmetry, the transformations commute with the time derivative, and therefore applying for example T to the action leads to

$$\begin{aligned} TS &= \int dt \frac{1}{2}(\partial_t \phi + \epsilon \partial_t \psi)^2 - \frac{1}{2}m^2(\phi + \epsilon \psi)^2 \\ &\quad + (\psi^\dagger + \epsilon(i\partial_t \phi - m\phi))(i\partial_t - m)\psi \\ &= S - im\epsilon \int dt \partial_t(\phi\psi). \end{aligned} \quad (1.2.2)$$

Imposing appropriate boundary conditions the action is indeed invariant. This invariance depends crucially on the fact that both particle fields ϕ and ψ have the same mass m , a general property of supersymmetric field theories.

Actually, an action more general than (1.2.1) is invariant under the supersymmetry variations in table 1.1:

$$S = \int dt \frac{1}{2}(\partial_t \phi)^2 + \psi^\dagger (i\partial_t - F''(\phi))\psi - \frac{1}{2}F'(\phi)^2, \quad (1.2.3)$$

where the prime denotes differentiation with respect to ϕ . The *superpotential* $F(\phi)$ not only carries the information on the mass of the fermion

Φ	$T\Phi$	$T^\dagger\Phi$
ϕ	$\phi + \epsilon\psi$	$\phi - \epsilon^\dagger\psi^\dagger$
ψ	ψ	$\psi - \epsilon^\dagger(i\partial_t\phi + m\phi)$
$\psi^\dagger$	$\psi^\dagger + \epsilon(i\partial_t\phi - m\phi)$	$\psi^\dagger$

Table 1.1: The $N = 2$ on-shell supersymmetry transformations of supersymmetric quantum mechanics. Note that writing the transformations in this way e.g. the real field ϕ is transformed into a complex field $\phi + \epsilon\psi$: the physically meaningful transformations are given by $T + T^\dagger$ and $iT - iT^\dagger$, mapping real fields to real fields, etc.

and boson fields, but also on the interactions between those fields. For example, taking $F(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{3}g\phi^3$ leads to

$$\begin{aligned}
S = \int dt \frac{1}{2}(\partial_t\phi)^2 - \frac{1}{2}m^2\phi^2 + \psi^\dagger(i\partial_t - m)\psi - 2g\phi\psi^\dagger\psi \\
- mg\phi^3 - \frac{1}{2}g^2\phi^4,
\end{aligned} \tag{1.2.4}$$

which not only shows that both particle fields have the same mass, but also that supersymmetry invariance leads to very specific coupling constants between the fields.

There is more to supersymmetry than just an extra invariance of an action in a field theory model. Supersymmetry can be seen as an extension of the ordinary space-time symmetries, consisting of Lorentz or Galilei boosts, rotations and translations. Moreover, due to the Coleman-Mandula theorem [27], supersymmetry algebras are the only possible non-trivial extensions of the ordinary space-time algebras [112]. In the relativistic case the non-supersymmetric algebra is called the Poincaré algebra.

Supersymmetric quantum mechanics is here formulated as a relativistic field theory. Since there are no rotations in one dimension, the Poincaré algebra in this case is generated by just the translations, i.e. by the Hamiltonian $H = i\partial_t$. To be able to identify the algebra underlying the supersymmetry transformations as given in table 1.1, an auxiliary bosonic field D needs to be introduced. The real scalar field D is not given a kinetic term and it serves to incorporate the mass and interaction terms of the

Φ	$s\Phi$	$s^\dagger\Phi$
ϕ	ψ	$\psi^\dagger$
ψ	0	$i\partial_t\phi - D$
$\psi^\dagger$	$i\partial_t\phi + D$	0
D	$-i\partial_t\psi$	$i\partial_t\psi^\dagger$

Table 1.2: The $N = 2$ off-shell supersymmetry variations of supersymmetric quantum mechanics.

scalar field ϕ . In terms of ϕ, ψ and D the *off-shell* action of supersymmetric quantum mechanics is given by

$$S = \int dt \frac{1}{2}(\partial_t\phi)^2 + \psi^\dagger(i\partial_t - F''(\phi))\psi + \frac{1}{2}D^2 + DF'(\phi). \quad (1.2.5)$$

Integrating out the auxiliary field D by using its algebraic equation of motion, $D = -F'$, gives back the *on-shell* action in the form (1.2.3).

To make sure the action in this new form is invariant under the supersymmetry transformations, D has to be given a transformation behaviour as well. In table 1.2, instead of the supersymmetry transformations, the supersymmetry variations are given, as is more common. The supersymmetry variations s and $s^\dagger$ are related to the transformations T and $T^\dagger$, respectively, by $T \equiv e^{\epsilon s} = 1 + \epsilon s$, $T^\dagger \equiv e^{-\epsilon^\dagger s^\dagger} = 1 - \epsilon^\dagger s^\dagger$. In checking directly from table 1.2 that s and $s^\dagger$ are indeed each others Hermitian conjugate, the fermionic character of the supersymmetry variations is used. They are fermionic in character since they map bosonic fields onto fermionic fields and vice versa. Indeed, e.g.

$$(s\psi^\dagger)^\dagger = (i\partial_t\phi + D)^\dagger = -i\partial_t\phi + D = -s^\dagger\psi. \quad (1.2.6)$$

Furthermore, because of their fermionic character they obey the following product rule:

$$s^{(\dagger)}[f_1 f_2] = [s^{(\dagger)} f_1] f_2 + (-1)^{|f_1|} f_1 [s^{(\dagger)} f_2], \quad (1.2.7)$$

where the grading of a function f , $|f|$, is 0 for bosonic f and 1 for fermionic f . Under these variations the action is mapped to zero up to total deriva-

tive terms as in equation (1.2.2), i.e. the action is invariant under the supersymmetry transformations.

Using, as before, the assumption that the variations commute with the time derivative, they are seen from table 1.2 to obey the following algebra:

$$\{s, s\} = \{s^\dagger, s^\dagger\} = 0, \quad \{s, s^\dagger\} = 2i\partial_t, \quad [s, \partial_t] = [s^\dagger, \partial_t] = 0. \quad (1.2.8)$$

This is the $N = 2$ supersymmetry algebra of supersymmetric quantum mechanics. This algebra is indeed an extension of the Poincaré algebra in $0 + 1$ dimensions. In higher dimensions the situation is essentially the same. Apart from having commutation relations with the operators that generate the translation in the different directions (the coordinate derivatives), the supersymmetry variations also have a commutation relation with the generators of the Lorentz transformations. The supersymmetry variations, already being fermionic in character, transform as spinors under the Lorentz transformations, as is required by e.g. $s\phi = \psi$. The supersymmetry algebra in higher dimensions can most transparently be discussed in terms of superspace, therefore the further discussion of the algebra is postponed.

In principle all the essentials of supersymmetry have now been introduced along the lines of supersymmetric quantum mechanics. However, it is so far absolutely not clear how to obtain or construct supersymmetric actions. In the next section the concept of superspace is introduced, again along the lines of supersymmetric quantum mechanics. Superspace is a useful tool in constructing field theory actions that have supersymmetry.

Superspace

Translations and Lorentz transformations are generated by operators that can be expressed in terms of derivatives. For example, the Hamiltonian $H = i\partial_t$ generates translations via $e^{-iaH}f(t) = f(t+a)$. The infinitesimal variation of $f(t)$ is written as $\delta f(t) = -iaHf(t) = a\partial_t f(t)$. The main idea behind superspace is to write the supersymmetry variations using differential operators as well. Generally the differential operators that generate the supersymmetry variations s_A are called supercharges and are denoted by Q_A . To make it possible to work with such operators, extra Grassmann coordinates have to be introduced. Superspace is a generalized space-time having as its coordinates the ordinary bosonic coordinates x^μ and additional *fermionic* coordinates θ^α . The coordinates therefore satisfy

the following algebra

$$[x^\mu, x^\nu] = 0, \quad \{\theta^\alpha, \theta^\beta\} = 0, \quad [x^\mu, \theta^\alpha] = 0. \quad (1.2.9)$$

How many fermionic coordinates are introduced depends on the number of supersymmetry variations in the theory one wants to describe, and on the spinor representation chosen in the (bosonic) space-time at hand. For example, in the case of supersymmetric quantum mechanics, there are two supersymmetries that can be represented using a complex supercharge, and therefore one complex supercoordinate θ and its conjugate $\theta^\dagger$ are used. In general the number of fermionic coordinates coincides with the number of supercharges. The number of supercharges depends strongly on the spinor representation used in space-time.

The supercharges are taken to satisfy the same algebra as the supersymmetry variations. The algebra of supersymmetric quantum mechanics is thus given by

$$\{Q, Q\} = \{Q^\dagger, Q^\dagger\} = 0, \quad \{Q, Q^\dagger\} = 2H, \quad [Q, H] = [Q^\dagger, H] = 0, \quad (1.2.10)$$

where Q and its Hermitian conjugate represent the complex supercharges and H the Hamiltonian. A superspace representation of this algebra is given by

$$H = i\partial_t, \quad Q = \partial_\theta + i\theta^\dagger\partial_t, \quad Q^\dagger = \partial_{\theta^\dagger} + i\theta\partial_t, \quad (1.2.11)$$

where indeed the supercharges are expressed in terms of derivatives, like the generators of the ordinary space-time symmetries. That the supercharges indeed generate the supersymmetry transformations will be shown later.

The general supersymmetry algebra, written in a very generic way, reads

$$\{Q_A, Q_B\} = f_{AB}^\mu P_\mu + Z_{AB}, \quad [Q_A, P_\mu] = 0, \quad [Q_A, M_{\mu\nu}] = (\sigma_{\mu\nu})_A^B Q_B, \quad (1.2.12)$$

plus the conventional space-time symmetry algebra of the momentum operators in the μ coordinate directions, P_μ , and of the generators of the Lorentz or Galilei boosts and rotations, $M_{\mu\nu}$. The Z_{AB} are the so-called central charges that commute with all other operators, including themselves. The $\sigma_{\mu\nu}$ are such that the supercharges transform as spinors under

the Lorentz or Galilei transformations. The f_{AB}^μ also take into account the spinorial character of the Q_A , but (as is the case with the $\sigma_{\mu\nu}$) the specific form of the f_{AB}^μ depends strongly on the dimensionality of spacetime and of the spinorial representation chosen.

Furthermore, an internal symmetry between the supercharges, an *R-symmetry*, is allowed. The generators of this symmetry, written as R_s , obey

$$[Q_A, R_s] = (r_s)_A^B Q_B, \quad (1.2.13)$$

and commute with all operators other than the supercharges.

Like Lie algebras, the supersymmetry algebras are also required to satisfy a Jacobi identity. Being a graded algebra, any triple O_1, O_2, O_3 of operators in the algebra has to satisfy the following equation:

$$[[O_1, O_2], O_3] + [[O_3, O_1], O_2] + [[O_2, O_3], O_1] = 0, \quad (1.2.14)$$

where $[\ , \]$ denotes the anticommutator $\{ \ , \ }$ if both operators are fermionic, and the commutator $[\ , \]$ otherwise.

The supersymmetry algebra clearly is an extension of the ordinary space-time symmetry algebra of the P_μ and $M_{\mu\nu}$, which in the relativistic case is the Poincaré algebra. For more details on supersymmetry algebras, see e.g. [112, 113, 116].

Functions or fields on superspace are called superfunctions or -fields. Because of the Grassmann nature of the fermionic coordinates a Taylor expansion in those coordinates terminates after a finite number of terms. The most general superfield in the superspace of supersymmetric quantum mechanics is given by

$$F(t, \theta, \theta^\dagger) = f_0(t) + \theta f_1(t) + \theta^\dagger f_2(t) + \theta\theta^\dagger f_{12}(t), \quad (1.2.15)$$

where f_0 and f_{12} are bosonic (or fermionic) fields and f_1 and f_2 fermionic (or bosonic) fields if F is taken to be bosonic (or fermionic, respectively). The superfields of the theory at hand encapsulate the fermionic and bosonic fields of the theory, thereby showing unification of the bosonic and fermionic fields.

Using Wigner's method of induced representations [118], the irreducible representations of e.g. the Poincaré algebra can be found. The irreducible representations of the Poincaré algebra or of a supersymmetry algebra are in one-to-one correspondence with the possible particle

content of the theory at hand. Wigner's method also applies to the supersymmetry algebras and therefore yields the possible particle content of supersymmetric theories, see e.g. [112, 113, 116]. To make sure that the field content of the superfields, the supersymmetric multiplet, matches the particle content of the algebra it may be necessary to impose constraints on the superfields of the theory. Furthermore, physical considerations can lead to the study of restricted particle contents. Constraints that are usually imposed are Hermiticity and chirality. For example, the superfield of supersymmetric quantum mechanics is the following Hermitian superfield:

$$\Phi(t, \theta, \theta^\dagger) = \phi(t) + i\theta\psi(t) + i\theta^\dagger\psi^\dagger(t) + \theta\theta^\dagger D(t). \quad (1.2.16)$$

It indeed contains two real bosonic fields, ϕ and D , and one complex Dirac fermion ψ , as follows from studying the $N = 2$ supersymmetry algebra in one dimension [82, 93, 107].

Chirality constraints are imposed using the *superderivatives* D_A of the theory. A chirality constraint on a superfield is for example $D_A F = 0$. The superderivatives are defined by the requirement that they anticommute with the supercharges and satisfy the same algebra as the supercharges, but up to a minus sign for the bosonic coordinate derivatives. For supersymmetric quantum mechanics they are therefore defined by

$$\{D, D\} = \{D^\dagger, D^\dagger\} = 0, \quad \{D, D^\dagger\} = -2H, \quad \{Q^{(\dagger)}, D^{(\dagger)}\} = 0. \quad (1.2.17)$$

In this case they are given by

$$D = \partial_\theta - i\theta^\dagger\partial_t, \quad D^\dagger = \partial_{\theta^\dagger} - i\theta\partial_t. \quad (1.2.18)$$

A chirality constraint like $D\Phi = 0$ is usually not imposed in supersymmetric quantum mechanics.

As claimed, the supersymmetry variations are generated by the supercharges like e.g. the ordinary translations are generated by the bosonic coordinate derivative operators. Indeed, the supersymmetry variations are generated by acting with the supercharges on the fermionic and bosonic coordinates. In the case of supersymmetric quantum mechanics the variations due to Q are given by

$$\delta\theta \equiv \epsilon Q\theta = \epsilon, \quad \delta\theta^\dagger \equiv \epsilon Q\theta^\dagger = 0, \quad \delta t \equiv \epsilon Q t = i\epsilon\theta^\dagger, \quad (1.2.19)$$

and due to $Q^\dagger$ by

$$\delta^\dagger \theta \equiv -\epsilon^\dagger Q^\dagger \theta = 0, \quad \delta^\dagger \theta^\dagger \equiv -\epsilon^\dagger Q^\dagger \theta^\dagger = -\epsilon^\dagger, \quad \delta^\dagger t \equiv -\epsilon^\dagger Q^\dagger t = -i\epsilon^\dagger \theta. \quad (1.2.20)$$

Knowing how the coordinates vary under supersymmetry, the variations of the superfields and actions are also known. The δ variation of a superfield F is for example given by

$$\begin{aligned} \delta F(t, \theta, \theta^\dagger) &\equiv F(t + \delta t, \theta + \delta \theta, \theta^\dagger + \delta \theta^\dagger) - F(t, \theta, \theta^\dagger) \\ &= \epsilon Q F(t, \theta, \theta^\dagger), \end{aligned} \quad (1.2.21)$$

where the last equality sign holds due to the product rule of differentiation. Acting with the δ variation on the Hermitian superfield of supersymmetric quantum mechanics and comparing the result with the original superfield enables one to find the supersymmetry variations s of the component fields as in table 1.2. Indeed, comparing $\Phi(t, \theta, \theta^\dagger)$ with

$$\begin{aligned} \delta \Phi(t, \theta, \theta^\dagger) &= \epsilon Q \left(\phi(t) + i\theta\psi(t) + i\theta^\dagger\psi^\dagger(t) + \theta\theta^\dagger D(t) \right) \\ &= i\epsilon\psi + i\theta^\dagger(-\epsilon\partial_t\phi + i\epsilon D) + \theta\theta^\dagger\epsilon\partial_t\psi \\ &= \delta\phi(t) + i\theta\delta\psi(t) + i\theta^\dagger\delta\psi^\dagger(t) + \theta\theta^\dagger\delta D(t). \end{aligned} \quad (1.2.22)$$

leads to $\delta\phi = i\epsilon\psi$, et cetera. That is, the supersymmetry variations s are recovered up to a factor of i for each term. Of course, for the second supersymmetry variation $\delta^\dagger = -\epsilon^\dagger Q^\dagger$ the situation is completely similar: the supersymmetry variations for $s^\dagger$ are also readily obtained, however up to a factor of $-i$ for each term.

In the superspace context it is an almost trivial matter to write down supersymmetry invariant actions. Any functional of superfields and superderivatives of superfields integrated over both the bosonic and fermionic coordinates gives a supersymmetric action. For example, working in the superspace of supersymmetric quantum mechanics any functional G of any superfields F_i and superderivatives thereof defines a supersymmetric action by

$$S = \int dt d\theta^\dagger d\theta G(F_i, D^\dagger F_i, D F_i, (D^\dagger)^2 F_i, D^\dagger D F_i, \dots). \quad (1.2.23)$$

In a superspace context therefore the problem is not so much how to construct supersymmetric actions, but to find supersymmetric actions that

are physically relevant. Since the derivatives of bosonic fields ϕ should appear only as $(\partial_\mu \phi)^2$ and the derivatives of fermionic fields only as $\psi^\dagger \partial_\mu \psi$, physically interesting actions usually have a term like $D_A F_i D_B F_i$ plus a functional of the superfields alone, $G(F_i)$, called the superpotential. Furthermore there are the physical constraints on the field content. The action of supersymmetric quantum mechanics for example is given by

$$S = \int dt d\theta^\dagger d\theta \frac{1}{2} (D\Phi)^\dagger D\Phi + F(\Phi), \quad (1.2.24)$$

where $F(\Phi)$ is the superpotential. This action is exactly the same as the one in equation (1.2.3) (note the difference in the argument of the superpotential), as can be verified by integrating over the Grassmann variables:

$$\int d\theta = 0, \quad \int d\theta \theta = 1, \quad \int d\theta_1 \dots d\theta_n \theta_n \dots \theta_1 = 1. \quad (1.2.25)$$

This *Berezin integration* is defined in such a way that the integral is invariant under a translation $\theta \rightarrow \theta + \epsilon$, like ordinary integration over a bosonic variable is.

It still remains to be shown that any action of the form (1.2.23) is indeed supersymmetry invariant. This will be shown in detail for one single supersymmetry variation, δ . The proof of supersymmetry invariance under the other variation as well as the proof of invariance of a general action like (1.2.23) in a more general superspace context is completely analogous.

Following equation (1.2.21), the variation of the action is given by

$$\begin{aligned} \delta S &\equiv \int dt d\theta^\dagger d\theta \{G(F_i + \delta F_i, D_A(F_i + \delta F_i)) - G(F_i, D_A F_i)\} \\ &= \int dt d\theta^\dagger d\theta \{G(F_i + \delta F_i, D_A F_i + \delta D_A F_i) - G(F_i, D_A F_i)\}, \end{aligned} \quad (1.2.26)$$

The second step can be made due to the fact that the superderivatives anticommute with the supercharges, and thus commute with δ . Making use of the product rule of differentiation, this can be rewritten as

$$\delta S = \int dt d\theta^\dagger d\theta \delta G(F_i, D_A F_i) = \int dt d\theta^\dagger d\theta \epsilon (\partial_\theta + i\theta^\dagger \partial_t) G(F_i, D_A F_i). \quad (1.2.27)$$

Using the properties of Berezin integration, the ∂_θ term will give zero, because the ∂_θ derivative removes all θ dependence of $G(F_i, D_A F_i)$, since this functional depends at most linearly on θ .

Concerning the remaining term, expanding $G(F_i, D_A F_i)$ in the fermionic coordinates as in equation (1.2.15) makes clear that multiplying it by $\theta^\dagger$ and integrating over θ and $\theta^\dagger$ selects its θ component, $g_1(F_i, D_A F_i)$, as in (1.2.15). The variation of the action is therefore equal to

$$\delta S = - \int dt \partial_t g_1(F_i, D_A F_i). \quad (1.2.28)$$

The action is thus invariant upon imposing appropriate boundary conditions.

So far supersymmetric field theory without gauge fields has been discussed. In the remainder of this section an example of a gauged supersymmetric field theory in two dimensions is presented.

A Two Dimensional Gauged Example

An example of a supersymmetry algebra in two space dimensions is given by

$$\{s, s_\mu\} = -P_\mu, \quad \{\tilde{s}, s_\mu\} = \epsilon_{\mu\nu} P_\nu, \quad (1.2.29)$$

where only the nonzero anticommutators are shown, and $P_\mu = i\partial_\mu$. This algebra is used in section 1.4 when discussing the super BF model.

It is the algebra of *twisted* $N = D = 2$ supersymmetry [33–35, 122]: the supersymmetry variations s , s_μ and $\tilde{s}$ can be obtained from the conventional ones by twisting. The twisted supersymmetry variations are no longer spinors: s is a scalar, s_μ a vector and $\tilde{s}$ a pseudo-scalar. More on twisting will be said later.

As an example of a gauged supersymmetric field theory the twisted formulation of super Yang-Mills having $N = 2$ supersymmetry in two dimensions is presented [35]. Although not the simplest model, this is the one that is subject of the link approach discussed in chapter 1.5. Since in that discussion only the component formulation of the model is needed, a superspace discussion is omitted here.

	s	$\tilde{s}$	s_μ
A_ν	$-\frac{i}{2}\lambda_\nu$	$-\frac{i}{2}\epsilon_{\nu\rho}\lambda_\rho$	$+\frac{i}{2}\epsilon_{\mu\nu}\tilde{\rho} - \frac{i}{2}\delta_{\mu\nu}\rho$
$\phi^{(\nu)}$	$+\frac{1}{2}\lambda_\nu$	$-\frac{1}{2}\epsilon_{\nu\rho}\lambda_\rho$	$+\frac{1}{2}\epsilon_{\mu\nu}\tilde{\rho} + \frac{1}{2}\delta_{\mu\nu}\rho$
K	$-\frac{i}{2}[D_\mu^+, \lambda_\mu]$	$-\frac{i}{2}\epsilon_{\rho\sigma}[D_\rho^+, \lambda_\sigma]$	$\frac{i}{2}[D_\mu^+, \rho] - \frac{i}{2}\epsilon_{\mu\nu}[D_\nu^-, \tilde{\rho}]$
λ_ν	0	0	$i[D_\mu^+, D_\nu^-] +$ $\delta_{\mu\nu}(K + i[D_\mu, \phi^{(\mu)}])$
ρ	$-i[D_\mu, \phi^{(\mu)}] - K$	$\frac{i}{2}\epsilon_{\rho\sigma}[D_\rho^-, D_\sigma^-]$	0
$\tilde{\rho}$	$-\frac{i}{2}\epsilon_{\rho\sigma}[D_\rho^+, D_\sigma^+]$	$i[D_\mu, \phi^{(\mu)}] - K$	0

Table 1.3: Supersymmetry variation rules for twisted $N = D = 2$ SYM; where $D_\mu^\pm = D_\mu \pm \phi^{(\mu)}$

The action is given by

$$\begin{aligned}
S = \int d^2x \operatorname{Tr} & \left[\frac{1}{2} F_{\mu\nu} F_{\mu\nu} + [D_\nu, \phi^{(\mu)}][D_\nu, \phi^{(\mu)}] \right. \\
& + K^2 + i[\phi^{(\mu)}, \lambda_\mu]\rho + i\tilde{\rho}\epsilon_{\mu\nu}[\phi^{(\mu)}, \lambda_\nu] \\
& \left. - \frac{1}{2}[\phi^{(\mu)}, \phi^{(\nu)}][\phi^{(\mu)}, \phi^{(\nu)}] + i[D_\mu, \lambda_\mu]\rho - i\tilde{\rho}\epsilon_{\mu\nu}[D_\mu, \lambda_\nu] \right],
\end{aligned} \tag{1.2.30}$$

where $F_{\mu\nu} \equiv i[D_\mu, D_\nu]$ denotes the ordinary gauge field strength with $D_\mu \equiv \partial_\mu - iA_\mu$, the gauge covariant derivative. The fields $\phi^{(\mu)}$ ($\mu = 1, 2$) represent two scalar fields in the $N = D = 2$ twisted SYM multiplet. They can be seen as the remnants of the two additional gauge fields when the $N = D = 2$ model is obtained from $N = 1$ super Yang-Mills in four dimensions via dimensional reduction [35]. Furthermore, the field K is an auxiliary bosonic field, and the fermionic fields of the model are given by λ_μ ($\mu = 1, 2$), ρ and $\tilde{\rho}$. All fields transform in the adjoint representation of the gauge group, which is unspecified here, but could for example be taken to be $SU(N)$.

The supersymmetry variations of the fields are given in table 1.3. From these variations, the non-zero anticommutators of the gauged supersym-

metry algebra can be read off to be

$$\{s, s_\mu\}(\varphi) = -i[D_\mu, \varphi], \quad \{\tilde{s}, s_\mu\}(\varphi) = +i\epsilon_{\mu\nu}[D_\nu, \varphi], \quad (1.2.31)$$

where φ denotes any component of the multiplet $(A_\mu, \phi^{(\mu)}, \rho, \tilde{\rho}, \lambda_\mu, K)$. In this algebra the derivative has been replaced by the covariant derivative acting in the adjoint representation.

Finally, note that in the light of the transformations presented in table 1.3, the fields $(\lambda_\nu, \rho, \tilde{\rho})$ can be identified via

$$\begin{aligned} \lambda_\nu &= is(A_\nu - i\phi^{(\nu)}) = -\epsilon_{\nu\rho}i\tilde{s}(A_\rho + i\phi^{(\rho)}), \\ \rho &= is_\mu(A_\mu - i\phi^{(\mu)}), \\ \tilde{\rho} &= -i\epsilon_{\mu\nu}s_\mu(A_\nu + i\phi^{(\nu)}), \\ K &= \frac{1}{2}s_\mu\lambda_\mu. \end{aligned} \quad (1.2.32)$$

More on the continuum formulation of this model, for example on the matching of bosonic and fermionic degrees of freedom, can be found in [79].

1.3 Supersymmetry on the Lattice

When supersymmetric field theories are defined on a lattice there unfortunately is a very persistent problem. Basically, defining a supersymmetric theory on a lattice automatically breaks the supersymmetry. This fact can most easily be explained in the context of supersymmetric quantum mechanics where scalar fields ϕ and D and the fermion field ψ are defined to live on the lattice sites.

The problem comes down to the failure of the product rule of differentiation, the Leibniz rule, on a lattice [43, 57, 59]. On a lattice derivative operators are replaced by finite difference operators, and these do not obey the Leibniz rule. For example, on a rectangular lattice an often used difference operator is given by the *forward difference operator* Δ_μ defined by

$$\Delta_\mu f(\mathbf{x}) = \frac{1}{|\mathbf{n}_\mu|} (f(\mathbf{x} + \mathbf{n}_\mu) - f(\mathbf{x})), \quad (1.3.1)$$

where $\mathbf{n}_\mu$ corresponds to the shift of one lattice spacing in the μ -direction. When this operator acts on a product of functions the following identity holds,

$$\Delta_\mu [f_1(\mathbf{x})f_2(\mathbf{x})] = [\Delta_\mu f_1(\mathbf{x})] f_2(\mathbf{x}) + f_1(\mathbf{x} + \mathbf{n}_\mu) [\Delta_\mu f_2(\mathbf{x})], \quad (1.3.2)$$

as can be seen by evaluating both sides. Due to the shift in the argument of one of the functions the ordinary Leibniz rule is violated.

Because the ordinary Leibniz rule no longer holds, one can not satisfy the supersymmetry algebra on the lattice. At least not in a straightforward way by just replacing the derivative operators by finite difference operators. For example, the supersymmetry variations s and $s^\dagger$ of supersymmetric quantum mechanics satisfy $\{s, s^\dagger\} = 2i\partial_t$, and on the lattice one would like them to satisfy $\{s, s^\dagger\} = 2i\Delta_t$. It is no problem to satisfy this property for single fields, as can be seen by simply replacing all ∂_t 's in table 1.2 by Δ_t . One runs into problems when it comes to satisfying this property for products of fields. As an example, $\{s, s^\dagger\}$ acts on ϕ^2 in the following way in the continuum:

$$\begin{aligned} \{s, s^\dagger\}\phi^2 &= s(2\phi\psi^\dagger) + s^\dagger(2\phi\psi) \\ &= 2(\psi\psi^\dagger + \phi(i\partial_t\phi + D)) + 2(\psi^\dagger\psi + \phi(i\partial_t\phi - D)) \\ &= 4\phi i\partial_t\phi = 2i\partial_t(\phi^2). \end{aligned} \tag{1.3.3}$$

On the lattice, the supersymmetry algebra is clearly broken, because there an identification like $2\phi\Delta_t\phi = \Delta_t(\phi^2)$ can not be made.

Investigating (1.3.3), it is seen that two different product rules are used. Twice the product rule for the variations s and $s^\dagger$, according to equation (1.2.7), and once for the difference operator¹. In principle the product rule for the supersymmetry variations is completely independent from the product rule for the difference operator, the Leibniz rule. While the latter follows directly from the definition of the difference operator (as in (1.3.2)), the first follows from the transformation behaviour of single fields. For example, under the supersymmetry transformation $\phi \rightarrow \phi + \epsilon s\phi$, ϕ^2 transforms according

$$\phi^2 \longrightarrow (\phi + \epsilon s\phi)(\phi + \epsilon s\phi) \longrightarrow \phi^2 + \epsilon(s\phi)\phi + \phi\epsilon s\phi = \phi^2 + 2\epsilon(s\phi)\phi. \tag{1.3.4}$$

But at the same time ϕ^2 by definition transforms according to

$$\phi^2 \longrightarrow \phi^2 + \epsilon s\phi^2. \tag{1.3.5}$$

¹In this thesis the product rule for supersymmetry variations is called ‘product rule’, whereas that one for derivatives or difference operators and supercharges is called ‘Leibniz rule’.

Repeating this argument in general, which holds on the lattice equally well as in the continuum, the product rule for the supersymmetry variations, equation (1.2.7), follows.

It is therefore clearly the failure of the Leibniz rule that causes the problem. From the fact that it is in principle no problem to satisfy the product rule of the supersymmetry variations on the lattice, it is clear that one would have no problem in putting the supersymmetry algebra on the lattice if no derivatives or difference operators would appear in the algebra. Unfortunately this is not the case due to the property of supersymmetry that its variations square to give the translations. However, when one deals with extended supersymmetry it might be possible to apply a change of variables (like going from one complex to two real variables in the appendix to this chapter) in such a way that one or more of the supersymmetry variations are free of derivatives. Of course this causes extra derivatives to appear in the other supersymmetry variations, making it even harder to put them on the lattice. This idea has successfully been applied to put one of the two supersymmetry variations of supersymmetric quantum mechanics exactly on the lattice [20], and reference therein.

The failure of the Leibniz rule on the lattice not only causes a problem in defining the supersymmetry algebra on the lattice, it also makes it problematic to define a supersymmetric lattice action, since in these actions products of fields appear. For instance, in showing supersymmetry invariance of the action of supersymmetric quantum mechanics in equation (1.2.2) the Leibniz rule is used for writing the variation of the action as an integral over a total derivative term.

Also in the superspace context the use of the Leibniz rule is evident. In determining the supersymmetry variation of a (function of a) superfield as well as in showing supersymmetry invariance of actions, in particular in equation (1.2.21) and in equations (1.2.26)-(1.2.28), the Leibniz rule is used. The lattice analogue of an integral over a total derivative is the sum over a ‘total difference’, which is straightforwardly seen to vanish as well.

In this context it is clear that the failure of the Leibniz rule for the difference operator implies the failure of the Leibniz rule for the supercharges. However, if one somehow manages to fix the Leibniz rule for the latter, one is able to have supersymmetry on the lattice in a superspace context, since for this having a Leibniz rule for the supercharges, not for the difference operator, is important. This is the main idea of the non-

commutativity approach due to D’Adda and Kawamoto et al. [32–34], to be discussed in full detail in the next chapter.

The failure of the Leibniz rule for difference operators on the lattice can equivalently be formulated by saying that (in the relativistic case) the Poincaré algebra is broken on the lattice. Replacing the derivative operators by finite difference operators the Poincaré algebra is seen to hold up to terms of the order of the lattice spacing. The breaking of the Poincaré symmetry on the lattice is obvious because of the nature of the lattice: only translation invariance over the lattice spacing and rotation invariance over right angles holds (in case of a square lattice). The same argument holds in the non-relativistic case. Since the supersymmetry algebras are extensions of the ordinary space-time symmetry algebras, also the supersymmetry algebras are necessarily broken on the lattice.

Because classically all the supersymmetry breaking terms are of the order of the lattice spacing, it is no problem to recover a supersymmetric theory if one takes the continuum limit, classically. However, due to quantum mechanical regularization and renormalization effects supersymmetry breaking terms generically become of order one. Taking the continuum limit one thus generically does not end up with a supersymmetric continuum theory. Non-supersymmetric field theories usually do not suffer from such a problem, because the discrete translation and cubic rotation invariance, which as a subset of the full space-time invariance are left unbroken on the lattice, are sufficient to prohibit the appearance of space-time invariance breaking terms in the continuum limit. Unfortunately, there is usually no supersymmetry analog of the discrete translation and cubic rotation invariance, and the latter themselves are usually not enough to guarantee supersymmetry in the continuum limit. In the supersymmetric case one therefore typically has to supplement the lattice action by the addition of large numbers of supersymmetry breaking terms such that in the continuum limit they cancel together with the already present terms that break supersymmetry. One has to deal with *fine tuning* of the couplings of all these supersymmetry violating terms to achieve a supersymmetric continuum limit. This is typically a very difficult problem.

Another problem when dealing with lattice supersymmetry is the *fermion doubling problem* [70, 92]. This problem states that when naively defining fermions on a d dimensional lattice the number of fermionic degrees of free-

dom is multiplied by a factor 2^d , and this effect persists in the continuum limit. This thus spoils supersymmetry since this requires the matching of the numbers of bosonic and fermionic degrees of freedom in a model. One therefore has to investigate what happens to supersymmetry when the doublers are removed (which potentially also occur in the bosonic sector [23, 58]). However, no further attention is paid to this problem in this thesis, as the approaches discussed here will be shown to be inconsistent on more fundamental grounds.

Review of Approaches

Already since the late 1970's and early 1980's effort has been put into latticizing supersymmetry, [4, 6, 43, 45, 46, 98]. The idea to use fine tuning to control the supersymmetry violating terms is due to Curci and Veneziano [31]. Other references to this approach are e.g. [48–51].

In more recent years there has been a revived interest in lattice supersymmetry due to the development of different techniques to have some of the supersymmetries of extended supersymmetry exactly on the lattice. The philosophy behind this is that it should reduce the problem of fine tuning. One of the ideas relies on the twisting procedure due to Witten [122] and the use of Dirac-Kähler fermions [80, 81]. This twisting is used to rewrite the action in such a way that it is exact with respect to a nilpotent supercharge Q : $S = Q(\dots)$. Keeping the nilpotency of Q , i.e. $Q^2 = 0$, on the lattice then guarantees a lattice action that is invariant with respect to Q , $QS = 0$. More on this approach due to amongst others Catterall and Sugino can be found in e.g. [17–22, 24, 102–104].

Kaplan et al. introduced an approach in which a lattice action with some exact supersymmetries is constructed by orbifolding a higher dimensional supersymmetric matrix model [25, 26, 47, 75–78]. Although at first sight this construction appears quite different from the one by Catterall and Sugino et al., they have been shown to be intimately related [108].

Based on an idea due to Golterman and Petcher [61], Feo has been able to have full exact lattice supersymmetry for the Wess-Zumino model by adding physically irrelevant terms to the action and by modifying the supersymmetry transformations accordingly [12]. The irrelevant terms are non-local and can only be determined by an expansion in powers of the coupling constant. This is the only successful attempt to have exact supersymmetry on the lattice so far.

Giedt et al. and Catterall et al. have studied supersymmetric quantum

mechanics formulated as a field theory on the lattice keeping one of the two supersymmetries exact on the lattice in a nilpotent and derivative-free way, see e.g. [20,23,58]. With its minimal field content and its dependency on only one coordinate this system is a simple toy model to test non-perturbative features of supersymmetry.

By introducing a noncommutativity between the bosonic and fermionic coordinates of superspace D’Adda and Kawamoto et al. [32–34] are able to ensure that the supercharges on the lattice do obey a normal Leibniz rule. They have been able to apply their method to supersymmetric field theories without gauge symmetry, having twisted $N = D = 2$ or 4 supersymmetry and using Dirac-Kähler fermions. The claim is that lattice theories constructed in this way exhibit invariance under the *full* set of continuum supersymmetries. Giving a link interpretation to the non-commutativity these ideas have been applied in a different formulation to supersymmetric gauge theories too [35–37]. Unfortunately both methods suffer from an inconsistency as was explained in [13–15]. The descriptions of both methods and the explanation of their inconsistencies are the main topics of the next two sections.

An approach that should not go unnoticed is one due to Kaku [71–73]. When one works on a lattice consisting of randomly placed points instead of on a square rectangular lattice as is usually done, one can have full Poincaré invariance by taking an average over a large number of random lattices. Since the breaking of the Poincaré invariance is the main obstacle towards having supersymmetry on the lattice, this seems a very fruitful approach. However this approach is no longer pursued since it is computationally much more expensive than the other approaches [74].

Unfortunately an extensive review article dealing in detail with the fundamental problems and approaches of supersymmetry on the lattice does not exist. However, in many articles problems and approaches are mentioned and discussed in more or less detail, see e.g. [12, 20, 43, 48, 49, 53, 76], and references therein.

1.4 The Noncommutativity Approach

In this chapter it is argued that the noncommutativity approach due to D’Adda and Kawamoto et al. to have fully supersymmetric field theories on the lattice suffers from an inconsistency. Supersymmetric quantum mechanics is worked out in this formalism and the inconsistency is shown

both in general and explicitly for that system.

The obstacle against having supersymmetry on the lattice is the failure of the Leibniz rule for the difference operator. This also spoils the Leibniz rule for the supercharges which is extensively used to write down supersymmetric actions in a superspace context. The noncommutativity approach comes down to introducing a noncommutativity between the bosonic and fermionic coordinates in a superspace representation in such a way that the Leibniz rule for the supercharges is restored. Having achieved this, it is in principle possible to define fully supersymmetric lattice actions in a straightforward way, merely copying what is done in the continuum. Although a noncommutativity is introduced, component fields are treated in this approach as ordinary functions of ordinary bosonic coordinates, as is suitable for simulations.

By D'Adda and Kawamoto et al. this method was seen to work for two supersymmetric algebras, namely twisted $N = D = 2$ and $N = D = 4$. In this thesis it is shown that the approach can also be used to define a lattice version of supersymmetric quantum mechanics, treated as an $N = 2$ one dimensional supersymmetric field theory. The aim is thus to preserve the full supersymmetry on the lattice, whereas Catterall et al. and Giedt et al. were able to have only one of the two supersymmetries of supersymmetric quantum mechanics exactly on the lattice [20, 23, 58].

However, in the second part of this chapter the approach is shown to suffer from an inconsistency. Having the Leibniz rule restored for the supercharges, a supersymmetry invariant lattice action in the superspace language is readily constructed. However, the physically interesting actions are the ones in terms of component fields, where the auxiliary superspace has been integrated out. In the present context the transformation rules of the component fields have to be derived using the noncommutative lattice superspace approach. It turns out however that the lattice action after integrating out the auxiliary superspace is no longer supersymmetry invariant. Supersymmetry invariance has to be shown by using the transformation behaviour of the individual component fields plus the corresponding product rule. Due to the noncommutativity, a supersymmetry variation of a given polynomial of component fields depends on the order in which the product of the fields in it is taken. Since the fields still normally (anti)commute amongst themselves, the definition of the supersymmetry variations and hence the answer to the question whether a given expression (like the action) is supersymmetry invariant, is not unique.

Supersymmetric Quantum Mechanics on the Lattice

The noncommutativity approach to supersymmetry on the lattice [32–34] is first introduced by means of supersymmetric quantum mechanics, which is formulated here in terms of Majorana fermions using imaginary time, following the conventions of [14]. Imaginary time is used having lattice simulations in mind. Since supersymmetric quantum mechanics was introduced using complex Dirac fermions and real time, the transformation between the two descriptions is discussed in detail in appendix 1.A.

The fermion fields are now given by real fields ψ_1 and ψ_2 and the action reads

$$S = \int dt \left[\frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\psi_1 \partial_t \psi_1 + \psi_2 \partial_t \psi_2) - i(m + 3g\phi^2)\psi_1 \psi_2 - \frac{1}{2}D^2 - D(m\phi + g\phi^3) \right], \quad (1.4.1)$$

when the superpotential is taken to be $F(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{4}g\phi^4$. The supersymmetry variations are given in table 1.4 as $\delta_1 = \epsilon^1 s_1$, $\delta_2 = \epsilon^2 s_2$.

In a superspace formulation, the supercoordinates are two real Grassmann variables θ^1 and θ^2 . The superspace action reads

$$S = \int dt d\theta^1 d\theta^2 \frac{1}{2} D_2 \Phi D_1 \Phi + iF(\Phi), \quad (1.4.2)$$

where the superfield is given by

$$\Phi = \Phi(t, \theta^1, \theta^2) = \phi(t) + i\theta^1 \psi_1(t) + i\theta^2 \psi_2(t) + i\theta^2 \theta^1 D(t). \quad (1.4.3)$$

This action is equal to (1.4.1) upon integrating over the Grassmannian variables and choosing the same superpotential.

