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Chapter 3

A Wiener Lemma for the discrete Heisenberg group

Invertibility criteria and applications to algebraic dynamics[1]

Abstract

We present a Wiener Lemma for the group algebra $\ell^1(\mathbb{H}, \mathbb{C})$ and group C^*-algebra $C^*(\mathbb{H})$ of the discrete Heisenberg group $\mathbb{H}$. In particular, we reduce the problem of deciding on invertibility in these Banach algebras to the study of invertibility of associated elements in rotation algebras.

Moreover, we will apply Wiener's Lemma to decide on expansiveness of certain principal algebraic actions of $\mathbb{H}$.

The proof of Wiener's Lemma for $\mathbb{H}$ can be generalised to countable nilpotent groups Γ. We will analyse invertibility in $\ell^1(\Gamma, \mathbb{C})$ and $C^*(\Gamma)$ and explain the important role of monomial representations for these investigations.

[1] **This chapter is based on:**
M. Göll, K. Schmidt and E. Verbitskiy, *A Wiener Lemma for the discrete Heisenberg group: Invertibility criteria and applications to algebraic dynamics*, (2014).

3.1 Introduction

Let Γ be a countably infinite discrete group. The aim of this article is to find a verifiable criterion – a Wiener Lemma – for invertibility in the group algebra

$$\ell^1(\Gamma, \mathbb{C}) := \left\{ (f_\gamma)_{\gamma \in \Gamma} \ : \ \sum_{\gamma \in \Gamma} |f_\gamma| < \infty \right\},$$

in particular for the case where Γ is the discrete Heisenberg group $\mathbb{H}$.

Our main motivation to study this problem is an application in the field of algebraic dynamics which we introduce first. An *algebraic* Γ*-action* is a homomorphism $\alpha \colon \Gamma \longrightarrow \mathrm{Aut}(X)$ from Γ to the group of automorphisms of a compact metrisable abelian group X [29].

We are especially interested in *principal actions* which are defined as follows. Let f be an element in the *integer group ring* $\mathbb{Z}[\Gamma]$, i.e., the ring of functions $\Gamma \longrightarrow \mathbb{Z}$ with finite support. The Pontryagin dual of the discrete abelian group $\mathbb{Z}[\Gamma]/\,\mathbb{Z}[\Gamma]f$ will be denoted by $X_f \subseteq \mathbb{T}^\Gamma$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ (which will be identified with the unit interval $(0,1]$). Pontryagin's duality theory of locally compact abelian groups tells us that X_f can be identified with the annihilator of the principal left ideal $\mathbb{Z}[\Gamma]f$, i.e.,

$$X_f = (\mathbb{Z}[\Gamma]f)^\perp = \left\{ x \in \mathbb{T}^\Gamma \ : \ \sum_{\gamma \in \Gamma} f_\gamma x_{\gamma'\gamma} = 0 \text{ for every } \gamma' \in \Gamma \right\}. \tag{3.1.1}$$

The *left shift-action* λ on $\mathbb{T}^\Gamma$ is defined by $(\lambda^\gamma x)_{\gamma'} = x_{\gamma^{-1}\gamma'}$ for every $x \in \mathbb{T}^\Gamma$ and $\gamma, \gamma' \in \Gamma$. Denote by α_f the restriction of λ on $\mathbb{T}^\Gamma$ to X_f. The pair (X_f, α_f) forms an algebraic dynamical system which we call *principal* Γ*-action* – since it is defined by a principal ideal (cf. (3.1.1)).

Since a principal Γ-action (X_f, α_f) is completely determined by an element $f \in \mathbb{Z}[\Gamma]$, one should be able to express its dynamical and topological properties in terms of properties of f. Expansiveness is such a dynamical property which allows a nice algebraic interpretation. Let (X, α) be an algebraic dynamical system and d a translation invariant metric on X. The Γ-action α is *expansive* if there exists a constant $\varepsilon > 0$ such that

$$\sup_{\gamma \in \Gamma} d(\alpha^\gamma x, \alpha^\gamma y) > \varepsilon\,,$$

for all pairs of distinct elements $x, y \in X$. We know from [7, Theorem 3.2] that (X_f, α_f) is expansive if and only if f is invertible in $\ell^1(\Gamma, \mathbb{R})$. This result was proven already in the special cases $\Gamma = \mathbb{Z}^d$ and for groups Γ which are nilpotent in [29] and in [8], respectively. Although, this result is a complete characterisation of expansiveness, it is general hard to check whether f is invertible in $\ell^1(\Gamma, \mathbb{R})$ or not.

3.1.1 Outline of the article

In Section 3.2 we will recall known criteria for invertibility in symmetric Banach algebras $\mathcal{A}$. In particular, we will discuss the equivalence of the existence of an inverse a^{-1} of $a \in \mathcal{A}$, and the invertibility of the operators $\pi(a)$ in the C^*-algebra of bounded operators on the representation space $\mathcal{H}_\pi$ for every unitary representations π (up to unitary equivalence) of $\mathcal{A}$. Moreover, we will describe the dual of $\mathbb{H}$ (cf. Section 3.2 for a definition) and discuss how its complicated structure makes the study of invertibility in the group algebras of $\mathbb{H}$ so tremendously difficult.

Theorem 3.2.11 is the main result of this paper which will allow us to restrict the attention to certain 'nice' canonical irreducible representations for questions concerning invertibility in the group algebra of the discrete Heisenberg group $\mathbb{H}$.

The proof of Theorem 3.2.11 can be found in the Sections 3.3 and 3.4.

In Section 3.5 we will generalise Theorem 3.2.11 to countable discrete nilpotent groups.

The Sections 3.6 and 3.8 contain applications of Theorem 3.2.11, in particular:

- invertibility of $f \in \mathbb{Z}[\mathbb{H}]$ in $\ell^1(\mathbb{H}, \mathbb{R})$ can be verified with the help of the finite-dimensional irreducible unitary representations of $\mathbb{H}$;

- criteria for non-invertibility for 'linear' elements in $f \in \mathbb{Z}[\mathbb{H}]$.

In Section 3.7 we will explore a connection to Time-Frequency Analysis and give an alternative proof of Wiener's Lemma for twisted convolution algebras, which only uses the representation theory of $\mathbb{H}$. Theorem 3.7.4 – which is based on a result of Linnell (cf. [22]) – gives a full description of the spectrum of the operators $\pi(f)$ acting on $L^2(\mathbb{R}, \mathbb{C})$, where π is a Stone-von Neumann representation (cf. (3.7.1) for a definition) and $f \in \mathbb{Z}[\mathbb{H}]$.

3.2 Wiener's Lemma

We start our discussion with a survey of Wiener's Lemma in its classical form. Let us denote by $\mathcal{A}(\mathbb{T})$ the Banach algebra of functions with absolutely convergent Fourier series on $\mathbb{T}$.