Furthermore, the supercharges and superderivatives are given by

$$Q_i = \partial_{\theta^i} + \theta^i \partial_t, \quad D_i = \partial_{\theta^i} - \theta^i \partial_t. \quad (1.4.4)$$

The supercharges satisfy the following algebra:

$$\{Q_1, Q_1\} = 2H, \quad \{Q_2, Q_2\} = 2H, \quad \{Q_1, Q_2\} = 0, \quad (1.4.5)$$

where $H = \partial_t$. Also in this formulation the superderivatives anti-commute with the supercharges, and satisfy the same algebra as the supercharges up to a minus sign for the terms involving the Hamiltonian.

Φ	$\delta_1 \Phi$	$\delta_2 \Phi$
ϕ	$i\epsilon^1 \psi_1$	$i\epsilon^2 \psi_2$
ψ_1	$i\epsilon^1 \partial_t \phi$	$\epsilon^2 D$
ψ_2	$-\epsilon^1 D$	$i\epsilon^2 \partial_t \phi$
D	$\epsilon^1 \partial_t \psi_2$	$-\epsilon^2 \partial_t \psi_1$

Table 1.4: Supersymmetry variations in the continuum. The same set of variations is valid for the (noncommutative) supersymmetry on the lattice replacing ∂_t by $\Delta_{\pm}$.

The noncommutativity approach towards supersymmetry on the lattice can be introduced as follows. Naturally on the lattice the derivative ∂_t is replaced by the forward difference operator Δ_+ or by the backward one Δ_- ,

$$\Delta_{\pm} f(t) = \pm \frac{1}{2n} (f(t \pm 2n) - f(t)), \quad (1.4.6)$$

where n corresponds to the shift of one lattice spacing. Another obvious candidate would be to work with the symmetric difference operator Δ_S ,

$$\Delta_S f(t) = \frac{1}{2n} (f(t+n) - f(t-n)). \quad (1.4.7)$$

However this difference operator is not suitable for the noncommutativity approach, as will be clarified later. Also why the difference is taken over two lattice spacings will become clear later.

The forward or backward difference operators do not obey the conventional Leibniz rule, but rather a ‘modified Leibniz rule’, see (1.3.2):

$$\Delta_{\pm} [f_1(t) f_2(t)] = [\Delta_{\pm} f_1(t)] f_2(t) + f_1(t \pm 2n) [\Delta_{\pm} f_2(t)]. \quad (1.4.8)$$

Taking the lattice supercharges to be $Q_i = \partial_{\theta^i} + \theta^i \Delta_{\pm}$ it is obvious that they do not obey the Leibniz rule either and a naive approach would run into problems defining supersymmetric actions on the lattice.

The main idea of the noncommutativity approach is to introduce a noncommutativity between the bosonic and Grassmann coordinates of

superspace in such a way that both supersymmetry transformations $\epsilon^1 Q_1$ and $\epsilon^2 Q_2$ obey the normal Leibniz rule and then to proceed along the lines of continuum supersymmetry.

Indeed, when $\epsilon^1 Q_1$ and $\epsilon^2 Q_2$ obey the normal product rule, it follows immediately that the naive discretization of the action of supersymmetric quantum mechanics,

$$S = n \sum_t \int d\theta^1 d\theta^2 \frac{1}{2} D_2 \Phi D_1 \Phi + iF(\Phi), \quad (1.4.9)$$

is supersymmetry invariant. Supersymmetry invariance can be shown by simply operating with the supercharges on the lattice action, as was shown in equations (1.2.26)-(1.2.28). The superfield Φ is taken as in the continuum, equation (1.4.3), with t now labelling the lattice points. The lattice superderivatives are obtained from the continuum ones like the supercharges are.

When the lattice supercharges $Q_i = \partial_{\theta^i} + \theta^i \Delta_{\pm}$ act on products of functions similar to equation (1.4.8), θ^i has to be ‘pulled through’ the first function f_1 and this can be used to restore the Leibniz rule for $\theta^i \Delta_{\pm}$ if one demands

$$\theta^i f(t) = f(t - a_i) \theta^i \Leftrightarrow [t, \theta^i] = a_i \theta^i \quad (\text{no sum}), \quad (1.4.10)$$

with $a_i = \pm 2n$. However, this new relation implies another noncommutativity between ∂_{θ^i} and t , with the opposite sign

$$[t, \partial_{\theta^i}] = -a_i \partial_{\theta^i} \quad (\text{no sum}), \quad (1.4.11)$$

which makes it impossible to obtain the Leibniz rule for both terms in the supercharges at the same time.

In contrast, the supersymmetry variations $\epsilon^i Q_i$ (no sum) can obey the Leibniz rule, provided the same noncommutativity is introduced for the variation parameters ϵ^i

$$[t, \epsilon^i] = a_i \epsilon^i \quad (\text{no sum}). \quad (1.4.12)$$

Now the Leibniz rule is obvious for the terms $\epsilon^i \partial_{\theta^i}$, because the shift due to ∂_{θ^i} is canceled by the shift due to ϵ^i . For the terms $\epsilon^i \theta^i \Delta_{\pm}$, ϵ^i and θ^i have to cooperate to cancel the shift due to $\Delta_{\pm}$, leading to the demand $2a_i = \pm 2n$.

Working on the other hand with the symmetric difference operator, the following Leibniz rule would have to be fixed for either Q_i or the $\epsilon^i Q_i$:

$$\Delta_S [f_1(t)f_2(t)] = [\Delta_S f_1(t)] f_2(t+n) + f_1(t-n) [\Delta_S f_2(t)]. \quad (1.4.13)$$

This can clearly not be done by imposing a noncommutativity between ϵ^i , θ_i and t , since the shifts of $-n$ and of $+n$ can not be corrected for at the same time. Therefore, the symmetric difference operator is not useful in the noncommutativity approach.

Altogether, the supercharges $Q_i = \partial_{\theta^i} + \theta^i \Delta_{\pm}$ fulfill the algebra

$$\{Q_1, Q_1\} = 2\Delta_{\pm}, \quad \{Q_2, Q_2\} = 2\Delta_{\pm}, \quad \{Q_1, Q_2\} = 0, \quad (1.4.14)$$

where the sign choice for Q_1 is independent from that for Q_2 . Since furthermore both supercharges commute with the difference operator, the same supersymmetry algebra as in the continuum is obtained. The supersymmetry variations $\epsilon^1 Q_1$ and $\epsilon^2 Q_2$ obey the Leibniz rule, provided the noncommutativity relations (1.4.10), (1.4.11) and (1.4.12) hold with

$$a_1 = \pm n, \quad a_2 = \pm n, \quad (1.4.15)$$

again with the same sign choice as before. The Grassmannians θ^i and ϵ^i still anticommute. Taking into account this noncommutativity, the Jacobi identities have been checked to give no further constraints.

Having a normal Leibniz rule for the supersymmetry variations, one can proceed to build a supersymmetric lattice action. As mentioned, the Hermitian superfield is taken as in the continuum with t now labeling the lattice points. The component fields are treated as ordinary functions of ordinary variables. Acting with the supercharges on it, one immediately sees that table 1.4 is also valid on the lattice upon replacing ∂_t by $\Delta_{\pm}$. But the noncommutativity shows up when moving the θ 's through the fields, compare e.g.

$$\Phi = \phi(t) - i\psi_1(t-a_1)\theta^1 - i\psi_2(t-a_2)\theta^2 + iD(t-a_1-a_2)\theta^2\theta^1. \quad (1.4.16)$$

with equation (1.4.3). This fact has to be taken into account when bringing the θ 's to the left in the computation of the component field action such that the integration over the θ 's can be performed.

From (1.4.16) it can be seen that $t-a_i = t \pm n$ needs to be a lattice point, which is why the difference is taken over two lattice spacings in equation (1.4.6).

In order to construct the kinetic term for this multiplet, the lattice superderivatives $D_i = \partial_{\theta^i} - \theta^i \Delta_{\pm}$ (with the sign choice for D_i as for Q_i) are used. For reasons that will become clear in a moment a relative shift α in $D_2 \Phi D_1 \Phi$ is allowed for:

$$\begin{aligned} n \sum_t \int d\theta^1 d\theta^2 \frac{1}{2} D_2 \Phi(t - \alpha) D_1 \Phi(t) = \\ \frac{n}{2} \sum_t [\Delta_{\pm} \phi(t + a_1 - \alpha) \Delta_{\pm} \phi(t) - D(t + a_2 - \alpha) D(t) \\ - (\psi_1(t) \Delta_{\pm} \psi_1(t - \alpha) + \psi_2(t + a_1 + a_2 - \alpha) \Delta_{\pm} \psi_2(t))]. \end{aligned} \quad (1.4.17)$$

This action is supersymmetry invariant for both supersymmetry variations for any choice of the shift α . This can be seen either by acting with any of the $\epsilon^i Q_i$ on the action using the fact that the $\epsilon^i Q_i$ obey the Leibniz rule, or by applying the supersymmetry variations s_i of the component fields, as can be read of in table 1.4, to this action. When taking this last, more natural approach² it must be realized that the supersymmetry variations s_i now obey a modified Leibniz rule, as will be explained in the next sections, see e.g. equation (1.4.35) below (taking $A = 1$ or 2 , and using that independently $a_1 = \pm n$ and $a_2 = \pm n$). Put differently, the combination $\epsilon^1 s_1$ or $\epsilon^2 s_2$ obeys a normal product rule, but the ϵ^i have a noncommutative behaviour.

At this point two physical arguments are invoked. When integrating out the auxiliary field D later, it is desirable to have an equation of motion like $D(t) = f[\phi]$ which comes out only if the corresponding term in the action reads $D^2(t)/2$, i.e. if the shift α is fixed to a_2 (all other choices would lead to a non-local equation $D(t + T) + D(t - T) = f[\phi]$ with some T).

Moreover, having lattice simulations in mind it is desirable to obtain a strictly positive kinetic term $(\Delta\phi(t))^2/2$ for the bosonic component. This gives another constraint depending on the choice of sign for the difference operators in the Q_i . By virtue of $\Delta_+ f(t) = \Delta_- f(t + 2n)$ and changes in the summation variable t all solutions to these constraints are equivalent to taking just the plus signs in the definition of the Q_i , a_i and D_i . This

²Superspace in the end is just a tool to obtain a supersymmetry invariant action, the real physics is contained in the transformation rules of the component fields and the action after the fermionic coordinates have been integrated out.

leads to the following kinetic part of the action

$$\frac{n}{2} \sum_t [(\Delta_+ \phi(t))^2 - D^2(t) - (\psi_1(t+n) \Delta_+ \psi_1(t) + \psi_2(t+n) \Delta_+ \psi_2(t))]. \quad (1.4.18)$$

Observe that the argument $t+n$ of the spinors $\psi_{1,2}$ is right in the middle of the two lattice points used in the difference operator.

For the mass and interaction terms there seems to be no particular reason to introduce relative shifts and the lattice analogue of the full continuum action (1.4.1) is found to be

$$\begin{aligned} S = n \sum_t & \left[\frac{1}{2} (\Delta_+ \phi)^2 - \frac{1}{2} [\psi_1(t+n) \Delta_+ \psi_1 + \psi_2(t+n) \Delta_+ \psi_2] \right. \\ & \left. - i[m + 3g\phi^{(2)}][\psi_1(t+n)\psi_2 - \psi_2(t+n)\psi_1]/2 \right. \\ & \left. - \frac{1}{2} D^2 - D[m\phi^{(1)} + g\phi^{(3)}] \right], \end{aligned} \quad (1.4.19)$$

where unshifted arguments (t) are not displayed for readability. The $\phi^{(a)}$ have been used to abbreviate the lattice terms that converge to the corresponding powers ϕ^a in the continuum limit

$$\phi^{(1)}(t) = [\phi(t+2n) + \phi(t)]/2, \quad (1.4.20)$$

$$\begin{aligned} \phi^{(2)}(t) &= [[\phi(t+2n) + \phi(t+n)][\phi(t+n) + \phi(t)] \\ &\quad + \phi^2(t+2n) + \phi^2(t)]/6, \end{aligned} \quad (1.4.21)$$

$$\phi^{(3)}(t) = [\phi^2(t+2n) + \phi^2(t)][\phi(t+2n) + \phi(t)]/4. \quad (1.4.22)$$

While the kinetic part of the lattice action (1.4.19) only differs by one shift from the naive translation of the continuum, the complicated structure of the $\phi^{(a)}$'s is the nontrivial result of the noncommutativity approach and is needed for supersymmetry invariance. However, the arguments of the fields in the $\phi^{(a)}$'s are never shifted over more than two lattice spacings.

Like in the continuum, the field D can be replaced by $-m\phi^{(1)} - g\phi^{(3)}$. This action will be considered in slightly more detail now, in order to compare to different existing supersymmetric quantum mechanics lattice actions. With Dirac spinors $\psi = (\psi_1 + i\psi_2)/\sqrt{2}$ it reads

$$\begin{aligned} S^{\text{on}} = n \sum_t & \left[\frac{1}{2} (\Delta_+ \phi)^2 + \frac{1}{2} (m\phi^{(1)} + g\phi^{(3)})^2 - \psi^\dagger(t+n) \Delta_+ \psi \right. \\ & \left. - \frac{1}{2} (m + 3g\phi^{(2)}) (\psi^\dagger(t+n)\psi(t) + \psi^\dagger(t)\psi(t+n)) \right]. \end{aligned} \quad (1.4.23)$$

Note the small difference in convention with respect to the Dirac spinors as compared with section 1.2. Without interactions the action can be written in the bilinear form

$$S_{g=0}^{\text{on}} = n \sum_{t,t'} \frac{1}{2} \phi(t) B_{tt'} \phi(t') + \psi^\dagger(t) F_{tt'} \psi(t'), \quad (1.4.24)$$

with the following bosonic and fermionic matrix:

$$\begin{aligned} B_{tt'} &= \frac{1}{2} \left(\frac{1}{n^2} + m^2 \right) \delta_{tt'} + \frac{1}{4} \left(-\frac{1}{n^2} + m^2 \right) (\delta_{t+2,t'} + \delta_{t-2,t'}), \\ F_{tt'} &= \frac{1}{2} \left(\frac{1}{n} - m \right) \delta_{t-1,t'} + \frac{1}{2} \left(-\frac{1}{n} - m \right) \delta_{t+1,t'}. \end{aligned} \quad (1.4.25)$$

The supersymmetry of the action expresses itself by the fact that $B = F^T F$ such that the corresponding determinants in the path integral cancel each other.

It can straightforwardly be seen that in the noninteracting case this action amounts to the one given in [58] by equation (2.18) putting $r/a = -m$, up to a relative minus sign for the fermion terms (here the conventions of [28] are followed). However, for the case of a Φ^4 -interaction that action gives a diagonal term coupling the fermions to ϕ^2 , while in the case presented here there will always be terms like $\phi(\cdot)\phi(\cdot)\psi^\dagger(t \pm n)\psi(t)$.

Although the superspace form of the lattice action is supersymmetric due to the fact that there is a Leibniz rule for the $\epsilon^i Q_i$, it is unclear whether there will be a consistent supersymmetry invariance at the level of the component fields, i.e. after the supercoordinates have been integrated out. As it turns out, the supersymmetry invariance is ambiguous, as will be discussed in the following sections.

The General Noncommutativity Approach and its Inconsistency

In this section it will be shown that the noncommutativity approach to lattice supersymmetry suffers from an inconsistency, independent of the dimensionality of space-time and the number of supersymmetries. Before doing so, the general formalism will first briefly be reviewed. The formalism presented here is equivalent to the one presented in [32–34], however formulated in a different way.

In the general noncommutativity approach the Lorentz transformations are ignored: from the supersymmetry algebra only the purely supersymmetric part and the translations are taken into account [34]. This simplification makes the problem of lattice supersymmetry less complicated.

Like in the case of supersymmetric quantum mechanics, in higher dimensional spaces the derivative operator ∂_μ is replaced on the lattice by the difference operator $\Delta_{\pm\mu}$ defined by

$$\Delta_{\pm\mu}f(\mathbf{x}) = \pm \frac{1}{2|\mathbf{n}_\mu|} (f(\mathbf{x} \pm 2\mathbf{n}_\mu) - f(\mathbf{x})), \quad (1.4.26)$$

where $\mathbf{n}_\mu$ corresponds to the shift of one lattice spacing in the μ -direction. Each $\Delta_{\pm\mu}$ obeys the modified Leibniz rule (1.3.2).

In a superspace representation with coordinates (x^μ, θ^A) the supercharges can generically be written as

$$Q_A = \partial_{\theta^A} + \frac{1}{2} f_{AB}^\mu \theta^B \Delta_{\pm\mu}, \quad (1.4.27)$$

on the lattice, where $f_{AB}^\mu = f_{BA}^\mu$. The general supersymmetry algebra reads

$$\{Q_A, Q_B\} = f_{AB}^\mu \Delta_{\pm\mu}, \quad [Q_A, \Delta_{\pm\mu}] = 0. \quad (1.4.28)$$

In complete analogy to the case of supersymmetric quantum mechanics, it is not possible to satisfy the Leibniz rule for the supercharges by introducing a noncommutativity, but it is possible for the supersymmetry variations $\delta_A = \epsilon^A Q_A$, provided the noncommutativities

$$[\mathbf{x}, \theta^A] = \mathbf{a}_A \theta^A, \quad [\mathbf{x}, \epsilon^A] = \mathbf{a}_A \epsilon^A \quad (\text{no sum}) \quad (1.4.29)$$

hold. The shift parameters $\mathbf{a}_A$ (now vectors) do not destroy the Leibniz rule for the first term in $\epsilon^A Q_A$, but they do fix it for the second term upon imposing the following conditions

$$\mathbf{a}_A + \mathbf{a}_B = \pm 2\mathbf{n}_\mu \text{ for } f_{AB}^\mu \neq 0. \quad (1.4.30)$$

As in the one dimensional case, the Jacobi identities have been checked to give no further constraints.

In [34] a solution to these relations was found for the twisted $N = 2$ supersymmetry algebra in two dimensions as well as for the twisted $N = 4$ supersymmetry algebra in four dimensions. As explained above, the

relations can also be satisfied for the $N = 2$ supersymmetry algebra in one dimension.

At this point one can proceed as in the continuum with building a supersymmetric action out of (chiral) superfields. The superderivatives

$$D_A = \partial_{\theta_A} - \frac{1}{2} f_{AB}^{\mu} \theta^B \Delta_{\pm\mu} \quad (1.4.31)$$

obey the algebra

$$\{D_A, D_B\} = -f_{AB}^{\mu} \Delta_{\pm\mu}, \quad \{D_A, Q_B\} = 0. \quad (1.4.32)$$

The transformation rules $\epsilon^A s_A$ for the component fields present in the theory at hand (like in table 1.4) can be derived by comparing the θ -expansions of the superfields Φ_i of the theory with their supersymmetry variations $\epsilon^A Q_A \Phi_i$ or, equivalently, by doing the same with for example the superfields $D_A \Phi_i$ or $D_A D_B \Phi_i$. Other superfields should not be considered for obtaining the transformation rules of the component fields: it is precisely due to the fact that the supercharges and superderivatives anticommute that the indicated superfields Φ_i , $D_A \Phi_i$, etc. all lead to the same variation rules for the component fields.

As mentioned before, the component fields are treated as ordinary functions, as is necessary having numerical simulations in mind.

However, this noncommutativity approach is spoiled by an inconsistency. At the heart of it lies the fact that the supersymmetry variations s_A obey a modified product rule when acting on a product of component fields. This can be seen by evaluating the transformations of a product of fields:

$$\begin{aligned} f_1(\mathbf{x}) f_2(\mathbf{x}) &\rightarrow (f_1(\mathbf{x}) + \epsilon^A s_A f_1(\mathbf{x})) (f_2(\mathbf{x}) + \epsilon^A s_A f_2(\mathbf{x})) \\ &= f_1(\mathbf{x}) f_2(\mathbf{x}) + [\epsilon^A s_A f_1(\mathbf{x})] f_2(\mathbf{x}) + f_1(\mathbf{x}) [\epsilon^A s_A f_2(\mathbf{x})] + O(\epsilon^2). \end{aligned} \quad (1.4.33)$$

At the same time it holds by definition that

$$f_1(\mathbf{x}) f_2(\mathbf{x}) \rightarrow f_1(\mathbf{x}) f_2(\mathbf{x}) + \epsilon^A s_A [f_1(\mathbf{x}) f_2(\mathbf{x})] + O(\epsilon^2). \quad (1.4.34)$$

From this the normal product rule for the supersymmetry variations $\epsilon^A s_A$ follows. For the variations s_A , ϵ^A is brought to the left using (1.4.29):

$$s_A [f_1(\mathbf{x}) f_2(\mathbf{x})] = [s_A f_1(\mathbf{x})] f_2(\mathbf{x}) + (-1)^{|f_1|} f_1(\mathbf{x} + \mathbf{a}_A) [s_A f_2(\mathbf{x})], \quad (1.4.35)$$

where ϵ_A has been omitted.

Because the product of component fields $f_1 f_2$ is equal to $(-1)^{|f_1||f_2|} f_2 f_1$, one would expect the corresponding supersymmetry variations to agree, too. However, because of the different shifts induced by ϵ^A , one obtains by doing the calculation again, this time starting with $f_2 f_1$

$$s_A \left[(-1)^{|f_1||f_2|} f_2(\mathbf{x}) f_1(\mathbf{x}) \right] = [s_A f_1(\mathbf{x})] f_2(\mathbf{x} + \mathbf{a}_A) + (-1)^{|f_1|} f_1(\mathbf{x}) [s_A f_2(\mathbf{x})]. \quad (1.4.36)$$

This means that *the supersymmetry variations do not evaluate to the same expression when acting on the product $f_1 f_2$ and the equivalent expression $(-1)^{|f_1||f_2|} f_2 f_1$, respectively.* Instead, the difference of the two can be written as

$$[s_A f_1(\mathbf{x})] (f_2(\mathbf{x} + \mathbf{a}_A) - f_2(\mathbf{x})) + (-1)^{|f_1|} (f_1(\mathbf{x}) - f_1(\mathbf{x} + \mathbf{a}_A)) [s_A f_2(\mathbf{x})], \quad (1.4.37)$$

which is not a total difference, i.e. it does not lead to a boundary term, in which case the difference would be irrelevant.

Because all shifts are proportional to the lattice spacing, the ambiguity disappears in the continuum limit, but to keep exact supersymmetry at finite lattice spacing remains a problem³.

In the derivation the natural point of view was taken that the supersymmetry variations of all combinations of component fields follow – via (1.4.34) – from the variations of the individual fields, table 1.4. Having obtained an action and the supersymmetry variations of component fields (plus the noncommutativity of t and ϵ) one should be allowed to neglect the auxiliary superspace formalism.

Supersymmetry variations of products of component fields derived from products of superfields, e.g. by θ -expanding Φ^2 and $\delta_A \Phi^2$, should agree with the ones used so far for consistency. If *other* supersymmetry rules for the products of component fields are obtained in this way, then an unacceptable memory to superspace is introduced and, what is more, products of fields are treated like new independent fields. This would then mean a second inconsistency in the noncommutativity approach. However, the supersymmetry variations obtained in this way are equivalent and therefore also show the inconsistency.

³From the corresponding modified Leibniz rule for the difference operator one might be tempted to conclude that this operator suffers from the same problem, but its definition $\pm 2n \Delta_{\pm} [f_1 f_2] = [f_1 f_2](t \pm 2n) - [f_1 f_2](t)$ is invariant under interchanging f_1 and f_2 .

Indeed, also in this approach it is possible to derive the inconsistency in the supersymmetry variations. Any of the superfields Φ_i , $D_A\Phi_i$, $D_AD_B\Phi_i$, etc. can be written as

$$F(\mathbf{x}) = f(\mathbf{x}) + \sum_B \theta^B s_B f(\mathbf{x}) + O(\theta^2), \quad (1.4.38)$$

because then $\epsilon^A Q_A F = \epsilon^A s_A f + O(\theta)$ gives the correct supersymmetry variation for f . This first component field f runs over all component fields present in the theory when taking all superfields F of the form Φ_i , $D_A\Phi_i$, etc. As mentioned before, other superfields should not be considered.

Studying the product

$$\begin{aligned} F_1(\mathbf{x})F_2(\mathbf{x}) &= f_1(\mathbf{x})f_2(\mathbf{x}) + \sum_B \theta^B \{[s_B f_1(\mathbf{x})]f_2(\mathbf{x}) + \\ &\quad (-1)^{|f_1|} f_1(\mathbf{x} + \mathbf{a}_B)[s_B f_2(\mathbf{x})]\} + O(\theta^2), \end{aligned} \quad (1.4.39)$$

the variation of which is given by

$$\begin{aligned} \epsilon^A Q_A [F_1(\mathbf{x})F_2(\mathbf{x})] &= \\ \epsilon^A \{[s_A f_1(\mathbf{x})]f_2(\mathbf{x}) + (-1)^{|f_1|} f_1(\mathbf{x} + \mathbf{a}_A)[s_A f_2(\mathbf{x})]\} &+ O(\theta), \end{aligned} \quad (1.4.40)$$

equation (1.4.35) for the supersymmetry variation of the product of any two component fields is read off from the first components of (1.4.39) and (1.4.41).

In general, the product of two superfields depends on their order (because of the noncommutativity (1.4.29)), but the first component is always the same (up to a possible sign due to fermionic behaviour). Therefore, looking in the same manner at the variation of

$$\begin{aligned} F_2(\mathbf{x})F_1(\mathbf{x}) &= f_2(\mathbf{x})f_1(\mathbf{x}) + \sum_B \theta^B \{[s_B f_2(\mathbf{x})]f_1(\mathbf{x}) + \\ &\quad (-1)^{|f_2|} f_2(\mathbf{x} + \mathbf{a}_B)[s_B f_1(\mathbf{x})]\} + O(\theta^2) \end{aligned} \quad (1.4.41)$$

and multiplying with the factor $(-1)^{|f_1||f_2|}$ one arrives at equation (1.4.36) for the supersymmetry variation of the product of two component fields with the opposite order. Hence, the inconsistency is rederived, because it is the disagreement between (1.4.35) and (1.4.36).

As a consequence, there is *an ambiguity in showing supersymmetry invariance of lattice actions*. Indeed, suppose starting with a term $f_1 f_2$ in

a lattice action S one can show supersymmetry invariance of this action, i.e. $s_A S = 0$. The same action will not be supersymmetry invariant when writing this term as $(-1)^{|f_1||f_2|} f_2 f_1$, because now $s_A S$ will be equal to a difference term of the form (1.4.37), which does not vanish. Therefore, the noncommutativity approach is seen to be inconsistent.

Explicit Calculations

In this section it is demonstrated how the inconsistency emerges when checking the supersymmetry invariance of the proposed lattice action of supersymmetric quantum mechanics (1.4.19). This is first done by explicitly calculating the supersymmetry variation of the mass terms, which of course have to be invariant on their own, under $\delta_1 = \epsilon^1 s_1$. The inconsistency is also illustrated by explicitly deriving how a particular product of component fields transforms, as in equation (1.4.34).

Secondly, the inconsistency is illustrated in a different way in the abelian super BF model.

Supersymmetric Quantum Mechanics

For the variation under δ_1 one can group the four terms proportional to m in the action (1.4.19) into two pairs M and N (which, however, get mixed under δ_2)

$$M = D(t)\phi(t) - i\psi_2(t+n)\psi_1(t), \quad (1.4.42)$$

$$N = D(t)\phi(t+2n) + i\psi_1(t+n)\psi_2(t). \quad (1.4.43)$$

Using the Leibniz rule for δ_1 and the supersymmetry variations of table 1.4 one gets

$$\begin{aligned} \delta_1 M &= -\epsilon^1 \Delta_+ \psi_2(t)\phi(t) + iD(t)\epsilon^1 \psi_1(t) \\ &\quad - i\epsilon^1 D(t+n)\psi_1(t) + \psi_2(t+n)\epsilon^1 \Delta_+ \phi(t). \end{aligned} \quad (1.4.44)$$

Moving ϵ^1 to the left results in two shifts (and a sign change)

$$\begin{aligned} \delta_1 M &= \epsilon^1 [-\Delta_+ \psi_2(t)\phi(t) + iD(t+n)\psi_1(t) \\ &\quad - iD(t+n)\psi_1(t) - \psi_2(t+2n)\Delta_+ \phi(t)] \\ &= -\epsilon^1 \Delta_+ [\psi_2(t)\phi(t)]. \end{aligned} \quad (1.4.45)$$

Since the total difference vanishes under the sum over t , this term is supersymmetry invariant as it should. The various terms cancel as in the continuum.

However, when $D(t)$ and $\phi(t)$ are interchanged, which does not change M ,

$$\tilde{M} = \phi(t)D(t) - i\psi_2(t+n)\psi_1(t), \quad (1.4.46)$$

ϵ^1 induces different shifts leading to

$$\begin{aligned} \delta_1 \tilde{M} &= \epsilon^1 [i\psi_1(t)D(t) - \phi(t+n)\Delta_+\psi_2(t) \\ &\quad - iD(t+n)\psi_1(t) - \psi_2(t+2n)\Delta_+\phi(t)] \\ &\neq \epsilon^1 \cdot \text{total difference.} \end{aligned} \quad (1.4.47)$$

Hence, written in this form, the term turns out not to be supersymmetry invariant.

Actually, the other mass term N is not invariant as it stands, but

$$\tilde{N} = \phi(t+2n)D(t) + i\psi_1(t+n)\psi_2(t) \quad (1.4.48)$$

is.

Since $M = \tilde{M}$ and $N = \tilde{N}$ (the fields are treated as (anti)commuting functions in a path integral), the definition of the supersymmetry transformations of the mass terms is not unique, nor is the question whether (the sum of) these terms are supersymmetry invariant.

Of course this problem does not occur for squares (like $(\Delta_+\phi)^2$ in (1.4.19)). It can be checked that it neither appears for the kinetic term of the spinors. However, the inconsistency is present for the mass term and persists for the on-shell action when D is integrated out. This is because the on-shell action is not just a sum of squares, as can be seen from the second line in (1.4.19), which as a fermionic term does not change when integrating out D .

To illustrate the inconsistency in a different way, it is investigated explicitly how products of component fields transform, as in equation (1.4.34). Considering for example ϕ and ψ_1 , one will conclude from the way in which the product $\phi\psi_1$ transforms (see table 1.4)

$$\begin{aligned} \phi(t)\psi_1(t) &\rightarrow (\phi(t) + \epsilon^1 s_1 \phi(t))(\psi_1(t) + \epsilon^1 s_1 \psi_1(t)) \\ &= (\phi(t) + i\epsilon^1 \psi_1(t))(\psi_1(t) + i\epsilon^1 \Delta\phi(t)) \\ &= \phi(t)\psi_1(t) + i\epsilon^1 \phi(t+n)\Delta\phi(t), \end{aligned} \quad (1.4.49)$$

that the supersymmetry variation of the product is given by

$$s_1(\phi(t)\psi_1(t)) = i\phi(t+n)\Delta\phi(t). \quad (1.4.50)$$

However, the following is equally valid

$$\begin{aligned} \psi_1(t)\phi(t) &\rightarrow (\psi_1(t) + i\epsilon^1\Delta\phi(t))(\phi(t) + i\epsilon^1\psi_1(t)) \\ &= \psi_1(t)\phi(t) + i\epsilon^1\Delta\phi(t)\phi(t) + i\epsilon^1\psi_1(t+n)\psi_1(t), \end{aligned} \quad (1.4.51)$$

and this leads to the conclusion that

$$s_1(\psi_1(t)\phi(t)) = i\phi(t)\Delta\phi(t) + i\psi_1(t+n)\psi_1(t). \quad (1.4.52)$$

Looking at the inconsistency in this way, it illustrates most clearly how the supersymmetry variation of a product of fields depends on the order of the product taken.

The $N = D = 2$ Abelian Super BF Model

As a second concrete example of the inconsistency, the Abelian super BF model in two dimensions is investigated. This model has been formulated using twisted $N = 2$ supersymmetry in [34]. Reformulating the algebra of $N = D = 2$ supersymmetry by twisting, it reads (see equation (1.2.29))

$$\{Q, Q_\mu\} = i\Delta_{+\mu}, \quad \{\tilde{Q}, Q_\mu\} = -i\epsilon_{\mu\nu}\Delta_{-\nu}, \quad (1.4.53)$$

where only the non-vanishing anticommutators are shown. Introducing the supercoordinates $\theta, \tilde{\theta}, \theta^\mu$, the supercharges read

$$\begin{aligned} Q &= \partial_\theta + \frac{i}{2}\theta^\mu\Delta_{+\mu}, \\ \tilde{Q} &= \partial_{\tilde{\theta}} - \frac{i}{2}\epsilon_{\mu\nu}\theta^\mu\Delta_{-\nu}, \\ Q_\mu &= \partial_{\theta^\mu} + \frac{i}{2}\theta\Delta_{+\mu} - \frac{i}{2}\epsilon_{\mu\nu}\tilde{\theta}\Delta_{-\nu}. \end{aligned} \quad (1.4.54)$$

Demanding that independently ϵQ , $\tilde{\epsilon}\tilde{Q}$, $\epsilon^1 Q_1$ and $\epsilon^2 Q_2$ satisfy the Leibniz rule, one is lead to the following non-commutativities

$$[\mathbf{x}, \theta] = \mathbf{a}\theta, [\mathbf{x}, \tilde{\theta}] = \tilde{\mathbf{a}}\tilde{\theta}, [\mathbf{x}, \theta^\nu] = \mathbf{a}^\nu\theta^\nu \text{ (no sum)}, \quad (1.4.55)$$

and similarly for the ϵ_A . The relations (1.4.30) can in this case be solved and written as

$$\mathbf{a} + \tilde{\mathbf{a}} + \mathbf{a}_1 + \mathbf{a}_2 = 0, \quad \mathbf{a} + \mathbf{a}_1 = \mathbf{n}_1, \quad \mathbf{a} + \mathbf{a}_2 = \mathbf{n}_2. \quad (1.4.56)$$

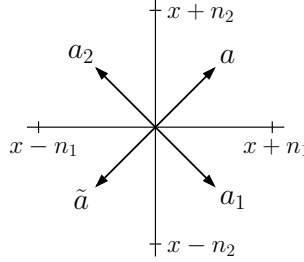


Figure 1.1: Symmetric choice of shift parameter $\mathbf{a}_A$ for twisted $N = D = 2$.

These conditions have a symmetric solution

$$\mathbf{a} = -\tilde{\mathbf{a}} = (1/2, 1/2), \quad \mathbf{a}_1 = -\mathbf{a}_2 = (1/2, -1/2), \quad (1.4.57)$$

as shown in figure 1.1. Working with this solution one is thus forced to introduce new lattice points at the half integer lattice sites, effectively doubling the lattice. Actually, one of the parameters is left undetermined by the equations (1.4.56), leading to a linear space of solutions. Another interesting solution is given by

$$\mathbf{a} = (0, 0), \quad \tilde{\mathbf{a}} = (-1, -1), \quad \mathbf{a}_1 = (1, 0), \quad \mathbf{a}_2 = (0, 1), \quad (1.4.58)$$

where one of the shift parameters vanishes and all others point to already existing lattice points.

Formulating the model on the component field level, there are thus four different supersymmetry transformations, $s, \tilde{s}, s_1, s_2$, each with its own shift parameter, respectively $\mathbf{a}, \tilde{\mathbf{a}}, \mathbf{a}_1, \mathbf{a}_2$. The field content of the abelian super BF model plus the transformation behaviour under s is given in table 1.5. This table has been written down using the conventions of [34] where the transformation s ‘carries a shift’, e.g. $s\bar{c}(\mathbf{x}) = -i\hbar(\mathbf{x} + \mathbf{a})$, whereas before the transformations s_i did not, e.g. $s_1\phi(x) = i\psi_1(x)$ in table 1.4. This difference in convention is due to a different way of defining the superfield, compare equation (4.29) or (5.1) in [34] with equation (1.2.16) in this thesis. Due to the fact that s carries a shift, and the ϵ parameters as well, s is seen to obey a product rule as in (1.4.35), however with a double shift: $\mathbf{a}_A$ in (1.4.35) equals $2\mathbf{a}$ (as in equation (18) in [33]).

Following the noncommutativity approach the lattice action for the

$\underline{\phi}$	$s\underline{\phi}$
$\underline{b}(\mathbf{x} + \mathbf{a})$	0
$\underline{\phi}(\mathbf{x} + \tilde{\mathbf{a}})$	$i\underline{\rho}(\mathbf{x} + \mathbf{a} + \tilde{\mathbf{a}})$
$\underline{\omega}_\nu(\mathbf{x} + \mathbf{a}_\nu)$	$\Delta_{+\nu}\underline{c}(\mathbf{x})$
$\underline{\lambda}(\mathbf{x} + \mathbf{a}_1 + \mathbf{a}_2)$	$\epsilon^{\rho\sigma}\Delta_{+\rho}\underline{\omega}_\sigma(\mathbf{x} + \mathbf{a}_\sigma)$
$\underline{\bar{c}}(\mathbf{x})$	$-i\underline{b}(\mathbf{x} + \mathbf{a})$
$\underline{\rho}(\mathbf{x} + \mathbf{a} + \tilde{\mathbf{a}})$	0
$\underline{c}(\mathbf{x})$	0

Table 1.5: The $N = 2$ twisted supersymmetry transformation s for the abelian super BF model, copied from [34]. The fields above the line are bosonic, the ones below fermionic.

abelian super BF model is given by

$$\begin{aligned}
S_{\text{BF}}^{\text{lat}} = \sum_{\mathbf{x}} \{ & \underline{\phi}(\mathbf{x} - \tilde{\mathbf{a}})\epsilon^{\mu\nu}\Delta_{+\mu}\underline{\omega}_\nu(\mathbf{x} + \mathbf{a}_\nu) + \underline{b}(\mathbf{x} - \mathbf{a})\Delta_{-\mu}\underline{\omega}_\mu(\mathbf{x} + \mathbf{a}_\mu) \\
& - i\underline{\bar{c}}(\mathbf{x})\Delta_{+\mu}\Delta_{-\mu}\underline{c}(\mathbf{x}) + i\underline{\rho}(\mathbf{x} - \mathbf{a} - \tilde{\mathbf{a}})\underline{\lambda}(\mathbf{x} - \mathbf{a} - \tilde{\mathbf{a}}) \}.
\end{aligned} \tag{1.4.59}$$

Applying s to the action and using the fact that it obeys a modified product rule, it follows that

$$\begin{aligned}
sS_{\text{BF}}^{\text{lat}} = & \sum_{\mathbf{x}} \left[i\underline{\rho}(\mathbf{x} + \mathbf{a} - \tilde{\mathbf{a}})\epsilon^{\mu\nu}\Delta_{+\mu}\underline{\omega}_\nu(\mathbf{x} + \mathbf{a}_\nu) + \underline{\phi}(\mathbf{x} - \tilde{\mathbf{a}} + 2\mathbf{a})\epsilon^{\mu\nu}\Delta_{+\mu}\Delta_{+\nu}\underline{c}(\mathbf{x}) \right. \\
& + \underline{b}(\mathbf{x} + \mathbf{a})\Delta_{+\mu}\Delta_{-\mu}\underline{c}(\mathbf{x}) - \underline{b}(\mathbf{x} + \mathbf{a})\Delta_{+\mu}\Delta_{-\mu}\underline{c}(\mathbf{x}) \\
& \left. - i\underline{\rho}(\mathbf{x} + \mathbf{a} - \tilde{\mathbf{a}})\epsilon^{\rho\sigma}\Delta_{+\rho}\underline{\omega}_\sigma(\mathbf{x} + \mathbf{a}_\sigma - \mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a} - \tilde{\mathbf{a}}) \right] \\
= & 0,
\end{aligned} \tag{1.4.60}$$

which equals 0 upon using the identities for the shift parameters and $\epsilon^{\mu\nu}\Delta_{+\mu}\Delta_{+\nu} = 0$. In other words, the action seems to be invariant under the supersymmetry transformation s .

Up to this point there is no disagreement with the authors of [34]. However, interchanging the first two fields in the action, i.e. writing the action as

$$S_{BF}^{\text{lat}} = \sum_{\mathbf{x}} [\epsilon^{\mu\nu} \Delta_{+\mu} \underline{\omega}_\nu(\mathbf{x} + \mathbf{a}_\nu) \underline{\phi}(\mathbf{x} - \tilde{\mathbf{a}}) + \text{rest unchanged}], \quad (1.4.61)$$

and applying the supersymmetry transformation s to this ‘new form’ of the action leads to

$$\begin{aligned} s S_{BF}^{\text{lat}} &= \sum_{\mathbf{x}} [\epsilon^{\mu\nu} \Delta_{+\mu} \Delta_{+\nu} \underline{c}(\mathbf{x}) \underline{\phi}(\mathbf{x} - \tilde{\mathbf{a}}) \\ &\quad + \epsilon^{\mu\nu} \Delta_{+\mu} \underline{\omega}_\nu(\mathbf{x} + \mathbf{a}_\nu + 2\mathbf{a})) i \underline{\rho}(\mathbf{x} + \mathbf{a} - \tilde{\mathbf{a}}) \\ &\quad + \underline{b}(\mathbf{x} + \mathbf{a}) \Delta_{+\mu} \Delta_{-\mu} \underline{c}(\mathbf{x}) - \underline{b}(\mathbf{x} + \mathbf{a}) \Delta_{+\mu} \Delta_{-\mu} \underline{c}(\mathbf{x}) \\ &\quad - i \underline{\rho}(\mathbf{x} + \mathbf{a} - \tilde{\mathbf{a}}) \epsilon^{\rho\sigma} \Delta_{+\rho} \underline{\omega}_\sigma(\mathbf{x} + \mathbf{a}_\sigma - \mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a} - \tilde{\mathbf{a}})] \\ &= \sum_{\mathbf{x}} i \underline{\rho}(\mathbf{x} + \mathbf{a} - \tilde{\mathbf{a}}) \epsilon^{\mu\nu} \Delta_{+\mu} (\underline{\omega}_\nu(\mathbf{x} + \mathbf{a}_\nu + 2\mathbf{a}) - \underline{\omega}_\nu(\mathbf{x} + \mathbf{a}_\nu)) \\ &\neq 0. \end{aligned} \quad (1.4.62)$$

Hence now the action appears not to be invariant under the supersymmetry transformation, thereby illustrating the inconsistency.

Some Observations

For the mass terms in supersymmetric quantum mechanics the two ‘good’ representations M and $\tilde{N}$, which are formally supersymmetry invariant, come from the superfield formulation without (anti)commuting fields on the way. This seems to be the general rule also for the interaction terms (then the corresponding terms in (1.4.19) have to be modified accordingly). It is conjectured that for all the terms that appeared in the noncommutativity approach so far (above and in the literature), there exists a ‘good’ representation which indeed is formally supersymmetry invariant.

After commuting fields one arrives at a ‘bad’ representation, like $\tilde{M}$, which is formally non-invariant. However, when commuting fields in all terms of the sum one can obtain a ‘good’ representation again. For instance,

$$\hat{M} = \phi(t) D(t) + i \psi_1(t) \psi_2(t + n) \quad (1.4.63)$$

which is obtained from the ‘good’ M by (anti)commuting the fields in both terms, is invariant again. An analogous effect has been seen in the interaction terms.

In the sum $M + \hat{M}$ all terms occur completely symmetrized (taking into account minus signs when flipping two fermions). This suggests to symmetrize every expression before applying the supersymmetry transformations s_A to it, in order to be independent of starting with a ‘good’ or ‘bad’ representation, e.g. symmetrizing the ‘bad’ $\tilde{M}$ also leads to the ‘good’ $\frac{1}{2}(M + \hat{M})$. On the level of the expression itself the symmetrization P is an identity operation, but new supersymmetry transformations might be defined by $\bar{s}_A = s_A P$. They do not suffer from the mentioned inconsistency, but will modify $\Delta_{\pm}$ on the right hand side of the supersymmetry algebra and, moreover, inherit another inconsistency from P , namely when acting on ‘composite fields’ they yield $\bar{s}_A[f_1(f_2 f_3)] \neq \bar{s}_A[f_1 f_2 f_3]$. Hence these formal manipulations do not remove the ambiguity in the supersymmetry transformations, but one might view them as indications that the noncommutativity approach will work in a more rigorous formulation.

1.5 The Link Approach

The authors that developed the noncommutativity approach, also developed the link approach to have fully supersymmetric gauge field theories on the lattice [35, 36]. This method has been applied to put super Yang-Mills theories on the lattice, both for the twisted $N = D = 2$ and $N = 4, D = 3$ supersymmetry algebras. In this chapter it is however shown that the link approach like the noncommutativity approach suffers from an inconsistency. This inconsistency is shown along the lines of the $N = D = 2$ super Yang-Mills model.

The link approach is closely related to the noncommutativity approach. Where in the latter the problems with the Leibniz rule at first sight were circumvented via the introduction of a noncommutativity between bosonic and fermionic coordinates in superspace, effectively shifting fields to neighbouring lattice sites, in the first the noncommutativity relation is replaced by a link character for the Grassmann coordinates, i.e. they are defined to live on the links instead of the sites in a lattice. Also the supercharges, supersymmetry transformations and the (component) fields get a link interpretation. When the lattice theory is set up along these lines, a similar ambiguity with respect to the supersymmetry

transformation of a product of fields occurs as in the noncommutativity approach, however only when the component fields in the product constitute a closed loop. Since the action of the model has to be built out of closed loops to ensure gauge invariance, it follows that the supersymmetry transformation of the action is ambiguous.

Before the link approach is introduced, it is explained why it is suitable for gauge field theories whereas the noncommutativity approach is not.

Contrary to scalars and fermions, in lattice field theory the gauge fields are not defined on the lattice sites. Instead of the gauge fields their exponentiated versions are defined on the *links* between the lattice sites [119] via

$$U(\mathbf{x}, \mathbf{x} + \mathbf{n}_\mu) \simeq \exp[i|\mathbf{n}|A_\mu(\mathbf{x})], \quad (1.5.1)$$

where $\mathbf{n}_\mu$ is the vector of length the lattice spacing in the μ -direction. The link field U is an element of the gauge group. In case of e.g. $SU(N)$ the inverse of U is thus given by $U^\dagger$.

Scalar and fermion fields are made to transform under an element G of the gauge group as

$$\phi(\mathbf{x}) \rightarrow G(\mathbf{x})\phi(\mathbf{x})G^{-1}(\mathbf{x}), \quad (1.5.2)$$

while the link gauge field transforms via

$$U(\mathbf{x}, \mathbf{x} + \mathbf{n}_\mu) \rightarrow G(\mathbf{x})U(\mathbf{x}, \mathbf{x} + \mathbf{n}_\mu)G^{-1}(\mathbf{x} + \mathbf{n}_\mu) \quad (1.5.3)$$

For example, the lattice Yang-Mills action is defined via a sum over square faces of the lattice, *plaquettes*. The action is proportional to tracing the link gauge fields around each of the squares of the lattice:

$$S_{YM}^{lattice} \simeq \sum_{\mathbf{x}, \mathbf{n}_\rho, \mathbf{n}_\sigma} \{1 - N^{-1} \text{ReTr} [U(\mathbf{x}, \mathbf{x} + \mathbf{n}_\rho)U(\mathbf{x} + \mathbf{n}_\rho, \mathbf{x} + \mathbf{n}_\rho + \mathbf{n}_\sigma)U(\mathbf{x} + \mathbf{n}_\rho + \mathbf{n}_\sigma, \mathbf{x} + \mathbf{n}_\sigma)U(\mathbf{x} + \mathbf{n}_\sigma, \mathbf{x})]\}, \quad (1.5.4)$$

for gauge group $SU(N)$. The essential point is that this formulation is both gauge invariant and also reproduces Yang-Mills theory in the continuum limit [40, 70, 119].

From these considerations it should be clear that the noncommutativity approach is not suitable for dealing with gauge fields on the lattice,

since it makes explicit use of having all fields defined on the lattice sites. Therefore the link approach [13, 35, 36] has been developed. Since the link approach incorporates link fields into the discretization of a theory, it forms a natural playground for lattice gauge theory. This construction again has the merit of appearing to retain *all* continuum supersymmetries on the lattice.

Instead of thinking in terms of non-commuting supercoordinates x^μ and θ^A , the θ^A are given a link variable interpretation. They now describe constant link variables associated to the links $(\mathbf{x}, \mathbf{x} + \mathbf{a}_A)$, θ^A being their common value:

$$x \bullet \xrightarrow{\theta^A} \bullet x + a_A \quad (1.5.5)$$

Objects on the right of θ_A are now forced to be at $\mathbf{x} + \mathbf{a}_A$, objects on the left at $\mathbf{x}$, thereby *replacing* the non-commutativity relation

$$[\mathbf{x}, \theta^A] = \mathbf{a}_A \theta^A \Leftrightarrow \mathbf{x} \theta^A = \theta^A (\mathbf{x} + \mathbf{a}_A) \quad (\text{no sum}). \quad (1.5.6)$$

In a similar way the ∂_{θ^A} are associated to constant link variables and can be viewed as the same link variable as θ^A , but with opposite link orientation. The standard algebraic properties of the θ^A and ∂_{θ^A} variables are now interpreted as link relations, for instance $\theta^A \theta^B = -\theta^B \theta^A$ reads

$$\theta_{\mathbf{x}+\mathbf{a}_A+\mathbf{a}_B, \mathbf{x}+\mathbf{a}_B}^A \theta_{\mathbf{x}+\mathbf{a}_B, \mathbf{x}}^B = -\theta_{\mathbf{x}+\mathbf{a}_B+\mathbf{a}_A, \mathbf{x}+\mathbf{a}_A}^B \theta_{\mathbf{x}+\mathbf{a}_A, \mathbf{x}}^A. \quad (1.5.7)$$

Via its modified Leibniz rule (1.3.2) also the difference operator $\Delta_{\pm\mu}$ gets the interpretation of a constant link variable, on the link $(\mathbf{x} + \mathbf{n}_\mu, \mathbf{x})$. It thus follows that the supercharges have a link interpretation as well. For example, in case of the twisted $N = D = 2$ algebra, the supercharge Q reads

$$Q_{\mathbf{x}, \mathbf{x}-\mathbf{a}} = (\partial_\theta)_{\mathbf{x}, \mathbf{x}-\mathbf{a}} + \frac{i}{2} (\theta^\mu)_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\mu} (\Delta_{+\mu})_{\mathbf{x}+\mathbf{a}_\mu, \mathbf{x}-\mathbf{a}}, \quad (1.5.8)$$

with both terms consistently pointing to $\mathbf{x} - \mathbf{a}$. The algebra itself reads

$$\begin{aligned} Q_{\mathbf{x}+\mathbf{n}_\mu, \mathbf{x}+\mathbf{a}_\mu} (Q_\mu)_{\mathbf{x}+\mathbf{a}_\mu, \mathbf{x}} + (Q_\mu)_{\mathbf{x}+\mathbf{n}_\mu, \mathbf{x}+\mathbf{a}} Q_{\mathbf{x}+\mathbf{a}, \mathbf{x}} &= i(\Delta_\mu)_{\mathbf{x}+\mathbf{n}_\mu, \mathbf{x}}, \\ \tilde{Q}_{\mathbf{x}, \mathbf{x}+\tilde{\mathbf{a}}} (Q_\mu)_{\mathbf{x}+\tilde{\mathbf{a}}, \mathbf{x}+\tilde{\mathbf{a}}+\mathbf{a}_\mu} + (Q_\mu)_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\mu} \tilde{Q}_{\mathbf{x}+\mathbf{a}_\mu, \mathbf{x}+\mathbf{a}_\mu+\tilde{\mathbf{a}}} &= -i\epsilon_{\mu\nu} (\Delta_{-\nu})_{\mathbf{x}-\mathbf{n}_\nu, \mathbf{x}}. \end{aligned} \quad (1.5.9)$$

The consistency conditions for the link approach are the same as for the noncommutativity approach, namely equation (1.4.56).

Unfortunately, a full consistent set of calculation rules for the link approach has never been presented. For instance, it is not clear how to consistently multiply two superfields. This should be contrasted with the noncommutative constructions where the calculation rules are the usual ones plus a noncommutativity between the bosonic and fermionic coordinates of superspace.

However, working on a component field level it will nevertheless be shown here that it is possible to define a product rule for taking supersymmetry variations of products of fields. This allows one to construct lattice actions which are formally invariant under the twisted lattice supersymmetries. The supersymmetry variations s_A are treated as link variables, like the supercharges Q_A . All component fields $\varphi_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\varphi}$ are treated as link fields, albeit including the degenerate case $\mathbf{a}_\varphi = 0$ in which case one is dealing with a site field. A supersymmetry transformation will thus map a site or a link field onto another site or link field in such a way that the supersymmetry algebra is preserved. The lattice action is then built out of the site and link component fields.

Twisted $N = D = 2$ Super Yang-Mills on the Lattice

In [35] the lattice formulation of twisted $N = D = 2$ super Yang-Mills theory has been treated precisely along the lines described above.

When the $N = D = 2$ algebra is gauged in the continuum, the derivative operator ∂_μ is replaced by a covariant derivative D_μ , as was discussed before, see equations (1.2.29) and (1.2.31). Following the link approach [35] the constant link variable $\Delta_{\pm\mu}$ on the lattice is upon gauging replaced by

$$(\Delta_{\pm\mu})_{\mathbf{x}\pm\mathbf{n}_\mu, \mathbf{x}} \rightarrow \mp(\mathcal{U}_{\pm\mu})_{\mathbf{x}, \mathbf{x}\pm\mathbf{n}_\mu}, \quad (1.5.10)$$

where the gauge link field also includes the adjoint fields $\phi^{(\mu)}$ that can be seen as remnants of gauge fields after dimensional reduction [35, 79]:

$$(\mathcal{U}_{\pm\mu})_{\mathbf{x}, \mathbf{x}\pm\mathbf{n}_\mu} \simeq (e^{\pm i(A_\mu \pm i\phi^{(\mu)})})_{\mathbf{x}\pm\mathbf{n}_\mu, \mathbf{x}}. \quad (1.5.11)$$

The links $\mathcal{U}_{\pm\mu}$ thus are to be thought of as exponentials of a complex vector potential whose imaginary part will give rise to the usual scalar fields $\phi^{(\mu)}$ of extended supersymmetry in the continuum limit.

The constant link variable Q_A is replaced by its gauged version ∇_A ,

$$(Q_A)_{\mathbf{x}, \mathbf{x}-\mathbf{a}_A} \rightarrow (\nabla_A)_{\mathbf{x}, \mathbf{x}+\mathbf{a}_A}. \quad (1.5.12)$$

The link fields $\mathcal{U}_{\pm\mu}$ and ∇_A are $\mathbf{x}$ (or better link) dependent elements of the gauge group, just like gauge links in ordinary lattice gauge theory. The gauge transformation of these link variables are given as in equation (1.5.3).

The following twisted $N = D = 2$ supersymmetry constraints follow from gauging the supersymmetry algebra (1.4.53),

$$\{\nabla, \nabla_\mu\}_{\mathbf{x}, \mathbf{x}+\mathbf{a}+\mathbf{a}_\mu} = +i(\mathcal{U}_{+\mu})_{\mathbf{x}, \mathbf{x}+\mathbf{n}_\mu}, \quad (1.5.13)$$

$$\{\tilde{\nabla}, \nabla_\mu\}_{\mathbf{x}, \mathbf{x}+\tilde{\mathbf{a}}+\mathbf{a}_\mu} = -i\epsilon_{\mu\nu} (\mathcal{U}_{-\nu})_{\mathbf{x}, \mathbf{x}-\mathbf{n}_\nu}, \quad (1.5.14)$$

where the left hand side of (1.5.13)–(1.5.14) should be understood as link anti-commutators, as in (1.5.7). The constraint equations (1.4.56) for the shift parameters are obviously crucial for consistency.

Supersymmetry transformations of link fields φ are now defined by

$$(s_A \varphi)_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\varphi+\mathbf{a}_A} = s_A(\varphi)_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\varphi} \equiv [\nabla_A, \varphi]_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\varphi+\mathbf{a}_A}, \quad (1.5.15)$$

where $(\varphi)_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\varphi}$ denotes a component field or a product of them. Commutator or anti-commutator brackets are used depending upon the bosonic or fermionic nature of the field φ :

$$\begin{aligned} (s_A \varphi)_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\varphi+\mathbf{a}_A} &= (\nabla_A)_{\mathbf{x}, \mathbf{a}+\mathbf{a}_A} \varphi_{\mathbf{x}+\mathbf{a}_A, \mathbf{x}+\mathbf{a}_A+\mathbf{a}_\varphi} \\ &\quad - (-)^{|\varphi|} \varphi_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\varphi} (\nabla_A)_{\mathbf{x}+\mathbf{a}_\varphi, \mathbf{x}+\mathbf{a}_\varphi+\mathbf{a}_A}. \end{aligned} \quad (1.5.16)$$

Pictorially this can be represented by

$$\begin{aligned} x \bullet \xrightarrow{s_A(\varphi)} x + a_A + a_\varphi &= x \bullet \xrightarrow{\nabla_A} \underset{x + a_A}{\bullet} \xrightarrow{\varphi} x + a_A + a_\varphi \\ &\quad - (-)^{|\varphi|} x \bullet \xrightarrow{\varphi} \underset{x + a_\varphi}{\bullet} \xrightarrow{\nabla_A} x + a_A + a_\varphi. \end{aligned} \quad (1.5.17)$$

For doing calculations one needs to know how a product of the (matrix valued) link fields transforms in terms of the transformations of single fields, in other words, one needs to know a product rule. It can easily be derived how something like

$$x \bullet \xrightarrow{\varphi_1} \underset{x + a_{\varphi_1}}{\bullet} \xrightarrow{\varphi_2} x + a_{\varphi_1} + a_{\varphi_2} \quad (1.5.18)$$

transforms, where the φ_i are single link fields. Using the definition (1.5.15), and by writing things out explicitly and adding and subtracting

$$(-1)^{|\varphi_1|} x \bullet \xrightarrow{\varphi_1} \underset{x + a_{\varphi_1}}{\bullet} \xrightarrow{\nabla_A} \underset{x + a_A + a_{\varphi_1}}{\bullet} \xrightarrow{\varphi_2} x + a_A + a_{\varphi_1} + a_{\varphi_2}, \quad (1.5.19)$$

at the same time, it follows that

$$\begin{aligned}
 & (s_A \varphi_1 \varphi_2)_{\mathbf{x}, \mathbf{x} + \mathbf{a}_A + \mathbf{a}_{\varphi_1} + \mathbf{a}_{\varphi_2}} \\
 &= (s_A \varphi_1)_{\mathbf{x}, \mathbf{x} + \mathbf{a}_A + \mathbf{a}_{\varphi_1}} (\varphi_2)_{\mathbf{x} + \mathbf{a}_A + \mathbf{a}_{\varphi_1}, \mathbf{x} + \mathbf{a}_A + \mathbf{a}_{\varphi_1} + \mathbf{a}_{\varphi_2}} \\
 &+ (-1)^{|\varphi_1|} (\varphi_1)_{\mathbf{x}, \mathbf{x} + \mathbf{a}_{\varphi_1}} (s_A \varphi_2)_{\mathbf{x} + \mathbf{a}_{\varphi_1}, \mathbf{x} + \mathbf{a}_A + \mathbf{a}_{\varphi_1} + \mathbf{a}_{\varphi_2}}.
 \end{aligned} \tag{1.5.20}$$

This is a product rule of the well known general form, but as in the noncommutativity construction, modified by shifts in the arguments of the fields - in this case the start and end points of link fields. Pictorially it can be expressed by

$$\begin{aligned}
 & s_A(x \bullet \xrightarrow[\mathbf{x} + \mathbf{a}_{\varphi_1}]{\varphi_1} \bullet \xrightarrow[\mathbf{x} + \mathbf{a}_{\varphi_1} + \mathbf{a}_{\varphi_2}]{\varphi_2} \bullet) \\
 &= x \bullet \xrightarrow[\mathbf{x} + \mathbf{a}_A + \mathbf{a}_{\varphi_1}]{s_A(\varphi_1)} \bullet \xrightarrow[\mathbf{x} + \mathbf{a}_A + \mathbf{a}_{\varphi_1} + \mathbf{a}_{\varphi_2}]{\varphi_2} \bullet \\
 &+ (-1)^{|\varphi_1|} x \bullet \xrightarrow[\mathbf{x} + \mathbf{a}_{\varphi_1}]{\varphi_1} \bullet \xrightarrow[\mathbf{x} + \mathbf{a}_A + \mathbf{a}_{\varphi_1} + \mathbf{a}_{\varphi_2}]{s_A(\varphi_2)} \bullet.
 \end{aligned} \tag{1.5.21}$$

Following the identifications (1.2.32) of the fields the remaining $N = D = 2$ Super Yang-Mills fields can be defined on the lattice via

$$\begin{aligned}
 -\epsilon_{\mu\nu}(\tilde{\rho})_{\mathbf{x}, \mathbf{x} - \tilde{\mathbf{a}}} &\equiv [\nabla_\mu, \mathcal{U}_{+\nu}]_{\mathbf{x}, \mathbf{x} + \mathbf{a}_\mu + \mathbf{n}_\nu}, \\
 -\delta_{\mu\nu}(\rho)_{\mathbf{x}, \mathbf{x} - \mathbf{a}} &\equiv [\nabla_\mu, \mathcal{U}_{-\nu}]_{\mathbf{x}, \mathbf{x} + \mathbf{a}_\mu - \mathbf{n}_\nu}, \\
 -\epsilon_{\mu\nu}(\lambda_\nu)_{\mathbf{x}, \mathbf{x} - \mathbf{a}_\nu} &\equiv \epsilon_{\mu\nu}[\nabla, \mathcal{U}_{-\nu}]_{\mathbf{x}, \mathbf{x} + \mathbf{a} - \mathbf{n}_\nu} = -[\tilde{\nabla}, \mathcal{U}_{+\mu}]_{\mathbf{x}, \mathbf{x} + \tilde{\mathbf{a}} + \mathbf{n}_\mu}, \\
 K_{x,x} &\equiv \frac{1}{2} \{ \nabla_\mu, \lambda_\mu \}_{\mathbf{x}, \mathbf{x}}.
 \end{aligned} \tag{1.5.22}$$

The full twisted $N = 2$ multiplet on the lattice is given by $(\mathcal{U}_\pm, \rho, \tilde{\rho}, \lambda_\mu, K)$, where the fields $\tilde{\rho}$, ρ and λ_ν are the $N = 2$ twisted fermions and where K is an auxiliary bosonic (site) field, needed for the supersymmetry algebra to close off-shell. By construction all the link and site fields in the multiplet are elements of the gauge group and transform in the same way under gauge transformations as the $\mathcal{U}_\pm$ and ∇_A , i.e. in the adjoint representation.

Using the definition of the supersymmetry transformations (1.5.15) and the twisted $N = D = 2$ constraints (1.5.13-1.5.14), one can deduce the supersymmetry transformations of the different component fields explicitly by using Jacobi identities. The result is shown in table 1.6

	s	$\tilde{s}$	s_μ
$\mathcal{U}_{+\nu}$	0	$+\epsilon_{\nu\rho}\lambda_\rho$	$-\epsilon_{\mu\nu}\tilde{\rho}$
$\mathcal{U}_{-\nu}$	$-\lambda_\nu$	0	$-\delta_{\mu\nu}\rho$
λ_ν	0	0	$-i[\mathcal{U}_{+\mu}, \mathcal{U}_{-\nu}] +$ $\delta_{\mu\nu}(K + \frac{i}{2}[\mathcal{U}_{+\rho}, \mathcal{U}_{-\rho}])$
ρ	$-\frac{i}{2}[\mathcal{U}_{+\rho}, \mathcal{U}_{-\rho}] - K$	$+\frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{-\rho}, \mathcal{U}_{-\sigma}]$	0
$\tilde{\rho}$	$-\frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{+\rho}, \mathcal{U}_{+\sigma}]$	$+\frac{i}{2}[\mathcal{U}_{+\rho}, \mathcal{U}_{-\rho}] - K$	0
K	$+\frac{i}{2}[\mathcal{U}_{+\rho}, \lambda_\rho]$	$-\frac{i}{2}\epsilon_{\rho\sigma}[\mathcal{U}_{-\rho}, \lambda_\sigma]$	$-\frac{i}{2}[\mathcal{U}_{+\mu}, \rho] - \frac{i}{2}\epsilon_{\mu\nu}[\mathcal{U}_{-\nu}, \tilde{\rho}]$

Table 1.6: Supersymmetry variations of the $N = D = 2$ lattice super Yang-Mills multiplet.

Arguing in a similar way, it can be seen that the following supersymmetry algebra must hold on the lattice

$$\begin{aligned}
\{s, s_\mu\}(\varphi)_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\varphi} &= +i[\mathcal{U}_{+\mu}, \varphi]_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\varphi+\mathbf{n}_\mu}, \\
\{\tilde{s}, s_\mu\}(\varphi)_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\varphi} &= -i\epsilon_{\mu\nu}[\mathcal{U}_{-\nu}, \varphi]_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\varphi-\mathbf{n}_\nu}, \\
s^2(\varphi)_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\varphi} &= \tilde{s}^2(\varphi)_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\varphi} = 0, \\
\{s, \tilde{s}\}(\varphi)_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\varphi} &= \{s_\mu, s_\nu\}(\varphi)_{\mathbf{x}, \mathbf{x}+\mathbf{a}_\varphi} = 0, \quad (1.5.23)
\end{aligned}$$

where φ denotes any (product of) component field(s) of the multiplet $(\mathcal{U}_{\pm\mu}, \rho, \tilde{\rho}, \lambda_\mu, K)$.

Finally the action of twisted $N = D = 2$ SYM on the lattice is given by

$$\begin{aligned}
S &\equiv \frac{1}{4} \sum_{\mathbf{x}} \text{Tr} \, s\tilde{s}\epsilon_{\mu\nu}s_\mu s_\nu \mathcal{U}_{+\mu}\mathcal{U}_{-\mu} \\
&= \sum_{\mathbf{x}} \text{Tr} \left[\left(\frac{i}{2}[\mathcal{U}_{+\mu}, \mathcal{U}_{-\mu}] - K \right)_{\mathbf{x}, \mathbf{x}} \left(-\frac{i}{2}[\mathcal{U}_{+\nu}, \mathcal{U}_{-\nu}] - K \right)_{\mathbf{x}, \mathbf{x}} \right. \\
&\quad - \frac{1}{4}\epsilon_{\mu\nu}\epsilon_{\rho\sigma}[\mathcal{U}_{+\mu}, \mathcal{U}_{+\nu}]_{\mathbf{x}, \mathbf{x}+\mathbf{n}_\mu+\mathbf{n}_\nu} [\mathcal{U}_{-\rho}, \mathcal{U}_{-\sigma}]_{\mathbf{x}+\mathbf{n}_\rho+\mathbf{n}_\sigma, \mathbf{x}} \\
&\quad - i[\mathcal{U}_{+\mu}, \lambda_\mu]_{\mathbf{x}, \mathbf{x}+\mathbf{a}}(\rho)_{\mathbf{x}+\mathbf{a}, \mathbf{x}} + \\
&\quad \left. - i(\tilde{\rho})_{\mathbf{x}, \mathbf{x}-\tilde{\mathbf{a}}}\epsilon_{\mu\nu}[\mathcal{U}_{-\mu}, \lambda_\nu]_{\mathbf{x}-\tilde{\mathbf{a}}, \mathbf{x}} \right], \quad (1.5.24)
\end{aligned}$$

where the summation over x should also cover the additional lattice sites introduced by the shift parameters a_A . For the symmetric choice of these parameters (figure 1.1) this means a sum over the integer sites (m_1, m_2) and over the half integer sites $(m_1 + \frac{1}{2}, m_2 + \frac{1}{2})$.

This action contains (products of) site fields, loops over one or two links that enclose no area, and also plaquettes as in the standard lattice Yang-Mills action (1.5.4). The fact that the action is given by the consecutive action of all the supersymmetry variations on $\mathcal{U}_{+\mu}\mathcal{U}_{-\mu}$ suggests that the action is by construction invariant under all the supersymmetries due to the nature of the lattice supersymmetry algebra. Furthermore, the closed loop nature of $\mathcal{U}_{+\mu}\mathcal{U}_{-\mu}$ plus the fact that the total shift of $s\tilde{s}\epsilon_{\mu\nu}s_\mu s_\nu$ is given by $a + \tilde{a} + a_1 + a_2 = 0$, makes sure that the action consists of a sum over closed loops, and is therefore manifestly gauge invariant.

Finally, the continuum limit of this lattice model can be taken by identifying $\mathcal{U}_{\pm\mu} \rightarrow \mp(\partial_\mu - iA_\mu \pm \phi^{(\mu)})$. In this way the continuum action (1.2.30) and supersymmetry transformations in table 1.3 satisfying the algebra (1.2.31) are obtained.

The Inconsistency

It can easily be seen from an expression like (1.5.20) that the Leibniz rule is not invariant under flipping the order of φ_1 and φ_2 , due to the shifts in the arguments of the fields, just as in the noncommutativity approach. However due to the link nature of the fields, an expression like

$$(\varphi_1)_{\mathbf{x}, \mathbf{x}+\mathbf{a}_{\varphi_1}} (\varphi_2)_{\mathbf{x}+\mathbf{a}_{\varphi_1}, \mathbf{x}+\mathbf{a}_{\varphi_1}+\mathbf{a}_{\varphi_2}} \quad (1.5.25)$$

can not be exchanged for

$$(\varphi_2)_{\mathbf{x}+\mathbf{a}_{\varphi_1}, \mathbf{x}+\mathbf{a}_{\varphi_1}+\mathbf{a}_{\varphi_2}} (\varphi_1)_{\mathbf{x}, \mathbf{x}+\mathbf{a}_{\varphi_1}}, \quad (1.5.26)$$

since the latter expression is not well defined and in general the two matrices φ_1 and φ_2 do not commute. Thus, in contrast to the noncommutativity formulation the product rule (1.5.20) seems to be well defined in the link approach. However, it is not.

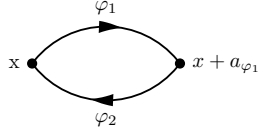
Consider the case of a closed loop, $\mathbf{a}_{\varphi_1} = -\mathbf{a}_{\varphi_2}$. Both

$$(\varphi_1)_{\mathbf{x}, \mathbf{x}+\mathbf{a}_{\varphi_1}} (\varphi_2)_{\mathbf{x}+\mathbf{a}_{\varphi_1}, \mathbf{x}} \quad (1.5.27)$$

and

$$(\varphi_2)_{\mathbf{x}+\mathbf{a}_{\varphi_1},\mathbf{x}}(\varphi_1)_{\mathbf{x},\mathbf{x}+\mathbf{a}_{\varphi_1}} \quad (1.5.28)$$

are well defined and are graphically represented by the same loop diagram



$$(1.5.29)$$

Of particular interest are traces over closed loops since they occur in the action. Upon taking the trace these two expressions are equal up to a factor $(-1)^{|\varphi_1||\varphi_2|}$. The fact that they have a different starting point is of no importance as the action (1.5.25) sums over all lattice points.

Consider now the supersymmetry variation of the first ordering of the fields. It is found that

$$\begin{aligned} & s_A \text{Tr}[(\varphi_1)_{\mathbf{x},\mathbf{x}+\mathbf{a}_{\varphi_1}}(\varphi_2)_{\mathbf{x}+\mathbf{a}_{\varphi_1},\mathbf{x}}] \quad (1.5.30) \\ &= \text{Tr}[(s_A \varphi_1)_{\mathbf{x},\mathbf{x}+\mathbf{a}_A+\mathbf{a}_{\varphi_1}}(\varphi_2)_{\mathbf{x}+\mathbf{a}_A+\mathbf{a}_{\varphi_1},\mathbf{x}+\mathbf{a}_A}] \\ & \quad + (-1)^{|\varphi_1|} \text{Tr}[(\varphi_1)_{\mathbf{x},\mathbf{x}+\mathbf{a}_{\varphi_1}}(s_A \varphi_2)_{\mathbf{x}+\mathbf{a}_{\varphi_1},\mathbf{x}+\mathbf{a}_A}] \\ &= \text{Tr } x \begin{array}{c} \xrightarrow{s_A(\varphi_1)} \xrightarrow{\varphi_2} \\ x + a_A + a_{\varphi_1} \end{array} x + a_A + (-1)^{|\varphi_1|} \text{Tr } x \begin{array}{c} \xrightarrow{\varphi_1} \xrightarrow{s_A(\varphi_2)} \\ x + a_{\varphi_1} \end{array} x + a_A \end{aligned}$$

But supersymmetry transforming the second ordering leads to the expression (up to the factor $(-1)^{|\varphi_1||\varphi_2|}$)

$$\begin{aligned} & s_A \text{Tr}[(\varphi_2)_{\mathbf{x}+\mathbf{a}_{\varphi_1},\mathbf{x}}(\varphi_1)_{\mathbf{x},\mathbf{x}+\mathbf{a}_{\varphi_1}}] \quad (1.5.31) \\ &= \text{Tr}[(s_A \varphi_2)_{\mathbf{x}+\mathbf{a}_{\varphi_1},\mathbf{x}+\mathbf{a}_A}(\varphi_1)_{\mathbf{x}+\mathbf{a}_A,\mathbf{x}+\mathbf{a}_A+\mathbf{a}_{\varphi_1}}] \\ & \quad + (-1)^{|\varphi_2|} \text{Tr}[(\varphi_2)_{\mathbf{x}+\mathbf{a}_{\varphi_1},\mathbf{x}}(s_A \varphi_1)_{\mathbf{x},\mathbf{x}+\mathbf{a}_{\varphi_1}+\mathbf{a}_A}] \\ &= \text{Tr } x + a_{\varphi_1} \begin{array}{c} \xrightarrow{s_A(\varphi_2)} \xrightarrow{\varphi_1} \\ x + a_A \end{array} x + a_A + a_{\varphi_1} \\ & \quad + (-1)^{|\varphi_2|} \text{Tr } x + a_{\varphi_1} \begin{array}{c} \xrightarrow{\varphi_2} \xrightarrow{s_A(\varphi_1)} \\ x \end{array} x + a_A + a_{\varphi_1}. \end{aligned}$$

These expressions are clearly not identical. They do not consist of the same fields. In fact the first ordering leads to a field living on the link $(\mathbf{x}, \mathbf{x} + \mathbf{a}_A)$ while the second gives a field residing on the link $(\mathbf{x} + \mathbf{a}_{\varphi_1}, \mathbf{x} + \mathbf{a}_{\varphi_1} + \mathbf{a}_A)$ with a different product of fields. Therefore the supersymmetry variation of such a gauge invariant loop is not well defined.

The technical reason for the inconsistency comes from the link nature: when transforming the first field in a product like (1.5.30) or (1.5.31), the second field has to be shifted by the shift of s_A . Hence the supersymmetry variations treat fields in products according to their order and reversing the order then leads to the shown contradiction. It is easy to see that this inconsistency applies to closed loop products with two or more fields (i.e. closed loops with two or more legs).

A special case of the situation discussed above deserves a little more attention: $\mathbf{a}_{\varphi_1} = -\mathbf{a}_{\varphi_2} = 0$. Under consideration are now two matrix valued fields F and G that live on lattice sites. Following the product rule, the supersymmetry transformation of the product $F_{\mathbf{x},\mathbf{x}}G_{\mathbf{x},\mathbf{x}}$ is given by

$$\begin{aligned} s_A \text{Tr}[F_{\mathbf{x},\mathbf{x}}G_{\mathbf{x},\mathbf{x}}] &= \text{Tr}[(s_A F)_{\mathbf{x},\mathbf{x}+\mathbf{a}_A} G_{\mathbf{x}+\mathbf{a}_A,\mathbf{x}+\mathbf{a}_A}] \\ &\quad + (-1)^{|F|} \text{Tr}[F_{\mathbf{x},\mathbf{x}}(s_A G)_{\mathbf{x},\mathbf{x}+\mathbf{a}_A}]. \end{aligned} \quad (1.5.32)$$

But since a trace over F and G is taken, the order of the fields in the product can be flipped, and the following is obtained (up to the factor $(-1)^{|F||G|}$)

$$\begin{aligned} s_A \text{Tr}[G_{\mathbf{x},\mathbf{x}}F_{\mathbf{x},\mathbf{x}}] &= \text{Tr}[(s_A G)_{\mathbf{x},\mathbf{x}+\mathbf{a}_A} F_{\mathbf{x}+\mathbf{a}_A,\mathbf{x}+\mathbf{a}_A}] \\ &\quad + (-1)^{|G|} \text{Tr}[G_{\mathbf{x},\mathbf{x}}(s_A F)_{\mathbf{x},\mathbf{x}+\mathbf{a}_A}]. \end{aligned} \quad (1.5.33)$$

This is almost exactly the inconsistency as it was encountered in the non-commutativity approach, see e.g. equation (1.4.35).

This inconsistency affects all the supersymmetry variations associated with the symmetric solution for the shift parameters (1.4.57). Notice, however, that there is no inconsistency in the case when the shift parameter vanishes: $\mathbf{a}_A = 0$. This occurs for example for one of the supersymmetries of the solution given by equation (1.4.58). In this case the supersymmetry variation s maps a closed loop to a sum of closed loops, and it is easy to see using cyclic permutation of the trace that all orderings of the fields yield exactly the same expression for the supersymmetry variation.

Such a supersymmetry variation has a site character and is hence similar to the conserved supersymmetries which appear in both [25, 26, 47, 76–78] and [17–19, 21, 22, 24]. As in that work, a maximum of one supersymmetry variation can be implemented exactly in this lattice model. Furthermore such a variation behaves as a scalar under Lorentz transformations. This can be done for any of the four supersymmetries by working with a suitable choice of the shift parameters. The other three supersymmetries will have non-zero shift parameter, and will suffer from the inconsistency.

To summarize, removing the inconsistency from the link approach leads exactly to the preserved supersymmetries obtained in the literature by other methods too.

Of course an interesting question is how this inconsistency affects the supersymmetry transformation of the action. If one derives the action from its definition (1.5.24) by ignoring the inconsistency explained above, i.e. by just taking the fields in the order in which they come, not changing the order in a product using the trace cyclicity, the form of the action as presented in (1.5.25) is obtained (taking into account the constraint equations (1.4.56)). Transforming this form of the action under s gives the expected result, zero. However, using the cyclicity of the trace to write the action in the form given in [35],

$$\begin{aligned}
S = \sum_{\mathbf{x}} \text{Tr} & \left[\frac{1}{4} [\mathcal{U}_{+\mu}, \mathcal{U}_{-\mu}]_{\mathbf{x}, \mathbf{x}} [\mathcal{U}_{+\nu}, \mathcal{U}_{-\nu}]_{\mathbf{x}, \mathbf{x}} + K_{\mathbf{x}, \mathbf{x}}^2 \right. \\
& - \frac{1}{4} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} [\mathcal{U}_{+\mu}, \mathcal{U}_{+\nu}]_{\mathbf{x}, \mathbf{x} + \mathbf{n}_\mu + \mathbf{n}_\nu} [\mathcal{U}_{-\rho}, \mathcal{U}_{-\sigma}]_{\mathbf{x} + \mathbf{n}_\rho + \mathbf{n}_\sigma, \mathbf{x}} \\
& - i [\mathcal{U}_{+\mu}, \lambda_\mu]_{\mathbf{x}, \mathbf{x} + \mathbf{a}} (\rho)_{\mathbf{x} + \mathbf{a}, \mathbf{x}} + \\
& \left. - i (\tilde{\rho})_{\mathbf{x}, \mathbf{x} - \tilde{\mathbf{a}}} \epsilon_{\mu\nu} [\mathcal{U}_{-\mu}, \lambda_\nu]_{\mathbf{x} - \tilde{\mathbf{a}}, \mathbf{x}} \right], \tag{1.5.34}
\end{aligned}$$

and transforming this form of the action under s , leads to the following

$$\begin{aligned}
sS = \frac{1}{2} \sum_{\mathbf{x}} \text{Tr} & \left[\frac{1}{2} [\mathcal{U}_{+\mu}, \lambda_\mu]_{\mathbf{x}, \mathbf{x} + \mathbf{a}} [\mathcal{U}_{+\nu}, \mathcal{U}_{-\nu}]_{\mathbf{x} + \mathbf{a}, \mathbf{x} + \mathbf{a}} \right. \\
& - \frac{1}{2} [\mathcal{U}_{+\mu}, \mathcal{U}_{-\mu}]_{\mathbf{x}, \mathbf{x}} [\mathcal{U}_{+\nu}, \lambda_\nu]_{\mathbf{x}, \mathbf{x} + \mathbf{a}} + \\
& \left. i K_{\mathbf{x}, \mathbf{x}} [\mathcal{U}_{+\mu}, \lambda_\mu]_{\mathbf{x}, \mathbf{x} + \mathbf{a}} - i [\mathcal{U}_{+\mu}, \lambda_\mu]_{\mathbf{x}, \mathbf{x} + \mathbf{a}} K_{\mathbf{x} + \mathbf{a}, \mathbf{x} + \mathbf{a}} \right]. \tag{1.5.35}
\end{aligned}$$

In the case $\mathbf{a} \neq 0$, the situation here is thus very much like that encountered in the noncommutativity approach [14, 15], where the action written without interchanging fields in a product was naively invariant, but no longer upon changing the order. In the case $\mathbf{a} = 0$, the expression (1.5.35) does reduce to zero, and thus, in this situation, there is again no inconsistency associated with the supersymmetry s . However, the other supersymmetries will now have non-zero shift parameters and variation with respect to them will be ill-defined.