Theorem 3.2.1 (Wiener's Lemma)**.** *An element $f \in \mathcal{A}(\mathbb{T})$ is invertible, i.e. $1/f \in \mathcal{A}(\mathbb{T})$, if and only if $f(s) \neq 0$ for all $s \in \mathbb{T}$.*

This result appears in a seemingly unrelated context. The convolution algebra $\ell^1(\mathbb{Z}, \mathbb{C})$ is isomorphic to $\mathcal{A}(\mathbb{T})$ and hence $\ell^1(\mathbb{Z}, \mathbb{C})$ can be embedded in $C(\mathbb{T}, \mathbb{C})$ in a natural way. Let $f \in \ell^1(\mathbb{Z}, \mathbb{C})$ and $C_f h = f * h$ be the convolution operator on $\ell^2(\mathbb{Z}, \mathbb{C})$. Then the following holds:

Theorem 3.2.2. *If $f \in \ell^1(\mathbb{Z}, \mathbb{C})$, then the convolution operator C_f is invertible on $\ell^2(\mathbb{Z}, \mathbb{C})$ if and only if f is invertible in $\ell^1(\mathbb{Z}, \mathbb{C})$. Moreover, if f is invertible, then C_f is invertible on all ℓ^p-spaces with $1 \leq p \leq \infty$.*

Before we start our review of more general results let us mention the concept of *inverse-closedness* which originates from Wiener's Lemma as well. The fact that $f \in \mathcal{A}(\mathbb{T})$ is invertible in $\mathcal{A}(\mathbb{T})$ if and only if f is invertible in the larger Banach algebra of continuous functions $C(\mathbb{T}, \mathbb{C})$ leads to the question: *for which pairs of nested unital Banach algebras $\mathcal{A}, \mathcal{B}$ with $\mathcal{A} \subseteq \mathcal{B}$ and with the same multiplicative identity element does the following implication hold:*

$$a \in \mathcal{A} \quad \text{and} \quad a^{-1} \in \mathcal{B} \implies a^{-1} \in \mathcal{A}\,. \tag{3.2.1}$$

In the literature a pair of Banach algebras which fulfils (3.2.1) is called a *Wiener pair*.

Wiener's Lemma was the starting point of Gelfand's theory of commutative Banach algebras. Gelfand's theory links the question of invertibility in a commutative Banach algebra $\mathcal{A}$ to the study of its irreducible representations and the compact space of maximal ideals $\mathrm{Max}(\mathcal{A})$. We collect in the following theorem several criteria for invertibility in unital commutative Banach algebras.

Theorem 3.2.3 (cf. [10]). *Suppose $\mathcal{A}$ is a unital commutative Banach algebra. The set of irreducible representations of $\mathcal{A}$ is isomorphic to the set of multiplicative functionals, and isomorphic to the compact space of maximal ideals* $\mathrm{Max}(\mathcal{A})$. *Furthermore, the following statements are equivalent*

1. *$a \in \mathcal{A}$ is invertible;*
2. *$a \notin m$ for all $m \in \mathrm{Max}(\mathcal{A})$;*
3. *$\Phi_m(a)$ is invertible in $\mathcal{A}/m$ for all $m \in \mathrm{Max}(\mathcal{A})$, where $\Phi_m \colon \mathcal{A} \longrightarrow \mathcal{A}/m \cong \mathbb{C}$ is the canonical projection map;*
4. *$\Phi_m(a) \neq 0$ for all $m \in \mathrm{Max}(\mathcal{A})$;*
5. *$\pi(a)v \neq 0$ for every one-dimensional irreducible representation π of $\mathcal{A}$ and $v \in \mathbb{C} \smallsetminus \{0\}$.*

In this article we concentrate on the harmonic analysis of rings associated with a countably infinite group Γ furnished with the discrete topology. Beside $\mathbb{Z}[\Gamma]$ and $\ell^1(\Gamma, \mathbb{C})$ we are interested in $C^*(\Gamma)$, the *group-C^*-algebra* of Γ, i.e., the enveloping C^*-algebra of $\ell^1(\Gamma, \mathbb{C})$.

We write a typical element $f \in \ell^\infty(\Gamma, \mathbb{C})$ as a formal sum $\sum_{\gamma \in \Gamma} f_\gamma \cdot \gamma$, where $f_\gamma = f(\gamma)$. The *involution* $f \mapsto f^*$ is defined by $f^* = \sum_{\gamma \in \Gamma} \bar{f}_{\gamma^{-1}} \cdot \gamma$. The product of $f \in$

$\ell^1(\Gamma,\mathbb{C})$ and $g\in\ell^\infty(\Gamma,\mathbb{C})$ is given by the *convolution*

$$fg=\sum_{\gamma,\gamma'\in\Gamma} f_\gamma g_{\gamma'}\cdot\gamma\gamma'=\sum_{\gamma,\gamma'\in\Gamma} f_\gamma g_{\gamma^{-1}\gamma'}\cdot\gamma'\,. \tag{3.2.2}$$

Let $\mathcal{A}$ be a Banach algebra with multiplicative identity element $1_{\mathcal{A}}$. The *spectrum* of $a\in\mathcal{A}$ is the set of elements $c\in\mathbb{C}$ such that $a-c1_{\mathcal{A}}$ is not invertible in $\mathcal{A}$ and will be denoted by $\sigma(a)$.

3.2.1 Representation theory

We recall at this point some relevant definitions and results from representation theory, which will be used later. Moreover, we will recall results for symmetric Banach-*-algebras which are in the spirit of Wiener's Lemma.

Unitary Representations

Let $\mathcal{H}$ be a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. We denote by $\mathcal{B}(\mathcal{H})$ the algebra of bounded linear operators on $\mathcal{H}$, furnished with the strong operator topology. Further, denote by $\mathcal{U}(\mathcal{H})\subset\mathcal{B}(\mathcal{H})$ the group of unitary operators on $\mathcal{H}$. If Γ is a countable group, a *unitary representation* π of Γ is a homomorphism $\gamma\mapsto\pi(\gamma)$ from Γ into $\mathcal{U}(\mathcal{H})$ for some complex Hilbert space $\mathcal{H}$. Every unitary representation π of Γ extends to a $*$-representation of $\ell^1(\Gamma,\mathbb{C})$, which is again denoted by π, and which is given by the formula $\pi(f)=\sum_{\gamma\in\Gamma}f_\gamma\pi(\gamma)$ for $f=\sum_{\gamma\in\Gamma}f_\gamma\cdot\gamma\in\ell^1(\Gamma,\mathbb{C})$. Clearly, $\pi(f^*)=\pi(f)^*$. The following theorem was probably first published in [11] but we refer to [26, Theorem 12.4.1].

Theorem 3.2.4. *Let Γ be a discrete group. Then there are bijections between*

- *the class of unitary representations of Γ;*
- *the class of non-degenerate[2] *-representations of $\ell^1(\Gamma,\mathbb{C})$;*
- *the class of non-degenerate *-representations of $C^*(\Gamma)$.*

Moreover, these bijections respect unitary equivalence and irreducibility.

Hence the representation theories of Γ, $\ell^1(\Gamma,\mathbb{C})$ and $C^*(\Gamma)$ coincide. In consideration of this result we will use the same symbol for a unitary representation of Γ and its corresponding *-representations of the group algebras $\ell^1(\Gamma,\mathbb{C})$ and $C^*(\Gamma)$.

[2] A representation π of a Banach *-algebra $\mathcal{A}$ is called *non-degenerate* if there is no non-zero vector $v\in\mathcal{H}_\pi$ such that $\pi(a)v=0$ for every $a\in\mathcal{A}$.

States and the GNS construction

Suppose that $\mathcal{A}$ is a unital C^*-algebra. A positive linear functional $\phi\colon \mathcal{A} \longrightarrow \mathbb{C}$ is a *state* if $\phi(1_{\mathcal{A}}) = 1$. We denote by $\mathcal{S}(\mathcal{A})$ the space of states of $\mathcal{A}$, which is a weak*-compact convex subset of the dual space of $\mathcal{A}$. The extreme points of $\mathcal{S}(\mathcal{A})$ are called *pure states*.