As a final illustration of the inconsistency, one term in the action is considered in more detail:

$$\begin{aligned}
 & -i \sum_{\mathbf{x}} \text{Tr} [(\tilde{\rho})_{\mathbf{x}+\tilde{\mathbf{a}},\mathbf{x}} (\mathcal{U}_{-1})_{\mathbf{x},\mathbf{x}-\mathbf{n}_1} (\lambda_2)_{\mathbf{x}-\mathbf{n}_1,\mathbf{x}+\tilde{\mathbf{a}}}] \\
 & = -i \sum_{\mathbf{x}} \text{Tr} \quad \begin{array}{c} \begin{array}{ccc} x-n_1 & \xleftarrow{\mathcal{U}_{-1}} & x \\ & \searrow \lambda_2 & \nearrow \tilde{\rho} \\ & x+\tilde{\mathbf{a}} \end{array} \end{array} \quad (1.5.36)
 \end{aligned}$$

Transforming the above loop as it stands under s leads to

$$\begin{aligned}
 & i \sum_{\mathbf{x}} \text{Tr} [(\tilde{\rho})_{\mathbf{x}+\tilde{\mathbf{a}},\mathbf{x}} (\lambda_1)_{\mathbf{x},\mathbf{x}-\mathbf{a}_1} (\lambda_2)_{\mathbf{x}-\mathbf{a}_1,\mathbf{x}+\mathbf{a}+\tilde{\mathbf{a}}} + \quad (1.5.37) \\
 & \quad \frac{i}{2} \epsilon_{\rho\sigma} [\mathcal{U}_{+\rho}, \mathcal{U}_{+\sigma}]_{\mathbf{x}+\tilde{\mathbf{a}},\mathbf{x}+\mathbf{a}} (\mathcal{U}_{-1})_{\mathbf{x}+\mathbf{a},\mathbf{x}-\mathbf{a}_1} (\lambda_2)_{\mathbf{x}-\mathbf{a}_1,\mathbf{x}+\mathbf{a}+\tilde{\mathbf{a}}} \Big],
 \end{aligned}$$

which is a link from $\mathbf{x} + \tilde{\mathbf{a}}$ to $\mathbf{x} + \mathbf{a} + \tilde{\mathbf{a}}$. Using the trace cyclicity, (1.5.36) can equivalently be written as

$$-i \sum_{\mathbf{x}} \text{Tr} [(\mathcal{U}_{-1})_{\mathbf{x},\mathbf{x}-\mathbf{n}_1} (\lambda_2)_{\mathbf{x}-\mathbf{n}_1,\mathbf{x}+\tilde{\mathbf{a}}} (\tilde{\rho})_{\mathbf{x}+\tilde{\mathbf{a}},\mathbf{x}}]. \quad (1.5.38)$$

Transforming this expression leads to

$$\begin{aligned}
 & i \sum_{\mathbf{x}} \text{Tr} [(\lambda_1)_{\mathbf{x},\mathbf{x}-\mathbf{a}_1} (\lambda_2)_{\mathbf{x}-\mathbf{a}_1,\mathbf{x}+\mathbf{a}+\tilde{\mathbf{a}}} (\tilde{\rho})_{\mathbf{x}+\mathbf{a}+\tilde{\mathbf{a}},\mathbf{x}+\mathbf{a}} \quad (1.5.39) \\
 & \quad + (\mathcal{U}_{-1})_{\mathbf{x},\mathbf{x}-\mathbf{n}_1} (\lambda_2)_{\mathbf{x}-\mathbf{n}_1,\mathbf{x}+\tilde{\mathbf{a}}} \left(\frac{i}{2} \right) \epsilon_{\rho\sigma} [\mathcal{U}_{+\rho}, \mathcal{U}_{+\sigma}]_{\mathbf{x}+\tilde{\mathbf{a}},\mathbf{x}+\mathbf{a}} \Big],
 \end{aligned}$$

which is a link from $\mathbf{x}$ to $\mathbf{x} + \mathbf{a}$. Choosing $\mathbf{a} = 0$, the expressions (1.5.37) and (1.5.39) are two equivalent ways of writing the same loop, and there is no inconsistency for s . Choosing $\mathbf{a} \neq 0$, the two expressions are different links, and the inconsistency is present.

1.6 Discussion

After an introduction to supersymmetry, putting emphasis on those aspects that are important for supersymmetry on the lattice, the basic problems with lattice supersymmetry have been explained.

In a series of papers D’Adda et al. [34, 35] have developed novel approaches to discretizing certain supersymmetric theories with the goal of preserving *all* continuum (twisted) supersymmetries. Two approaches have been followed, both of which modify the supersymmetry variation of products of fields so as to make it compatible with a modified lattice product rule. In the non-commutativity approach described in [34] this property is ensured by introducing a non-commutativity between superspace coordinates, and in the link approach described in [35] it is ensured via the introduction of a link nature to fields and supersymmetry variations.

First the noncommutativity formalism to put supersymmetric quantum mechanics on the lattice, including masses and typical field-theoretic interactions, has been presented. At first sight the approach keeps all supersymmetries exact. Indeed, showing the invariance of the lattice action in terms of superfields (the analogue of equation (1.2.24)) works as in the continuum, using the Leibniz rule of $\epsilon^A Q_A$. However, an inconsistency has been pointed to, namely that the supersymmetry variation of products of component fields is not uniquely defined. Therefore, the supersymmetry invariance of the lattice action in terms of component fields is ambiguous. This problem has been shown to be present in the noncommutativity approach in general and has also been demonstrated extensively for (the mass term of) supersymmetric quantum mechanics and the abelian super BF model.

The inconsistency can be thought of as a remainder of the noncommutative algebras like $[t, \epsilon] \neq 0$. This structure is somewhat hidden in the lattice actions, which seem to contain only ordinary functions of ordinary numbers (integers, e.g. t), as is very useful for numerical simulations. However, the bosonic coordinates like t cannot be just numbers, as any number commutes with everything in any algebra. It seems to be the too naive treatment of (fields as functions of) the coordinates which spoils the definition of supersymmetry in such systems.

A natural step to establish a more profound formalism would be to promote all objects to $T \times T$ -matrices (T being the number of lattice sites), to replace the sum over t by the trace and so on. It has been tried to represent

the component fields by diagonal matrices and ϵ (and θ) by an off-diagonal one, in order to realize the noncommutativity. However, the supersymmetry variations turned out to be either trivially vanishing or to suffer from the same inconsistency. One attempt at such a formulation is given in [2]. Although it is claimed it does not, this lattice matrix formulation of supersymmetric quantum mechanics also suffers from the same inconsistency. In this formulation, as in the discussion of the noncommutativity approach in this thesis, it is clear that supersymmetry invariance on the superspace level is well defined, whereas it is not on the component field level. D'Adda et al. [2] take the point of view that all that is relevant is the supersymmetry invariance at the superspace level, and therefore deny the inconsistency, both in case of the noncommutativity approach and in the case of their lattice matrix formulation of supersymmetric quantum mechanics, although they also mention more research is needed to clarify the relation between superspace and component field lattice supersymmetry. Their point of view is strongly opposed in this thesis: superspace is nothing but a calculational tool for constructing supersymmetric actions, and of true physical significance is what happens at the component field level. Since in that sense supersymmetry invariance is ambiguous, as is also admitted in [2], the inconsistency is present, in both cases.

For obtaining a way out of the inconsistency using a matrix formulation of supersymmetric quantum mechanics, the only way seems to be to use non-sparse matrices, which increases the number of degrees of freedom considerably (and slows down numerical simulations). An alternative route to follow could be to define a modified product of fields, like the Moyal product, as is typical for functions over noncommutative spaces.

After the discussion of the noncommutativity approach, the link approach was presented. In this approach to gauged lattice supersymmetry the fields live either on links or on sites of a lattice and transform under supersymmetry variation into fields of opposite Grassmann character living on neighbouring links or sites. The supersymmetry variations also either have a link or a site character. The precise link or site nature of the fields and the supersymmetry variations depends on the choice of shift parameters, and ensures that a closed lattice supersymmetry algebra can be constructed incorporating lattice difference operators [35]. In this thesis this construction is summarized and another apparent inconsistency of this approach which is similar to that encountered with the noncommu-

tative construction, is pointed to. While the arguments are quite general they are illustrated by referring to the lattice action for $N = 2$ super Yang-Mills theory described in [35].

The problem is best exposed by considering the supersymmetry variation of a gauge invariant loop. If a supersymmetry variation has a link character, it is shown that the result of supersymmetry variation of this closed loop is not invariant under an initial cyclic permutation of the fields in the loop. Since the action is built out of traces over such loops, this inconsistency is visible at the level of the action – different cyclic permutations of terms in the action yield the same action written in different ways whose supersymmetry variations are not equal: they differ from zero to non-zero.

In the $N = D = 2$ model the shift parameters are constrained such that at most one of them is zero. Choosing one of the shift parameters zero, the corresponding supersymmetry variation has a site character and can be placed on the lattice consistently, similar to what has been achieved in [25, 26, 47, 76–78] and [17–19, 21, 22, 24]. The other three have a link character and their action on gauge invariant loops is not well defined. Choosing none of them zero, all will suffer from the inconsistency.

Furthermore, if a supersymmetry variation has a link character, the variation of the action under this transformation will always be a link variable. Since link variables are not gauge invariant, this means that the expectation value of the supersymmetry variation of the action will be zero, regardless of the way the action is written. This, in turn, means that the partition function will actually be supersymmetry invariant. At first glance this reasoning appears to be a way out of the inconsistency, but it is not. Indeed, this observation actually implies that the supersymmetry variation of *any* gauge invariant lattice action will have vanishing expectation value – clearly a non-physical result.

As a final remark, the component field transformations described here and in [35] do not contain an infinitesimal Grassmann parameter ϵ_A - it is claimed in [36, 37, 87] that the introduction of such a parameter with link character alleviates the inconsistency problems.

1.A Majorana Formulation of SUSY QM

In this thesis two formulations of supersymmetric quantum mechanics are used: a formulation using Dirac or complex fermions and real time and a formulation using Majorana or real fermions and imaginary time. In this appendix both formulations are recaptured and related to each other.

In the first formulation superspace is defined by a time coordinate t and a complex Grassmann coordinate $\theta, \theta^\dagger$ obeying $[t, \theta] = 0, \{\theta, \theta^\dagger\} = 0$. The $N = 2$ supersymmetry in this superspace is generated by a complex supercharge Q

$$Q = \partial_\theta + i\theta^\dagger \partial_t, \quad Q^\dagger = \partial_{\theta^\dagger} + i\theta \partial_t, \quad (1.A.1)$$

that obeys the algebra

$$Q^{(\dagger)^2} = 0, \quad \{Q, Q^\dagger\} = 2H, \quad [Q^{(\dagger)}, H] = 0, \quad (1.A.2)$$

where $H = i\partial_t$ is the Hamiltonian or generator of the time translation symmetry.

The action of supersymmetric quantum mechanics is given by

$$\begin{aligned} S &= \int dt d\theta^\dagger d\theta \frac{1}{2} (D\Phi)^\dagger D\Phi + F(\Phi) \\ &= \int dt \frac{1}{2} (\partial_t \phi)^2 + \psi^\dagger (i\partial_t - F''(\phi)) \psi + \frac{1}{2} D^2 + DF'(\phi), \end{aligned} \quad (1.A.3)$$

with the superfield Φ given by

$$\Phi = \Phi(t, \theta, \theta^\dagger) = \phi(t) + i\theta\psi(t) + i\theta^\dagger\psi^\dagger(t) + \theta\theta^\dagger D(t), \quad (1.A.4)$$

and the superderivative by $D = \partial_\theta - i\theta^\dagger \partial_t$. The superderivatives D and $D^\dagger$ anti-commute with the supercharges, and satisfy the same algebra as them, however up to a minus sign for terms in which the Hamiltonian appears.

The supersymmetry variations ϵQ and $\epsilon^\dagger Q^\dagger$ under which the action is invariant are generated by acting with the supercharges on the superfield. The result can be phrased in terms of transformations of the component fields s and $s^\dagger$ as was done in table 1.2. The algebra obeyed by s and $s^\dagger$ is the following:

$$\{s, s\} = \{s^\dagger, s^\dagger\} = 0, \quad \{s, s^\dagger\} = 2i\partial_t, \quad [s, \partial_t] = [s^\dagger, \partial_t] = 0. \quad (1.A.5)$$

From this formulation, the formulation in terms Majorana fermions and imaginary time, which is also used in [14], can be found following a set of transformations.

On a component field level the transformation from complex to real variables is done by defining the real spinors fields ψ_1 and ψ_2 by

$$\psi = \frac{1}{\sqrt{2}}(\psi_2 + i\psi_1), \quad \psi^\dagger = \frac{1}{\sqrt{2}}(\psi_2 - i\psi_1). \quad (1.A.6)$$

The supersymmetry variations s and $s^\dagger$ are redefined in terms of two imaginary variations s_1 and s_2 as follows:

$$s = \frac{1}{\sqrt{2}}(s_1 - is_2), \quad s^\dagger = -\frac{1}{\sqrt{2}}(s_1 + is_2). \quad (1.A.7)$$

Introducing two real ϵ parameters, ϵ^1 and ϵ^2 , now leads to two Hermitian supersymmetry variations $\epsilon^1 s_1$ and $\epsilon^2 s_2$. Furthermore, the transformation to imaginary time amounts to

$$t \rightarrow it, \quad \partial_t \rightarrow -i\partial_t, \quad S \rightarrow iS. \quad (1.A.8)$$

After these transformations the action of supersymmetric quantum mechanics reads

$$\begin{aligned} S = \int dt \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\psi_1 \partial_t \psi_1 + \psi_2 \partial_t \psi_2) - iF''(\phi)\psi_1 \psi_2 \\ - \frac{1}{2}D^2 - DF'(\phi). \end{aligned} \quad (1.A.9)$$

Taking the superpotential for example to be $F(\phi) = \frac{1}{2}m\phi^2 + \frac{1}{4}g\phi^4$, the action boils down to

$$\begin{aligned} S = \int dt \frac{1}{2}(\partial_t \phi)^2 - \frac{1}{2}(\psi_1 \partial_t \psi_1 + \psi_2 \partial_t \psi_2) - i(m + 3g\phi^2)\psi_1 \psi_2 \\ - \frac{1}{2}D^2 - D(m\phi + g\phi^3). \end{aligned} \quad (1.A.10)$$

The new supersymmetry variations are given in table 1.4 as $\delta_1 = \epsilon^1 s_1$ and $\delta_2 = \epsilon^2 s_2$. These can be obtained following the transformations (1.A.6)-(1.A.8). Either using the transformations (1.A.7) or reading them off

directly from table 1.4, the supersymmetry algebra in terms of s_1 and s_2 is found to be:

$$\{s_1, s_1\} = \{s_2, s_2\} = -2\partial_t, \quad \{s_1, s_2\} = 0, \quad [\partial_t, s_1] = [\partial_t, s_2] = 0. \quad (1.A.11)$$

As might be expected, also this Majorana formulation of supersymmetric quantum mechanics can be formulated in terms of a superspace. Demanding the superfield to be of the following form

$$\begin{aligned} \Phi &= \Phi(t, \theta, \theta^\dagger) = \phi(t) + i\theta\psi(t) + i\theta^\dagger\psi^\dagger(t) + \theta\theta^\dagger D(t) \\ &= \Phi(t, \theta^1, \theta^2) = \phi(t) + i\theta^1\psi_1(t) + i\theta^2\psi^2(t) + i\theta^2\theta^1 D(t), \end{aligned} \quad (1.A.12)$$

where θ^1 and θ^2 are real Grassmann variables, it follows that

$$\theta^1 = \frac{i}{\sqrt{2}}(\theta - \theta^\dagger), \quad \theta^2 = \frac{1}{\sqrt{2}}(\theta + \theta^\dagger). \quad (1.A.13)$$

The real superspace thus consists of two real anti-commuting supercoordinates θ^1 and θ^2 . Furthermore there is the now imaginary time coordinate. Following these coordinate transformations, the supercharges and superderivatives can be transformed as well:

$$Q = \frac{1}{\sqrt{2}}(iQ_1 + Q_2), \quad Q^\dagger = \frac{1}{\sqrt{2}}(-iQ_1 + Q_2), \quad Q_i = \partial_{\theta^i} + \theta^i\partial_t, \quad (1.A.14)$$

and similarly $D_i = \partial_{\theta^i} - \theta^i\partial_t$. The superderivatives still anti-commute with the supercharges, and satisfy the same algebra as the supercharges up to a minus sign for the Hamiltonian terms:

$$\{Q_1, Q_1\} = 2H, \quad \{Q_2, Q_2\} = 2H, \quad \{Q_1, Q_2\} = 0, \quad (1.A.15)$$

where $H = \partial_t$. The supersymmetry variations as given in table 1.4 can now be generated by letting $\epsilon_1 Q_1$ and $\epsilon_2 Q_2$ act on the superfield.

Finally, the action of supersymmetric quantum mechanics in this Majorana superspace formulation is given by

$$S = \int dt d\theta^1 d\theta^2 \frac{1}{2} D_2 \Phi D_1 \Phi + iF(\Phi), \quad (1.A.16)$$

which upon integrating over the Grassmannian variables is equal to (1.A.9).

CHAPTER 2

THE NON-LINEAR SCHRÖDINGER MODEL

2.1 Introduction

In this chapter the symmetries of the non-linear Schrödinger model are studied. This model is a non-relativistic field theory in $d + 1$ dimensions:

$$S = \int dt d^d \mathbf{x} \, i\Psi^* \partial_t \Psi + \frac{1}{2m} \Psi^* \Delta \Psi - \frac{g^2}{n} (\Psi^* \Psi)^n. \quad (2.1.1)$$

Systems described by such an action appear frequently in the context of condensed matter systems [30], where the corresponding non-linear Schrödinger equation

$$i\partial_t \Psi = -\frac{1}{2m} \Delta \Psi + g^2 |\Psi|^{2n-2} \Psi, \quad (2.1.2)$$

is also known as the Gross-Pitaevskii equation [62, 95]. After quantization the field Ψ describes a gas of weakly interacting bosonic particles with repulsive hardcore (i.e. δ -function) interactions, at zero temperature and in an infinite volume. However, only little attention is paid to applications of this model here.

In this thesis the main interest lies in the breaking of its conformal symmetries. The non-interacting model, i.e. having $g = 0$, is a non-relativistic conformal field theory in all dimensions, both classically and

quantum mechanically. It has next to translation and rotation invariance, Galilei invariance, scale invariance and special conformal invariance. The interacting non-linear theory, i.e. having $g \neq 0$, however only preserves the scale and special conformal symmetries in specific numbers of dimensions for specific values of n at the classical level. The interacting model has these invariances only for a $|\Psi|^6$ interaction in $1 + 1$ dimensions and a $|\Psi|^4$ interaction in $2 + 1$ dimensions. The quantization of this latter case is studied in detail in this chapter, showing that due to renormalization effects also in this case the scale and special conformal symmetries are broken.

2.2 The Classical Theory and Its Symmetries

The Symmetries of the Free Schrödinger Model

The action of the free Schrödinger theory is given by

$$S = \int dt d^d \mathbf{x} \, i\Psi^* \partial_t \Psi + \frac{1}{2m} \Psi^* \Delta \Psi. \quad (2.2.1)$$

It is invariant under the action of the Schrödinger group [63, 91], which is a set of space and time transformations including space and time translations, spatial rotations, Galilei boosts, dilatations and special conformal transformations, acting on the coordinates $(\mathbf{x}, t)$ as

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta}, \quad \mathbf{x}' = \frac{\mathbf{R} \cdot \mathbf{x} + \mathbf{u}t + \mathbf{a}}{\gamma t + \delta}. \quad (2.2.2)$$

The transformations are parametrized by $(\alpha, \beta, \gamma, \delta)$, which are real scalar parameters restricted by $\alpha\delta - \beta\gamma = 1$, by $\mathbf{R}$, which is a d -dimensional orthogonal matrix, and by $(\mathbf{u}, \mathbf{a})$, which are d -dimensional vectors.

The subgroup of transformations with $\alpha = \delta = 1$ and $\gamma = 0$ consist of time translations parametrized by β , space translations parametrized by $\mathbf{a}$, spatial rotations parametrized by $d(d-1)/2$ parameters $\omega_{ab} = -\omega_{ba}$ such that

$$R_{ij} = \left(e^{-\frac{1}{2} \omega_{ab} L_{ab}} \right)_{ij}, \quad (L_{ab})_{ij} = \delta_{ia} \delta_{jb} - \delta_{ib} \delta_{ja}, \quad (2.2.3)$$

and Galilean boosts parametrized by $\mathbf{u}$.

The remaining transformations rescale time in a non-trivial way. Dilatations or scale transformations are generated by taking $\alpha = 1/\delta = e^\eta$,

$\beta = \gamma = 0$. Special conformal transformations are generated by $\alpha = \delta = 1$ and $\beta = 0$.

The transformations (2.2.2) can be realized on the complex scalar fields Ψ and Ψ^* such that the action (2.2.1) remains invariant. This realization includes an additional $U(1)$ phase transformation. For rotations and space and time translations, the transformation rule for the scalar field is by definition given by

$$\Psi'(\mathbf{x}', t') = \Psi(\mathbf{x}, t). \quad (2.2.4)$$

However, under Galilean boosts parametrized by $\mathbf{u}$, the field transforms with an additional space-time dependent phase factor,

$$\Psi'(\mathbf{x}', t') = e^{im(\mathbf{u} \cdot \mathbf{x} + \frac{1}{2} \mathbf{u}^2 t)} \Psi(\mathbf{x}, t). \quad (2.2.5)$$

Under dilatations the field has a dimension-dependent non-zero Weyl weight, which also turns up in the special conformal transformations. Taking $\alpha = 1/\delta = e^\eta$, $\beta = \gamma = 0$ the dilatations are realized on Ψ as

$$\Psi'(\mathbf{x}', t') = \alpha^{-d/2} \Psi(\mathbf{x}, t) = e^{-d\eta/2} \Psi(\mathbf{x}, t), \quad (2.2.6)$$

The special conformal transformations with $\alpha = \delta = 1$ and $\beta = 0$ take the form

$$\Psi'(\mathbf{x}', t') = (1 + \gamma t)^{d/2} e^{-m \frac{i}{2} \frac{\gamma \mathbf{x}^2}{(1 + \gamma t)}} \Psi(\mathbf{x}, t). \quad (2.2.7)$$

Finally, the $U(1)$ phase transformations leave the space-time coordinates invariant, and take the standard form on the complex scalar field:

$$\Psi'(\mathbf{x}', t') = \Psi'(\mathbf{x}, t) = e^{i\theta} \Psi(\mathbf{x}, t). \quad (2.2.8)$$

The infinitesimal form of these transformations is given by

$$\begin{aligned} \delta\Psi(\mathbf{x}, t) &= \Psi'(\mathbf{x}, t) - \Psi(\mathbf{x}, t) \\ &= -\beta\partial_t\Psi - \mathbf{a} \cdot \nabla\Psi - \omega_{ij} x_i \nabla_j\Psi - \mathbf{u} \cdot (t\nabla - im\mathbf{x})\Psi \\ &\quad - \eta \left(2t\partial_t + \mathbf{x} \cdot \nabla + \frac{d}{2} \right) \Psi + i\theta\Psi \\ &\quad - \gamma \left(-t^2\partial_t - t \left(\mathbf{x} \cdot \nabla + \frac{d}{2} \right) + \frac{i}{2} m \mathbf{x}^2 \right) \Psi. \end{aligned} \quad (2.2.9)$$

Following Noether's theorem a set of conserved charges is associated to the invariance of the free action (2.2.1) under the transformations of the Schrödinger group. They are given by

a. the Hamiltonian

$$H_0 = \frac{1}{2m} \int d^d x |\nabla \Psi|^2, \quad (2.2.10)$$

induced by invariance under time translations;

b. the momentum

$$\mathbf{P} = \frac{i}{2} \int d^d x \Psi^* \overleftrightarrow{\nabla} \Psi, \quad (2.2.11)$$

following from the invariance under space translations;

c. the angular momentum

$$M_{ij} = \frac{i}{2} \int d^d x \left(x_i (\Psi^* \overleftrightarrow{\nabla}_j \Psi) - x_j (\Psi^* \overleftrightarrow{\nabla}_i \Psi) \right), \quad (2.2.12)$$

following from rotation invariance;

d. the Galilean boost operator

$$\mathbf{G} = t\mathbf{P} + m \int d^d x \mathbf{x} |\Psi|^2, \quad (2.2.13)$$

following from invariance under the Galilei transformations;

e. the scaling operator

$$D_0 = 2tH_0 + \frac{i}{2} \int d^d x \mathbf{x} \cdot (\Psi^* \overleftrightarrow{\nabla} \Psi), \quad (2.2.14)$$

induced by the dilatation or scale invariance;

f. the special conformal charge

$$K_0 = t^2 H_0 - t D_0 - \frac{m}{2} \int d^d x \mathbf{x}^2 |\Psi|^2, \quad (2.2.15)$$

induced by the invariance under the special conformal transformations;

g. the particle number

$$N = \int d^d x |\Psi|^2, \quad (2.2.16)$$

following from the invariance under $U(1)$ phase transformations.

The connection between the constants of motion and the transformations of the Schrödinger group can nicely be illustrated by introducing Poisson brackets. This is done by identifying Ψ as a generalized coordinate, and the canonical momentum variable associated to it as given by $p_\Psi = i\Psi^*$. This identification paves the way towards introducing an equal time Poisson bracket which is canonically given by

$$\{\Psi(\mathbf{x}, t), \Psi^*(\mathbf{y}, t)\} = -i\delta^d(\mathbf{x} - \mathbf{y}), \quad (2.2.17)$$

and can be extended for general functionals F and G to:

$$\{F, G\} = -i \int d^d \mathbf{x} \left[\frac{\delta F}{\delta \Psi(\mathbf{x}, t)} \frac{\delta G}{\delta \Psi^*(\mathbf{x}, t)} - \frac{\delta G}{\delta \Psi(\mathbf{x}, t)} \frac{\delta F}{\delta \Psi^*(\mathbf{x}, t)} \right]. \quad (2.2.18)$$

Letting the conserved charges of the Schrödinger group act on the field Ψ via Poisson's bracket leads to the infinitesimal transformations they generate:

$$\begin{aligned} \{\Psi, H_0\} &= \partial_t \Psi, & \{\Psi, M_{ij}\} &= (x_i \nabla_j - x_j \nabla_i) \Psi, \\ \{\Psi, \mathbf{P}\} &= \nabla \Psi, & \{\Psi, \mathbf{G}\} &= (t \nabla - i m \mathbf{x}) \Psi, \\ \{\Psi, N\} &= -i \Psi, & \{\Psi, D_0\} &= \left(2t \partial_t + \mathbf{x} \cdot \nabla + \frac{d}{2} \right) \Psi, \\ & & \{\Psi, K_0\} &= \left(-t^2 \partial_t - t \left(\frac{d}{2} + \mathbf{x} \cdot \nabla \right) + \frac{i}{2} m \mathbf{x}^2 \right) \Psi, \end{aligned} \quad (2.2.19)$$

upon invoking Schrödinger's equation of motion $\partial_t \Psi = i \Delta \Psi / 2m$. As a re-

sult they realize the Schrödinger algebra extended with a central charge N :

$$\begin{aligned}
\{M_{ij}, M_{kl}\} &= \delta_{il}M_{jk} - \delta_{ik}M_{jl} - \delta_{jl}M_{ik} + \delta_{jk}M_{il}, \\
\{H_0, D_0\} &= 2H_0, \quad \{H_0, \mathbf{G}\} = \mathbf{P}, \\
\{\mathbf{P}, D_0\} &= \mathbf{P}, \quad \{P_i, M_{jk}\} = \delta_{ij}P_k - \delta_{ik}P_j, \\
\{H_0, K_0\} &= -D_0, \quad \{P_i, G_j\} = m\delta_{ij}N, \\
\{\mathbf{G}, D_0\} &= -\mathbf{G}, \quad \{G_i, M_{jk}\} = \delta_{ij}G_k - \delta_{ik}G_j, \\
\{K_0, D_0\} &= -2K_0, \quad \{K_0, \mathbf{P}\} = \mathbf{G},
\end{aligned} \tag{2.2.20}$$

all other brackets vanishing.

Instead of defining the theory by its action (2.2.1) and associated equation of motion, it can equivalently be defined by the Hamiltonian (2.2.10) generating the time evolution of phase-space functions via the Poisson bracket $F[\Psi, \Psi^*; t]$:

$$\frac{dF}{dt} = \left. \frac{\partial F}{\partial t} \right|_{\Psi, \Psi^*} + \{F, H_0\}, \tag{2.2.21}$$

where the partial time derivative refers to time dependence other than through the fields Ψ and Ψ^* . For the constants of motion this implies directly that

$$\{\mathbf{P}, H_0\} = \{M_{ij}, H_0\} = \{N, H_0\} = 0. \tag{2.2.22}$$

The explicit time dependence in the definition of $\mathbf{G}$, D_0 and K_0 implies

$$\{\mathbf{G}, H_0\} = -\mathbf{P}, \quad \{D_0, H_0\} = -2H_0, \quad \{K_0, H_0\} = D_0, \tag{2.2.23}$$

in agreement with the Schrödinger algebra (2.2.20).

The conformal symmetry of the Schrödinger model has direct relevance

for its physical content. In particular, consider the quantities [56]

$$\begin{aligned}\mathbf{I} &\equiv m \int d^d x \, \mathbf{x} \Psi^* \Psi = \mathbf{G} - t\mathbf{P}, \\ I_1 &\equiv \frac{m}{2} \int d^d x \, \mathbf{x}^2 \Psi^* \Psi = t^2 H_0 - tD_0 - K_0, \\ I_2 &\equiv \frac{i}{2} \int d^d x \, \mathbf{x} \cdot \Psi^* \overleftrightarrow{\nabla} \Psi = D_0 - 2tH_0.\end{aligned}\tag{2.2.24}$$

As $(H_0, \mathbf{P}, \mathbf{G}, D_0, K_0)$ are constants of motion, it follows that

$$\frac{d\mathbf{I}}{dt} = -\mathbf{P}, \quad \frac{dI_1}{dt} = -I_2, \quad \frac{dI_2}{dt} = -2H_0.\tag{2.2.25}$$

Therefore, if there would be any *static* (i.e. time-independent) solutions of the theory, such that $\mathbf{I}$, I_1 and I_2 are constant themselves, these solutions would necessarily have zero energy and momentum. Of course, such solutions do not exist in the present case: all the solutions of the free Schrödinger theory represent scattering states, superpositions of δ -function normalizable plane waves with a continuous energy spectrum.

Conformal Symmetry in the Non-Linear Schrödinger Model

Introducing a non-linear interaction term of the form

$$S_{int} = -\frac{g^2}{n} \int dt d^d x \, (\Psi^* \Psi)^n,\tag{2.2.26}$$

where g is a coupling constant, obviously preserves the invariance under time and space translations, spatial rotations, Galilei boosts and the $U(1)$ -invariance. However, the conformal transformations are broken unless

$$2 + d = nd.\tag{2.2.27}$$

Indeed, it can be read off from equations (2.2.19) that the Weyl weight of the time coordinate is 2, that of the space coordinates 1, and that of the field $d/2$. Assuming the coupling constant to be dimensionless, dilatation invariance requires that the dimension of $|\Psi|^{2n}$ matches that of the space-time integration measure: $2 + d = 2n \cdot d/2$. Below it will be shown that under this condition not only the dilatation symmetry is preserved, but

also the special conformal symmetry. In the case $d = 1$ this condition is satisfied for $n = 3$, in the case $d = 2$ for $n = 2$; also, for $d \rightarrow \infty$ one gets $n \rightarrow 1$. In all other cases n must take on non-integer values.

The non-linear theory defined by equation (2.1.1) is still invariant under time and space translations, spatial rotations, Galilei transformations, and $U(1)$ phase transformations. As a result the generators $(\mathbf{P}, M_{ij}, \mathbf{G}, N)$ as in (2.2.11)-(2.2.16) also represent constants of motion in the interacting theory. The Hamiltonian and the generators of scale and special conformal transformations are however replaced by

$$H_n = \int d^d x \frac{1}{2m} |\nabla \Psi|^2 + \frac{g^2}{n} (\Psi^* \Psi)^n, \quad (2.2.28)$$

$$D_n = 2tH_n + I_2, \quad (2.2.29)$$

$$K_n = t^2 H_n - t D_n - I_1,$$

with I_1 and I_2 defined by the explicit first expressions in equation (2.2.24). The quantities D_n and K_n indeed generate the infinitesimal transformations:

$$\begin{aligned} \{\Psi, D_n\} &= \left(2t\partial_t + \mathbf{x} \cdot \nabla + \frac{d}{2} \right) \Psi, \\ \{\Psi, K_n\} &= \left(-t^2\partial_t - t \left(\frac{d}{2} + \mathbf{x} \cdot \nabla \right) + \frac{i}{2} m \mathbf{x}^2 \right) \Psi. \end{aligned} \quad (2.2.30)$$

However, D_n and K_n are constants of motion only if n and d are related by (2.2.27). This was already explained for D_n . To show this formally, and for K_n as well, the results

$$\begin{aligned} \{\mathbf{I}, H_n\} &= -\mathbf{P}, \quad \{I_1, H_n\} = -I_2, \\ \{I_2, H_n\} &= -2H_n + \frac{(2 + d - nd)}{n} \int d^d x g^2 (\Psi^* \Psi)^n. \end{aligned} \quad (2.2.31)$$

are noted. The first of these equations implies the conservation of $\mathbf{G}$ for all n and d ; the other two equations imply

$$\frac{dK_n}{dt} = -t \frac{dD_n}{dt}, \quad \frac{dD_n}{dt} = \frac{(2 + d - nd)}{n} \int d^d x g^2 (\Psi^* \Psi)^n. \quad (2.2.32)$$

Equation (2.2.27) is thus indeed the necessary and sufficient condition for the right-hand side of these equations to vanish.

When this condition is satisfied one obtains an interacting classical field theory invariant under the full Schrödinger group of transformations. For these theories one can derive equations similar to (2.2.25):

$$\frac{d\mathbf{I}}{dt} = -\mathbf{P}, \quad \frac{dI_1}{dt} = -I_2, \quad \frac{dI_2}{dt} = -2H_n, \quad (2.2.33)$$

and conclude again that any static solutions of the interacting theories necessarily have zero energy and momentum. Now it is obvious that both the free Hamiltonian H_0 as well as H_n in the interacting theory with $g^2 > 0$ are non-negative. Therefore the only zero-energy solution is $\Psi = 0$; it follows immediately that the conformal models have no non-trivial static solutions, neither the $|\Psi|^6$ -model in $d = 1$, nor the $|\Psi|^4$ -model in $d = 2$. As in the free theory, the physical solutions represent scattering states with a continuous energy spectrum.

2.3 Quantization of the Free Schrödinger Model

Quantization of a classical physical theory is done in the following way. First, the *states* of a quantum system are described by vectors $|\psi\rangle$ living in a Hilbert space $\mathcal{H}$. A physical quantity A in the classical theory is replaced by a Hermitian operator $\hat{A}$, an *observable*, acting on the states in this Hilbert space. The result obtained when A is measured is one of the eigenvalues of $\hat{A}$. For example, when observing A in a system that is in the state $|\psi\rangle$, the expectation value of the result is given by

$$\langle A \rangle = \frac{\langle \psi | \hat{A} | \psi \rangle}{\langle \psi | \psi \rangle}. \quad (2.3.1)$$

Furthermore, the actual quantization step can be done by replacing the Poisson bracket $\{ , \}$ in the classical theory by the commutator bracket $-i[,]$, with $[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$ (in units where $\hbar = 1$). The fundamental commutation relations of the theory are defined in terms of the generalized coordinates $\hat{q}_i$ and momenta $\hat{p}_i$ via equal time commutators:

$$[\hat{q}_i(\mathbf{x}, t), \hat{p}_j(\mathbf{y}, t)] = i\delta_{ij}(\mathbf{x} - \mathbf{y}). \quad (2.3.2)$$

Along these lines Hamilton's equation of motion (2.2.21) is replaced by Heisenberg's equation of motion:

$$\frac{d\hat{Q}}{dt} = \partial_t \hat{Q} - i[\hat{Q}, \hat{H}]. \quad (2.3.3)$$

This equation describes the time evolution of the quantum system.

Following the outlined prescription, the free Schrödinger model is quantized by introducing a commutator bracket for the free Schrödinger field Ψ_0 from the Poisson bracket:

$$[\Psi_0(\mathbf{x}, t), \Psi_0^\dagger(\mathbf{y}, t)] = \delta(\mathbf{x} - \mathbf{y}). \quad (2.3.4)$$

The dagger $\dagger$ here denotes the Hermitian conjugate; the circumflex accents denoting operators are omitted, it should be clear from the context whether one is dealing with a classical object or a quantum operator. In view of the definition of their commutator, the fields Ψ_0 and $\Psi_0^\dagger$ are now interpreted as annihilation and creation operators respectively. The Hilbert space of this field theory is constructed as a Fock space using this interpretation of Ψ_0 and $\Psi_0^\dagger$. The vacuum state $|0\rangle$ in the theory is defined as that state in the theory that is annihilated by all the annihilation operators:

$$\Psi_0(\mathbf{x}, t)|0\rangle = 0. \quad (2.3.5)$$

All the other states in the theory are built out of this vacuum state by repeated operation with the creation operators, e.g.

$$\Psi_0^\dagger(\mathbf{x}, t)|0\rangle. \quad (2.3.6)$$

This state is interpreted as representing the probability amplitude for a particle at position $\mathbf{x}$ and at time t .

The constants of motion of the classical theory need to be replaced by Hermitian operators after quantization. A priori, there are different ways of defining a Hermitian operator out of a classical quantity. Due to the introduction of the noncommutativity between fields, one has to be careful in defining the conserved operators generating symmetries: at the classical level the fields commute and their ordering is no issue, however this of course is no longer the case at the quantum level. In other words, at the quantum level there are for each conserved quantity a priori many

equivalent Hermitian options that all lead to the same unique quantity at the classical level. However, there is a further principle for defining observables at the quantum level: the observables should be defined in such a way that the ground state of the theory has zero energy, momentum, et cetera. This is obtained by writing all the creation operators on the left of the annihilation operators: *normal ordering*.

Following this principle, the quantum expressions of the classical conserved charges are precisely as in equations (2.2.10)-(2.2.16), only having the $*$ replaced by $\dagger$. They now act on the field Ψ_0 via the commutator bracket:

$$\begin{aligned}
 [\Psi_0, H_0] &= i\partial_t \Psi_0, & [\Psi_0, \mathbf{P}] &= i\nabla \Psi_0, \\
 [\Psi_0, M_{ij}] &= i(x_i \nabla_j - x_j \nabla_i) \Psi_0, & [\Psi_0, \mathbf{G}] &= i(t\nabla - im\mathbf{x}) \Psi_0, \\
 [\Psi_0, D_0] &= i\left(2t\partial_t + \mathbf{x} \cdot \nabla + \frac{d}{2}\right) \Psi_0, & [\Psi_0, N] &= \Psi_0, \\
 [\Psi_0, K_0] &= i\left(t^2\partial_t - t\left(\mathbf{x} \cdot \nabla + \frac{d}{2}\right) + \frac{i}{2}m\mathbf{x}^2\right) \Psi_0.
 \end{aligned} \tag{2.3.7}$$

Furthermore, in these terms, having the Poisson bracket replaced by the commutator bracket, the centrally extended Schrödinger algebra is satisfied as before in equation (2.2.20). Finally, the time evolution of any operator $F[\Psi_0, \Psi_0^\dagger; t]$ is described by Heisenberg's equation of motion (2.3.3). Using this identity it is straightforward to verify that as in the classical theory the full set of charges F of the Schrödinger group is conserved: $dF/dt = 0$.

When calculating physically interesting quantities, it is necessary to know the explicit solutions of the free Schrödinger equation. Its solutions are superpositions of plane waves of momentum $\mathbf{k}$ and energy $\mathbf{k}^2/2m$, and can be expressed as

$$\Psi_0(\mathbf{x}, t) = \frac{1}{(2\pi)^{d/2}} \int d^d \mathbf{k} \, a(\mathbf{k}) e^{i\mathbf{k} \cdot \mathbf{x} - i(\mathbf{k}^2/2m)t}, \tag{2.3.8}$$

$$a(\mathbf{k}) = \frac{1}{(2\pi)^{d/2}} \int d^d \mathbf{x} \, \Psi_0(\mathbf{x}, 0) e^{-i\mathbf{k} \cdot \mathbf{x}}. \tag{2.3.9}$$

The coefficient $a(\mathbf{k})$ is the momentum representation of the field Ψ_0 .

Following this solution the commutation relation between Ψ_0 and $\Psi_0^\dagger$ can be translated into a commutation relation between the time-independent coefficients $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$. The result is given by

$$[a(\mathbf{k}_1), a^\dagger(\mathbf{k}_2)] = \delta(\mathbf{k}_1 - \mathbf{k}_2). \quad (2.3.10)$$

In view of this, also the operator $a^\dagger$ can be interpreted as a creation operator, this time of a state or a particle with momentum $\mathbf{k}$:

$$|\mathbf{k}\rangle = a^\dagger(\mathbf{k})|0\rangle, \quad (2.3.11)$$

and likewise $a(\mathbf{k})$ will be the annihilation operator of a state with momentum $\mathbf{k}$. Successive operation with either the $\Psi^\dagger$ or $a^\dagger$ operators on the vacuum will build all states in the Hilbert space. These are just two equivalent representations of the same Hilbert space, one in position representation and one in momentum representation.

As an example of a physically interesting quantity in quantum field theory, the *n-point functions* describing the probability of propagation are considered. A particle located at a space-time point $x_1 = (\mathbf{x}_1, t_1)$ is quantum mechanically described by

$$|x_1\rangle = \Psi_0^\dagger(x_1)|0\rangle. \quad (2.3.12)$$

The probability amplitude of finding this particle at a space-time point x_2 , or equivalently the probability amplitude of propagation from x_1 to x_2 , is naively given by

$$\langle x_2|x_1\rangle = \langle 0|\Psi_0(x_2)\Psi_0^\dagger(x_1)|0\rangle, \quad (2.3.13)$$

with the probability given by $|\langle x_2|x_1\rangle|^2$. However, to make sure that propagation from $\mathbf{x}_1$ to $\mathbf{x}_2$ only happens when t_2 is at a later time than t_1 , a time ordering operator needs to be included in the expression for propagation:

$$\langle x_2|x_1\rangle = \langle 0|T\left(\Psi_0(x_2)\Psi_0^\dagger(x_1)\right)|0\rangle. \quad (2.3.14)$$

The operator T makes sure that the field at the later time is written to the left, thus implying that $\langle 0|T\left(\Psi_0(x_2)\Psi_0^\dagger(x_1)\right)|0\rangle$ vanishes when t_1 is later than t_2 .