A representation π of $\mathcal{A}$ is *cyclic* if there exists a vector $v \in \mathcal{H}_\pi$ such that the set $\{\pi(a)v \,:\, a \in \mathcal{A}\}$ is dense in $\mathcal{H}_\pi$, in which case v is called a cyclic vector. The Gelfand-Naimark-Segal (GNS) construction links the cyclic representations of $\mathcal{A}$ and the states of $\mathcal{A}$ in the following way. If π is a cyclic representation with a cyclic unit vector v, then $\phi_{\pi,v}$, defined by

$$\phi_{\pi,v}(a) = \langle \pi(a)v, v\rangle$$

for every $a \in \mathcal{A}$, is a state of $\mathcal{A}$. If π is irreducible, then $\phi_{\pi,v}$ is a pure state. Moreover, for every state ϕ of $\mathcal{A}$ there is a cyclic representation $(\pi_\phi, \mathcal{H}_\phi)$ and a cyclic unit vector $v_\phi \in \mathcal{H}_\phi$ such that $\phi(a) = \langle \pi_\phi(a)v_\phi, v_\phi\rangle$ for every $a \in \mathcal{A}$. The pure states of $\mathcal{A}$ correspond to irreducible representations of $\mathcal{A}$ (up to unitary equivalence) via the GNS construction.

Type I groups

The *commutant* of a subset N of $\mathcal{B}(\mathcal{H})$ is the set

$$N^{'} := \{T \in \mathcal{B}(\mathcal{H}) \,:\, TS = ST \text{ for all } S \in N\}\,.$$

A *von Neumann algebra* $\mathcal{N}$ is a *-subalgebra of $\mathcal{B}(\mathcal{H})$ which fulfils $\mathcal{N} = (\mathcal{N}^{'})^{'}$. The von Neumann algebra $\mathcal{N}_\pi$ generated by a unitary representation π of a group Γ, is the smallest von Neumann algebra which contains $\pi(\Gamma)$.

We call a representation π a *factor* if $\mathcal{N}_\pi \cap \mathcal{N}_\pi^{'} = \mathbb{C}\cdot 1_{\mathcal{B}(\mathcal{H}_\pi)}$. A group is of *Type I* if every factor representation is a direct sum of copies of an irreducible representation.

Induced and monomial representations

Let Γ be countably infinite and H a subgroup of Γ. Let σ be a unitary representation of H with representation space $\mathcal{H}_\sigma$. The *induced representation* Ind_H^Γ of σ is the set of functions in $L^2(\Gamma, \mathcal{H}_\sigma)$ which satisfy:

$$f(h\gamma) = \sigma(h)f(\gamma)\,,$$

for every $h \in H$ and $\gamma \in \Gamma$ and where Γ acts via the right regular representation.

A representation of Γ is called *monomial* if it is unitary equivalent to a representation induced from a one-dimensional representation of a subgroup of Γ.

Theorem 3.2.5 ([14]). *If Γ is a nilpotent group of Type I, then all its irreducible representations are monomial.*

3.2.2 Symmetric Banach-*-algebras

In order to study invertibility in $\ell^1(\Gamma, \mathbb{C})$ and $C^*(\Gamma)$ in the non-abelian setting we will try to find criteria similar to those described in Theorem 3.2.3. For this purpose the following definition will play a key role.

Definition 3.2.1. A unital Banach-*-algebra $\mathcal{A}$ is *symmetric* if for every element $a \in \mathcal{A}$ the spectrum of a^*a is non-negative, i.e., $\sigma(a^*a) \subseteq [0, \infty)$.

Typical examples of symmetric Banach-*-algebras are C^*-algebras.
We turn to the study of nilpotent groups and their associated group algebras.

Theorem 3.2.6 ([16])**.** *Let Γ be a countably infinite discrete nilpotent group. Then the Banach-*-algebra $\ell^1(\Gamma, \mathbb{C})$ is symmetric.*

The reason why it is convenient to restrict to the study of invertibility in symmetric unital Banach-*-algebra is demonstrated by the following three theorems, which show similarities to Wiener's Lemma and Theorem 3.2.3, respectively.

Theorem 3.2.7 ([24])**.** *An element a in a symmetric unital Banach-*-algebra $\mathcal{A}$ is not left invertible in $\mathcal{A}$ if and only if there exists a pure state ϕ with $\phi(a^*a) = 0$. Equivalently, a is not left invertible if and only if there exists an irreducible representation π of $\mathcal{A}$ and a unit vector $u \in \mathcal{H}_\pi$ such that $\pi(a)u = 0$.*

This result should be compared with Gelfand's theory for commutative Banach algebras. Wiener's Lemma for $\ell^1(\mathbb{Z}, \mathbb{C})$ says that an element $f \in \ell^1(\mathbb{Z}, \mathbb{C})$ is invertible if and only if the Fourier-transform of f does not vanish on $\mathbb{T}$, i.e., $(\mathcal{F}f)(s) \neq 0$ for all $s \in \mathbb{T}$.[3] The Fourier-transform of f, evaluated at the point $\theta \in \mathbb{T}$, can be viewed as the evaluation of the one-dimensional irreducible unitary representation $\pi_\theta \colon n \mapsto e^{2\pi i n\theta}$ of $\mathbb{Z}$ at f, i.e.,

$$(\mathcal{F}f)(\theta) = \Big(\sum\nolimits_{n\in\mathbb{Z}} f_n \pi_\theta(n)\Big)1 = \pi_\theta(f)1\,.$$

For symmetric Banach-*-algebras one obtains an important result concerning inverse-closedness.

Theorem 3.2.8 ([26, Theorem 11.4.1 and Corollary 12.4.5])**.** *If $\ell^1(\Gamma, \mathbb{C})$ is a symmetric Banach-*-algebra, then*

1. *$\ell^1(\Gamma, \mathbb{C})$ is semisimple, i.e., the intersection of the kernels of all the irreducible representations of $\ell^1(\Gamma, \mathbb{C})$ is trivial.*

[3] To fix notation: for $f \in L^2(\mathbb{T}, \lambda_\mathbb{T})$ (where $\lambda_\mathbb{T}$ is Lebesgue measure on $\mathbb{T}$), the Fourier transform $\hat{f} \colon \mathbb{Z} \longrightarrow \mathbb{C}$ is defined by $\hat{f}_n = \int_\mathbb{T} f(s)e^{-2\pi i n s}\, d\lambda_\mathbb{T}(s)$. The Fourier transform $(\mathcal{F}g) \colon \mathbb{T} \longrightarrow \mathbb{C}$ of $g \in \ell^2(\mathbb{Z}, \mathbb{C})$ is defined by $(\mathcal{F}g)(s) = \sum_{n\in\mathbb{Z}} g_n e^{2\pi i n s}$.

2. *$\ell^1(\Gamma,\mathbb{C})$ and its enveloping C^*-algebra $C^*(\Gamma)$ form a Wiener pair.*

We now turn to the spectral invariance of convolution operators. We already saw in the introduction of Wiener's Lemma that invertibility of $f \in \ell^1(\mathbb{Z},\mathbb{C})$ can be validated by studying invertibility of the convolution operator C_f acting on the Hilbert space $\ell^2(\mathbb{Z},\mathbb{C})$. Moreover, the spectrum of C_f is independent of the domain, i.e., the spectrum of the operator $C_f\colon \ell^p(\mathbb{Z},\mathbb{C}) \longrightarrow \ell^p(\mathbb{Z},\mathbb{C})$ is the same for all $p \in [1,\infty]$. As the following theorem shows, this result is true for a large class of groups, in particular, for all finitely generated nilpotent groups.

Theorem 3.2.9 ([3]). *Let $f \in \ell^1(\Gamma,\mathbb{C})$ and C_f the associated convolution operator on $\ell^p(\Gamma,\mathbb{C})$. For all $1 \le p \le \infty$ we get $\sigma_{\mathcal{B}(\ell^p(\Gamma,\mathbb{C}))}(C_f) = \sigma_{\mathcal{B}(\ell^2(\Gamma,\mathbb{C}))}(C_f)$ if and only if Γ is amenable and $\ell^1(\Gamma,\mathbb{C})$ is a symmetric Banach-*-algebra.*

Finally, let us state a very general form of Wiener's Lemma for $\ell^1(\Gamma,\mathbb{C})$, where Γ is an arbitrary discrete countably infinite group.

Theorem 3.2.10 ([7, Theorem 3.2]). *Let $f \in \ell^1(\Gamma,\mathbb{C})$, then f is invertible in $\ell^1(\Gamma,\mathbb{C})$ if and only if*

$$K_\infty(f) \coloneqq \{v \in \ell^\infty(\Gamma,\mathbb{C}) \,:\, C_f v = 0\} = \{0\}\,.$$

This theorem says that it is enough to check if 0 is in the discrete spectrum of the left convolution operator $C_f\colon \ell^\infty(\Gamma,\mathbb{C}) \longrightarrow \ell^\infty(\Gamma,\mathbb{C})$ in order to determine whether f is invertible or not (cf. (3.2.2)).

3.2.3 The discrete Heisenberg group

The discrete Heisenberg group $\mathbb{H}$ is generated by $S = \{x, x^{-1}, y, y^{-1}\}$, where

$$x = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad y = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

The center of $\mathbb{H}$ is generated by

$$z = xyx^{-1}y^{-1} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

The elements x, y, z satisfy the following commutation relations

$$xz = zx, \;\; yz = zy, \;\; x^k y^l = y^l x^k z^{kl}, \;\; k,l \in \mathbb{Z}. \tag{3.2.3}$$

The discrete Heisenberg group is nilpotent and hence amenable.

The dual of a discrete group

Let Γ be a countable discrete group. In this subsection we will discuss what kind of problems can occur when Γ is not of Type I. Denote by $\widehat{\Gamma}$ the *dual* of Γ, i.e., the set of all unitary equivalence classes of irreducible unitary representations of Γ.

Definition 3.2.2. Let $\mathcal{A}$ be a C^*-algebra. A closed two-sided ideal $\mathcal{I}$ of $\mathcal{A}$ is *primitive* if there exists an irreducible representation π of $\mathcal{A}$ such that $\ker \pi = \mathcal{I}$. The set of primitive ideals of $\mathcal{A}$ is denoted by Prim$(\mathcal{A})$.

Suppose that the group Γ is not of Type I. Then certain pathologies arise:

- The map $\widehat{\Gamma} \longrightarrow \mathrm{Prim}(C^*(\Gamma))$ given by $\pi \mapsto \ker(\pi)$ is not injective. In other words, if $\pi_1, \pi_2 \in \widehat{\Gamma}$, then $\ker \pi_1 = \ker \pi_2$ does not necessarily imply that π_1 and π_2 are unitary equivalent.
- $\widehat{\Gamma}$ is not behaving nicely neither as a topological space nor as a measurable space in its natural topology or Borel structure, respectively (cf. [10, Chapter 7] for an overview).

Furthermore, there are examples where the direct integral decomposition of a representation is not unique, in the sense that there are disjoint measures μ, ν on $\widehat{\Gamma}$ such that $\int_{\widehat{\Gamma}}^{\oplus} \pi d\mu$ and $\int_{\widehat{\Gamma}}^{\oplus} \pi d\nu$ are unitary equivalent. Moreover, we cannot assume that all irreducible representations are induced from one-dimensional representations of finite-index subgroups like it is the case for nilpotent groups of Type I by Theorem 3.2.5.

The discrete Heisenberg group $\mathbb{H}$ does not possess an abelian normal subgroup of finite index, and is therefore not of Type I (cf. [30]). In fact, one is able to construct uncountably many unitary inequivalent irreducible representations of $\mathbb{H}$ for every irrational $\theta \in \mathbb{T}$. These representations arise from a special class of measures. This fact is well-known to specialists, but details are not easily accessible in the literature. Since these results are important for our understanding of invertibility, we present this construction in some detail for the convenience of the reader. We would like to mention first that Moran announces in [23] a construction of unitary represention of $\mathbb{H}$ using the same approach as presented here. These results were not published as far as we know. Moreover, Brown gives examples of unitary irreducible representations of the discrete Heisenberg group which are not monomial in [5].

Let $(X, \mathfrak{B}, \mu)$ be a measure space, where X is a compact metric space, $\mathfrak{B}$ is a Borel σ-algebra, and μ a finite measure.

Definition 3.2.3. A probability measure μ is *quasi-invariant* with respect to a homeomorphism $\phi\colon X \longrightarrow X$ if $\mu(B) = 0$ if and only if $\mu(\phi B) = 0$, for $B \in \mathfrak{B}$. A quasi-invariant measure μ is ergodic if

$$B \in \mathfrak{B} \quad \text{and} \quad \phi B = B \implies \mu(B) \in \{0, 1\}.$$

In [20] uncountably many inequivalent ergodic quasi-invariant measures for every irrational rotation of the circle were constructed. Later it was shown in [19] that a homeomorphism ϕ on a compact metric space X has uncountably many inequivalent non-atomic ergodic quasi-invariant measures if and only if ϕ has a recurrent point x, i.e., $\phi^n(x)$ returns infinitely often to any punctured neighbourhood of x.

We use the measures found in [20] to construct unitary irreducible representations of $\mathbb{H}$. Let $\mathbb{Z}$ act on $\mathbb{T}$ via rotations

$$R_\theta \colon t \mapsto t + \theta \mod 1 \tag{3.2.4}$$

by an irrational angle $\theta \in \mathbb{T}$. Let μ be an ergodic R_θ-quasi-invariant probability measure on $\mathbb{T}$. Let $T_{\theta,\mu} \colon L^2(\mathbb{T},\mu) \longrightarrow L^2(\mathbb{T},\mu)$ be the unitary operator defined by

$$(T_{\theta,\mu} f)(t) = \sqrt{\frac{d\mu(t+\theta)}{d\mu(t)}} f(t+\theta) = \sqrt{\frac{d\mu(R_\theta t)}{d\mu(t)}} f(R_\theta t)\,, \tag{3.2.5}$$

for every $f \in L^2(\mathbb{T},\mu)$ and $t \in \mathbb{T}$. The operator $T_{\theta,\mu}$ is well-defined because of the quasi-invariance of μ. Consider also the unitary operator M_μ defined by

$$(M_\mu f)(t) = e^{2\pi i t} f(t)\,, \tag{3.2.6}$$

for every $f \in L^2(\mathbb{T},\mu)$ and $t \in \mathbb{T}$.

We will show that the representation $\pi_{\theta,\mu}$ of $\mathbb{H}$ defined by

$$\pi_{\theta,\mu}(x) \coloneqq T_{\theta,\mu}\,, \qquad \pi_{\theta,\mu}(y) \coloneqq M_\mu \quad \text{and} \quad \pi_{\theta,\mu}(z) \coloneqq e^{2\pi i\theta} \tag{3.2.7}$$

is irreducible. Obviously, $T_{\theta,\mu} M_\mu = e^{2\pi i\theta} M_\mu T_{\theta,\mu} = \pi_{\theta,\mu}(z) M_\mu T_{\theta,\mu}$.

Lemma 3.2.4. *The unitary representation $\pi_{\theta,\mu}$ of $\mathbb{H}$ given by* (3.2.7) *is irreducible.*

Proof. Every element in $L^2(\mathbb{T},\mu)$ can be approximated by linear combinations of elements in the set

$$\{M_\mu^n 1 \,:\, n \in \mathbb{Z}\} = \{t \mapsto e^{2\pi i n t} \,:\, n \in \mathbb{Z}\}\,.$$

A bounded operator A on $L^2(\mathbb{T},\mu)$, which commutes with all operators of the form M_μ^n, $n \in \mathbb{Z}$, and hence with multiplication with any L^∞-function, must be a multiplication operator, i.e., $Af(t) = g(t) \cdot f(t)$ for some $g \in L^\infty(\mathbb{T},\mu)$.