The expression within the vacuum states can be rewritten as

$$\begin{aligned}
 T(\Psi_0(x_2)\Psi_0^\dagger(x_1)) & \quad (2.3.15) \\
 &= \theta(t_2 - t_1)\Psi_0(x_2)\Psi_0^\dagger(x_1) + \theta(t_1 - t_2)\Psi_0^\dagger(x_1)\Psi_0(x_2) \\
 &= \Psi_0^\dagger(x_1)\Psi_0(x_2) + \theta(t_2 - t_1)[\Psi_0(x_2), \Psi_0^\dagger(x_1)] \\
 &\equiv \Psi_0^\dagger(x_1)\Psi_0(x_2) + D_+(\mathbf{x}_2 - \mathbf{x}_1, t_2 - t_1),
 \end{aligned}$$

where $\theta(t)$ is the step function, i.e. equal to 1 for positive t and equal to 0 for negative t . Calculating the *unequal* times commutator leads to

$$[\Psi_0(x_2), \Psi_0^\dagger(x_1)] = \frac{1}{(2\pi)^d} \int d^d \mathbf{k} e^{i\mathbf{k} \cdot (\mathbf{x}_2 - \mathbf{x}_1) - i(\mathbf{k}^2/2m)(t_2 - t_1)}. \quad (2.3.16)$$

Using the Fourier representation of the step function,

$$\theta(t) = \frac{i}{2\pi} \int_{-\infty}^{\infty} dk_0 \frac{e^{-ik_0 t}}{k_0 + i\varepsilon}, \quad (2.3.17)$$

it is seen that the probability amplitude for propagation can be expressed as

$$\begin{aligned}
 D_+(\mathbf{x}_2 - \mathbf{x}_1, t_2 - t_1) &= \langle 0 | T \left(\Psi_0(x_2) \Psi_0^\dagger(x_1) \right) | 0 \rangle \quad (2.3.18) \\
 &= \frac{i}{(2\pi)^{d+1}} \int dk_0 d^d \mathbf{k} \frac{e^{i\mathbf{k} \cdot (\mathbf{x}_2 - \mathbf{x}_1) - i(k_0 + \mathbf{k}^2/2m)(t_2 - t_1)}}{k_0 + i\varepsilon} \\
 &= \frac{i}{(2\pi)^{d+1}} \int dk_0 d^d \mathbf{k} \frac{e^{i\mathbf{k} \cdot (\mathbf{x}_2 - \mathbf{x}_1) - ik_0(t_2 - t_1)}}{k_0 - \mathbf{k}^2/2m + i\varepsilon}.
 \end{aligned}$$

In the momentum representation the probability amplitude for propagation or the *propagator* is expressed diagrammatically via

$$\frac{i}{(2\pi)^{d+1}} \frac{1}{k_0 - \mathbf{k}^2/2m + i\varepsilon} = \text{---} \overrightarrow{\hspace{1.5cm}} \text{---} \quad (2.3.19)$$

k

The probability amplitude of propagation is also called a *two-point function*. The definition of the *n-point function* is a straightforward extension of that of the two-point function:

$$G^{(2n)}(x_{2n}, \dots, x_1) = \langle 0 | T \left(\Psi_0(x_{2n}) \dots \Psi_0(x_{n+1}) \Psi_0^\dagger(x_n) \dots \Psi_0^\dagger(x_1) \right) | 0 \rangle, \quad (2.3.20)$$

where in the free theory the term involving the exponent of the interaction Hamiltonian is absent. It represents the probability amplitude of propagation of n particles located at the positions $x_1, \dots, x_n$ into n particles located at $x_{n+1}, \dots, x_{2n}$. This can be understood via Wick's theorem [94, 117]. This theorem says the following. Let the contraction $\underbrace{AB}$ of two fields A and B be defined by the difference between their time and normal ordering:

$$\underbrace{AB} = T(AB) - N(AB). \quad (2.3.21)$$

The time ordering of a general product of fields is then given by

$$\begin{aligned} T(A_1 \dots A_i) &= N(A_1 \dots A_i) + \underbrace{A_1 A_2} N(A_3 \dots A_i) + \dots \\ &\quad + \underbrace{A_1 A_2 A_3 A_4} N(A_5 \dots A_i) + \dots \\ &\quad + \text{all possible contractions.} \end{aligned} \quad (2.3.22)$$

The contraction of for example $\Psi_0(x_2)$ and $\Psi_0^\dagger(x_1)$ is given by $D_+(x_2 - x_1)$, and since two Ψ_0 or two $\Psi_0^\dagger$ fields commute, the contraction between them equals zero, as can be understood by studying the derivation in (2.3.15). The non-zero contributions to $G^{(2n)}(x_{2n}, \dots, x_1)$ are thus obtained when all of the n Ψ_0 fields are contracted with the n $\Psi_0^\dagger$ fields. This can be done in many ways: when e.g. the $\Psi_0^\dagger(x_i)$ are contracted pairwise precisely with the $\Psi_0(x_{i+n})$, the probability amplitude the particle at x_1 propagates towards x_{n+1} , the particle at x_2 propagates towards x_{n+2} etc. is obtained, because all the contractions and thus individual probabilities amplitudes in (2.3.22) are multiplied. Summing the probability amplitudes of all the $n!$ possible ways of propagation leads to the total probability amplitude that the n particles at $x_1, \dots, x_n$ propagate into n particles located at $x_{n+1}, \dots, x_{2n}$.

The n -point functions play an important role when studying the quantum solutions of the non-linear interacting Schrödinger model in the next section.

2.4 Quantization of the Non-Linear Schrödinger Model

In this section the quantum non-linear Schrödinger model is only considered for $n = 2$ such that the interaction term is given by $|\Psi|^4$. It is

quantized canonically by imposing

$$[\Psi(\mathbf{x}, t), \Psi^\dagger(\mathbf{y}, t)] = \delta(\mathbf{x} - \mathbf{y}). \quad (2.4.1)$$

on the fields Ψ and its Hermitian conjugate, like in equation (2.3.4). Here Ψ now represents an operator solution of the non-linear Schrödinger equation:

$$i\partial_t \Psi = -\frac{1}{2m} \Delta \Psi + g^2 |\Psi|^2 \Psi. \quad (2.4.2)$$

The vacuum state of this model, denoted by $|\Omega\rangle$, is canonically defined by

$$\Psi(\mathbf{x}, t)|\Omega\rangle = 0. \quad (2.4.3)$$

As when quantizing the free model, the Hamiltonian and other generators of the Schrödinger group are taken to be the normal-ordered and Hermitian operator versions of the corresponding classical ones. There exist conserved operators $\mathbf{P}$, M_{ij} , $\mathbf{G}$ and N , defined by the expressions (2.2.11)-(2.2.16) with the classical fields Ψ and Ψ^* replaced by the quantum fields Ψ and $\Psi^\dagger$. However, the Hamiltonian takes up an additional bilinear term $\mu\Psi^\dagger\Psi$:

$$H_\mu[\Psi, \Psi^\dagger] = \int d^d x \frac{1}{2m} \nabla \Psi^\dagger \cdot \nabla \Psi + \mu \Psi^\dagger \Psi + \frac{g^2}{2} \Psi^\dagger{}^2 \Psi^2. \quad (2.4.4)$$

This term arises from the operator-ordering ambiguity in going from the classical to the quantum theory. The $|\Psi|^4$ term can be written quantum mechanically like $\Psi^\dagger\Psi^\dagger\Psi\Psi$, but equally well like $\Psi^\dagger\Psi\Psi^\dagger\Psi$, since both expressions are Hermitian and annihilate the ground state. The most general quantum Hamiltonian is given by a linear combination of the two expressions, and after commuting fields this is parametrized by the $\mu\Psi^\dagger\Psi$ term.

It follows immediately, that one can subtract a constant term μN from the energy and define another μ -independent conserved energy operator

$$H_s = H - \mu N = \int d^d x \frac{1}{2m} \nabla \Psi^\dagger \cdot \nabla \Psi + \frac{g^2}{2} \Psi^\dagger{}^2 \Psi^2. \quad (2.4.5)$$

Equivalently, one can redefine the fields:

$$\Phi(\mathbf{x}, t) = e^{-i\mu t} \Psi(\mathbf{x}, t), \quad \Phi^\dagger(\mathbf{x}, t) = e^{i\mu t} \Psi^\dagger(\mathbf{x}, t), \quad (2.4.6)$$

with the same equal-time commutation relations

$$\left[\Phi(\mathbf{x}, t), \Phi^\dagger(\mathbf{y}, t) \right] = \delta^d(\mathbf{x} - \mathbf{y}). \quad (2.4.7)$$

In terms of these fields the Hamiltonian (2.4.4) becomes

$$H[\Phi, \Phi^\dagger] = \int d^d x \frac{1}{2m} \nabla \Phi^\dagger \cdot \nabla \Phi + \frac{g^2}{2} \Phi^{\dagger 2} \Phi^2. \quad (2.4.8)$$

More generally, a shift in the value of μ in the Hamiltonian (2.4.4) can be compensated by a multiplicative field renormalization of the type (2.4.6).

Assuming such a (finite) renormalization of the fields and the μ term to have been performed, the theory is considered as defined by (2.4.7) and (2.4.8). In addition to the conserved operators of the Galilean transformations $\mathbf{P}$, M_{ij} , $\mathbf{G}$ and N , which take the same form in terms of the new fields Φ and $\Phi^\dagger$, one can then also construct the conformal operators

$$D = 2tH + I_2, \quad (2.4.9)$$

$$K = t^2 H - tD - I_1,$$

where I_1 and I_2 are the normal-ordered expressions (3.5.30).

The transformations of the field operators Φ and $\Phi^\dagger$ under the full Schrödinger algebra are obtained by computing their commutator brackets with all the generators. The result is as in the free model (2.3.7), having Ψ_0 replaced by Φ . Furthermore, in these terms the centrally extended Schrödinger algebra is also satisfied as before (2.2.20).

Following Heisenberg's equation, for the full set of Schrödinger operators to be conserved, $dF/dt = 0$, additional constraints have to be imposed, as in the classical theory. The operators $\mathbf{P}$, M_{ij} , $\mathbf{G}$ and N are conserved directly. However for the conformal operators the results (2.2.32) are reobtained as normal-ordered operator equations:

$$\frac{dK}{dt} = -t \frac{dD}{dt}, \quad \frac{dD}{dt} = \frac{(2-d)}{2} g^2 \int d^d x \Phi^{\dagger 2} \Phi^2. \quad (2.4.10)$$

In the next chapter renormalization effects in the quantum theory will show that even in $d = 2$ dimensions, D and K are no longer conserved.

The coupling constant g^2 is allowed to become negative. In this case the potential term has a negative sign with respect to the kinetic term

in the Hamiltonian (2.4.8). Since the potential is therefore unbounded from below, one could expect the solutions one finds in a quantum theory to be unstable. This would be true in relativistic quantum field theory. However, the non-linear Schrödinger model is a non-relativistic model, and thus preserves particle number. Therefore at fixed particle number one is dealing with a non-relativistic N -body Schrödinger particle problem. When the coupling constant has a negative sign this indicates the particles are subject to an attractive potential. Due to the fixed particle number there will be no instability in the solutions due to the form of the potential, it merely signifies an attractive interaction of a finite number of particles. When the coupling constant has a positive sign this indicates a repulsive interaction.

Perturbation Theory

Unlike in the free model, where the Schrödinger equation can be solved to find the most general solution for the quantum field Ψ_0 , this is not possible in the non-linear model. This presents difficulties when one wants to calculate interesting physical quantities like the n -point functions. A way around this problem is obtained by studying the interaction or non-linear part of the Hamiltonian as a correction or *perturbation* to the free part. This provides a way to express the n -point function of the non-linear model in terms of the known vacuum and fields of the free model plus the interaction Hamiltonian. Following [94] this is done as follows for the 2-point function:

$$\langle \Omega | T(\Phi(x_2)\Phi^\dagger(x_1)) | \Omega \rangle = \frac{\langle 0 | T \left\{ \Phi_0(x_2)\Phi_0^\dagger(x_1)U_I \right\} | 0 \rangle}{\langle 0 | U_I | 0 \rangle}, \quad (2.4.11)$$

and similarly for the general n -point functions. The field Φ_0 is the field of the free field theory upon the phase-shift to get rid of the $\mu\Psi^\dagger\Psi$ term, and obeys the commutation relation of the free model:

$$[\Phi_0(\mathbf{x}, t), \Phi_0^\dagger(\mathbf{y}, t)] = \delta(\mathbf{x} - \mathbf{y}). \quad (2.4.12)$$

Furthermore

$$U_I = T \exp \left(i \int_{-\infty}^{\infty} dt H_I(t) \right), \quad H_I = \int d^d x \frac{g^2}{2} \Phi_0^{\dagger 2} \Phi_0^2. \quad (2.4.13)$$

In practice, the n -point functions can be now calculated step by step by Taylor expanding U_I and by using Wick's theorem; under the assumption that the interaction Hamiltonian presents a *small* correction to the free Hamiltonian, i.e. by assuming the coupling constant g^2 to be relatively small.

As an example the 2-point function is expanded to first order in g^2 . First of all the expression $\langle 0|U_I|0\rangle$ is studied. At lowest order this expression trivially produces 1. At first order it is given by

$$i\frac{g^2}{2}\langle 0|T\int_{-\infty}^{\infty}dt\int d^dz\Phi^\dagger\Phi^\dagger\Phi\Phi|0\rangle, \quad (2.4.14)$$

where from now on the subscript 0 is omitted. The only non-zero contributions to this expression are obtained when each $\Phi^\dagger$ is contracted with a Φ , which can be done in 2 ways. Thus leading to

$$ig^2\int_{-\infty}^{\infty}dt\int d^dzD_+(0)D_+(0). \quad (2.4.15)$$

The space-time integral only naively leads to an infinite contribution. More precisely the space-time integral contributes a factor which is the (infinite) volume of space-time. Formally one should work in a finite volume and only at the very end take the volume of space-time to infinity. This then leads to a factor of $(2\pi)^{d+1}$:

$$ig^2\frac{1}{(2\pi)^{d+1}}\int dk\frac{i}{k_0-\mathbf{k}^2/2m+i\varepsilon}\int dp\frac{i}{p_0-\mathbf{p}^2/2m+i\varepsilon}, \quad (2.4.16)$$

where the integration over dk means integration over both dk_0 and $d^d\mathbf{k}$. Diagrammatically this can be expressed as


(2.4.17)

The fact that propagation takes place from a space-time point towards itself, $D_+(0)$, is represented by the closed loops. Vice versa, the integral expression can be obtained from the diagram by following the *Feynman*

Rules of the non-linear Schrödinger model in momentum space:

a. Each propagator line comes with a term

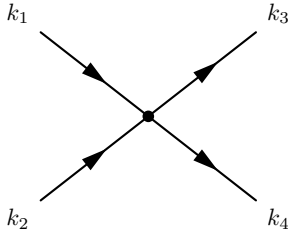
$$\text{---}\overrightarrow{\text{---}}\text{---}_k = \frac{1}{(2\pi)^{d+1}} \frac{i}{k_0 - \mathbf{k}^2/2m + i\varepsilon}, \quad (2.4.18)$$

where k is the momentum running through that propagator.

b. Propagator loops are integrated over:

$$\int dk \equiv \int dk_0 \int d^d \mathbf{k} \quad (2.4.19)$$

c. Each vertex brings momentum conservation and a factor of $i(2\pi)^{d+1}g^2/2$:



$$= \frac{ig^2}{2} (2\pi)^{d+1} \delta(k_1 + k_2 - k_3 - k_4). \quad (2.4.20)$$

d. Combinatorial factors take into account the number of ways of building the diagram and the $1/n!$ factors that arise when Taylor expanding the exponent in U_I .

Like the propagator has a physical interpretation as a particle moving from one space-time point to another, also the interaction vertex has a clear intuitive physical interpretation. It represents two incoming particles that annihilate to produce two new outgoing particles.

In a relativistic quantum field theory, however, many more interpretations can be assigned to the $|\Phi|^4$ -vertex. This difference corresponds to the fact that there are no anti-particles in a non-relativistic theory. Indeed, in this non-relativistic theory the Schrödinger field Φ is Fourier expanded just in terms of the annihilation operators a . The field Φ thus describes the annihilation of a particle, and the field $\Phi^\dagger$ describes the creation of a particle. This leads to the interpretation of the $|\Phi|^4$ -vertex as describing the simultaneous annihilation of two particles and creation of two new ones. However, when a relativistic quantum field Φ' being a solution of the Klein-Gordon equation is expanded, this is done in terms of both the annihilation operators a and the creation operators $a^\dagger$. It follows that Φ' describes either the annihilation of a particle or the creation of

an anti-particle, and $\Phi'^\dagger$ describes either the creation of a particle or the annihilation of an anti-particle. In the relativistic theory the $|\Phi'|^4$ -vertex thus describes 2^4 possible physical processes. This vertex can for example describe the annihilation of two particles and two anti-particles, or the annihilation of a particle and two anti-particles followed by the creation of an anti-particle, to give just two examples.

It furthermore automatically follows that the number of possible Feynman diagrams built out of various interaction vertices and propagators in the non-relativistic Φ^4 -model is much smaller than the number in a relativistic Φ^4 -model.

The next step in the calculation of the 2-point function is to expand the expression

$$\langle 0|T \left\{ \Phi(x_2)\Phi^\dagger(x_1)U_I \right\} |0\rangle \quad (2.4.21)$$

to first order in g^2 . The lowest order contribution is just the expression in the free theory:

$$D_+(x_2 - x_1) = \langle 0|T(\Phi(\mathbf{x}_2, t_2)\Phi^\dagger(\mathbf{x}_1, t_1))|0\rangle. \quad (2.4.22)$$

The first order contribution is given by

$$i\frac{g^2}{2} \int d^d\mathbf{z} \int dt \langle 0|T \left(\Phi(x_2)\Phi^\dagger(x_1)\Phi^\dagger(z)\Phi^\dagger(z)\Phi(z)\Phi(z) \right) |0\rangle. \quad (2.4.23)$$

The time ordering will give a potentially non-zero result in various ways. Contracting $\Phi(x_2)$ with $\Phi^\dagger(x_1)$ will lead to $D_+(x_2 - x_1)$ multiplied by the first order result obtained when expanding the denominator of (2.4.11), the diagram '8'.

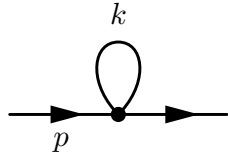
Another potentially non-zero result is obtained when $\Phi(x_2)$ is connected with a $\Phi^\dagger(z)$, and $\Phi^\dagger(x_1)$ with a $\Phi(z)$, and the remaining fields $\Phi^\dagger(z), \Phi(z)$ amongst each other. This can be done in 4 different ways. Working this out leads to

$$4i\frac{g^2}{2} \int d^d\mathbf{z} \int dt D_+(x_2 - z)D_+(z - z)D_+(z - x_1), \quad (2.4.24)$$

which after Fourier transforming reads in momentum language

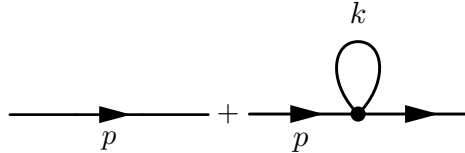
$$4i \frac{g^2}{2} \left(\frac{1}{(2\pi)^{d+1}} \right)^{3-1} \frac{ie^{-ip \cdot x_1}}{p_0 - \mathbf{p}^2/2m + i\varepsilon} \frac{ie^{ip \cdot x_2}}{p_0 - \mathbf{p}^2/2m + i\varepsilon} \times \int dk \frac{i}{k_0 - \mathbf{k}^2/2m + i\varepsilon} \quad (2.4.25)$$

where $p \cdot x = \mathbf{p} \cdot \mathbf{x} - p_0 t$. Diagrammatically this integral can be expressed as


(2.4.26)

The integral expression (2.4.25) can be read of from the diagram following the Feynman rules a-d plus the additional rule that e. Each incoming propagator comes with a factor $e^{-ip \cdot x}$ and each outgoing one with a factor $e^{ip \cdot x}$.

Finally, summing all these diagrammatic expressions and dividing by the expression for $\langle 0|U_I|0\rangle$, leads to


(2.4.27)

Observe that the terms involving so-called *vacuum bubble diagrams*, like the diagram ‘8’, that arise when all fields in the interaction Hamiltonian are contracted amongst themselves, i.e. no incoming or outgoing particles, all cancel out. This is something that happens in general, and not just to first order.

Path Integral Quantization

Feynman has developed an alternative formulation of quantum mechanics [52]. Instead of formulating it in terms of a Schrödinger equation and a wave function that are used to calculate transition amplitudes between different quantum states, these transition amplitudes are expressed directly in terms of a *path integral* over possible quantum states of the exponent

of the action of the system. Field theory can be quantized precisely along these lines. For example, the 2-point function describing the transition between different quantum position states, is in these terms given by [94]

$$\langle \Omega | T(\Phi(x_2)\Phi^\dagger(x_1)) | \Omega \rangle = \frac{\int \mathcal{D}\Phi \Phi(x_2)\Phi^\dagger(x_1) e^{iS}}{\int \mathcal{D}\Phi e^{iS}}. \quad (2.4.28)$$

The path integral quantization method is particularly convenient when quantizing gauge theories.

When dealing with a free theory, the action S is a quadratic function of the fields Φ and $\Phi^\dagger$. This means that the path integral can be performed exactly as a Gaussian integral. In this way it can be verified [94] that the path integral expression (2.4.28) indeed leads to the previously obtained result for the propagator (2.4.18).

As it turns out, the correct expression for the propagator in momentum space can in this formalism directly be read off from the action after Fourier transforming the fields:

$$\Phi(\mathbf{x}, t) = \frac{1}{(2\pi)^{(d+1)/2}} \int dk_0 d^d \mathbf{k} a(k) e^{i\mathbf{k} \cdot \mathbf{x} - i k_0 t}. \quad (2.4.29)$$

In these terms the free part of the action reads

$$S_0 = \int dk_0 d^d \mathbf{k} a^\dagger(k) \left(k_0 - \frac{1}{2m} \mathbf{k}^2 + i\varepsilon \right) a(k), \quad (2.4.30)$$

where a small $i\varepsilon$ -term has been added to the action to make the path integral over $\exp(iS)$ converge. In this way precisely the inverse of the expression (2.4.18) for the propagator is obtained (up to some factors).

Also in case of an interacting theory, the path integral method leads to the same results as before. In this case the action can be split into a free and an interacting part:

$$S = S_0 + S_{int}, \quad S_{int} = - \int dt H_{int}, \quad (2.4.31)$$

where the interacting part of the action equals minus the time integral of the interaction Hamiltonian. The theory can now again be solved perturbatively by Taylor expansion, this time of $\exp(iS_{int})$. Performing the path integral of the Taylor expanded action leads to the same result as the expansion of U_I (2.4.13) followed by Wick contracting the fields. In

terms of Feynman diagrams the same rules and diagrammatic expressions are obtained [94].

As a last remark on the path integral quantization method, there is another method for computing the n -point functions and obtaining the Feynman diagrammatic expansion. The basic object is the generating functional $Z[J]$ which is defined via

$$Z[J] = \int \mathcal{D}\Phi \exp \left(iS + i \int dt d^2\mathbf{x} J^\dagger \Phi + \Phi^\dagger J \right). \quad (2.4.32)$$

The function J acts as a source for the particle field Φ . By taking derivatives with respect to the source, the n -point functions can be derived. For example, the 2-point function is given by

$$\langle 0|T \left(\Phi(x_2)\Phi^\dagger(x_1) \right) |0\rangle = \frac{1}{Z[0]} \left(\frac{-i\delta}{\delta J^\dagger(x_2)} \right) \left(\frac{-i\delta}{\delta J(x_1)} \right) Z[J]_{J=0}. \quad (2.4.33)$$

By taking functional derivatives in this way all n -points functions can be derived.

Although in this chapter the quantization of the non-linear Schrödinger model is studied using the formalism of canonical quantization, the quantization of the Jackiw-Pi model which involves gauge fields, will later be studied in the path integral formalism.

2.5 Renormalization Effects

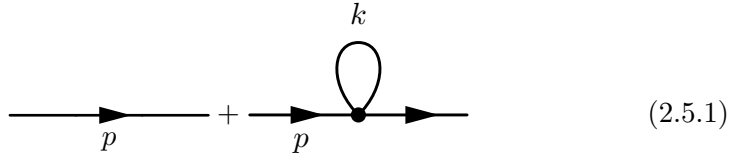
Physically the terms that appear when expanding the 2-point function can be interpreted as correction terms to the free propagator, as is clear from the diagrammatic expansions in the previous section. When all terms in the expansion are taken into account correctly, the propagator of the full interacting theory is obtained. Since the propagator is defined via a Fourier transformation of the field Φ and depends on the mass m of the field, the corrections to the free propagator can also be interpreted as corrections to the field and its mass. In a similar way part of the terms that arise from the expansion of the 4-point function can be interpreted as causing corrections to the coupling constant g^2 . Indeed, the 4-point function describes processes with two incoming and two outgoing particles, and thus includes diagrams like the interaction vertex and corrections to it.

However, when calculating these correction terms in general infinities arise, leading naively to e.g. an infinite effective coupling constant. The same effect applies to the correction terms of the propagator, leading in general to an infinite field and mass in the interacting theory. These infinities are unphysical and are dealt with by assuming *a posteriori* that the original or bare field, mass and coupling constant are actually the ones that are infinite, and in such a way that all infinities that arise in the calculation of the correction terms are absorbed by the infinities in the bare parameters, resulting in a finite effective field, mass and coupling constant. It are these parameters that are the physical ones and can be measured. When such a redefinition or *renormalization* of the parameters of the theory is possible, the theory is called *renormalizable*.

In the following the renormalization of Φ and m are studied in all dimensions, however that of g^2 only in $d = 2$ dimensions.

The Renormalization of Φ and m .

Diagrammatically the corrections to the propagator to second order in g^2 are given by



$$\text{---}\!\!\!\!\!\rightarrow_p + \text{---}\!\!\!\!\!\rightarrow_p \text{---}\!\!\!\!\!\rightarrow_k \text{---}\!\!\!\!\!\rightarrow_p \quad (2.5.1)$$

However, all the contributions to all orders in g^2 vanish, as will be explained shortly, and thus the propagator in the interacting theory is exactly the one in the free theory. At the same time it is concluded that the physical field and mass are just the bare ones: there is no renormalization of these. This result holds in all dimensions d .

Indeed, the loop in the *seagull diagram* (2.5.1) represents the contraction of two field operators at the same point in space-time, $D_+(0)$. In such an equal-time contraction, the time ordering of the operators $\Phi^\dagger(t)$ and $\Phi(t)$ is equal to their normal ordering, following [1]. Hence their contraction vanishes by definition (2.3.21). Alternatively, one may evaluate the loop in the diagram in

$$\begin{aligned} D_+(0) &= \frac{i}{(2\pi)^{d+1}} \int dk_0 d^d \mathbf{k} \frac{1}{k_0 - \frac{1}{2} \mathbf{k}^2 + i\varepsilon} \\ &= \frac{1}{2(2\pi)^d} \int d^d \mathbf{k}, \end{aligned} \quad (2.5.2)$$

where the dk_0 integral is performed via contour integration in the complex plane. The remaining $d^d\mathbf{k}$ integral formally leads to an infinity that has to be dealt with by employing a rotation-invariant regularization scheme such as dimensional regularization [11, 106]. This scheme was developed originally to handle diverging momentum-integrals in relativistic quantum field theory in $3 + 1$ dimensions. Here dimensional regularization will be used to deal with the $d^d\mathbf{k}$ part of the momentum integrals only: dk_0 is treated separately. Following the dimensional regularization scheme, which is explained in detail in appendix 2.A, it is again seen that the diagram leads to zero:

$$D_+(0) = 0. \quad (2.5.3)$$

It is thus concluded that at first order in g^2 the corrections to the propagator vanish. Since higher-order corrections consist of insertions of these first order seagull type propagator corrections, as can be understood by continuing the Taylor expansion of the 2-point function, it is concluded that to all orders the propagator receives no corrections. All possible corrections will vanish by the argument above.

As a result of this analysis of propagator corrections, neither the fields or the chemical potential μ , nor the mass, are renormalized by quantum loop effects. This also shows that one can without loss of generality take $\mu = 0$ from the start, as was done in (2.4.6) by switching to the fields Φ and $\Phi^\dagger$.

The 1-Loop Renormalization of g^2 .

Next the expansion of the 4-point function is considered. This will give corrections to the coupling constant g^2 . The terms that can be interpreted as corrections to the coupling constant are given in equation (2.5.4). The lower right diagram connects one of the outgoing lines of one vertex to one of the incoming lines of the other vertex and vice versa, thus leading to the twist in the diagram. Assuming that one of the vertices is at a slightly earlier or later time than the other, the twist has the physical consequence that one of the internal lines runs backward in time, which is impossible in a non-relativistic theory. The expression corresponding to this diagram will indeed lead to zero, as will be calculated below.

$$(2.5.4)$$

The full expression corresponding to this sum of diagrams is

$$\frac{1}{(2\pi)^9} \delta^{d+1}(p_1 + p_2 - q_1 - q_2) A^{(1)}(p_i, q_j) \times \prod_{i=1}^2 \frac{ie^{-ip_i \cdot z_i}}{p_{i,0} - \frac{1}{2m} \mathbf{p}_i^2 + i\varepsilon} \frac{ie^{iq_i \cdot z_i}}{q_{i,0} - \frac{1}{2m} \mathbf{q}_i^2 + i\varepsilon}, \quad (2.5.5)$$

with taking all the combinatorial factors into account correctly,

$$A^{(1)} = i \frac{g^2}{2} + \frac{\frac{4}{2}(ig^2/2)^2}{(2\pi)^{d+1}} \int dk B_1(k) + \frac{\frac{32}{2}(ig^2/2)^2}{(2\pi)^{d+1}} \int dk B_2(k), \quad (2.5.6)$$

$$B_1 = \frac{i}{k_0 - \frac{1}{2m} \mathbf{k}^2 + i\varepsilon} \frac{i}{(p_1)_0 + (p_2)_0 - k_0 - \frac{1}{2m} (\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{k})^2 + i\varepsilon}, \quad (2.5.7)$$

$$B_2 = \frac{i}{k_0 - \frac{1}{2m} \mathbf{k}^2 + i\varepsilon} \frac{i}{k_0 + (p_2)_0 - (q_2)_0 - \frac{1}{2m} (\mathbf{k} + \mathbf{p}_2 - \mathbf{q}_2)^2 + i\varepsilon}. \quad (2.5.8)$$

To evaluate these expressions, first the dk_0 -integrals are performed via contour integration in the complex plane. Since B_2 has two poles but both in the lower half plane, the dk_0 integral of B_2 gives zero as can be seen by closing the integration contour in the upper half plane. The

integral over B_1 will give a non-zero contribution since B_1 has one pole in the lower half plane, and one in the upper. When the integration contour is closed in the upper half plane, and when the integration variable $\mathbf{k}$ is shifted over $(\mathbf{p}_1 + \mathbf{p}_2)/2$, it is seen that upon taking the incoming particles on-shell (i.e. $p_{0,i} = \mathbf{p}_i^2/2m$), the integral of B_1 is given by

$$\frac{1}{(2\pi)^{d+1}} \int dk B_1(k) = \frac{im}{(2\pi)^d} \int d^d \mathbf{k} \frac{1}{\mathbf{k}^2 - (\mathbf{p}_1 - \mathbf{p}_2)^2/4}. \quad (2.5.9)$$

Evaluating the remaining integral using dimensional regularization invoking $d = 2 - \epsilon$ dimensions, as explained in appendix 2.A, it follows that

$$\frac{1}{(2\pi)^{d+1}} \int dk B_1(k) = \frac{\Lambda^{-\epsilon} \Gamma(\epsilon/2)}{(4\pi)^{1-\epsilon/2}} [-(\mathbf{p}_1 - \mathbf{p}_2)^2/4\Lambda^2]^{-\epsilon/2}, \quad (2.5.10)$$

where the dimension of momentum Λ has been extracted. It thus follows that,

$$A^{(1)} = i \frac{g^2}{2} - \frac{ig^4}{2} \frac{m}{4\pi} \left(\frac{\Lambda^{-\epsilon}}{(4\pi)^{-\epsilon/2}} \Gamma(\epsilon/2) [-(\mathbf{p}_1 - \mathbf{p}_2)^2/4\Lambda^2]^{-\epsilon/2} \right). \quad (2.5.11)$$

Clearly, this expression diverges in the limit $\epsilon \rightarrow 0$ to $d = 2$ dimensions:

$$A^{(1)} = i \frac{g^2}{2} - \frac{ig^4}{2} \frac{m\Lambda^{-\epsilon}}{4\pi} \left(\frac{2}{\epsilon} - \gamma_E - i\pi - \ln((\mathbf{p}_1 - \mathbf{p}_2)^2/16\pi\Lambda^2) \right). \quad (2.5.12)$$

The term $-i\pi$ follows from the negative logarithm, which is a multi-valued function: $\ln(-1) = (2n+1)i\pi$ with n an integer. In this case $\ln(-1) = -i\pi$, as required by unitarity of the scattering matrix. This issue is discussed in depth later.

Noting that $mg^2\Lambda^{-\epsilon}$ is dimensionless for any d , the divergence can be ‘repaired’ by introducing a finite renormalized coupling constant g_R^2 via

$$mg_R^2 = mg^2\Lambda^{-\epsilon} - \frac{m^2}{2\pi\epsilon} g^4 \Lambda^{-2\epsilon}, \quad (2.5.13)$$

where mg_R^2 is dimensionless. Using these terms, $A^{(1)}$ is finite and given by

$$A^{(1)} = i \frac{g_R^2}{2} \left(1 + \frac{mg_R^2}{4\pi} [\gamma_E + i\pi + \ln((\mathbf{p}_1 - \mathbf{p}_2)^2/16\pi\Lambda^2)] \right) + \mathcal{O}(g_R^6), \quad (2.5.14)$$

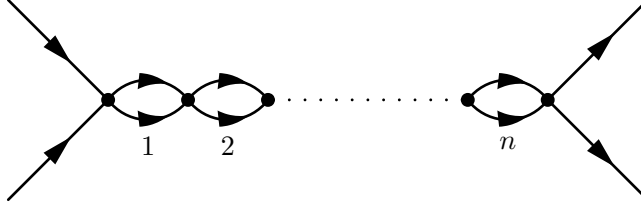


Figure 2.1: The n -loop contribution of order $(g_R^2)^{n+1}$.

upon taking the limit $\epsilon \rightarrow 0$.

It thus observed that an infinity in the correction terms to g^2 arises in $d = 2$ dimensions. This infinity can be absorbed by switching to the renormalized coupling constant g_R^2 .

The Exact Renormalization of g^2

Because the seagull diagrams vanish, the only surviving n -loop contribution to the 4-vertex is given in figure 2.1. This is very much in contrast with the relativistic Φ^4 -model where many more diagrams are possible. The expression associated to this n -loop diagram is simply the product of n times the 1-loop expression:

$$A^{(n)} = i \frac{g^2}{2} \left(\frac{ig^2}{(2\pi)^{d+1}} \int dk B_1(k) \right)^n. \quad (2.5.15)$$

Adding up the loop contributions at all orders, the full result for A is thus given by

$$A^{total} = i \frac{g^2}{2} \sum_{n=0}^{\infty} \left(\frac{ig^2}{(2\pi)^{d+1}} \int dk B_1(k) \right)^n, \quad (2.5.16)$$

$$= i \frac{g^2}{2} \left(1 - \frac{ig^2}{(2\pi)^{d+1}} \int dk B_1(k) \right)^{-1}, \quad (2.5.17)$$

where the summation has been performed as a geometric series. Plugging in the previously obtained result for the integral leads to

$$A^{total} = i \frac{g^2}{2} \frac{1}{1 + g^2 \frac{m\Lambda^{-\epsilon}}{4\pi} \left(\frac{2}{\epsilon} - \gamma_E - i\pi - \ln((\mathbf{p}_1 - \mathbf{p}_2)^2 / 16\pi\Lambda^2) \right)} \quad (2.5.18)$$

or equivalently

$$A^{total} = \frac{i\Lambda^\epsilon/2m}{\frac{1}{mg^2\Lambda^{-\epsilon}} + \frac{1}{4\pi} \left(\frac{2}{\epsilon} - \gamma_E - i\pi - \ln((\mathbf{p}_1 - \mathbf{p}_2)^2/16\pi\Lambda^2) \right)}. \quad (2.5.19)$$

Again an infinity is seen to arise when taking the limit $\epsilon \rightarrow 0$ to $d = 2$ dimensions. A finite renormalized coupling constant can in this case be introduced by

$$\frac{1}{mg_R^2} = \frac{1}{mg^2\Lambda^{-\epsilon}} + \frac{1}{2\pi\epsilon}. \quad (2.5.20)$$

Expanding this result to second order in g^2/ϵ gives back the definition (2.5.13) following the 1-loop calculation.

With this renormalization a finite expression for A^{total} is given by

$$A^{total} = i\frac{g_R^2}{2} \frac{1}{1 - g_R^2 \frac{m}{4\pi} (\gamma_E + i\pi + \ln((\mathbf{p}_1 - \mathbf{p}_2)^2/16\pi\Lambda^2))}, \quad (2.5.21)$$

which consistently leads to the 1-loop expression when it is expanded to second order in g_R^2 .

The Unitarity of the Scattering Matrix

Having obtained the result that a single momentum loop from a space-time point to itself makes a Feynman diagram vanish, like the seagull diagram (2.5.1), it has become a simple exercise to write down the full 4-point function

$$G^{(4)}(x_4, x_3, x_2, x_1) = \langle 0 | T \left(\Phi(x_4) \Phi(x_3) \Phi^\dagger(x_2) \Phi^\dagger(x_1) U_I \right) | 0 \rangle, \quad (2.5.22)$$

which represents two incoming particles at x_1 and x_2 that scatter into two outgoing particles at x_3 and x_4 .

The lowest order term is given by the contraction of $\Phi^\dagger(x_1)$ with either $\Phi(x_3)$ or $\Phi(x_4)$ multiplied by the contraction of $\Phi^\dagger(x_2)$ with the remaining field. Assuming for simplicity that $t_1 < t_2 < t_3 < t_4$, this means that the particle at x_1 moves to either x_3 or x_4 , and the other particle to the other space-time point. Diagrammatically this can be represented by

$$\begin{array}{ccc} x_1 \longrightarrow & x_3 & x_1 \longrightarrow x_4 \\ & + & \\ x_2 \longrightarrow & x_4 & x_2 \longrightarrow x_3 \end{array} \quad (2.5.23)$$

Under the assumption that $t_1 < t_2 < t_3 < t_4$, the contraction of, say, $\Phi^\dagger(x_1)$ with $\Phi(x_3)$, is given by

$$D_+(x_3 - x_1) = \int \frac{d^2\mathbf{k}}{(2\pi)^2} e^{i\mathbf{k} \cdot (\mathbf{x}_3 - \mathbf{x}_1) - i(\mathbf{k}^2/2m)(t_3 - t_1)}, \quad (2.5.24)$$

see equations (2.3.16) and (2.3.18). Using this fact, the lowest order term of the 4-point function can be rewritten as

$$2(2\pi)^4 \int \prod_{i=1}^2 \frac{d^2\mathbf{p}_i}{(2\pi)^2} \frac{d^2\mathbf{q}_i}{(2\pi)^2} I(\mathbf{p}_j, \mathbf{q}_j) \times \quad (2.5.25)$$

$$\exp \left[i\mathbf{p}_1 \cdot \mathbf{x}_1 - i\mathbf{p}_1^2 t_1/2m + i\mathbf{p}_2 \cdot \mathbf{x}_2 - i\mathbf{p}_2^2 t_2/2m \right]$$

$$\exp \left[-i\mathbf{q}_1 \cdot \mathbf{x}_3 + i\mathbf{q}_1^2 t_3/2m - i\mathbf{q}_2 \cdot \mathbf{x}_4 + i\mathbf{q}_2^2 t_4/2m \right],$$

where as before the $\mathbf{p}_i$ are the momenta of the incoming particles and the $\mathbf{q}_i$ those of the outgoing ones. The expression $I(\mathbf{p}_j, \mathbf{q}_j)$ is given by

$$I(\mathbf{p}_j, \mathbf{q}_j) = \frac{1}{2} \left[\delta^2(\mathbf{p}_1 - \mathbf{q}_1) \delta^2(\mathbf{p}_2 - \mathbf{q}_2) + \delta^2(\mathbf{p}_1 - \mathbf{q}_2) \delta^2(\mathbf{p}_2 - \mathbf{q}_1) \right] \quad (2.5.26)$$

When in the free theory the transition probability from the two particle state

$$|\mathbf{p}_1, \mathbf{p}_2\rangle = \frac{1}{\sqrt{2}} a^\dagger(\mathbf{p}_1) a^\dagger(\mathbf{p}_2) |0\rangle \quad (2.5.27)$$

into the two particle state $|\mathbf{q}_1, \mathbf{q}_2\rangle$ is calculated, the result is given by $I(\mathbf{p}_j, \mathbf{q}_j)$:

$$I(\mathbf{p}_j, \mathbf{q}_j) = \langle \mathbf{q}_1, \mathbf{q}_2 | \mathbf{p}_1, \mathbf{p}_2 \rangle. \quad (2.5.28)$$

This is an illustration of the fact that the transition probabilities in position space, in this case the 4-point function, are related to the transition probabilities in momentum space, in this case the expression I , via Fourier transformation as in equation (2.5.25).