Indeed, if A commutes with multiplication by $h \in L^\infty(\mathbb{T},\mu)$, then

$$Ah = h \cdot A1 = hg\,,$$

say. Denote by $\|\cdot\|_{op}$ the operator norm, then

$$\|hg\|_{L^2(\mathbb{T},\mu)} = \|Ah\|_{L^2(\mathbb{T},\mu)} \le \|A\|_{op} \|h\|_{L^2(\mathbb{T}\,\mu)}\,, \tag{3.2.8}$$

which implies that $g \in L^\infty(\mathbb{T},\mu)$ (otherwise one would be able to find a measurable set B with positive measure on which g is strictly larger than $\|A\|_{op}$, and the indicator function 1_B would lead to a contradiction with (3.2.8)).

The ergodicity of μ with respect to R_θ implies that only constant functions in $L^\infty(\mathbb{T},\mu)$ are R_θ-invariant μ-a.e.. Hence, if A commutes with $T_{\theta,\mu}$ as well, then we can conclude that A is multiplication by a constant $c \in \mathbb{C}$. By Schur's Lemma, the operators $T_{\theta,\mu}, M_\mu \in \mathcal{B}(L^2(\mathbb{T},\mu))$ define an irreducible representation $\pi_{\theta,\mu}$ of $\mathbb{H}$. □

Suppose that $\theta \in \mathbb{T}$ is irrational, and that μ and ν are two ergodic R_θ-quasi-invariant measures on $\mathbb{T}$. Let $\pi_{\theta,\mu}$ and $\pi_{\theta,\nu}$ be the corresponding irreducible unitary representations constructed above.

Lemma 3.2.5. *The representations $\pi_{\theta,\mu}$ and $\pi_{\theta,\nu}$ are unitary equivalent if and only if μ and ν are equivalent.*

Proof. Assume $\pi_{\theta,\mu}$ and $\pi_{\theta,\nu}$ are unitary equivalent. Then there exists a unitary operator $U\colon L^2(\mathbb{T},\mu) \longrightarrow L^2(\mathbb{T},\nu)$ such that

$$U\pi_{\theta,\mu}(\gamma) = \pi_{\theta,\nu}(\gamma)U \tag{3.2.9}$$

for every $\gamma \in \mathbb{H}$.

Denote multiplication by a function $h \in C(\mathbb{T},\mathbb{C})$ by A_h. The set of trigonometric polynomials, which is spanned by $\{M_\mu^n 1 \,:\, n \in \mathbb{Z}\}$, is dense in $C(\mathbb{T},\mathbb{C})$. This implies that (3.2.9) holds for all $h \in C(\mathbb{T},\mathbb{C})$, i.e., that $UA_h = A_h U$ for any $h \in C(\mathbb{T},\mathbb{C})$.

Since U is an isometry we get that

$$\begin{aligned}
\int |h|^2 1^2 d\mu &= \langle A_h 1, A_h 1\rangle_\mu && (3.2.10)\\
&= \langle A_h U(1), A_h U(1)\rangle_\nu && (3.2.11)\\
&= \int |h|^2 |U(1)|^2 d\nu\,, && (3.2.12)
\end{aligned}$$

where $\langle\cdot,\cdot\rangle_\rho$ is the standard inner product on the Hilbert space $L^2(\mathbb{T},\rho)$. Using the same argument for U^{-1} we get, for every $h \in C(\mathbb{T},\mathbb{C})$,

$$\int |h|^2 1^2 d\nu = \int |h|^2 |U^{-1}(1)|^2 d\mu\,. \tag{3.2.13}$$

Define, for every positive finite measure σ on $\mathbb{T}$, a linear functional

$$I_\sigma\colon C(\mathbb{T},\mathbb{C}) \longrightarrow \mathbb{C} \quad \text{by} \quad I_\sigma(h) = \int h\, d\sigma\,.$$

Since $I_\mu(h) = I_{|U(1)|^2\nu}(h)$ and $I_\nu(h) = I_{|U^{-1}(1)|^2\mu}(h)$ for all positive continuous functions h by (3.2.10) – (3.2.13), we conclude from the Riesz representation theorem that μ

and ν are equivalent.

Conversely, if μ and ν are equivalent, then the linear operator

$$U\colon L^2(\mathbb{T},\mu) \longrightarrow L^2(\mathbb{T},\nu) \quad \text{given by} \quad Uf = \sqrt{\frac{d\mu}{d\nu}}f$$

for every $f \in L^2(\mathbb{T},\mu)$, is unitary and satisfies that $U\pi_{\theta,\mu}(\gamma) = \pi_{\theta,\nu}(\gamma)U$ for every $\gamma \in \mathbb{H}$. □

In this way one obtains uncountably many inequivalent irreducible unitary representation of $\mathbb{H}$ for a given irrational rotation number $\theta \in \mathbb{T}$.

Theorem 3.2.7 states that in order to decide on invertibility of $f \in \ell^1(\mathbb{H},\mathbb{C})$, one has to check invertibility of $\pi(f)$ for every irreducible representations π of $\mathbb{H}$, and in particular, for every $\pi_{\theta,\mu}$ as above. (In fact, every irreducible unitary representation π of $\mathbb{H}$ is unitary equivalent to $\pi_{\theta,\mu}$ for some probability measure μ on $\mathbb{S}$ which is quasi-invariant and ergodic with respect to some circle rotation.)

Since the space (of equivalence classes) of probability measures, which are quasi-invariant and ergodic under rotations, is extremely complicated and has no nice Borel structure, the problem of deciding on invertibility of f via Theorem 3.2.7 becomes impractical. However, the problem becomes easier if one is able to restrict oneself to unitary representations arising from *rotation invariant* probability measures. This is exactly our main result.

Before formulating this result we write down the relevant representations explicitly.

3.2.4 The representations $\pi_\theta^{(s,t)}$.

Take $\theta \in \mathbb{T}$, and consider the corresponding rotation $R_\theta\colon \mathbb{T} \longrightarrow \mathbb{T}$ given by (3.2.4). If θ is irrational, the Lebesgue measure $\lambda = \lambda_{\mathbb{T}}$ on $\mathbb{T}$ is the unique R_θ-invariant probability measure, and the representation $\pi_{\theta,\lambda}$ on $L^2(\mathbb{T},\lambda)$ defined in (3.2.7) is irreducible. One can modify this representation by setting, for every $s,t \in \mathbb{T}$,

$$\pi_\theta^{(s,t)}(x) = e^{2\pi is}\pi_{\theta,\lambda}(x), \quad \pi_\theta^{(s,t)}(y) = e^{2\pi it}\pi_{\theta,\lambda}(y), \quad \pi_\theta^{(s,t)}(z) = e^{2\pi i\theta}. \tag{3.2.14}$$

Then $\pi_\theta^{(s,t)}$ is obviously again an irreducible unitary representation of $\mathbb{H}$ on $\mathcal{H}_{\pi_\theta^{(s,t)}} = L^2(\mathbb{T},\lambda)$.