Since the seagull type diagrams vanish, it is not hard to verify that all higher order contributions to the 4-point function are given by the sum of diagrams of the form in figure 2.1. Using the assumption that $t_1 < t_2 < t_3 < t_4$, the first order contribution is given by

$$\frac{ig_R^2}{2} \int_{t_2}^{t_3} dt_z d^2\mathbf{z} \langle 0 | \Phi(x_4) \Phi(x_3) \Phi^{\dagger 2}(z) \Phi^2(z) \Phi^\dagger(x_2) \Phi^\dagger(x_1) | 0 \rangle. \quad (2.5.29)$$

When sending t_1 and t_2 to minus infinity and t_3 and t_4 to plus infinity, this expression can like the lowest order expression be rewritten into

$$4 \frac{ig_R^2}{2} (2\pi)^3 \int \prod_{i=1}^2 \frac{d^2 \mathbf{p}_i}{(2\pi)^2} \frac{d^2 \mathbf{q}_i}{(2\pi)^2} \Delta(\mathbf{p}_j, \mathbf{q}_j) \exp[\dots] \exp[\dots], \quad (2.5.30)$$

$$\Delta(\mathbf{p}_j, \mathbf{q}_j) \equiv \delta^2(\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{q}_1 - \mathbf{q}_2) \delta([\mathbf{p}_1^2 + \mathbf{p}_2^2 - \mathbf{q}_1^2 - \mathbf{q}_2^2]/2m),$$

where the first δ -function represents momentum conservation and the second energy conservation. They are obtained via the integration over t and z . The exponents are as before, equation (2.5.25).

Since the remaining higher order contributions have already been examined when discussing the renormalization of g^2 , the full expression for the 4-point function is readily found to be

$$2(2\pi)^4 \int \prod_{i=1}^2 \frac{d^2 \mathbf{p}_i}{(2\pi)^2} \frac{d^2 \mathbf{q}_i}{(2\pi)^2} S(\mathbf{p}_j, \mathbf{q}_j) \exp[\dots] \exp[\dots], \quad (2.5.31)$$

where the *two particle scattering matrix* S is given by

$$S(\mathbf{p}_j, \mathbf{q}_j) = I(\mathbf{p}_j, \mathbf{q}_j) + \frac{2}{2\pi} A^{total}([\mathbf{p}_1 - \mathbf{p}_2]^2) \Delta(\mathbf{p}_j, \mathbf{q}_j), \quad (2.5.32)$$

where A^{total} is a function of the incoming momenta via $[\mathbf{p}_1 - \mathbf{p}_2]^2$, as can be verified in equation (2.5.21).

To make sure the transition probabilities are well-defined, the scattering matrix is required to be a unitary matrix. In momentum space this requirement reads

$$\int \prod_{i=1}^2 d^2 \mathbf{k}_i S(\mathbf{p}_j, \mathbf{k}_i) S^*(\mathbf{k}_i, \mathbf{q}_j) = I(\mathbf{p}_j, \mathbf{q}_j). \quad (2.5.33)$$

It is straightforward to show that in the case without interactions, i.e. $A^{total} = 0$, this condition is satisfied,

$$\int \prod_{i=1}^2 d^2 \mathbf{k}_i I(\mathbf{p}_j, \mathbf{k}_i) I^*(\mathbf{k}_i, \mathbf{q}_j) = I(\mathbf{p}_j, \mathbf{q}_j). \quad (2.5.34)$$

This means that the additional contributions due to A^{total} in the calculation of unitarity of the full scattering matrix actually have to vanish, as

is required by the *optical theorem* [94]. That this indeed happens depends crucially on the choice $\ln(-1) = -i\pi$ made in deriving A^{total} before.

Calculating the additional contributions, that due to the cross-term “ $IA^* + AI$ ” is found to be given by

$$\frac{2}{2\pi} 2 \operatorname{Re} \left[A^{total}([\mathbf{p}_1 - \mathbf{p}_2]^2) \right] \Delta(\mathbf{p}_j, \mathbf{q}_j). \quad (2.5.35)$$

The contribution of “ AA^* ” leads to

$$\begin{aligned} & \frac{4}{(2\pi)^2} \int d^2 \mathbf{k}_1 A^{total}([\mathbf{p}_1 - \mathbf{p}_2]^2) A^{*total}([2\mathbf{k}_1 - \mathbf{p}_1 - \mathbf{p}_2]^2) \times \\ & \delta([\mathbf{k}_1^2 + (\mathbf{p}_1 + \mathbf{p}_2 - \mathbf{k}_1)^2 - \mathbf{p}_1^2 - \mathbf{p}_2^2]/2m) \Delta(\mathbf{p}_j, \mathbf{q}_j). \end{aligned} \quad (2.5.36)$$

Shifting the integration variable over $(\mathbf{p}_1 + \mathbf{p}_2)/2$, an angular integral contributing 2π can straightforwardly be done. The final result

$$\frac{2}{2\pi} m |A^{total}([\mathbf{p}_1 - \mathbf{p}_2]^2)|^2 \Delta(\mathbf{p}_j, \mathbf{q}_j) \quad (2.5.37)$$

follows from the remaining integral over $|\mathbf{k}_1|$.

Since the choice $\ln(-1) = -i\pi$ ensures that

$$m |A^{total}|^2 = -2 \operatorname{Re}(A^{total}) \quad (2.5.38)$$

is satisfied, the sum of the contributions due to “ $IA^* + AI$ ” and “ AA^* ” is indeed seen to vanish. This implies unitarity of the scattering matrix.

Please note that a factor i has been absorbed into the definition of A , see e.g. equations (2.5.21) and (2.5.32), whereas this is usually not done. This implies the optical theorem [94] is satisfied by taking the real part of A^{total} as in equation (2.5.38), instead of the imaginary part.

The Running of the Coupling

The defining equation for A^{total} (2.5.16) does not depend on the dimension of momentum or equivalently the momentum scale Λ in $d = 2$ dimensions. Therefore any choice of Λ that is made to get to equation (2.5.21) to repair the infinity in A^{total} in $d = 2$ dimensions is principally equally good, and should not affect the final expression for it. However, in equation (2.5.21) A^{total} naively seems to have become a function of Λ . This apparent paradox is resolved by noting that the definition of the renormalized coupling constant (2.5.20) provides a relation between g^2 , g_R^2 and Λ . Since g^2 is

independent of Λ , g_R^2 is a function of Λ , and in such a way that A^{total} is Λ -independent as well. The precise dependence of g_R^2 on Λ is given by the β -function:

$$\beta(g_R^2) \equiv \Lambda \frac{\partial g_R^2}{\partial \Lambda} = \frac{m}{2\pi} g_R^4. \quad (2.5.39)$$

Remarkably this result is obtained both in perturbation theory to all orders via equation (2.5.20), and at 1-loop level via equation (2.5.13).

It is now an easy exercise to check that

$$\Lambda \frac{dA^{total}}{d\Lambda} = \Lambda \frac{\partial A^{total}}{\partial \Lambda} + \beta(g_R^2) \frac{\partial A^{total}}{\partial g_R^2} = 0. \quad (2.5.40)$$

Alternatively, one can say that the scale invariance of A has been broken by a term proportional to the β -function. Furthermore, the same result holds already at 1-loop level: the expression for A given by equation (2.5.14) is independent of Λ via the Λ -dependence of g_R^2 via equation (2.5.13).

However, the exact perturbation theory result is not fully equivalent to the 1-loop result. At 1-loop level it holds that $\epsilon g^2 \rightarrow m g_R^4 / 2\pi$ when $\epsilon \rightarrow 0$, whereas the exact result leads to the conclusion that $\epsilon g^2 \rightarrow 0$ in this limit.

In [9] a similar β -function is computed in $d = 3$ dimensions.

Solving the β -function equation leads to the coupling constant g_R^2 as a function of the momentum scale Λ . To solve for the running coupling yet another momentum scale Λ' needs to be taken:

$$g_R^2(\Lambda) = \frac{g_R^2(\Lambda')}{1 - g_R^2(\Lambda') \frac{m}{4\pi} \ln(\Lambda/\Lambda')^2}. \quad (2.5.41)$$

This expression shows that the theory is infrared free and has a Landau-type singularity for

$$\Lambda_s = e^{2\pi/mg_R^2(\Lambda')} \Lambda' \quad (2.5.42)$$

The behaviour of $g_R^2(\Lambda)$ has been plotted in figure 2.2.

A natural reference scale Λ' is given by the scale at which the energy in the system approaches relativistic values: $E \simeq mc^2$. At this point one expects the non-relativistic model to lose its physical validity. This energy scale corresponds to a momentum scale $\Lambda' \simeq mc$.

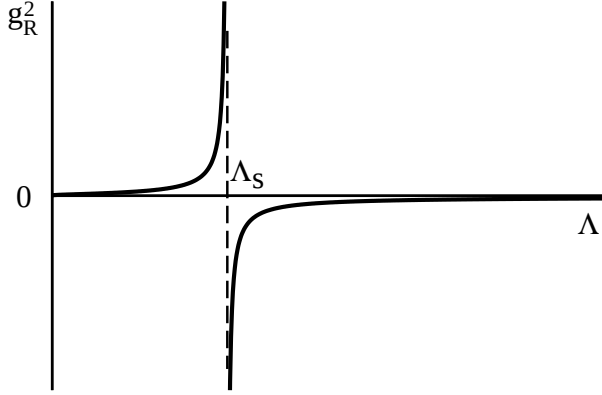


Figure 2.2: Plot of g_R^2 as a function of the momentum scale Λ : showing the running of the coupling of the non-linear Schrödinger model having $|\Phi|^4$ interaction.

For $0 < \Lambda < e^{-2\pi/m} \Lambda_s$ the theory is in the perturbative regime $0 < g_R^2 < 1$. In this regime it represents an effective theory for a weakly interacting Bose-gas having repulsive hardcore (i.e. δ -function) interactions. It is to be noted that there also exists a perturbative regime $|g_R^2| < 1$ for large momentum scales $\Lambda > e^{2\pi/m} \Lambda_s$. In this regime the coupling constant is negative: the interactions are attractive. Such an attractive interaction is relevant in the Jackiw-Pi model, an extension of the non-linear Schrödinger model with Chern-Simons gauge interactions [64,67]. Indeed, the Jackiw-Pi model preserves the conformal symmetries of the Schrödinger group at tree level, and possesses static classical solutions with zero energy in the regime of negative coupling. This model will be studied in the next chapter of this thesis.

Observe, that replacing the bare coupling $g^2 \rightarrow -\bar{g}^2$ after renormalization leads to a negative β -function:

$$\beta(\bar{g}_R^2) = -m \frac{\bar{g}_R^4}{2\pi}. \quad (2.5.43)$$

The upshot is just a change of sign $g_R^2 \rightarrow -\bar{g}_R^2$, such that

$$\bar{g}_R^2(\Lambda) = \frac{\bar{g}_R^2(\Lambda')}{1 + \bar{g}_R^2(\Lambda') \frac{m}{4\pi} \ln(\Lambda/\Lambda')^2}, \quad (2.5.44)$$

The behaviour of this function is as in figure 2.2, but reflected across the Λ -axis by interchanging the positive and negative g_R^2 -values. This leads to exactly the same physics: repulsive interactions, now corresponding to $\bar{g}_R^2 < 0$, in the low momentum regime below $\Lambda_s = e^{-2\pi/m\bar{g}_R^2(\Lambda')}\Lambda'$, and attractive interactions with $\bar{g}_R^2 > 0$ in the high-momentum domain above Λ_s .

Of course, as a non-relativistic theory the model is not expected to be valid on scales of the order $\Lambda \simeq mc$, where non-relativistic effects become relevant. However, if $\Lambda_s \ll 1$, there still could exist a perturbative regime for negative coupling $e^{2\pi/m}\Lambda_s < \Lambda < mc$. Finally it is observed, that in the limit $\Lambda \rightarrow \infty$ the running coupling vanishes, as does the β -function.

The Fate of the Conformal Symmetries

As is clear from the discussion in the previous section the renormalized coupling constant g_R^2 is a function of the momentum scale Λ as dictated by the β -function. This fact makes it clear directly that due to renormalization the scale invariance of the model has been broken, and thereby the special conformal invariance as well. In other words, these symmetries are anomalous.

This fact can be illustrated as follows. The conservation equation in the quantum theory (2.4.10) reads when working in $d = 2 - \epsilon$ dimensions

$$\frac{dK}{dt} = -t \frac{dD}{dt}, \quad \frac{dD}{dt} = \frac{\epsilon}{2} g^2 \int d^d x \Phi^{\dagger 2} \Phi^2 = \epsilon H_{int}, \quad (2.5.45)$$

where H_{int} is the interaction part of the Hamiltonian:

$$H_{int}(t) = \frac{g^2}{2} \int d^d x \Phi^{\dagger 2}(t, \mathbf{x}) \Phi^2(t, \mathbf{x}). \quad (2.5.46)$$

Using the expression (2.4.9) for D , this can be rewritten in the form

$$t \frac{dD}{dt} = \frac{\epsilon}{2} g^2 \frac{\partial D}{\partial g^2}. \quad (2.5.47)$$

On the other hand, classically the conserved scaling charge $D = D(t, g^2)$ is independent of the scale Λ . Since time t has scaling dimension 2, the combination $\Lambda^2 t$ is scale-independent. In the renormalized theory the dimensionless scaling charge D can be a function only of the renormalized coupling g_R^2 and the dimensionless time-parameter $y = \Lambda^2 t$:

$$D = D(t, g^2(g_R^2, \Lambda)) = D_R(y, g_R^2). \quad (2.5.48)$$

Now as D can not depend on the renormalization scale Λ , it is seen that

$$\Lambda \frac{dD}{d\Lambda} = \Lambda \frac{dg_R^2}{d\Lambda} \frac{\partial D_R}{\partial g_R^2} + 2y \frac{\partial D_R}{\partial y} = 0. \quad (2.5.49)$$

It then follows that

$$t \frac{dD}{dt} = y \frac{\partial D_R}{\partial y} = -\frac{\beta}{2} \frac{\partial D_R}{\partial g_R^2}. \quad (2.5.50)$$

Comparing this with equation (2.5.47) it is found that consistency requires that

$$\frac{\epsilon g^2}{2} \frac{\partial D}{\partial g^2} = -\frac{\beta}{2} \frac{\partial D_R}{\partial g_R^2}. \quad (2.5.51)$$

Indeed, when equation (2.5.20) is rewritten in the form

$$g^2 = \frac{g_R^2 \Lambda^\epsilon}{1 - \frac{mg_R^2}{2\pi\epsilon}}, \quad (2.5.52)$$

it is straightforward to show that in the limit $\epsilon \rightarrow 0$

$$\epsilon g^2 \frac{\partial}{\partial g^2} = \epsilon g_R^2 \left(1 - \frac{mg_R^2}{2\pi\epsilon}\right) \frac{\partial}{\partial g_R^2} = -\beta \frac{\partial}{\partial g_R^2}, \quad (2.5.53)$$

which proves the identity (2.5.51).

Since the β -function is non-vanishing in the limit $\epsilon \rightarrow 0$, it is thus concluded there is a conformal anomaly proportional to the β -function:

$$\begin{aligned} \frac{dK}{dt} &= -t \frac{dD}{dt}; \\ \frac{dD}{dt} &= -\frac{1}{2} \beta(g_R^2) \int d^2x \Phi^\dagger{}^2 \Phi^2. \end{aligned} \quad (2.5.54)$$

The charges K and D are no longer conserved and the scale and special conformal symmetries are broken.

So far the conformal anomaly has been shown to exist on the level of operator identities. However, it also appears in observable quantities. For example, consider the correlation function

$$\begin{aligned} &\langle \Phi(z_1) \Phi(z_2) \Phi^*(z_3) \Phi^*(z_4) \frac{dD}{dt} \rangle \\ &\equiv \mathcal{N}_c \langle 0 | T \left(\Phi(z_1) \Phi(z_2) \Phi^*(z_3) \Phi^*(z_4) \frac{dD}{dt} e^{-i \int dt H_{int}} \right) | 0 \rangle. \end{aligned}$$

The proof that it is non-zero shows that $dD/dt \neq 0$, thereby showing the conformal anomaly. The easiest way to do this is to use the result (2.5.45) and integrate the equation over time:

$$\begin{aligned}
 & \int dt \langle \Phi(z_1) \Phi(z_2) \Phi^*(z_3) \Phi^*(z_4) \frac{dD}{dt} \rangle \\
 &= \epsilon \int dt \langle \Phi(z_1) \Phi(z_2) \Phi^*(z_3) \Phi^*(z_4) H_{int} \rangle \\
 &= i\epsilon g^2 \frac{\partial}{\partial g^2} \langle \Phi(z_1) \Phi(z_2) \Phi^*(z_3) \Phi^*(z_4) \rangle \\
 &= -i\beta \frac{\partial}{\partial g_R^2} G_4(z_1, z_2, z_3, z_4).
 \end{aligned} \tag{2.5.55}$$

The step from the first to the second line is not quite trivial; more precisely, one finds

$$\begin{aligned}
 & i\epsilon g^2 \frac{\partial}{\partial g^2} \langle \Phi(z_1) \Phi(z_2) \Phi^*(z_3) \Phi^*(z_4) \rangle \\
 &= i\epsilon \mathcal{N}_c g^2 \frac{\partial}{\partial g^2} \langle 0|T \left(\Phi(z_1) \Phi(z_2) \Phi^*(z_3) \Phi^*(z_4) e^{-i \int dt H_{int}} \right) |0 \rangle \\
 &\quad - \epsilon \langle \Phi(z_1) \Phi(z_2) \Phi^*(z_3) \Phi^*(z_4) \rangle \int dt \langle H_{int} \rangle \\
 &= \epsilon \int dt \langle \Phi(z_1) \Phi(z_2) \Phi^*(z_3) \Phi^*(z_4) H_{int} \rangle,
 \end{aligned} \tag{2.5.56}$$

where it has been used that $\langle H_{int} \rangle = 0$. Equation (2.5.56) shows explicitly that the scaling charge D is not conserved in the full quantum theory, the anomaly being proportional to the β -function.

2.6 Conclusion

In summary, the main results of this chapter are that the classical scale and conformal symmetries of the non-linear Schrödinger model in $2 + 1$ dimensions having $|\Psi|^4$ interaction are broken by quantum mechanical renormalization effects. This fact has been shown at the operator level via a derivation of the conservation equation for the scale charge D , but has also been shown to manifest itself via a correlation function that in principle can be related to observable quantities.

Furthermore, it has been shown that the field and mass in this model do not renormalize; only the coupling constant g^2 does. Via these renormalization effects the model determines its range where perturbation theory can be trusted. Indeed, via the β -function the coupling constant g_R^2 depends on the energy scale Λ , which puts constraints on the allowed values of Λ such that the model is in the perturbative regime, i.e. g_R^2 smaller than one. The result is that only for small and for very large energy scales the model can be used in a perturbative expansion.

In the next chapter a similar study will be performed in case of the Jackiw-Pi model, which is an extension of the non-linear Schrödinger model in $2 + 1$ dimensions with Chern-Simons gauge fields. Classically this model is invariant under exactly the same group of symmetries as the non-linear Schrödinger model is. The detailed derivation of the quantum mechanical fate of dD/dt given in this chapter translates directly to that model, and a similar breaking of the conformal symmetries is observed for most values of the coupling constants. However, an explicit example in terms of correlation functions will be omitted there.

2.A Dimensional Regularization

In this appendix the dimensional regularization technique to control diverging momentum integrals that are encountered when calculating Feynman diagrams, is explained. An excellent discussion of this method can be found in [39].

Momentum loop integrals over $\mathbf{k}$ that are encountered can show diverging behaviour due to ill convergence of the integrand when $\mathbf{k}^2 \rightarrow \infty$. Such converging or diverging behaviour depends strongly on the number of dimensions in which such an integral is calculated. For example, the canonical form in which the Feynman momentum integrals appear, is given by

$$\int \frac{d^d \mathbf{k}}{(2\pi)^d} \frac{(\mathbf{k}^2)^\alpha}{(\mathbf{k}^2 + M^2)^\beta} = \frac{\pi^{d/2}}{(2\pi)^d} \frac{\Gamma(\alpha + d/2)\Gamma(\beta - \alpha - d/2)}{(M^2)^{\beta - \alpha - d/2}\Gamma(\beta)\Gamma(d/2)}, \quad (2.A.1)$$

where d is the number of dimensions. For large values of $\mathbf{k}^2$ this integral behaves like

$$\int d|\mathbf{k}| |\mathbf{k}|^{2(\alpha - \beta) + d - 1}. \quad (2.A.2)$$

This integral will thus diverge logarithmically for example for $d = 2, \beta = 1, \alpha = 0$. And indeed, in this case also on the right hand side of (2.A.1) a divergence is encountered in $\Gamma(\beta - \alpha - d/2)$, which for these values becomes $\Gamma(0)$. However this integral converges for example when $d = 1$.

The main idea of the dimensional regularization scheme is to calculate any integral in an arbitrary number of dimensions, meaning also non-integer ones, and to take the limit towards the dimension one is interested in only at the very end of the calculation. In this way diverging behaviour of an integral can be controlled.

For example, when an integral shows logarithmically diverging behaviour, the integral converges in all dimensions smaller than the one of interest. Taking the example given above for $d = 2, \beta = 1, \alpha = 0$, the integral converges when d is taken to be $d = 2 - \epsilon$ where ϵ is a small positive number. The divergence of the integral when $d = 2$ or $\epsilon \rightarrow 0$ can now be expressed as a power series in ϵ . Indeed, it follows that

$$\int \frac{d^{2-\epsilon} \mathbf{k}}{(2\pi)^{2-\epsilon}} \frac{1}{(\mathbf{k}^2 + M^2)^2} = \frac{1}{4\pi} \frac{\Lambda^{-\epsilon}}{(4\pi)^{-\epsilon/2}} \frac{\Gamma(\epsilon/2)}{(M^2/\Lambda^2)^{\epsilon/2}}, \quad (2.A.3)$$

where the momentum scale Λ (i.e. $\mathbf{k} = \Lambda \mathbf{k}'$ where $\mathbf{k}'$ is dimensionless) has been extracted, such that M^2/Λ^2 is dimensionless. When this expression is expanded in ϵ using $\Gamma(\epsilon/2) = 2/\epsilon - \gamma_E$, where γ_E is Euler's number, it is seen that

$$\int \frac{d^2 \mathbf{k}}{(2\pi)^2} \frac{1}{(\mathbf{k}^2 + M^2)^2} = \lim_{\epsilon \rightarrow 0} \frac{1}{4\pi} \left(\frac{2\Lambda^{-\epsilon}}{\epsilon} - \gamma_E - \ln[M^2/4\pi\Lambda^2] \right). \quad (2.A.4)$$

The diverging behaviour of the integral in 2 dimensions is now fully controlled via the $1/\epsilon$ term. This result is for example used in equations (2.5.10) and (3.6.24)-(3.6.25).

Using this technique, one can possibly extract interesting physical information from diverging integrals. Suppose, as an example, that a diverging integral $\sim 1/\epsilon + 1 + \epsilon + \dots$ in a certain number of dimensions, is multiplied by a quantity that is 0 in this dimension in the following way, $\sim \epsilon + \epsilon^2 + \dots$. The dimensional regularization technique now provides a way to control the $0 \cdot \infty$:

$$(\epsilon + \epsilon^2 + \dots)(1/\epsilon + 1 + \epsilon + \dots) = 1 + 2\epsilon + \dots \quad (2.A.5)$$

The lore of the dimensional regularization scheme can thus be summarized by saying that any integral should first be calculated in arbitrary dimension, and that the result in the dimension one is interested

in should be obtained via a power expansion in a small parameter ϵ that parametrizes the deviation away from this dimension. Only at the very end of *all* calculations the limit $\epsilon \rightarrow 0$ should be taken.

Finally, the dimensional regularization scheme also provides a way to handle integrals that are more badly divergent than logarithmically. For example, the remaining $d^d \mathbf{k}$ integral describing the first order propagator correction in equation (2.5.2), can be calculated via

$$\int d^d \mathbf{k} = \lim_{\beta \rightarrow 0} \int d^d \mathbf{k} \frac{1}{(\mathbf{k}^2 + M^2)^\beta}. \quad (2.A.6)$$

In doing this it is important that the $\beta \rightarrow 0$ limit is taken before taking the limit to the dimension one is interested in. It now follows that

$$\begin{aligned} \int d^d \mathbf{k} &= \lim_{\beta \rightarrow 0} \int d^d \mathbf{k} \frac{1}{(\mathbf{k}^2 + M^2)^\beta} \\ &= \lim_{\beta \rightarrow 0} \pi^{n/2} \frac{1}{(M^2)^{1-d/2}} \frac{\Gamma(1-d/2)}{\Gamma(\beta)} \\ &= \lim_{\beta \rightarrow 0} \pi^{n/2} \frac{\beta}{(M^2)^{1-d/2}} \frac{\Gamma(1-d/2)}{\Gamma(\beta+1)} = 0, \end{aligned} \quad (2.A.7)$$

as used in equation (2.5.2).

CHAPTER 3

THE JACKIW-PI MODEL

3.1 Introduction

The Jackiw-Pi model and its symmetries are studied in this chapter. This model is an extension of the non-linear Schrödinger model in $2 + 1$ dimensions with $U(1)$ Chern-Simons gauge fields [44, 64, 67–69]. Its action is given by

$$S = \int dt d^2\mathbf{x} \frac{\sigma}{2} \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma + i\Psi^* D_t \Psi - \frac{1}{2m} |\mathbf{D}\Psi|^2 - \frac{g^2}{2} (\Psi^* \Psi)^2. \quad (3.1.1)$$

The Chern-Simons kinetic term for gauge fields in $2 + 1$ dimensions has been used to describe massive gauge particles when coupled to an ordinary Maxwell kinetic term, but also to describe matter particles with anyonic statistics. It has also found its applications in the context of the Quantum Hall Effect [44], [64].

The Jackiw-Pi model can be seen as a variation on the Abelian Higgs-model where the Maxwell kinetic term has been replaced by a Chern-Simons term and where the matter Lagrangian is taken to be non-relativistic. In the Abelian Higgs-model vortices arise when studying topological defects. The study of vortices plays an important role in modern theoretical physics, both in condensed matter physics and in high energy physics.

For a specific choice of the coupling constants e and g^2 assuming time-independence and self-duality, also in the Jackiw-Pi model vortices arise, the properties of which are studied in detail in this chapter.

Like the non-linear Schrödinger model, a variation on the Jackiw-Pi model has been used to study the physics of gasses of bosonic particles [5].

The Jackiw-Pi model has also been considered in the context of Aharonov-Bohm scattering [8], which essentially is the scattering of charged particles in the plane by a magnetic flux tube. In reference [8] the renormalization of g^2 is studied. In this chapter the full renormalization at 1-loop level is presented, and at this order of perturbation theory only g^2 is seen to renormalize. Like the non-linear Schrödinger model, the classical Jackiw-Pi model has next to translation, rotation, Galilei and gauge invariance, scale invariance and special conformal invariance. Due to the 1-loop renormalization effects the conformal symmetries of the Jackiw-Pi model are seen to break precisely as in the non-linear Schrödinger model. However, this breaking does not occur when the coupling constants e and g^2 are fixed to the value that allows the vortex solutions: in this case not only the conformal symmetries survive quantization, but also the time-independent self-dual vortices themselves.

3.2 Chern-Simons Gauge Fields

Maxwell's theory for describing electromagnetism and its later generalization to Yang-Mills theory for describing the electroweak and strong interactions, only show their true beautiful structure and richness in the context of fibre bundles in differential geometry. In this context the gauge field must be understood as a connection in a principal fibre bundle having the gauge group as its fibre. The field strength is identified with the curvature associated with this connection. The base manifold of the principal bundle is the physical space-time.

When mathematicians started to study obstructions which prevent a fibre bundle from being a trivial one, and thereby their classification, they introduced the concept of characteristic classes and along with them the Chern-Simons forms of the characteristic classes [86, 88]. Although in mathematics the term Chern-Simons form is broadly used, in physics it is reserved to denote the Chern-Simons form of one type of characteristic classes, the Chern characters.

For example, a Chern-Simons form is given by the 3-form

$$Q_3 = \text{Tr} \left(AdA + \frac{2}{3} A^3 \right), \quad (3.2.1)$$

where A is the connection or gauge field associated with the field strength F , and d denotes the exterior derivative. Choosing the gauge group to be $U(1)$ this expression reduces in local Cartesian coordinates to

$$\epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma, \quad (3.2.2)$$

up to some numerical factors, where α, β and γ run from 1 to 3; $\epsilon^{\alpha\beta\gamma}$ is the totally anti-symmetric tensor such that $\epsilon^{123} = 1$. Similarly another $U(1)$ Chern-Simons form can be written as

$$\epsilon^{\mu\nu\rho\sigma\tau} A_\mu \partial_\nu A_\rho \partial_\sigma A_\tau, \quad (3.2.3)$$

where the indices now run from 1 to 5.

It was only realized after this mathematical development that the Chern-Simons forms can actually be used as an action for gauge fields in physical theories, thus returning to the playground where it all started. It can either supplement the Maxwell or Yang-Mills action or even replace it completely. Originally the first was done, and in this way the Chern-Simons term was seen to generate a mass for the gauge fields [41].

For example, demanding rotation invariance, the Chern-Simons form (3.2.2) can be used as an action in two space and one time dimensions. This is what is done in the Jackiw-Pi model where the action of the gauge fields is given by

$$S_{CS} = \int dt d^2 \mathbf{x} \frac{\sigma}{2} \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma. \quad (3.2.4)$$

The constant $\sigma = \pm 1$ is used to control the relative sign of the Chern-Simons term with respect to terms involving the matter fields in the full action of the Jackiw-Pi model.

Under a parity transformation from a left-handed to a right-handed coordinate system, which in two spatial dimensions is given by $(x_1, x_2) \rightarrow (x_1, -x_2)$, σ changes sign. Since a parity transformation acts on the gauge fields as $(A_1, A_2) \rightarrow (A_1, -A_2)$, it follows that the Chern-Simons Lagrangian is parity-invariant, whereas the full action changes sign due to the transformation property $d^2 \mathbf{x} \rightarrow -d^2 \mathbf{x}$ of the integration measure.

The pure Chern-Simons action (3.2.4) does not involve a metric, or in other words is a topological action. Since in general the Chern-Simons terms do not depend upon the metric they are both Galilei and Lorentz invariant. Furthermore, the energy momentum tensors of the pure Chern-Simons actions vanish. This implies that the Chern-Simons gauge fields alone will show no dynamics. Dynamics has to be inherited from coupling to dynamical matter fields or from adding the standard Maxwell or Yang-Mills terms to the action.

It is interesting to notice that the equation of motion of the pure Chern-Simons action (3.2.4)

$$\epsilon^{\alpha\beta\gamma} F_{\beta\gamma} = 0 \quad (3.2.5)$$

implies that the field strength vanishes, another sign that interesting dynamics has to come from elsewhere. Pure Chern-Simons electric and magnetic fields are thus zero when there is no matter in the system. This is in contrast with Maxwellian electrodynamics where in the absence of matter the electric and magnetic fields describe the electromagnetic waves (e.g. light). In mathematical terms the pure Chern-Simons gauge fields describe flat connections, i.e. having vanishing curvature.

As a final remark, the equation of motion is gauge invariant, because the field strength is. However, the action is seen to change under a gauge transformation $A_\mu \rightarrow A_\mu + \partial_\mu \omega$ by

$$S_{CS} \rightarrow S_{CS} + \int dt d^2 \mathbf{x} \frac{\sigma}{4} \epsilon^{\alpha\beta\gamma} \partial_\alpha \omega F_{\beta\gamma}. \quad (3.2.6)$$

Using the Bianchi identity $\epsilon^{\alpha\beta\gamma} \partial_\alpha F_{\beta\gamma} = 0$, this additional term is found to be a total derivative term, which vanishes under appropriate boundary conditions.

3.3 Introduction to the Jackiw-Pi Model

The Jackiw-Pi model combines the Chern-Simons kinetic term (3.2.4) for $U(1)$ gauge fields with the action of the non-linear Schrödinger model for a complex matter field Ψ in Euclidean $2 + 1$ dimensions. The action is thus given by

$$S = \int dt d^2 \mathbf{x} \frac{\sigma}{2} \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma + i \Psi^* D_t \Psi - \frac{1}{2m} |\mathbf{D}\Psi|^2 - \frac{g^2}{2} (\Psi^* \Psi)^2, \quad (3.3.1)$$

where $D_\alpha = \partial_\alpha - ieA_\alpha$ is the gauge covariant derivative. The indices in the Chern-Simons term run from 0 to 2 ($\epsilon^{012} = 1$), 0 denoting the temporal component of the gauge field and 1 and 2 its spatial components.

Following the discussion in the previous section this action is straightforwardly seen to be $U(1)$ gauge invariant under the transformations

$$\Psi \rightarrow e^{ie\omega}\Psi, \quad A_\mu \rightarrow A_\mu + \partial_\mu\omega, \quad (3.3.2)$$

for any function ω .

The equations of motion of this model are given by

$$-i\partial_t\Psi = \frac{1}{2m}\mathbf{D}^2\Psi + eA_0\Psi - g^2(\Psi^*\Psi)\Psi, \quad (3.3.3)$$

$$\sigma(\nabla_1 A_2 - \nabla_2 A_1) + e\Psi^*\Psi = 0, \quad (3.3.4)$$

$$\partial_t A_i - \nabla_i A_0 = \sigma \frac{ie}{2m} \epsilon^{ij} (\Psi^* D_j \Psi - (D_j \Psi)^* \Psi). \quad (3.3.5)$$

The equation of motion (3.3.3) is a gauged Schrödinger equation for the matter field. When the usual definitions for the magnetic field B , electric field E_i , charge density ρ and charge current J^i in two spatial dimensions are used,

$$B \equiv \nabla_1 A_2 - \nabla_2 A_1, \quad \mathbf{E} \equiv \partial_t \mathbf{A} - \nabla A_0, \quad (3.3.6)$$

$$\rho \equiv \Psi^*\Psi, \quad \mathbf{J} \equiv \frac{1}{2mi}(\Psi^* \mathbf{D}\Psi - (\mathbf{D}\Psi)^* \Psi). \quad (3.3.7)$$

the other two equations of motion (3.3.4) and (3.3.5) can be rewritten as

$$B = -\sigma e \rho, \quad E_i = -\sigma e \epsilon^{ij} J^j. \quad (3.3.8)$$

These equations relate the magnetic field B to the matter or charge density ρ via the so-called Gauss-Chern-Simons equation, and the electric field to the gauged matter or charge current J_i . The magnetic field is proportional to the matter density, and the electric field is perpendicular to the current, both with the factor $-\sigma e$.

Under a parity transformation the magnetic field changes sign, i.e. is a pseudo-scalar. Similarly $\mathbf{E}$ and $\mathbf{J}$ are vectors, since $\mathbf{E}$ and $\mathbf{J}$ transform as the coordinates do: $(E_1, E_2) \rightarrow (E_1, -E_2)$ and likewise for $\mathbf{J}$. The

density ρ is an ordinary scalar. Since σ also changes sign under a parity transformation, the equations in (3.3.8) remain satisfied.

Next to the equations of motion that follow from the action there is also the Bianchi identity for the electromagnetic field that follows from its definition as the curvature of a connection. This identity reads $\partial_\alpha F_{\beta\gamma} + \partial_\beta F_{\gamma\alpha} + \partial_\gamma F_{\alpha\beta} = 0$. Normally in four dimensions this identity leads to the two source-free Maxwell equations, however in $2+1$ dimensions it reduces to a single equation:

$$\partial_t B = \epsilon^{ij} \nabla_i E_j, \quad (3.3.9)$$

which in light of the equations above is nothing but the current conservation law:

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0. \quad (3.3.10)$$

The equations (3.3.8) and (3.3.10) show that the particles in the Jackiw-Pi model have to be interpreted as dyons, carrying both electric and magnetic charge. The equation relating ρ to B implies they carry magnetic charge in units of $-\sigma e$. From the equation that relates the current to the electric field it is interpreted that they carry electric charge as well, also in units of $-\sigma e$.

The notion that these particles carry magnetic charge is straightforward and clear because the magnetic field is simply proportional to the matter density via equation (3.3.4). However, one must exercise care when giving the interpretation that they carry electric charge as well. This is because the equations of motion due to the Chern-Simons term are completely different from the equations that would arise due to a Maxwell term. For comparison, if a Maxwell-like term $\frac{1}{2}(\mathbf{E}^2 - B^2)$ would have been written in the action instead, the equations of motion involving the electric and magnetic fields would have been

$$\nabla \cdot \mathbf{E} = -e\rho, \quad \epsilon^{ij} \nabla_j B - \partial_t E_i = eJ_i. \quad (3.3.11)$$

Thus showing that the electric and magnetic properties of matter are completely different when studying Chern-Simons instead of Maxwell electrodynamics. More on the interpretation of the electric and magnetic fields in the Jackiw-Pi model is said in section 3.4 where explicit solutions are studied.

As a last remark, the energy or Hamiltonian in the Jackiw-Pi model is given by

$$H = \int d^2\mathbf{x} \frac{1}{2m} |\mathbf{D}\Psi|^2 + \frac{g^2}{2} |\Psi|^4. \quad (3.3.12)$$

The Chern-Simons term for the gauge fields is topological and thus does not contribute to the Hamiltonian. As in the non-linear Schrödinger model the coupling constant g^2 is allowed to become negative. This signifies an attractive interaction, whereas a positive coupling constant signifies a repulsive interaction.

3.4 Time-Independent Self-Dual Solutions

The equations of motion (3.3.3)-(3.3.5) of the Jackiw-Pi model can be solved under certain restrictions [44, 64, 67–69].

At first, it needs to be assumed that the physical solutions to the equations are time-independent. This means that gauge dependent quantities are still allowed to depend on time, so called pure-gauge time dependence. For example, by writing the matter field and the gauge fields like

$$\Psi = e^{ie\omega} \sqrt{\rho}, \quad A_\mu = \partial_\mu \omega + \mathcal{A}_\mu, \quad (3.4.1)$$

ω is the function that determines the gauge. This function is still allowed to be time dependent, where as the gauge independent quantities ρ and $\mathcal{A}_\mu$ are taken to be time-independent.

The second assumption one needs to make is a so-called self-duality condition:

$$D_\pm \Psi \equiv (D_1 \pm iD_2) \Psi = 0, \quad (3.4.2)$$

which is a first order equation, whereas the equation of motion (3.3.3) is second order. Either the plus or the minus sign needs to be chosen.

As a third assumption the value of the coupling constant g^2 needs to be fixed to a very special value:

$$g^2 = \pm \sigma \frac{e^2}{m}, \quad (3.4.3)$$

where the sign choice depends on the sign choice in the self-duality condition.

These last two assumptions can be interpreted in the following way. The energy in the Jackiw-Pi Model can be rewritten using a Bogomolny trick,

$$\begin{aligned} E &= \int d^2\mathbf{x} \frac{1}{2m} |\mathbf{D}\Psi|^2 + \frac{g^2}{2} (\Psi^*\Psi)^2 \\ &= \int d^2\mathbf{x} \frac{1}{2m} |D_{\pm}\Psi|^2 \pm \frac{eB}{2m} \Psi^*\Psi + \frac{g^2}{2} (\Psi^*\Psi)^2, \end{aligned} \quad (3.4.4)$$

as can be seen using the fact that

$$\mathbf{D}^2 = D_{\mp} D_{\pm} \mp eB. \quad (3.4.5)$$

When the equation of motion (3.3.4) is utilized, the energy can be rewritten as

$$E = \int d^2\mathbf{x} \frac{1}{2m} |D_{\pm}\Psi|^2 + \frac{1}{2} \left(g^2 \mp \sigma \frac{e^2}{m} \right) (\Psi^*\Psi)^2. \quad (3.4.6)$$

Thus making it clear that the second and third assumptions imply that the energy vanishes.

The equations of motion can now be solved in the following way. Using the fact (3.4.5), the first equation of motion (3.3.3) can be rewritten as

$$i\partial_t\Psi + eA_0\Psi = \pm\sigma\frac{e^2}{2m}\rho\Psi, \quad (3.4.7)$$

under the second and third assumptions and upon using the equation of motion for the magnetic field (3.3.4). When the matter field is written as in equation (3.4.1), this equation can be reduced to

$$A_0 = \partial_t\omega \pm \sigma\frac{e}{2m}\rho, \quad (3.4.8)$$

when pure-gauge time dependence is assumed.

The assumption of self-duality (3.4.2) can be translated into

$$A_i = \nabla_i\omega \pm \epsilon^{ij}\frac{1}{2e}\nabla_j\log\rho. \quad (3.4.9)$$

When this expression is plugged into the equation for the magnetic field (3.3.4), the following equation is obtained:

$$(\nabla_1^2 + \nabla_2^2)\log\rho = \pm\sigma 2e^2\rho. \quad (3.4.10)$$

This so-called Liouville equation determines the particle density ρ . And the particle density in turn determines the gauge fields via equations (3.4.8)-(3.4.9) up to gauge dependence. Note that no gauge dependent quantities appear in the Liouville equation.

Only the third equation of motion (3.3.5) now remains to be studied more closely. However, having obtained definite equations for the gauge independent parts of the matter field and the gauge fields, this equation, being a gauge independent one, should be solved automatically and give no further conditions. Using the fact that the self duality condition expresses the current J^i in terms of a derivative of the particle density,

$$D_{\pm}\Psi = 0 \implies J^i = \mp \epsilon^{ij} \frac{1}{2m} \nabla_j \rho, \quad (3.4.11)$$

the equation (3.3.5) can be rewritten as

$$\mathbf{E} = \partial_t \mathbf{A} - \nabla A_0 = \mp \sigma \frac{e}{2m} \nabla \rho. \quad (3.4.12)$$

Plugging in the expressions for the gauge fields (3.4.8)-(3.4.9) it is easily seen that this equation is indeed identically satisfied.

Last but not least, it is an easy exercise to show that the Bianchi identity or equivalently the current conservation law (3.3.10) is satisfied.