If θ is rational we write it as $\theta = p/q$ with $0 \le p < q$ and $\gcd(p,q) = 1$ and consider the unitary representation $\pi_\theta^{(s,t)}$ of $\mathbb{H}$ on $\mathcal{H}_{\pi_\theta^{(s,t)}} = \mathbb{C}^q$ given by

$$\pi_\theta^{(s,t)}(x) = e^{2\pi is}\begin{pmatrix} 0 & I_{q-1} \\ 1 & 0 \end{pmatrix}, \tag{3.2.15}$$

$$\pi_\theta^{(s,t)}(y) = e^{2\pi i t} \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & e^{2\pi i\theta} & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & e^{2\pi i(q-2)\theta} & 0 \\ 0 & 0 & \dots & 0 & e^{2\pi i(q-1)\theta} \end{pmatrix} \quad \text{and} \quad \pi_\theta^{(s,t)}(z) = e^{2\pi i\theta} I_q \,, \tag{3.2.16}$$

with $s, t \in \mathbb{T}$, where I_{q-1} is the $(q-1) \times (q-1)$ identity matrix.

Every R_θ-invariant and ergodic probability measure μ on $\mathbb{T}$ is uniformly distributed on the set $\{t, 1/q + t, \dots, t + (q-1)/q\} \subset \mathbb{T}$ for some $t \in \mathbb{T}$; if we denote this measure by μ_t then $\mu_t = \mu_{t+k/q}$ for every $k = 0, \dots, q-1$.

With this notation at hand we can state our main result, the proof of which will be given in the following Sections 3.3 – 3.4.

Theorem 3.2.11. *An element $a \in \ell^1(\mathbb{H}, \mathbb{C})$ is invertible if and only if the linear operator $\pi_\theta^{(s,t)}(a)$ is invertible on the corresponding Hilbert space $\mathcal{H}_{\pi_\theta^{(s,t)}}$ for every $\theta, s, t \in \mathbb{T}$.*

The main advantage of Theorem 3.2.11 over Theorem 3.2.7 is that it is not necessary to check invertibility of $\pi(a)$ for *every* irreducible representation of $\mathbb{H}$, but that one can restrict oneself for this purpose to the 'nice' part of the dual of the non-Type I group $\mathbb{H}$. As we shall see later, one can make a further reduction if θ is irrational: in this case one only has to check invertibility of $\pi_\theta(a) = \pi_\theta^{(1,1)}(a)$ on $L^2(\mathbb{T}, \lambda)$.

3.3 Local principles

Let Γ be a countable discrete nilpotent group. In this section we will discuss so-called local principles in order to check invertibility of an element a in $\ell^1(\Gamma, \mathbb{C})$ or $C^*(\Gamma)$. The idea is to study invertibility of projections of a onto certain quotient algebras of $\ell^1(\Gamma, \mathbb{C})$ or $C^*(\Gamma)$ and to conclude from this information whether a is invertible or not. Therefore, the main task is to find a sufficient family $\mathcal{S}$ of ideals of $\ell^1(\Gamma, \mathbb{C})$ or $C^*(\Gamma)$ such that one can deduce the invertibility of a from the invertibility of the projections of a on $\ell^1(\Gamma, \mathbb{C})/\mathcal{I}$ for all $\mathcal{I} \in \mathcal{S}$.

3.3.1 Allan's local principle

We have used Allan's local principle already in [12] to study invertibility in $\ell^1(\mathbb{H}, \mathbb{C})$. However, in that paper we were not able to prove Theorem 3.2.11 with this approach.

Suppose $\mathcal{A}$ is a unital Banach algebra with non-trivial center

$$\mathcal{C}(\mathcal{A}) \coloneqq \left\{ c \in \mathcal{A} \,:\, cb = bc \;\text{ for all } b \in \mathcal{A} \right\}.$$

The commutative Banach subalgebra $\mathcal{C}(\mathcal{A})$ is closed and contains the identity $1_{\mathcal{A}}$. For every $m \in \mathrm{Max}(\mathcal{C}(\mathcal{A}))$ (the space of maximal ideals of $\mathcal{C}(\mathcal{A})$) we denote by $\mathcal{J}_m$ the smallest closed two-sided ideal of $\mathcal{A}$ which contains m and denote by $\Phi_m : a \mapsto \Phi_m(a)$ the canonical projection of an element $a \in \mathcal{A}$ to the quotient algebra $\mathcal{A}/\mathcal{J}_m$. The algebra $\mathcal{A}/\mathcal{J}_m$, furnished with the quotient norm

$$\|\Phi_m(a)\| \coloneqq \inf_{b\in\mathcal{J}_m} \|a+b\|_{\mathcal{A}} \tag{3.3.1}$$

becomes then a unital Banach algebra.

Theorem 3.3.1 ([1] Allan's local principle)**.** *An element $a \in \mathcal{A}$ is invertible in $\mathcal{A}$ if and only if $\Phi_m(a)$ is invertible in $\mathcal{A}/\mathcal{J}_m$ for every $m \in \mathrm{Max}(\mathcal{C}(\mathcal{A}))$.*

3.3.2 Local principles in C^*-algebras

In this section we will describe how the representation theory and local principles of a C^*-algebra are related.

As we have seen earlier, $\ell^1(\Gamma,\mathbb{C})$ is inverse-closed in $C^*(\Gamma)$. Hence we concentrate on the group C^*-algebra $C^*(\Gamma)$.

Let $\mathcal{A}$ be a unital C^*-algebra. For every two-sided closed ideal $\mathcal{J}$ of $\mathcal{A}$, denote by $\Phi_{\mathcal{J}}$ the canonical projection from $\mathcal{A}$ to the C^*-algebra $\mathcal{A}/\mathcal{J}$.

Lemma 3.3.1. *An element a in $\mathcal{A}$ is invertible if and only if the projections of a on $\mathcal{A}/\mathcal{I}$ are invertible for every two-sided closed ideals $\mathcal{I}$ of $\mathcal{A}$ contained in some primitive ideal of $\mathcal{A}$.*

Corollary 3.3.2. *If $\pi(a)$ is not invertible for an irreducible representation π, then for every two-sided closed ideal $\mathcal{I} \subseteq \ker(\pi)$ of $C^*(\mathbb{H})$, the element $\Phi_{\mathcal{I}}(a)$ is non-invertible in $\mathcal{A}/\mathcal{I}$.*

Proof of Lemma 3.3.1. If $a \in \mathcal{A}$ is not invertible, then by Theorem 3.2.7 there exists an irreducible unitary representation π of $\mathcal{A}$ such that $\pi(a)v = 0$ for some non-zero vector $v \in \mathcal{H}_\pi$. For every two-sided closed ideal $\mathcal{I} \subseteq \ker(\pi)$ of $\mathcal{A}$ the representation π induces a well defined irreducible representation $\pi_{\mathcal{I}}$ of the C^*-algebra $\mathcal{A}/\mathcal{I}$, i.e.,

$$\pi_{\mathcal{I}}(\Phi_{\mathcal{I}}(a)) = \pi(a)\,.$$

Moreover, for every two-sided closed ideals $\mathcal{I} \subseteq \ker(\pi)$ of $C^*(\mathbb{H})$, the element $\Phi_{\mathcal{I}}(a)$ is not invertible in $\mathcal{A}/\mathcal{I}$, since the operator $\pi_{\mathcal{I}}(\Phi_{\mathcal{I}}(a))$ has a non-trivial kernel in $\mathcal{H}_\pi$.

Suppose now that π is an irreducible unitary representation of $\mathcal{A}$. Let us assume that $\Phi_{\mathcal{I}}(a)$ is not invertible in the C^*-algebra $\mathcal{A}/\mathcal{I}$ for some two-sided closed ideal of $\mathcal{A}$ with $\mathcal{I} \subseteq \ker(\pi)$. Hence, there exists an irreducible representation ρ of $\mathcal{A}/\mathcal{I}$ such that

$$\rho(\Phi_{\mathcal{I}}(a))v = 0$$

for some vector $v \in \mathcal{H}_\rho$. The irreducible representation ρ can be extended to an irreducible representation $\tilde{\rho}$ of $\mathcal{A}$ which vanishes on $\mathcal{I}$ and which is given by $\tilde{\rho} = \rho \circ \Phi_{\mathcal{I}}$. Therefore, a is not invertible in $\mathcal{A}$. $\square$

From the proof of Lemma 3.3.1 we get the following corollary.