In summary, solving the equations of motion is thus reduced to solving the Liouville equation (3.4.10). This equation gives a time-independent matter or charge density ρ out of which the electric and magnetic fields and charge current can be calculated. This gives physical solutions that have no time dependence.

If necessary, the gauge freedom expressed in the function ω can be used to make the gauge functions A_0 and A_i regular. An example of this will be given later in this section when some explicit examples are discussed.

Similary, this gauge freedom can be used to gauge away one of the gauge fields. For example, in the gauge $A_0 = 0$ the solutions described above can still be found, it just comes down to choosing

$$\omega = \mp \sigma \frac{e}{2m} \rho t, \quad (3.4.13)$$

since ρ is time-independent.

Finally, the Liouville equation can be solved under a simple restriction of the choice of $\pm$. Solutions exist choosing the minus sign when $\sigma = +$,

and the plus sign when $\sigma = -$, such that always $\pm\sigma = -$. It can easily be verified that solutions of

$$\Delta \log \rho = -2e^2 \rho, \quad (3.4.14)$$

are given by

$$\rho = \frac{4}{e^2} \frac{|\partial_z f|^2}{(1 + |f|^2)^2}, \quad (3.4.15)$$

where f is a function of $z = x + iy$ only, of which the derivative with respect to z exists, a *holomorphic function*. However, isolated poles of the functions f are allowed. Thus, more precisely, it should be said that f is required to be a *meromorphic function*. A similar expression arises for the topological charge in the $O(3)$ -model [96].

In [64, 65] it is explained that the physically interesting solutions are given by f being equal to a ratio of polynomials:

$$\rho = \frac{4}{e^2} \frac{|\partial_z f|^2}{(1 + |f|^2)^2}, \quad f = \frac{a_m z^m + \dots + a_0}{b_n z^n + \dots + b_0}, \quad (3.4.16)$$

where of course it has been assumed that neither a_m nor b_n equals zero. These restrictions guarantee necessary positivity of the matter density ρ , a regular magnetic field and a finite total magnetic flux Φ (the magnetic field integrated over the plane).

Furthermore, it was already known in the literature (e.g. [67]) that under inversion of the function f , ρ is invariant:

$$f \rightarrow 1/f: \quad \rho \rightarrow \rho. \quad (3.4.17)$$

Writing down an obvious generalization of this transformation on f ,

$$f \rightarrow \frac{af + b}{cf + d}, \quad (3.4.18)$$

one can analyze what restrictions are placed on the coefficients a, b, c and d when it is demanded that ρ remains invariant under this transformation. It is not hard to verify that it is necessary and sufficient that these coefficients constitute a $U(2)$ matrix:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U(2). \quad (3.4.19)$$

As a final remark, under the sign choice allowing solutions of the Liouville equation, the relation (3.4.3) changes into

$$g^2 = -\frac{e^2}{m}. \quad (3.4.20)$$

This relation will turn out to play an important role when studying the renormalization effects in the Jackiw-Pi model in chapter 3.6.

Having obtained the general expression for classical self-dual solutions in the Jackiw-Pi model, next the physical content of two explicit solutions will be studied.

A One Vortex Example

As a first example, the solution following from the choice

$$f(z) = \frac{1}{z^n} \quad (3.4.21)$$

is studied more closely. This function leads to a matter density described by

$$\rho = 4n^2 \frac{1}{e^2} \frac{r^{2(n-1)}}{(1 + r^{2n})^2} \quad (3.4.22)$$

where r is the radial coordinate in the plane:

$$r = |z| = \sqrt{x^2 + y^2}. \quad (3.4.23)$$

The matter density behaves as $1/r^{2n+2}$ for large r and as $r^{2(n-1)}$ for small r . It has no singularities and is thus a regular function. For $n = 1$ it has a maximum at the origin. However, for $n > 1$ there is a zero at the origin of power $2(n - 1)$. This radially symmetric charge configuration is shown in figure 3.1 for $n = 1$ and in figure 3.2 for $n = 2$. The electric field is shown as well via equation (3.4.12):

$$\mathbf{E} = \frac{e}{2m} \nabla \rho, \quad (3.4.24)$$

where the choice of sign necessary to solve the Liouville equation has been implemented.

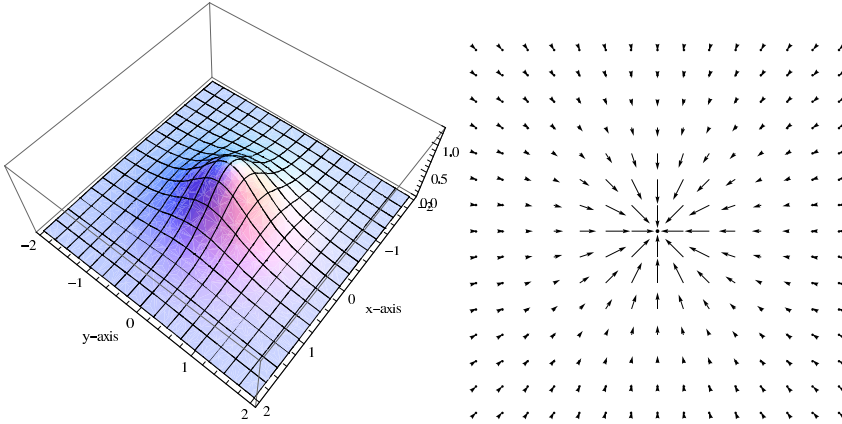


Figure 3.1: Plot of the matter density ρ or magnetic field B given by $f = 1/z^n$ for $n = 1$ on the left. There is a maximum at the origin. In the plot of the electric field on the right the horizontal axis is the x -axis and the vertical one the y -axis. These plots have been made assuming e to be positive and σ to be negative.

Following this equation, the electric field is seen to point in the direction of regions with high matter density when e is positive, and away when e is negative, as can also be seen in the figures. In this sense the matter would be interpreted in a Maxwellian setting to be charged as $-e$. In this time-independent self-dual solution the electric field is insensitive to the value of σ . However, it was suggested before (section 3.3) from the equation that relates the electric field to the matter current that the matter is actually electrically charged in units of $-\sigma e$. These remarks reveal unusual aspects in interpreting the coupling between the matter and electric fields in this model.

Furthermore, the particles in this model carry all the magnetic field, via $B = -\sigma e \rho$. In a sense they can thus be interpreted as pseudo-scalar magnetic monopoles. They are pseudo-scalar because the magnetic field changes sign under a parity transformation.

In the end, in this and all the other time-independent self-dual solutions in the Jackiw-Pi model the magnetic fields must be understood as proportional to the matter density with a factor $-\sigma e$, and the electric fields as proportional to the gradient of the matter density with a factor $e/2m$, see equations (3.3.8), (3.4.12) and (3.4.24).

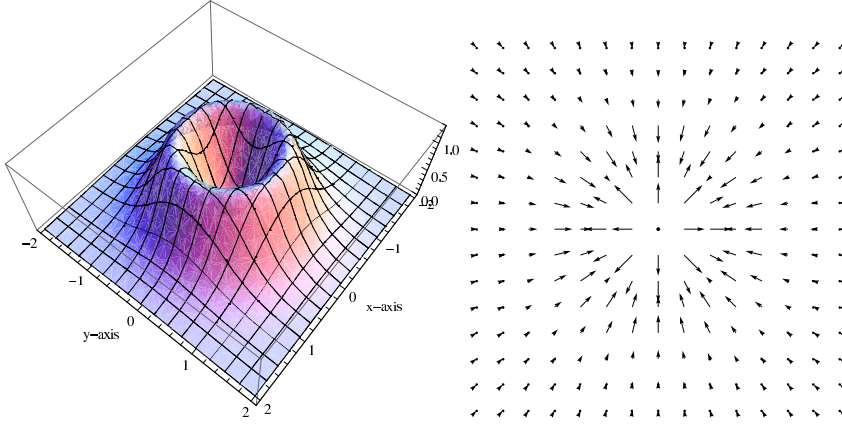


Figure 3.2: Plot of the matter density ρ or magnetic field B given by $f = 1/z^n$ for $n = 2$ on the left. There is a zero at the origin. In the plot of the electric field on the right the horizontal axis is the x -axis and the vertical one the y -axis. These plots have been made assuming e to be positive and σ to be negative.

It is an easy exercise to calculate the total magnetic flux Φ associated to the magnetic field B in this solution. This flux is given by

$$\Phi \equiv \int d^2\mathbf{x} B = -\sigma e \int d^2\mathbf{x} \rho \quad (3.4.25)$$

which leads to a radial integral that can straightforwardly be done. The result is

$$\Phi = -\sigma \frac{4n^2}{e} 2\pi \int_0^\infty dr r \frac{r^{2(n-1)}}{(1+r^{2n})^2} = -\sigma \frac{4\pi}{e} n. \quad (3.4.26)$$

This is in agreement with the general result obtained in [65], which says that for a configuration given by $f = P/Q$ for general polynomials P and Q such that the degree of P is smaller than the degree of Q (denoted by n), the flux is given by

$$\Phi = -\sigma \frac{4\pi}{e} n. \quad (3.4.27)$$

In the present case $P = 1$ and $Q = z^n$.

In this solution with $f = 1/z^n$ the matter density has a zero at the origin for $n > 1$ of power $2(n-1)$. This zero produces a singular contribution to

the vector potential via the derivative of the logarithm. Indeed, following (3.4.9) it is seen that

$$\mathbf{A} = \nabla\omega + \frac{\sigma}{e} \frac{\hat{\theta}}{r} \left(n - 1 - \frac{2n}{1 + r^{-2n}} \right), \quad (3.4.28)$$

where θ is the angular coordinate on the plane and $\hat{\theta}$ denotes its unit vector. Around the origin, the gauge field thus behaves like

$$\mathbf{A} = \nabla\omega + \frac{\sigma}{e} \frac{\hat{\theta}}{r} (n - 1), \quad (3.4.29)$$

showing singular behaviour there for $n > 1$. The singular term $(n - 1)/r$ derives from the term that causes the zero in ρ , $r^{2(n-1)}$, via $\frac{1}{2}\nabla \ln \rho$.

Making the Ansatz

$$\omega = -\frac{\sigma}{e} (n - 1)\theta \quad (3.4.30)$$

for the gauge function, the vector potential is given by:

$$\mathbf{A} = -\frac{\sigma}{e} \frac{\hat{\theta}}{r} \left(\frac{2n}{1 + r^{-2n}} \right) = -\frac{\sigma}{e} \left(\frac{r^{2n-1}}{1 + r^{2n}} \right) \hat{\theta}, \quad (3.4.31)$$

which is regular everywhere on the plane, as can be seen in figure 3.3. That this Ansatz is indeed the correct one can be checked by either calculating $B = \epsilon^{ij} \nabla_i A_j$ and verifying that it satisfies $-\sigma e \rho$, or by calculating the magnetic flux using Stokes' theorem:

$$\Phi = \int d^2\mathbf{x} B = \oint_{S_{r \rightarrow \infty}^1} \mathbf{A} \cdot d\mathbf{l}, \quad (3.4.32)$$

where the integration is done counterclockwise over a circle at infinity. Since in cylindrical coordinates the integration element $d\mathbf{l}$ is given by $r d\theta \hat{\theta}$, this integral reduces in the limit $r \rightarrow \infty$ to

$$\Phi = -2n \frac{\sigma}{e} \int_0^{2\pi} d\theta \frac{1}{1 + r^{-2n}} = -\sigma \frac{4\pi}{e} n, \quad (3.4.33)$$

where the regular expression for the gauge field (3.4.31) has been used. Indeed, the same result as before is obtained (3.4.26).

The Ansatz $\omega = 0$ for the gauge field leads to a different physical situation for $n \geq 2$. In this case the singular term $\hat{\theta}/r$ in the vector

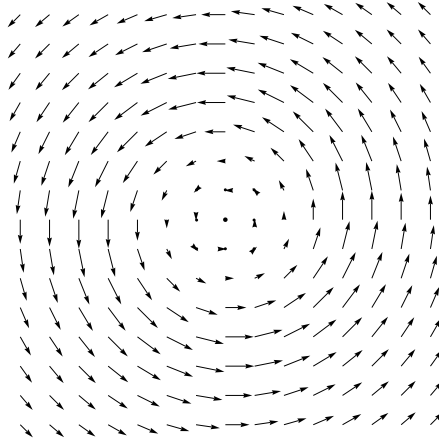


Figure 3.3: Plot of the gauge field $\mathbf{A}$ given by equation (3.4.31) that comes along with solution (3.4.22) or figure 3.1 or 3.2. A similar profile shows for each value of n . This gauge field has a vortex at the origin. Also here it has been assumed that e is positive and σ negative.

potential contributes a term proportional to $\delta(r)$ to the magnetic field, making it clear that the identity $B = -\sigma e\rho$ is no longer satisfied. The fact that different choices for the gauge function lead to different physical interpretations can be traced back to be due to the fact that the Ansatz (3.4.30) for ω is singular at the origin, whereas the Ansatz $\omega = 0$ is not. In case of the Jackiw-Pi model the Gauss-Chern-Simons equation $B = -\sigma e\rho$ dictates the correct choice for ω [67].

Observing the profile of the gauge field in figure 3.3, it is seen to wind around the origin, independent of the value of n . Such a point is here called a vortex, and it coincides with the point where f diverges. In this solution in the Jackiw-Pi model the magnetic field actually has a zero at the place where the vortex lives for $n \neq 1$, as can clearly be seen in e.g. figure 3.2 for $n = 2$. For $n = 1$ it so happens that the magnetic field has a maximum at the origin. The magnetic field is not located at the position of the vortex, but is spread out over space in both cases.

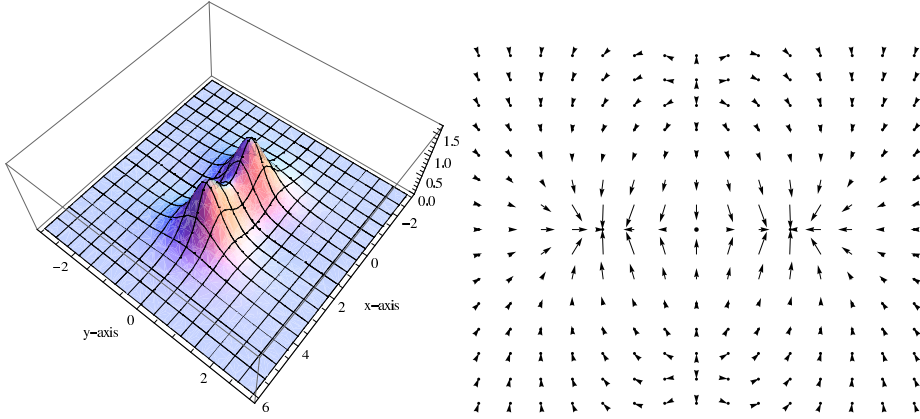


Figure 3.4: Plot of the matter density ρ or magnetic field B given by $f = 1/z^n + 1/(z - 3)^n$ for $n = 1$. There are zeros at $z = x + iy = 3/2(1 \pm i)$. In the plot of the electric field the horizontal axis is the x -axis, running from -1 to 4 , and the vertical one the y -axis, running from -2 to 2 . These plots have been made assuming e to be positive and σ to be negative.

A Multi-Vortex Example

As a second example two of the previously studied configurations are placed in the plane:

$$f(z) = \frac{1}{z^n} + \frac{c}{(z - a)^n}. \quad (3.4.34)$$

The total flux in this case is given by

$$\Phi = -\sigma \frac{4\pi}{e} 2n, \quad (3.4.35)$$

as can be understood via the discussion around equation (3.4.27). This expression does not depend on either c or a .

Based upon the previous example, one would interpret this configuration f as two vortices with a relative weight of c , one placed at the origin and the other one placed at a , the points where f diverges. However, a careful study will show that a different interpretation needs to be given.

In figures 3.4 and 3.5 the matter density profile that follows from this choice of the function f has been plotted for $n = 1$ and 2 assuming that $c = 1$ and $a = 3$. As expected it shows two nearby lumps of matter or magnetic field.

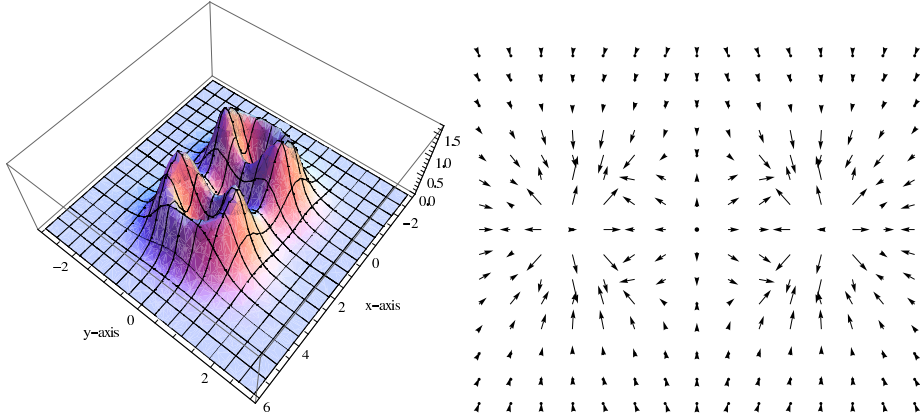


Figure 3.5: Plot of the matter density ρ or magnetic field B given by $f = 1/z^n + 1/(z-3)^n$ for $n = 2$. There are zeros at $z = x + iy = 0, 3, 3/2, 3/2(1 \pm i\sqrt{3})$. In the plot of the electric field the horizontal axis is the x -axis, running from -1 to 4 , and the vertical one the y -axis, running from -2 to 2 . These plots have been made assuming e to be positive and σ to be negative.

Calculating the zeros of both f (denoted by f_k for $k = 0, \dots, n-1$) and $\partial_z f$ (denoted by f'_l for $l = 0, \dots, n$)

$$f_k = \frac{a}{1 - c^{1/n} e^{i\pi(2k+1)/n}}, \quad f'_l = \frac{a}{1 - c^{1/(n+1)} e^{i\pi(2l+1)/(n+1)}}, \quad (3.4.36)$$

the expression (3.4.15) for the charge density can be rewritten as

$$\rho = n^2 |1 + c|^2 \frac{|z|^{2(n-1)} |z - a|^{2(n-1)} \prod_{l=0}^n |z - f'_l|^2}{\left(|z|^{2n} |z - a|^{2n} + |1 + c|^2 \prod_{k=0}^{n-1} |z - f_k|^2 \right)^2}. \quad (3.4.37)$$

For $n = 1$ this simplifies to

$$\rho = |1 + c|^2 \frac{|z - f'_0|^2 |z - f'_1|^2}{(|z|^2 |z - a|^2 + |1 + c|^2 |z - f_0|^2)^2}. \quad (3.4.38)$$

This shows that now also for $n = 1$ there appear zeros, since ρ has zeros at f'_0 and f'_1 . These zeros will of course give singular behaviour in the gauge field. As before, this singular behaviour can be cured by choosing the gauge function to be

$$\omega = -\sigma \arctan \left(\frac{\text{Im}(z - f'_0)}{\text{Re}(z - f'_0)} \right) - \sigma \arctan \left(\frac{\text{Im}(z - f'_1)}{\text{Re}(z - f'_1)} \right). \quad (3.4.39)$$

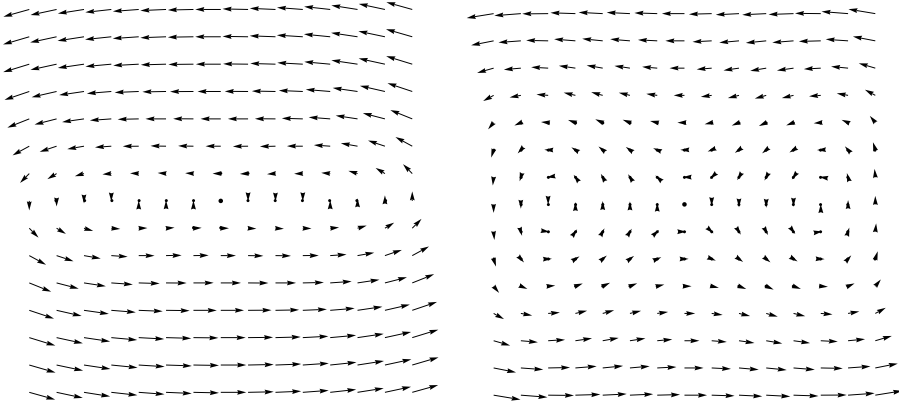


Figure 3.6: Plot of the gauge field that comes with solution (3.4.37) for $n = 1$ on the left and for $n = 2$ on the right. In both cases $c = 1$ and $a = 3$. The horizontal axes are the x -axes where x runs from $1/2$ to $5/2$, the vertical axes are the y -axes where y runs from -1 to 1 . Vortices are found at $(x, y) = (1, 0), (2, 0)$ for $n = 1$ and at $(2.2, 0), (0.8, 0)$ for $n = 2$. A zero is found at $(3/2, 0)$ both for $n = 1$ and $n = 2$. Again e is assumed to be positive and σ negative.

Close to each of the zeros this gauge function essentially behaves like $\sigma\theta$:

$$\omega(z \rightarrow f'_l) = -\sigma\theta + \dots, \quad (3.4.40)$$

where the dots represent contributions from the other singular point, which are smooth around this zero. For example, in the case $c = 1, a = 3$ the zeros of ρ live at $(x, y) = 3/2(1, \pm 1)$.

In the previous examples the position of the divergence of f coincided with the position of the center of vorticity of the gauge field. In this example the gauge field $\mathbf{A}$ has been plotted in figure 3.6 in the case $n = 1, c = 1, a = 3$ using the regular expression dictated by ω above. Vortices, naively expected to live at the points where f diverges, $(x, y) = (0, 0), (3, 0)$, in this case actually live at $(x, y) = (1, 0), (2, 0)$. Furthermore, the positions of the vortices do not coincide with the points where ρ has a maximum or a zero.

Studying ρ for general n (3.4.37), the positions 0 and a are also zeros in the matter density. Furthermore there are $n + 1$ more zeros, the zeros of f' , the f'_l , as before. A gauge function that cures the singular behaviour

due to the zeros is given by

$$\begin{aligned} \omega = & -\sigma(n-1) \arctan\left(\frac{\text{Im}(z)}{\text{Re}(z)}\right) - \sigma(n-1) \arctan\left(\frac{\text{Im}(z-a)}{\text{Re}(z-a)}\right) \\ & - \sigma \sum_{l=0}^n \arctan\left(\frac{\text{Im}(z-f'_l)}{\text{Re}(z-f'_l)}\right). \end{aligned} \quad (3.4.41)$$

In the case $n = 2$, $c = 1$ and $a = 3$ the zeros of ρ live at $(0, 0)$, $(3, 0)$ and $(3/2, \pm 3\sqrt{3}/2)$. Also in this case the gauge field has been plotted in figure 3.6. There are two vortices, living at approximately $(x, y) = (2.2, 0)$ and $(0.8, 0)$. These vortices do not live at the places where f diverges, $(0, 0)$, $(3, 0)$, or where ρ has a zero or a maximum. However in this case a zero in the gauge field is found at a zero of the matter density: at $(3/2, 0)$.

These examples illustrate the physics of vortices in case of the non-relativistic Chern-Simons electrodynamics in the Jackiw-Pi model. It is interesting to note that all the presented solutions have the same energy, namely $E = 0$, as follows from the three assumptions that enable one to find the solutions. In this model it thus also costs no energy to add extra vortices or magnetic flux to the plane. A question worth studying is the dynamics of these (so far time-independent, thus static) vortices. This is usually done in an adiabatic approximation by making the parameters of the static vortex solutions (i.e. in this case a and c) time-dependent, out of which an effective Hamiltonian or Lagrangean is derived [44, 84]. However, in case of the Jackiw-Pi model all static vortex solutions have zero energy, a property which remains when the coefficients are made time-dependent. Dynamics can in this case thus not be derived along these lines. However, other approaches are more successful [3, 66, 85].

3.5 The Symmetries of the Jackiw-Pi Model

The Jackiw-Pi Model has a rich invariance under symmetries. Next to being gauge invariant, it is invariant under exactly the same group of space-time symmetries as the non-linear Schrödinger model, the Schrödinger group, see chapter 2.2. On the space-time coordinates $\mathbf{x}$ and t this group acts as written down in equation (2.2.2). This group of transformations comprises space- and time-translations, rotations, Galilei transformations, scale transformations or dilatations and special conformal transformations.

Under the space and time translations, the transformation rule for the fields Ψ and A_μ are

$$\begin{aligned}\Psi'(\mathbf{x}', t') &= \Psi(\mathbf{x}, t), \\ A'_\mu(\mathbf{x}', t') &= A_\mu(\mathbf{x}, t).\end{aligned}\tag{3.5.1}$$

Under rotations they read

$$\begin{aligned}\Psi'(\mathbf{x}', t') &= \Psi(\mathbf{x}, t), \\ A'_0(\mathbf{x}', t') &= A_0(\mathbf{x}, t), \\ A'_i(\mathbf{x}', t') &= R_{ij}A_j(\mathbf{x}, t).\end{aligned}\tag{3.5.2}$$

Under Galilean boosts parametrized by $\mathbf{u}$, the field transforms with an additional space-time dependent phase factor

$$\begin{aligned}\Psi'(\mathbf{x}', t') &= e^{im(\mathbf{u} \cdot \mathbf{x} + \frac{1}{2}\mathbf{u}^2 t)} \Psi(\mathbf{x}, t), \\ A'_0(\mathbf{x}', t') &= A_0(\mathbf{x}, t) - \mathbf{u} \cdot \mathbf{A}(\mathbf{x}, t), \\ A'_i(\mathbf{x}', t') &= A_i(\mathbf{x}, t).\end{aligned}\tag{3.5.3}$$

Under dilatations the field has a dimension-dependent non-zero Weyl weight, which also turns up in the special conformal transformations. Taking $\alpha = 1/\delta = e^\eta$, $\beta = \gamma = 0$ the dilatations are realized on complex Ψ as

$$\begin{aligned}\Psi'(\mathbf{x}', t') &= e^{-\eta} \Psi(\mathbf{x}, t), \\ A'_0(\mathbf{x}', t') &= e^{-2\eta} A_0(\mathbf{x}, t), \\ A'_i(\mathbf{x}', t') &= e^{-\eta} A_i(\mathbf{x}, t),\end{aligned}\tag{3.5.4}$$

The special conformal transformations with $\alpha = \delta = 1$ and $\beta = 0$ take the form

$$\begin{aligned}\Psi'(\mathbf{x}', t') &= (1 + \gamma t) e^{-m \frac{i}{2} \frac{\gamma \mathbf{x}^2}{(1 + \gamma t)}} \Psi(\mathbf{x}, t), \\ A'_0(\mathbf{x}', t') &= (1 + \gamma t)^2 A_0(\mathbf{x}, t) + \gamma (1 + \gamma t) \mathbf{x} \cdot \mathbf{A}(\mathbf{x}, t), \\ A'_i(\mathbf{x}', t') &= (1 + \gamma t) A_i(\mathbf{x}, t),\end{aligned}\tag{3.5.5}$$

Finally, the $U(1)$ gauge transformations leave the space-time coordinates invariant, and take the standard form on the complex scalar field:

$$\begin{aligned}\Psi'(\mathbf{x}', t') &= \Psi(\mathbf{x}, t) = e^{ie\omega} \Psi(\mathbf{x}, t), \\ A'_\mu(\mathbf{x}', t') &= A'_\mu(\mathbf{x}, t) = A_\mu(\mathbf{x}, t) + \partial_\mu \omega.\end{aligned}\tag{3.5.6}$$

Under all these transformations the matter field Ψ transforms exactly as in the non-linear Schrödinger model. The gauge fields A_μ naturally transform exactly like the derivatives do. For example, under the special conformal transformations

$$t' = \frac{t}{1 + \gamma t}, \quad \mathbf{x}' = \frac{\mathbf{x}}{1 + \gamma t}, \quad (3.5.7)$$

the derivatives change as

$$\frac{\partial}{\partial t'} = (1 + \gamma t)^2 \frac{\partial}{\partial t} + \gamma(1 + \gamma t)x_i \frac{\partial}{\partial x_i}, \quad \frac{\partial}{\partial x'_i} = (1 + \gamma t) \frac{\partial}{\partial x_i}, \quad (3.5.8)$$

in agreement with (3.5.5).

The infinitesimal form of the transformation behaviour of Ψ is as in chapter 2.2 and can be found there, equation (2.2.9). For the gauge fields the infinitesimal forms are given by

$$\begin{aligned} \delta A_0(\mathbf{x}, t) &= A'_0(\mathbf{x}, t) - A_0(\mathbf{x}, t) \\ &= -\beta \partial_t A_0 - \mathbf{a} \cdot \nabla A_0 - \omega \epsilon_{ij} x_i \nabla_j A_0 - \mathbf{u} \cdot t \nabla A_0 - \mathbf{u} \cdot \mathbf{A} \\ &\quad - \eta (2t \partial_t + \mathbf{x} \cdot \nabla + 2) A_0 + \partial_t \omega \\ &\quad - \gamma (-t^2 \partial_t - t(\mathbf{x} \cdot \nabla + 2)) A_0 + \gamma \mathbf{x} \cdot \mathbf{A}, \end{aligned} \quad (3.5.9)$$

$$\begin{aligned} \delta A_i(\mathbf{x}, t) &= A'_i(\mathbf{x}, t) - A_i(\mathbf{x}, t) \\ &= -\beta \partial_t A_i - \mathbf{a} \cdot \nabla A_i - \omega \epsilon_{ij} x_i \nabla_j A_i - \omega \epsilon_{ij} A_j - \mathbf{u} \cdot t \nabla A_i \\ &\quad - \eta (2t \partial_t + \mathbf{x} \cdot \nabla + 1) A_i + \nabla_i \omega \\ &\quad - \gamma (-t^2 \partial_t - t(\mathbf{x} \cdot \nabla + 1)) A_i. \end{aligned} \quad (3.5.10)$$

Following Noether's theorem a set of conserved charges is associated to the

invariance of the free action under the transformations of the Schrödinger group. They are given by

a. the Hamiltonian

$$H = \int d^2\mathbf{x} \frac{1}{2m} |\mathbf{D}\Psi|^2 + \frac{g^2}{2} |\Psi|^4, \quad (3.5.11)$$

induced by invariance under time translations;

b. the momentum

$$\mathbf{P} = \frac{1}{2} \int d^2\mathbf{x} i\Psi^* \overleftrightarrow{\nabla} \Psi + \sigma A_2 \overleftrightarrow{\nabla} A_1, \quad (3.5.12)$$

following from the invariance under space translations;

c. the angular momentum

$$M = \int d^2\mathbf{x} \epsilon_{ij} x_i [i\Psi^* \nabla_j \Psi + \sigma A_2 \nabla_j A_1] + \frac{\sigma}{2} \mathbf{A}^2. \quad (3.5.13)$$

following from rotation invariance;

d. the Galilean boost operator

$$\mathbf{G} = t\mathbf{P} + m \int d^2\mathbf{x} \mathbf{x} |\Psi|^2, \quad (3.5.14)$$

following from invariance under the Galilei transformations;

e. the scaling operator

$$D = 2tH + \int d^2\mathbf{x} \mathbf{x} \cdot \left[\frac{i}{2} \Psi^* \overleftrightarrow{\nabla} \Psi + \frac{\sigma}{2} A_2 \overleftrightarrow{\nabla} A_1 \right], \quad (3.5.15)$$

induced by the dilatation invariance;

f. the special conformal charge

$$K = t^2 H - tD - \frac{m}{2} \int d^2\mathbf{x} \mathbf{x}^2 |\Psi|^2, \quad (3.5.16)$$

induced by the invariance under the special conformal transformations;

g. the particle number

$$N = \int d^2\mathbf{x} |\Psi|^2, \quad (3.5.17)$$

following from the invariance under $U(1)$ gauge transformations.

Poisson Bracket Formulation

As when discussing the non-linear Schrödinger model, the symmetries of the Jackiw-Pi model can be expressed using Poisson brackets. However, an obstruction for doing this is the fact that there is no term in which the time derivative of the gauge field A_0 appears, like in gauge theories with a Maxwell or Yang-Mills action. In other words, the canonical momentum associated to A_0 is equal to zero. This prevents one from straightforwardly defining a Poisson bracket involving A_0 .

Furthermore, the action of the Jackiw-Pi Model can be rewritten in such a guise by using partial integration that the gauge field A_0 does not appear with derivatives at all,

$$S = \int dt d^2\mathbf{x} \quad i\Psi^* \partial_t \Psi - \frac{1}{2m} |\mathbf{D}\Psi|^2 - \frac{g^2}{2} |\Psi|^4 + \sigma A_2 \partial_t A_1 \quad (3.5.18) \\ + A_0 (e|\Psi|^2 + \sigma B) .$$

It is clear from this expression that the gauge field A_0 is non-dynamical and plays the role of a Lagrange multiplier imposing the constraint

$$\mathcal{X} \equiv e\Psi^* \Psi + \sigma B \simeq 0, \quad (3.5.19)$$

the equation of motion due to A_0 . An original reference for dealing with constraints in a (quantum) theory is given by [42].

Integrating out A_0 , i.e. imposing its equation of motion, the effective action of the Jackiw-Pi model reads

$$S = \int dt d^2\mathbf{x} \quad i\Psi^* \partial_t \Psi - \frac{1}{2m} |\mathbf{D}\Psi|^2 - \frac{g^2}{2} |\Psi|^4 + \sigma A_2 \partial_t A_1, \quad (3.5.20)$$

where the $\mathcal{X} \simeq 0$ constraint must be understood. This action has a residual gauge invariance under the time-independent transformations

$$\Psi \rightarrow e^{ie\omega} \Psi, \quad A_i \rightarrow A_i + \nabla_i \omega. \quad (3.5.21)$$

When the action is gauge transformed in this way, invariance is however only obtained after imposing the $\mathcal{X} \simeq 0$ constraint.

Having integrated out the A_0 gauge field, it is now possible to introduce Poisson brackets. The fields Ψ and A_1 are identified as generalized coordinates and the following canonical momentum variables are introduced:

$$p_\Psi = i\Psi^*, \quad p_{A_1} = \sigma A_2, \quad (3.5.22)$$

with the Poisson brackets defined by

$$\begin{aligned}\{\Psi(\mathbf{x}, t), \Psi^*(\mathbf{y}, t)\} &= -i\delta(\mathbf{x} - \mathbf{y}), \\ \{A_i(\mathbf{x}, t), A_j(\mathbf{y}, t)\} &= \sigma\epsilon_{ij}\delta(\mathbf{x} - \mathbf{y}),\end{aligned}\quad (3.5.23)$$

and thus for general functionals F and G :

$$\{F, G\} = \int \left[-i \frac{\delta F}{\delta \Psi} \frac{\delta G}{\delta \Psi^*} + i \frac{\delta G}{\delta \Psi} \frac{\delta F}{\delta \Psi^*} + \sigma\epsilon_{\alpha\beta} \frac{\delta F}{\delta A_\alpha} \frac{\delta G}{\delta A_\beta} \right]. \quad (3.5.24)$$

With these brackets the infinitesimal transformations of the fields Ψ and A_i can be found, but as before (2.2.19) up to a minus sign: $\delta\Psi = -\{\Psi, F\}$ for the symmetry generated by the conserved charge F , and similarly for A_i .

Using these Poisson brackets, the time evolution of the system is generated by the Hamiltonian (3.5.11) via equation (2.2.21). Having imposed the constraint $\mathcal{X} \simeq 0$, the time evolution of the system has to respect this constraint. If this is not automatically the case, the *primary constraint* $\mathcal{X} \simeq 0$ leads to a *secondary constraint* $\partial_t \mathcal{X} \simeq 0$ to ensure it, and if necessary to yet another constraint to make sure the time evolution of the system respects the secondary constraint as well, and so on. However, in the present case the constraint $\mathcal{X} \simeq 0$ is automatically conserved in time,

$$\partial_t \mathcal{X} = \{\mathcal{X}, H_c\} = 0, \quad (3.5.25)$$

and thus does not imply further constraints. The constraint is therefore also *first class*. A constraint is called first class if its Poisson bracket with all the other constraint vanishes upon imposing the constraints (after calculating the brackets).

Because $\partial_t \mathcal{X} = 0$, it follows directly that for functions f that are time-independent,

$$\Gamma[f] \equiv - \int d^2\mathbf{x} \ f \mathcal{X} \simeq 0 \quad (3.5.26)$$

is a constant of motion. Indeed, $\Gamma[f]$ is conserved in time: $\{\Gamma[f], H\} = 0$, and this relation holds without imposing the constraint. The expression $\Gamma[f]$ plays an interesting role: it can be identified as the generator of the residual gauge transformations. Indeed,

$$\{\Psi, \Gamma[\omega]\} = i\omega\Psi, \quad \{A_i, \Gamma[\omega]\} = \nabla_i \omega. \quad (3.5.27)$$

A gauge invariant quantity can thus be identified by having zero bracket with $\Gamma[\omega]$.

Having integrated out A_0 has as a consequence that some of the symmetries of the Jackiw-Pi model are now only realized upon imposing the $\mathcal{X} \simeq 0$ constraint. The dilatation, space and time translation and rotation symmetries are still valid automatically. However the Galilei, special conformal and gauge symmetries are only valid upon imposing the constraint. This fact reflects itself in the algebra of the conserved quantities and in their time dependence. Denoting equality upon imposing the constraint by $\simeq$, the algebra is now given by

$$\begin{aligned}
\{H, D\} &= 2H, & \{H, \mathbf{G}\} &\simeq \mathbf{P}, \\
\{\mathbf{P}, D\} &= \mathbf{P}, & \{P_i, M\} &= \epsilon_{ij} P_j \\
\{H, K\} &\simeq D, & \{P_i, G_j\} &= m\delta_{ij} N, \\
\{\mathbf{G}, D\} &\simeq -\mathbf{G}, & \{G_i, M\} &= \epsilon_{ij} G_j \\
\{K, D\} &\simeq -2K, & \{K, \mathbf{P}\} &= \mathbf{G}, \\
\{\mathbf{G}, K\} &\simeq 0,
\end{aligned} \tag{3.5.28}$$

all other brackets equal to zero without having to impose the constraint. The particle density N still acts as a central charge. Furthermore, the generator of gauge transformations, $\Gamma[f]$, has the following brackets with the other constants of motion:

$$\begin{aligned}
\{\Gamma[f], H\} &= 0, & \{\Gamma[f], N\} &= 0 \\
\{\Gamma[f], \Gamma[f']\} &= 0, & \{\Gamma[f], P_i\} &= -\Gamma[\nabla_i f] \simeq 0, \\
\{\Gamma[f], G_i\} &= -t\Gamma[\nabla_i f] \simeq 0, & \{\Gamma[f], M\} &= -\Gamma[\epsilon_{ij} x_i \nabla_j f] \simeq 0, \\
\{\Gamma[f], D\} &= -t\Gamma[x_i \nabla_i f] \simeq 0, & \{\Gamma[f], K\} &= -t\Gamma[x_i \nabla_i f] \simeq 0.
\end{aligned} \tag{3.5.29}$$

This shows that only the Hamiltonian and the particle density are directly gauge invariant, the others upon imposing the constraint.

Similarly, using equation (2.2.21) to calculate the time dependence of the various charges shows that H , $\mathbf{P}$, M , D and N are time-independent without imposing the constraint, however $\mathbf{G}$ and K only upon imposing the constraint.

Along these lines it furthermore follows that by defining

$$\begin{aligned}\mathbf{I} &\equiv m \int d^d x \, \mathbf{x} \, \Psi^* \Psi = \mathbf{G} - t\mathbf{P}, \\ I_1 &\equiv \frac{m}{2} \int d^d x \, \mathbf{x}^2 \, \Psi^* \Psi = t^2 H - tD - K, \\ I_2 &\equiv \frac{i}{2} \int d^d x \, \mathbf{x} \cdot \Psi^* \overleftrightarrow{\mathbf{D}} \Psi = D - 2tH,\end{aligned}\tag{3.5.30}$$

as in the non-linear Schrödinger model, it can be shown that

$$\frac{d\mathbf{I}}{dt} \simeq -\mathbf{P}, \quad \frac{dI_1}{dt} \simeq -I_2, \quad \frac{dI_2}{dt} = -2H.\tag{3.5.31}$$

Therefore, as in the non-linear Schrödinger model, the conformal invariance of this model has the implication that if there would be any *static* (i.e. time-independent) solutions of the theory, such that $\mathbf{I}$, I_1 and I_2 are constant themselves, these solutions would necessarily have zero energy and momentum. Contrary to the non-linear Schrödinger model, where this condition was seen to imply that no non-trivial time-independent solutions exist, the Jackiw-Pi model does allow non-trivial time-independent solutions that have zero energy and momentum, as was seen in chapter 3.4.

3.6 Quantization and Renormalization Effects

In this chapter the Jackiw-Pi model is quantized using Feynman's path integral quantization method. This method is particularly convenient when quantizing gauge theories. Finding the propagator of the gauge field can be problematic due to gauge invariance. However, this problem can be circumvented in the path integral formulation where the gauge can be fixed via the Faddeev-Popov method.

Indeed, when the action of the Jackiw-Pi model is Fourier transformed to momentum space, a problem is presented by the kinetic term of the gauge field that represents the inverse of the propagator of the gauge field. Since three gauge fields are present, this kinetic term will be a 3×3 matrix in momentum space. However, due to gauge invariance leading to redundant degrees of freedom, this matrix can not be inverted. Via the method of Faddeev and Popov this problem can be solved by adding a

gauge fixing term to the action [94]. For example, the Coulomb gauge $\nabla \cdot \mathbf{A} = \nabla_i A_i = 0$ can be imposed by adding the term

$$S_{GaugeFix} = \frac{1}{\xi} \int dt d^2\mathbf{x} (\nabla \cdot \mathbf{A})^2. \quad (3.6.1)$$

With this term present the inverse of the kinetic term of the gauge fields in momentum space following from action (3.3.1) with the additional gauge fixing term, is given by

$$\frac{\sigma}{\mathbf{k}^4} \begin{pmatrix} \xi k_0^2 & ik_2 \mathbf{k}^2 + \sigma \xi k_1 k_0 & -ik_1 \mathbf{k}^2 + \sigma \xi k_2 k_0 \\ -ik_2 \mathbf{k}^2 + \sigma \xi k_1 k_0 & \sigma \xi k_1^2 & \sigma \xi k_1 k_2 \\ ik_1 \mathbf{k}^2 + \sigma \xi k_2 k_0 & \sigma \xi k_1 k_2 & \sigma \xi k_2^2 \end{pmatrix} \quad (3.6.2)$$

Specializing to the Landau gauge-fixing condition $\xi = 0$, the only nonvanishing component of the propagator of the gauge field is given by propagation from A_0 to either A_1 or A_2 and vice versa.

In [8] it is explained that there are no real gauge particles in the Jackiw-Pi model since the gauge fields are completely constrained due to the equations of motion (3.3.3)-(3.3.5). However, they can still be treated dynamically in internal lines. Another sign of this fact is that the propagator of the gauge field becomes independent of k_0 in the Coulomb gauge, meaning that the propagator in coordinate space is instantaneous in time.

Fixing the gauge using the Faddeev-Popov method introduces additional Grassmannian fields to the action, the *ghost fields*. However in a $U(1)$ gauge theory these fields do not couple to the matter or gauge fields in Coulomb- or Lorentz-type gauges, and can consequently be ignored.

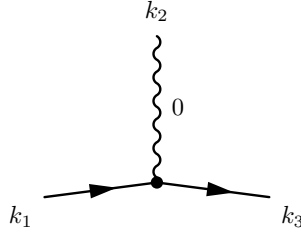
Writing the Feynman rules in momentum space down in agreement with the conventions used in the chapter on the non-relativistic Schrödinger model, it is seen that each Ψ -propagator comes with

$$\text{---}\blacktriangleright_k = \frac{1}{(2\pi)^3} \frac{i}{k_0 - \mathbf{k}^2/2m + i\varepsilon}; \quad (3.6.3)$$

and each $A_0 A_i$ - or $A_i A_0$ -propagator with

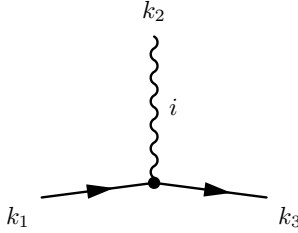
$$\overset{0}{\text{---}}\overset{i}{\text{---}}_k = -\overset{i}{\text{---}}\overset{0}{\text{---}}_k = \frac{\sigma}{(2\pi)^3} \frac{\epsilon^{ij} k_j}{\mathbf{k}^2}. \quad (3.6.4)$$

In the Jackiw-Pi model there are two 3-vertices, one involving the A_0 field and one involving either A_1 or A_2 . The Feynman rule for the first is given by



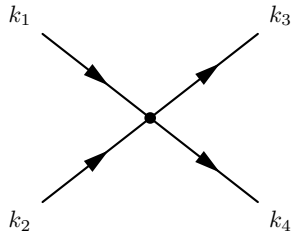
$$= -ie(2\pi)^{d+1}, \quad (3.6.5)$$

and for the latter by



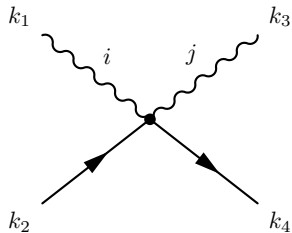
$$= -i\frac{e}{2m}(\mathbf{k}_1 + \mathbf{k}_3)(2\pi)^{d+1}. \quad (3.6.6)$$

Also there are two 4-vertices. Next to the one also present in the non-linear Schrödinger model,



$$= i\frac{g^2}{2}(2\pi)^{d+1}, \quad (3.6.7)$$

one involving twice the gauge field A_1 or A_2 :



$$= i\frac{e^2}{2m}\delta^{ij}(2\pi)^{d+1}. \quad (3.6.8)$$

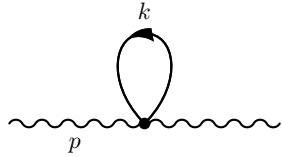
Finally, all vertices naturally come with momentum conservation.

Renormalization Effects

Along the lines set out when dealing with the non-linear Schrödinger model, the perturbative corrections to the tree-level propagators and coupling constants are calculated. However, this time only at 1-loop level. As it turns out, most Feynman diagrams vanish at 1-loop level. This can be traced back to be due to the fact that the propagator of the gauge field does not depend on k_0 and is an odd function of $\mathbf{k}$, equation (3.6.4).

Corrections to the A -propagator

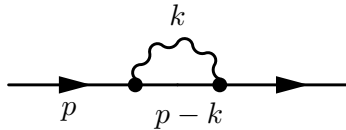
The only diagram that can be written down for the propagator of the gauge field at one loop level is given by


(3.6.9)

This *seagull diagram* is known to vanish, as was discussed when studying the non-linear Schrödinger model. It is thus concluded that the gauge field does not renormalize.

Corrections to the Ψ -propagator

Knowing that the seagull diagram vanishes, the only possible correction to the Ψ propagator comes from the diagram


(3.6.10)

where the photon propagator either runs from A_0 to A_i or vice versa. Since these contributions are equal up to a minus sign, the sum of these two diagrams vanishes. However, on their own the diagrams are also seen to vanish. Indeed, each of the diagrams is proportional to

$$\int dk_0 d^2\mathbf{k} \frac{2p_i - k_i}{p_0 - k_0 - \frac{1}{2m}(\mathbf{p} - \mathbf{k})^2 + i\varepsilon} \frac{\sigma \epsilon^{ij} k_j}{\mathbf{k}^2}. \quad (3.6.11)$$

After performing the k_0 integral, one is left with

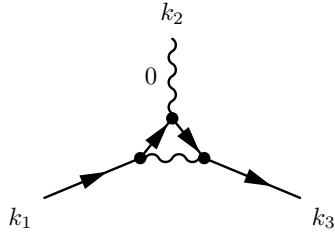
$$\pi i \int d^2 \mathbf{k} \frac{\sigma \epsilon^{ij} (2p_i - k_i) k_j}{\mathbf{k}^2} = 0, \quad (3.6.12)$$

which equals zero because $\epsilon^{ij} k_i k_j = 0$ and the oddness in $\mathbf{k}$ of the remaining term.

It thus follows that neither the Ψ -field nor the mass renormalize at this order in perturbation theory.

Corrections to the A_0 -3-vertex

The only possible diagram is given by


(3.6.13)

where the k_i are the three-momenta $(k_{i,0}, \mathbf{k}_i)$ running through the different external propagator lines.

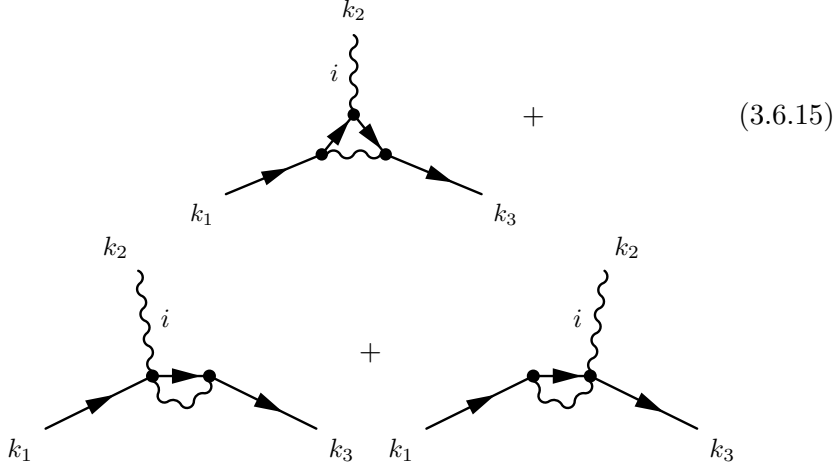
Because the propagator of the gauge field doesn't depend upon k_0 , the integral over k_0 is of the form

$$\int dk_0 \frac{1}{k_0 + \dots - i\varepsilon} \frac{1}{k_0 + \dots - i\varepsilon} = 0. \quad (3.6.14)$$

Performing the integral via closing of the integration contour in the complex plane, there are two poles, both in the upper half plane. Closing the contour in the lower half plane, the integral is directly seen to vanish. This means the whole diagram vanishes and that the coupling constant e of the A_0 -3-vertex, the electric charge, receives no corrections. For consistency this means that also the other 3-vertex and the 4-vertex with the gauge fields should receive no corrections, since they are proportional to e and e^2 respectively.

Corrections to the A_i -3-vertex

And indeed, also the corrections to the 3-vertex involving A_i are seen to vanish. The possible diagrams are given by

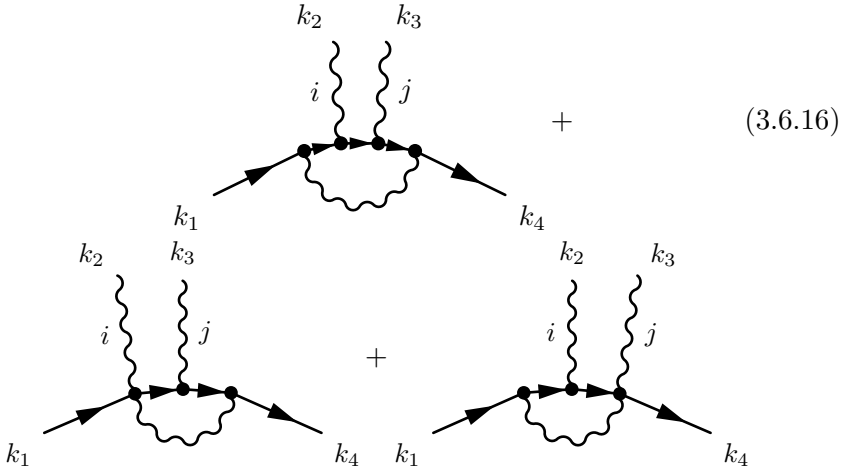


$$(3.6.15)$$

The first vanishes due to the k_0 integral in the complex plane, the second and third because after the k_0 integral an integral odd in $\mathbf{k}$ remains.

Corrections to the $A^2\Psi^2$ -4-vertex

In this case the one loop level diagrams are given by



$$(3.6.16)$$

All three are seen to vanish because the photon propagator does not depend on k_0 .

It is thus indeed consistently concluded from all the vertices that the electric charge e does not renormalize.

Corrections to the Ψ^4 -vertex

The only renormalization effects in the Jackiw-Pi model at one loop level appear when calculating the corrections to the coupling constant g^2 of the Ψ^4 -interaction.

At tree level the contributions to this process are given by

$$(3.6.17)$$

The expression corresponding to this sum of diagrams has been calculated in [8] working in the center of mass frame. Up to the terms that describe the external lines, as in equation (2.5.5), it is in the present conventions given by

$$A^0(p_i, q_j) = i \frac{g^2}{2} - \frac{\sigma e^2}{2m} \cot \theta, \quad (3.6.18)$$

where θ is here defined as the scattering angle between the colliding particles in the plane.

At one loop level the four contributing diagrams are given by

$$(3.6.19)$$

The upper right box diagram represents four different diagrams, depending on where the vertices involving A_0 are placed. The corresponding expressions have been examined in [8] and found to be finite. The total result is given by

$$A_{box}^{(1)} = \frac{e^4 i}{8\pi m} (\ln |2 \sin \theta| - i\pi). \quad (3.6.20)$$

Also the two triangle diagrams have been studied in [8]. In that paper their divergence is controlled via a cut-off in momentum space. Here they are studied using dimensional regularization as was done in chapter 2.

The bottom left triangle diagram reads

$$A_{triangle,1}^{(1)} = (-ie)^2 \left(\frac{ie^2}{2m} \right) \int \frac{dk_0 d^2 \mathbf{k}}{(2\pi)^3} B_{triangle,1} \quad (3.6.21)$$

$$B_{triangle,1} = \frac{\epsilon^{in} k_n \epsilon^{im} (q_2 - p_2 - k)_m}{\mathbf{k}^2 (\mathbf{q}_2 - \mathbf{p}_2 - \mathbf{k})^2} \times \frac{i}{p_{1,0} - k_0 - \frac{1}{2m}(\mathbf{p}_1 - \mathbf{k})^2 + i\varepsilon} \quad (3.6.22)$$

Performing the k_0 integral and writing the integrand into one denominator using Feynman's trick, leads to

$$A_{triangle,1}^{(1)} = \frac{e^4 i}{4m} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \int_0^1 dx \frac{\mathbf{k}^2 - x(1-x)(\mathbf{p}_2 - \mathbf{q}_2)^2}{[\mathbf{k}^2 + x(1-x)(\mathbf{p}_2 - \mathbf{q}_2)^2]^2}. \quad (3.6.23)$$

Using the technique of dimensional regularization, this integral becomes

$$A_{triangle,1}^{(1)} = \frac{e^4 i}{16\pi m} \Lambda^{-\epsilon} \left\{ \frac{2}{\epsilon} - \gamma_E - \frac{1}{4\pi} + 2 - \ln \left(\frac{(\mathbf{p}_2 - \mathbf{q}_2)^2}{4\pi\Lambda^2} \right) \right\}, \quad (3.6.24)$$

where the momentum scale has been extracted. Similarly, the other triangle diagram leads to

$$A_{triangle,2}^{(1)} = \frac{e^4 i}{16\pi m} \Lambda^{-\epsilon} \left\{ \frac{2}{\epsilon} - \gamma_E - \frac{1}{4\pi} + 2 - \ln \left(\frac{(\mathbf{p}_1 - \mathbf{q}_1)^2}{4\pi\Lambda^2} \right) \right\} \quad (3.6.25)$$

Using the previously obtained result (2.5.12) for the remaining diagram, the total result at one loop level is given by

$$\begin{aligned} A^{(1)} = & i \frac{g^2}{2} - \frac{\sigma e^2}{2m} \cot \theta + \frac{e^4 i}{8\pi m} \ln |2 \sin \theta| \\ & - \frac{im\Lambda^{-\epsilon}}{8\pi} \left[g^4 - \frac{e^4}{m^2} \right] \left(\frac{2}{\epsilon} - \gamma_E - i\pi \right) + \frac{im}{8\pi} g^4 \ln \frac{(\mathbf{p}_1 - \mathbf{p}_2)^2}{16\pi\Lambda^2} \\ & - \frac{im}{8\pi} \frac{e^4}{m^2} \left[\frac{1}{4\pi} - 2 + \frac{1}{2} \ln \frac{(\mathbf{p}_1 - \mathbf{q}_1)^2}{4\pi\Lambda^2} + \frac{1}{2} \ln \frac{(\mathbf{p}_2 - \mathbf{q}_2)^2}{4\pi\Lambda^2} \right]. \end{aligned} \quad (3.6.26)$$

This expression is the result of a subtle interplay between terms coming from the different diagrams. For example, the $2/\epsilon$ term comes from the triangle diagrams and the 1-loop diagram from the non-linear Schrödinger model, while the $i\pi$ term comes from this latter diagram and the box diagram, yet both $2/\epsilon$ and $i\pi$ come with the same coefficient.

The 1-loop result for A clearly diverges as $\epsilon \rightarrow 0$. Similar to what was done in the non-linear Schrödinger model, this divergence can be repaired by introducing a renormalized coupling constant:

$$mg_R^2 = mg^2 \Lambda^{-\epsilon} - \frac{m^2}{2\pi\epsilon} \left(g^4 - \frac{e^4}{m^2} \right) \Lambda^{-2\epsilon}, \quad (3.6.27)$$

where mg_R^2 is dimensionless. In terms of this renormalized coupling con-

stant g_R^2 , $A^{(1)}$ is finite and given by

$$\begin{aligned}
 A^{(1)} = & i \frac{g_R^2}{2} - \frac{\sigma e^2}{2m} \cot \theta + \frac{e^4 i}{8\pi m} \ln |2 \sin \theta| \\
 & + \frac{im\Lambda^{-\epsilon}}{8\pi} \left[g_R^4 - \frac{e^4}{m^2} \right] (\gamma_E + i\pi) + \frac{im}{8\pi} g_R^4 \ln \frac{(\mathbf{p}_1 - \mathbf{p}_2)^2}{16\pi\Lambda^2} \\
 & - \frac{im}{8\pi} \frac{e^4}{m^2} \left[\frac{1}{4\pi} - 2 + \frac{1}{2} \ln \frac{(\mathbf{p}_1 - \mathbf{q}_1)^2}{4\pi\Lambda^2} + \frac{1}{2} \ln \frac{(\mathbf{p}_2 - \mathbf{q}_2)^2}{4\pi\Lambda^2} \right],
 \end{aligned} \tag{3.6.28}$$

upon taking the limit $\epsilon \rightarrow 0$.

The Running of the Coupling

The β -function resulting from the renormalization of g^2 is found to be given by

$$\beta(g_R^2) \equiv \Lambda \frac{\partial g_R^2}{\partial \Lambda} = \frac{m}{2\pi} \left(g_R^4 - \frac{e^4}{m^2} \right), \tag{3.6.29}$$

as can be calculated from equation (3.6.27) by taking into account the finite limit of ϵg^2 as $\epsilon \rightarrow 0$.

As in the case of the non-linear Schrödinger model, it is an easy exercise to check that this β -function implies a Λ -dependence of g_R^2 in such a way that A remains Λ -independent:

$$\left[\Lambda \frac{\partial}{\partial \Lambda} + \beta(g_R^2) \frac{d}{dg_R^2} \right] A = 0. \tag{3.6.30}$$

Solving the $\beta(g_R^2)$ -function equation leads to the coupling constant g_R^2 as a function of the momentum scale Λ . Trivial solutions are clearly given by g_R^2 being constantly equal to $\pm e^2/m$:

$$g_R^2(\Lambda) \equiv \pm \frac{e^2}{m}, \tag{3.6.31}$$

which for the minus sign is precisely the condition necessary for the time-independent self-dual solutions (3.4.20). Abbreviating this constant by $\alpha \equiv e^2/m$, another solution of the differential equation is given by

$$g_R^2(\Lambda) = \alpha - \frac{2\alpha(g_R^2(\Lambda') - \alpha)}{g_R^2(\Lambda') - \alpha - (g_R^2(\Lambda') + \alpha) \exp \left[-\frac{\alpha m}{2\pi} \ln(\Lambda/\Lambda')^2 \right]}, \tag{3.6.32}$$

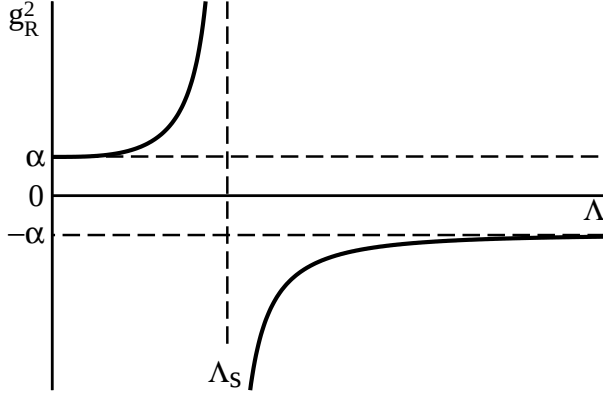


Figure 3.7: Plot of g_R^2 as a function of the momentum scale Λ for $|g_R^2(\Lambda')| > \alpha$.

expressed in terms of g_R^2 at some reference momentum scale Λ' . It has been plotted for various choices of $g_R^2(\Lambda')$ in figures 3.7 and 3.8. In the limit $\alpha \rightarrow 0$, this solution reduces to the solution (2.5.41) found for the β -function of the non-linear Schrödinger model. This is as expected since the Jackiw-Pi model reduces to that model when taking $e = 0$ leading to $\alpha = 0$.

From this solution it is clear that in the limit $\Lambda \rightarrow 0$, g_R^2 becomes equal to α , and furthermore, in the limit $\Lambda \rightarrow \infty$, g_R^2 becomes equal to $-\alpha$:

$$g_R^2(0) = \alpha, \quad g_R^2(\infty) = -\alpha. \quad (3.6.33)$$

When $g_R^2(\Lambda')$ is chosen such that $|g_R^2(\Lambda')| > \alpha$, there is a singularity for

$$\Lambda_s = e^{-\frac{\pi}{m\alpha} \ln[(g_R^2(\Lambda') - \alpha)/(g_R^2(\Lambda') + \alpha)]} \Lambda'. \quad (3.6.34)$$

Also this singularity reduces to the one of the non-linear Schrödinger model in the limit $\alpha \rightarrow 0$. This solution has been plotted in figure 3.7.

From this figure it is clear that when $\alpha \geq 1$ this solution implies that the Jackiw-Pi model never is in the perturbative regime. However, when $\alpha < 1$ there is a perturbative region both for small Λ and for very large Λ .

When $g_R^2(\Lambda')$ is chosen such that $|g_R^2(\Lambda')| < \alpha$, there is a zero in $g_R^2(\Lambda)$ for

$$\tilde{\Lambda}_s = e^{-\frac{\pi}{m\alpha} \ln[-(g_R^2(\Lambda') - \alpha)/(g_R^2(\Lambda') + \alpha)]} \Lambda'. \quad (3.6.35)$$

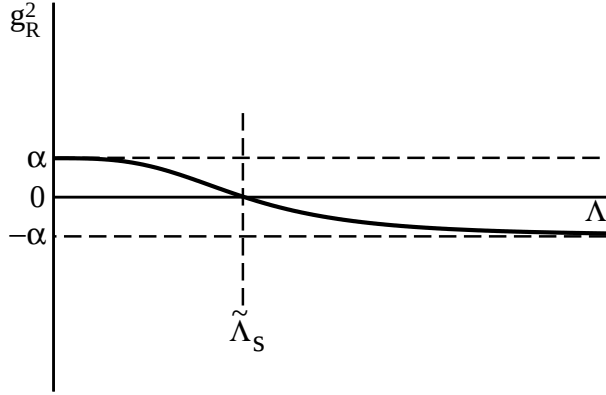


Figure 3.8: Plot of g_R^2 as a function of the momentum scale Λ for $|g_R^2(\Lambda')| < \alpha$

This solution has been plotted in figure 3.8. From this solution it is clear that the model is in the perturbative regime for intermediate values of Λ when $\alpha \geq 1$, respectively for all values of Λ when $\alpha < 1$.

Summarizing, the four possible solutions to the β -function equation are $g_R^2 = \pm\alpha$ and the solutions plotted in figure 3.7 respectively 3.8.

The Fate of the Conformal Symmetries

From the obtained results it is clear that only the coupling constant g^2 renormalizes, unless the coupling constants in the classical model are chosen such that $g^2 = \pm e^2/m$, in which case $g_R^2 = g^2$. In all other cases the renormalized coupling constant g_R^2 becomes a function of the scale Λ and the scale and special conformal symmetries are anomalous. Since only g^2 renormalizes, the argument given in chapter 2 makes it clear that the time-dependence of the corresponding charges D and K is given by

$$\begin{aligned} \frac{dK}{dt} &= -t \frac{dD}{dt}; \\ \frac{dD}{dt} &= \frac{1}{2} \beta(g_R^2) \int d^2x \Phi^{\dagger 2} \Phi^2 = \frac{m}{4\pi} \left(g_R^4 - \frac{e^4}{m^2} \right) \int d^2x \Phi^{\dagger 2} \Phi^2. \end{aligned} \quad (3.6.36)$$

From this expression it follows that the conformal symmetries indeed survive if and only if the condition $g^2 = g_R^2 = \pm e^2/m$ is satisfied.

3.7 Conclusion

It has been shown that at first order level in perturbation theory only the coupling constant g^2 renormalizes: neither the fields, nor the mass or electric charge renormalize. Furthermore, the renormalization effects of g^2 break the conformal symmetries of the Jackiw-Pi model in the same way as occurred in the non-linear Schrödinger model. However, when classically the coupling constants are fixed to satisfy the condition $mg^2 = \pm e^2$, also the coupling constant g^2 does not renormalize, and the conformal symmetries are seen to survive the quantization. It so happens that for the minus sign this condition is exactly the one the coupling constants need to satisfy such that the classical self-dual solutions exist (3.4.3, 3.4.20). In this special case, thus not only the conformal symmetries survive quantization, but also the classical self-dual solutions.

Assuming $\alpha = e^2/m$ to be smaller than one, and choosing g_R^2 smaller than α at some reference energy scale Λ' , the Jackiw-Pi model is seen to have the unusual property that the model is in the perturbative regime for all values of Λ , as shown in figure 3.8. For small values of Λ g_R^2 is seen to be positive, signifying a repulsive interaction between the particles, and for large values of Λ g_R^2 is negative, signifying an attractive interaction between the particles.

When in the case α smaller than one, g_R^2 is chosen larger than α at the reference energy scale, the model is in a perturbative regime either for small values of Λ or for very large values, as shown in figure 3.7. When α is larger than one, there only is a perturbative regime for intermediate values of Λ when g_R^2 is chosen smaller than α at the reference energy scale, shown in figure 3.8.

As a final remark, the classical self-dual solutions of the Jackiw-Pi model have been shown to have unusual electric and magnetic properties due to the Chern-Simons kinetic term for the gauge fields. The magnetic field is proportional to the matter density, and the electric field is proportional to the gradient of the matter density. These solutions furthermore also contain vortices in the gauge field.

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SAMENVATTING

De fysica van elementaire deeltjes, maar ook verschijnselen in de fysica van gecondenseerde materie zoals supergeleiding, worden in de theoretische natuurkunde beschreven met behulp van *quantumveldentheorie*. Symmetrie, ofwel de invariantie van een natuurkundig model onder een transformatie, is binnen de moderne veldentheorie een van de belangrijkste concepten [70, 94, 110, 111]. Verschillende soorten symmetrieën worden onderscheiden.

Een soort zijn de *interne symmetrieën*, zoals bijvoorbeeld ijksymmetrie die fysica beschrijft van deeltjes die in meerdere smaken voorkomen. De fysica van de leptonische deeltjes, zoals het elektron, muon, tauon en de neutrino's wordt beschreven door middel van een $SU(2) \times U(1)$ -ijksymmetrie die de *electromagnetische* en *zwakke wisselwerkingen* unificeert. De krachten in deze *elektrozwakke wisselwerking* worden overgedragen door middel van de W - en Z -bosonen en het foton. De quarks transformeren ook onder deze symmetrie, maar ook onder een $SU(3)$ -ijksymmetrie. De krachten in deze *sterke wisselwerking* worden overgedragen door middel van de gluonen.

Bij energieën die laag genoeg zijn is de fysica van de elementaire deeltjes niet meer invariant onder de volledige $SU(2) \times U(1)$ -ijksymmetrie, maar slechts onder een $U(1)$ -ijksymmetrie. Deze $U(1)$ -symmetrie beschrijft electromagnetisme. De energieschaal waarop de breking van deze symmetrie plaatsvindt, wordt de *elektrozwakke brekingsschaal* genoemd. Deze symmetriebreking wordt beschreven door middel van het *Higgs-mechanisme* waarin er een nieuwe veld aan het *Standaard Model* van de deeltjesfysica wordt toegevoegd, namelijk het *Higgs-veld*. De energiepotentiaal van dit veld is bij energieën onder de elektrozwakke brekingsschaal slechts invariant onder $U(1)$ -transformaties en dit leidt tot de genoemde breking van de $SU(2) \times U(1)$ -symmetrie. Dit mechanisme, ook bekend als *spontane symmetrie-breking*, staat in contrast met de *dynami-*

sche symmetrie-breking die later in deze samenvatting aan de orde komt.

Een andere soort symmetrie wordt gevormd door de *ruimte-tijd symmetrieën*. Deze combineren transformaties van de coördinaten van de ruimte-tijd met transformaties op de velden in het model op een zodanige wijze dat het model als geheel invariant blijft. Aan natuurkundige modellen wordt bijvoorbeeld over het algemeen de eis gesteld dat ze invariant zijn onder ruimte- en tijd-translaties, hetgeen betekent dat de natuurkunde beschreven in dat model onafhankelijk is van waar en wanneer een experiment uitgevoerd wordt. Op een vergelijkbare wijze betekent rotatie-invariantie dat er geen voorkeursrichting in de ruimte is. Verder volgt uit Einsteins principe dat alle inertiaalstelsels fysisch equivalent zijn, dat relativistische theorieën invariant moeten zijn onder Lorentz-transformaties. Voor niet-relativistische theorieën geldt dat deze onder Galileï-transformaties invariant moeten zijn.

Een interessante klasse van de ruimte-tijd symmetrieën zijn de *conforme symmetrieën*. Hieronder valt ook de *schaalsymmetrie* van een model. Deze symmetrie betekent dat onder een herschaling van de ruimte- en tijd-coördinaten een model invariant is. Aangezien het inzoomen op de ruimte-tijd equivalent is met het bestuderen van een model bij hogere energieën, betekent dit dat de fysica van een schaalinvariant model onafhankelijk is van de energie-schaal waarop men het model bestudeert. In het algemeen geldt dat modellen die schaalinvariant zijn in een klassieke beschrijving, zoals bijvoorbeeld de sterke wisselwerking met massaloze quarks, deze invariantie verliezen als het model gequantiseerd wordt. Dit gebeurt via *renormalisatie*-effecten. Dit houdt het volgende in. Als er oneindigheden opduiken als in storingstheorie correcties op de velden, massa's en koppelingsconstanten worden berekend, dan worden die als volgt behandeld: er wordt aangenomen dat de originele velden, massa's en koppelingsconstanten degenen zijn die oneindig zijn, en wel op zodanige wijze dat de gecorrigeerde velden, massa's en koppelingconstanten eindig zijn. Als het mogelijk is dit te doen, wordt het model renormaliseerbaar genoemd. Via de *renormalisatiegroep-vergelijkingen* volgt nu dat de velden, massa's en koppelingsconstanten afhankelijk worden van de energieschaal [16, 94, 105, 120], wat een eventuele klassieke schaalinvariantie breekt. Omdat het toepassen van storingstheorie vereist dat de koppelingsconstanten in het model relatief klein zijn, volgt hieruit dat de breking van de schaalinvariantie restricties legt op de energieën waarbij

het model in storingstheorie toegepast kan worden.

Deze breking van de schaalinvariantie is een voorbeeld van een dynamische breking van een symmetrie door quantummechanische effecten. Als een symmetrie in een klassiek model door quantisatie gebroken wordt, dan wordt deze symmetrie een *anomalie* genoemd.

In dit proefschrift wordt precies zo'n effect waargenomen in het niet-relativistische Schrödinger-model, zoals gepubliceerd in [38], en in het Jackiw-Pi-model. In hoofdstuk 2 van dit proefschrift worden de symmetriën van het niet-relativistische Schrödinger-model bestudeerd. Dit model leeft in d ruimte-dimensies en in een tijd-dimensie. Zonder interacties is het model zowel klassiek als quantummechanisch invariant onder de Schrödinger-groep van translaties, rotaties, Galileï-boosts, schaal- en speciale conforme transformaties. Als er interacties aan het model toegevoegd worden, dan behoudt het klassieke model de invariantie onder de Schrödinger-groep, mits het een zespunts-interactie betreft in een ruimte-dimensie, en een vierpunts-interactie in twee ruimte-dimensies. In alle andere gevallen gaan de schaal- en speciale conforme invarianties verloren.

Van het Schrödinger-model met de vierpunts-interactie in twee ruimte-dimensies zijn in dit proefschrift na quantisatie de renormalisatie-effecten in storingstheorie tot alle ordes berekend. De velden en massa in dit model blijken niet te renormeren, maar de koppelingsconstante g^2 blijkt dit wel te doen. Ten gevolge hiervan wordt de koppelingsconstante een functie van de energieschaal. Dit breekt niet alleen de schaalsymmetrie op de hiervoor uitgelegde wijze, maar ook de speciale conforme symmetrie. Door de afhankelijk van de koppelingsconstante g^2 van de energieschaal expliciet uit te rekenen, is ten slotte aangetoond dat alleen bij kleine en bij hele grote energieën het model in storingstheorie toepasbaar is.

In hoofdstuk 3 is een vergelijkbare studie uitgevoerd in het geval van het Jackiw-Pi model. Het Jackiw-Pi model is een uitbreiding van het Schrödinger-model met vierpunts-interactie in twee ruimte- en een tijd-dimensie met Chern-Simons ijkvelden. Het klassieke model is naast ijk-invariant net als het Schrödinger-model invariant onder de Schrödinger-groep. Door het berekenen van renormalisatie-effecten tot op eerste orde in storingstheorie blijkt dat na quantisatie van het Jackiw-Pi model de materie- en ijkvelden, massa m , en de elektrische lading e waarmee de materie- en ijkvelden aan elkaar koppelen, niet renormeren. Opnieuw blijkt alleen de koppelingsconstante g^2 van de vierpunts-interactie te renorme-

ren, behalve als in het klassieke model aan de conditie $mg^2 = \pm e^2$ voldaan is. Hieruit volgt dat, zoals in het niet-lineaire Schrödinger-model, hierdoor de schaal- en speciale conforme invarianties breken, tenzij dus $mg^2 = \pm e^2$.

Door verder de expliciete afhankelijkheid van g^2 van de energieschaal te bepalen, is nagegaan bij welke energieschalen het Jackiw-Pi model in storingstheorie toepasbaar is. Dit blijkt sterk afhankelijk te zijn van de specifieke fysische situatie, zoals besproken in hoofdstuk 3.7.

Daarnaast heeft het klassieke Jackiw-Pi model tijdsafhankelijke en zelf-duale oplossingen, mits aan de conditie $mg^2 = -e^2$ is voldaan. In dit geval overleven dus niet alleen de schaal- en speciale conforme symmetrieën de quantisatie, maar ook deze oplossingen. De elektrische en magnetische eigenschappen van deze oplossingen, die vortices in het ijk-veld blijken te hebben, zijn in detail besproken in hoofdstuk 3.4.

Een derde soort symmetrie wordt gevormd door de *supersymmetrieën*. Supersymmetrie is zowel een vorm van interne symmetrie als van ruimte-tijd symmetrie. Het is een interne symmetrie omdat supersymmetrie in essentie een symmetrie is tussen bosonische en fermionische deeltjes waarbij elk bosonisch deeltje een fermionische partner krijgt en vice versa. Tegelijkertijd kan supersymmetrie gezien worden als de enige mogelijk niet-triviale uitbreiding van de translatie-, rotatie- plus Lorentz- of Galileï-symmetrieën [27].

Omdat supersymmetrie tot nu toe niet waargenomen is in de natuur, is het duidelijk dat supersymmetrie alleen kan bestaan bij energieën groter dan die waarbij tot nu toe experimenten zijn uitgevoerd. Als supersymmetrie boven een bepaalde energieschaal blijkt te bestaan, dan helpt dit bij het oplossen van vele theoretische problemen in de hoge-energie fysica [10, 70, 112, 113, 116]. Omdat supersymmetrische theorieën veel minder last hebben van de quantummechanische renormalisatie-effecten die bijvoorbeeld schaalsymmetrieën breken, zijn deze theorieën veel beter in staat om bepaalde vraagstukken aan te pakken. Zoals bijvoorbeeld het vraagstuk van de kleine waarde van de kosmologische constante; het hiërarchie-probleem van het grote gat tussen de schaal waarop de elektromagnetische en zwakke wisselwerkingen unificeren en de schaal waarop verwacht wordt dat dit met de sterke en elektrozwakke wisselwerkingen gebeurt; of de renormalisatie van quantummechanische modellen van de zwaartekracht.

Ervan uitgaande dat supersymmetrie in quantumveldentheorieën van

elementaire deeltjes bij voldoende hoge energie bestaat, maakt het noodzakelijk dat er een bepaald mechanisme is dat de supersymmetrie breekt. Deze breking kan bijvoorbeeld bewerkstelligd worden door expliciete brekingstermen in het model te introduceren. Echter, dit komt erop neer dat de massa's van de deeltjes en hun supersymmetrische partners met de hand geïntroduceerd worden in plaats van dat het model ze voorspelt. In principe wil men een dynamisch quantummechanisme om de supersymmetrie te breken, zoals bij de breking van de schaalsymmetrie. In het geval van supersymmetrische modellen in vier dimensies is er een theorema dat zegt dat supersymmetrie niet gebroken kan worden door quantummechanische effecten in storingstheorie als de supersymmetrie niet al direct gebroken wordt bij quantisatie [54]. Men is dus geïnteresseerd in niet-perturbatieve mechanismen om supersymmetrie te breken. Het bestuderen van een theorie zonder gebruik te maken van storingstheorie kan systematisch gedaan worden door gebruik te maken van computersimulaties. Omdat computers maar een eindige hoeveelheid data kunnen verwerken, is het hiervoor noodzakelijk om het model op een discreet ruimte-tijd-rooster te definiëren. De velden leven nu of op de roosterpunten, of op de links die deze roosterpunten verbinden. Dit verklaart de interesse die men heeft in rooster-supersymmetrie. Helaas is het erg moeilijk om supersymmetrie op een rooster te definiëren, omdat de discretisatie van de ruimte-tijd direct de supersymmetrie breekt [43, 45, 49, 57, 59, 76].

In hoofdstuk 1 van dit proefschrift worden twee methodes van aanpak van de problematiek van rooster-supersymmetrie in detail bestudeerd, de *niet-commutativiteits-methode* [32–34], en de *link-methode* [35–37]. Het grote probleem bij het definiëren van supersymmetrische veldentheorieën op een rooster komt neer op het feit dat op een rooster de produktregel voor differentiëren een alternatieve vorm aanneemt. De niet-commutativiteits-methode weet deze produktregel voor essentiële operatoren te repareren door een niet-commutativiteit te introduceren tussen de ruimte-tijd-coördinaten en de parameters die de supersymmetrie beschrijven. Op het eerste gezicht lijkt het erop dat inderdaad op deze wijze de volledige supersymmetrie op het rooster behouden blijft. Deze methode leidt er echter toe dat het voor de supersymmetrie-transformaties verschil maakt in welke volgorde het product van verschillende velden genomen wordt, terwijl het resultaat van dit product zelf niet verandert als de volgorde verandert wordt. Hierdoor zijn de supersymmetrie-transformaties ambigu gedefinieerd: de niet-commutativiteits-methode is inconsistent [14, 15].

De claim van de andere methode, de link-methode, is door op ingenieuze wijze de verschillende velden hetzij op de roosterpunten hetzij op de links tussen de roosterpunten te definiëren, de volledige supersymmetrie van een model op het rooster te behouden. Deze methode die slechts voor een paar modellen volledig ontwikkeld is, blijkt echter aan een zelfde inconsistentie te lijden [13]. Als een product van velden die op links tussen de roosterpunten leven, een lus langs de desbetreffende roosterpunten vormt, dan is de supersymmetrie-transformatie van dit product op ambigue wijze gedefinieerd. Echter, voor bepaalde keuzes van parameters in het model kan wel een deel van de supersymmetrie met deze methode op het rooster gedefinieerd worden, vergelijkbaar met wat door anderen bereikt is [17, 18, 24, 25, 76, 77].

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CURRICULUM VITAE

Mark Okker de Kok was born on Wednesday 4th October 1978 in Haarlem, The Netherlands. He attended Gymnasium (Dutch secondary school) at the Herbert Vischers College in Nieuw Vennep, where he graduated in 1997. He continued by studying physics and mathematics at Leiden University from 1997 until 2003, when he obtained both his M.Sc. degrees ‘cum laude’. During his studies he specialized in the field of elementary particle physics and cosmology, studying not only their experimental but also their theoretical and mathematical aspects. Both at Utrecht University and as a participant in the 2001 CERN Summer Student Program at the European Organization for Nuclear Research (CERN) in Geneva, he performed experimental research into elementary particle physics. In 2002 he wrote his M.Sc. thesis in mathematics in the field of differential geometry under supervision of prof. dr. H. Geiges at the University of Cologne. His M.Sc. thesis in theoretical high energy physics was written under the supervision of prof. dr. P.J. van Baal at the Instituut-Lorentz for Theoretical Physics of Leiden University in 2003.

During the period 2004-2008 he worked on his Ph.D. research under the shared supervision of prof. dr. P.J. van Baal and prof. dr. J.W. van Holten of VU University and the Dutch National Institute for Subatomic Physics (NIKHEF) in Amsterdam; also under the supervision of Dr. F. Bruckmann of, initially Leiden University, and later Regensburg University. The results of this research are described in this thesis.

In the autumn of 2007, the author was one of the participants of the Dutch National ThinkTank and wrote a report on the Dutch educational system under the supervision of McKinsey & Company.

ACKNOWLEDGMENTS

It goes without saying that without the support of many people this thesis would never have seen the light of day.

First of all I would like to express my gratitude to Peter Bongaarts for taking up the role of mentor at times when such assistance was crucially needed.

During the research I very much enjoyed and benefited from scientific discussions with numerous people, including Ana Achúcarro, Falk Bruckmann, Simon Catterall, Mboyo Esole, Peter Horváthy, Ernst-Michael Ilgenfritz, Roman Jackiw, Kareljan Schoutens, Jack Smith, Kepa Sousa, Gerben Stavenga, Pierre van Baal and Jan-Willem van Holten.

Furthermore, I would like to thank Michel Heijdra and Hans de Haan for being paranimf: it is a great honour for me that two of my closest friends and fellow physics students will be standing next to me at the defence.

And finally I would like to thank the many friends and family that supported me throughout the years that I worked on this project.

Mark de Kok
Leiden, March 2008