Corollary 3.3.3. *If* Γ *is a discrete nilpotent group, then* $a \in \ell^1(\Gamma, \mathbb{C})$ *is invertible if and only if* $\Phi_{\mathcal{I}}(a)$ *is invertible for every* $\mathcal{I} \in \mathrm{Prim}(C^*(\Gamma))$.

For the next corollary we only have to recall two facts:

1. For an arbitrary unital C^*-algebra $\mathcal{A}$ and an irreducible representation π of $\mathcal{A}$, the C^*-algebras $\pi(\mathcal{A})$ and $\mathcal{A}/\ker\pi$ are isomorphic.

2. Every two-sided maximal ideal is primitive (cf. [25, Theorem 4.1.9]).

We denote by $\mathrm{Max}(\mathcal{A})$ the set of two-sided maximal ideals of $\mathcal{A}$.

Corollary 3.3.4. *An element* a *in a* C^**-algebra* $\mathcal{A}$ *is invertible if and only if* $\pi(a)$ *has a bounded inverse* (*or equivalently, if* $\Phi_{\ker\pi}(a)$ *is invertible*) *for every irreducible representations* π *of* $\mathcal{A}$ *which fulfils* $\ker\pi \in \mathrm{Max}(\mathcal{A})$.

3.4 Wiener's Lemma for the discrete Heisenberg group

Let us apply the general observations made in the previous section to study invertibility in $\ell^1(\mathbb{H}, \mathbb{C})$ and $C^*(\mathbb{H})$. By Schur's Lemma, if π an irreducible unitary representation of $\mathbb{H}$, then $\pi(z) = e^{2\pi i\theta} 1_{\mathcal{B}(\mathcal{H}_\pi)}$ for some $\theta \in \mathbb{T}$. Since the two-sided closed ideal $\mathcal{J}_\theta = \overline{(z - e^{2\pi i\theta})C^*(\mathbb{H})}$ is a subset of $\ker(\pi)$ for every irreducible unitary representation π of $\mathbb{H}$ with $\pi(z) = e^{2\pi i\theta} 1_{\mathcal{B}(\mathcal{H}_\pi)}$, by Lemma 3.3.1 and Corollary 3.3.2 we have to check invertibility only in $\mathcal{Q}_\theta = C^*(\mathbb{H})/\mathcal{J}_\theta$ for all $\theta \in \mathbb{T}$. This is exactly the conclusion which one obtains when applying Allan's local principle to $C^*(\mathbb{H})$. Indeed, $\mathcal{C}(C^*(\mathbb{H})) \simeq C(\mathbb{T}, \mathbb{C})$, and the maximal ideals of $C(\mathbb{T}, \mathbb{C})$ are given by the sets

$$m_\theta := \{F \in C(\mathbb{T}) \,:\, F(\theta) = 0\}$$

and $\mathcal{J}_{m_\theta} = \mathcal{J}_\theta$. Hence, Allan's local principle can be viewed as an effective way to apply Lemma 3.3.1 in order to check invertibility.

If θ is rational the irreducible unitary representations vanishing on $\mathcal{J}_\theta$ are given by (3.2.15) – (3.2.16) and were determined in [4].

Now suppose θ is irrational. In order to study the representation theory of the C^*-algebra $\mathcal{Q}_\theta$ we have to understand the link to one of the most studied non-commutative

C^*-algebras — the irrational rotation algebras. We call a C^*-algebra *irrational rotation algebra* if it is generated by two unitaries U, V which fulfil the commutation relation

$$UV = e^{2\pi i\theta} VU\,, \tag{3.4.1}$$

for some irrational $\theta \in \mathbb{T}$. We already saw examples of irrational rotation algebras above, namely, the C^*-subalgebras of $\mathcal{B}(L^2(\mathbb{T}, \mu))$ which are generated by M_μ and $T_{\theta,\mu}$, where μ is a R_θ-quasi-invariant and ergodic measure. The reason why we call all C^*-algebras which fulfil (3.4.1) irrational rotation algebras with parameter θ is the following striking result which can be found in [6, Theorem VI.1.4].

Theorem 3.4.1. *If $\theta \in \mathbb{T}$ is irrational, then all C^*-algebras which are generated by two unitaries U, V satisfying* (3.4.1)*, are *-isomorphic.*

We will denote the irrational rotation algebra with parameter θ by $\mathcal{R}_\theta$ and will not distinguish between the different realisations of $\mathcal{R}_\theta$ because of the universal property described in Theorem 3.4.1. Let us further note that the proof of Theorem 3.4.1 is deduced from the simplicity of the universal irrational rotation algebra.

The C^*-algebra Q_θ is clearly a rotation algebra with parameter θ. The simplicity of $\mathcal{R}_\theta$ implies that $\mathcal{J}_\theta$ is a maximal two-sided ideal of $C^*(\mathbb{H})$. Hence, there exists an irreducible representation π of $\mathbb{H}$ such that $\ker \pi = \mathcal{J}_\theta$, since every two-sided maximal ideal is primitive (cf. [25, Theorem 4.1.9]). Moreover, all the irreducible representations π vanishing on $\mathcal{J}_\theta$ have the same kernel: otherwise we would get a violation of the maximality of $\mathcal{J}_\theta$. These representations are not all in the same unitary equivalence class (as we saw in Section 3.2), which is an indication of the fact that $\mathbb{H}$ is not of Type I.

Proof of Theorem 3.2.11. Combine Lemma 3.3.1 (i.e., Allan's local principle) for $C^*(\mathbb{H})$ with Theorem 3.4.1. Obviously, if θ is irrational, any representation of $\mathbb{H}$ which vanishes on $\mathcal{J}_\theta$ can be used to check invertibility in $\mathcal{Q}_\theta$. In particular we may use the representations $\pi_\theta^{(1,1)}$ as in (3.2.14). □

Remark 3.4.1. We should note here that for all realisations of the irrational rotation algebra the spectrum of $a \in \mathcal{R}_\theta$ is the same as a set. But this does *not* imply that an eigenvalue (or an element of the continuous spectrum) of a in one realisation is an eigenvalue (or an element of the continuous spectrum) of a in all the other realisations.

3.5 Invertibility in $C^*(\Gamma)$ for discrete nilpotent groups

In this section we aim to find results for countable discrete nilpotent groups similar to those presented in the previous section for the discrete Heisenberg group.

3.5.1 Monomial representations

The Heisenberg group

Denote by $\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})$ the representation of $\mathbb{H}$ induced from the normal subgroup

$$N := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,:\, a, b \in \mathbb{Z} \right\}$$

and the character $\sigma_{\theta,s}$ which is defined by

$$\sigma_{\theta,s}(z) = e^{2\pi i\theta} \quad \text{and} \quad \sigma_{\theta,s}(x) = e^{-2\pi i s}\,.$$

For the convenience of the reader we will write down $\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})$ for every $\theta, s \in \mathbb{T}$ explicitly

$$\left(\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})(x^k y^l z^m)F\right)(n) = e^{2\pi i(m\theta - k(n\theta+s))}F(n+l) \tag{3.5.1}$$

for all $k, l, m, n \in \mathbb{Z}$ and $F \in \ell^2(\mathbb{Z}, \mathbb{C})$.

The representations $\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})$ play a special role since they can be extended to the Stone-von Neumann representations of the *real* Heisenberg group $\mathbb{H}_{\mathbb{R}}$ consisting of all unipotent upper triangular matrices in $\mathrm{SL}(3,\mathbb{R})$. The Stone-von Neumann representations of $\mathbb{H}_{\mathbb{R}}$ are obtained from Mackey's induction procedure from the real analogue of N, i.e.,

$$N_{\mathbb{R}} := \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,:\, a, b \in \mathbb{R} \right\}$$

and its characters. The Stone-von Neumann representations are defined by modulation and translation operators on $L^2(\mathbb{R}, \mathbb{C})$.

It is easy to see that for irrational θ the representation $\pi_\theta^{(1,1)}$ in (3.2.14) is unitary equivalent (via Fourier transformation) to the representation $\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,1})$. Moreover, every irreducible finite dimensional representation of a nilpotent group Γ is induced from a one dimensional representation of a subgroup of Γ (cf. [5, Lemma 1]). Hence, for the discrete Heisenberg group the knowledge of the monomial representations is enough to decide on invertibility.

The natural question arises, whether one can always restrict oneself to the class of monomial representations of Γ when analysing invertibility in the corresponding group algebras, in case Γ is a countable discrete nilpotent group. We will show that the answer is positive.

The general case

Let Γ be a countable discrete nilpotent group. Define an equivalence relation on $\widehat{\Gamma}$ as follows:

$$\pi_1 \sim \pi_2 \iff \ker \pi_1 = \ker \pi_2 \,,$$

where π_1, π_2 are irreducible unitary representations of Γ. This equivalence relation is the same as the notion of weak equivalence according to [9].

The next theorem was established by Howe in [15, Proposition 5].

Theorem 3.5.1. *Suppose that Γ is a countable discrete nilpotent group. Then every irreducible unitary representation is weakly equivalent to an irreducible monomial representation of Γ.*

Combining this result with Lemma 3.3.1 leads to the following theorem:

Theorem 3.5.2. *An element $f \in C^*(\Gamma)$ is non-invertible if and only if there exists an irreducible monomial representation π such that $\pi(f)$ has no bounded inverse.*

For convenience of the reader we explain the ideas once more.

Proof. If f is not invertible, then there exists an irreducible unitary representation π and a non-zero vector $v \in \mathcal{H}_\pi$ such that $\pi(f)v = 0$. This implies that $\Phi_{\ker \pi}(f)$ is not invertible in $C^*(\Gamma)/\ker \pi$. Moreover, there exists an irreducible monomial representation ρ with $\ker \rho = \ker \pi$ (cf. Theorem 3.5.1) and hence

$$\pi(C^*(\Gamma)) \simeq C^*(\Gamma)/\ker \pi = C^*(\Gamma)/\ker \rho \simeq \rho(C^*(\Gamma)) \,.$$

Therefore, $\Phi_{\ker \rho}(f)$ and $\rho(f)$ are not invertible.

On the other hand, if $\pi(f)$ is not invertible for an irreducible monomial representation π, then $\Phi_{\ker \pi}(f)$ is not invertible in the C^*-algebra $C^*(\Gamma)/\ker(\pi)$. Hence there exists an irreducible representation ρ of $C^*(\Gamma)/\ker(\pi)$ such that $\rho(\Phi_{\ker \pi}(f))$ has a non-trivial kernel. Moreover, ρ can be extended to a representation $\tilde{\rho}$ of $C^*(\Gamma)$ vanishing on $\ker \pi$. Therefore, f is not invertible. $\square$

3.5.2 Maximality of primitive ideals

In the previous subsection we saw that we can restrict our attention to irreducible monomial representations for questions about invertibility. Unfortunately, this subclass of irreducible representations might still be quite big. We will use another general result about the structure of $\mathrm{Prim}(C^*(\Gamma))$ to make the analysis of invertibility in $C^*(\Gamma)$ easier.

Theorem 3.5.3 ([27])**.** *Let Γ be a discrete nilpotent group. Then*

$$\mathrm{Prim}(C^*(\Gamma)) = \mathrm{Max}(C^*(\Gamma)) \,,$$

i.e., every primitive ideal of $C^*(\Gamma)$ is maximal.

The simplification in the study of invertibility in $C^*(\mathbb{H})$ was due to the simplicity of the irrational rotation algebras $\mathcal{R}_\theta$, which is equivalent to the maximality of the two-sided closed ideal $\mathcal{J}_\theta$. We should note here that Theorem 3.4.1 is usually proved by the construction of a unique trace on $\mathcal{R}_\theta$, which is rather complicated. Alternatively, let $\theta \in \mathbb{T}$ be irrational. Then it easily follows from Theorem 3.5.3 and the fact that $\pi_\theta^{(s,t)}$ is an irreducible representation (cf. Lemma 3.2.4) with $\ker(\pi_\theta^{(s,t)}) = \mathcal{J}_\theta$ that $\mathcal{J}_\theta$ is maximal. This is exactly the statement of Theorem 3.4.1.

3.5.3 Examples

The first example shows how to establish a Wiener Lemma for $\mathbb{H}$ from the general observation made in this Section.

Example 3.5.1. Consider the monomial representations $\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})$ of $\mathbb{H}$ as defined in (3.5.1) for irrational θ and arbitrary $s \in \mathbb{T}$. Obviously, one has $\ker(\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})) = \mathcal{J}_\theta$ for all $s \in \mathbb{T}$.

We will show that there is no bounded operator on $\ell^2(\mathbb{Z},\mathbb{C})$ which commutes with $\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})(x)$ and $\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})(y)$ except multiples of the identity operator. Let $\{\delta_k : k \in \mathbb{Z}\}$ be the standard basis of $\ell^2(\mathbb{Z},\mathbb{C})$ and $C = (c_{n,k})_{n,k\in\mathbb{Z}}$ an operator which commutes with $\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})(x)$ and $\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})(y)$. From the equations

$$\begin{aligned} e^{-2\pi i s}\sum_{n\in\mathbb{Z}} c_{n,k}e^{-2\pi i\theta n}\delta_n &= C\left(\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})(x)\delta_k\right) \\ &= \mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})(x)(C\delta_k) \\ &= e^{-2\pi i s}e^{-2\pi i\theta k}\sum_{n\in\mathbb{Z}} c_{n,k}\delta_n \end{aligned}$$

and the fact that θ is irrational we can conclude that $c_{n,k} = 0$ for all $n,k \in \mathbb{Z}$ with $n \neq k$. On the other hand, for $k \in \mathbb{Z}$

$$\begin{aligned} c_{k,k}\delta_{k+1} &= \mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})(y)(C\delta_k) \\ &= C\left(\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})(y)\delta_k\right) \\ &= c_{k+1,k+1}\delta_{k+1}\,. \end{aligned}$$

Therefore, the only operators in the commutant of $\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})(\mathbb{H})$ are scalar multiples of the identity, which is equivalent to the irreducibility of the representation $\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})$ by Schur's Lemma. Hence, the kernel of the irreducible monomial representation $\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})$ is a maximal two-sided ideal (cf. Theorem 3.5.3) given by $\mathcal{J}_\theta$.

For every irreducible representation π of $\mathbb{H}$ with $\mathcal{J}_\theta \subseteq \ker(\pi)$ one has $\ker(\pi) = \mathcal{J}_\theta$ due to the maximality of $\mathcal{J}_\theta$ which we deduce from the irreducibility of $\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s})$.

Consider $\theta = \frac{n}{d}$ with relatively prime $n, d \in \mathbb{N}$. We note that analysing invertibility in $\mathcal{Q}_\theta$ reduces to the study of monomial representations as well. Set

$$\mathbb{H}/Z(d) := \left\{ \begin{pmatrix} 1 & a & \bar{b} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad a, c \in \mathbb{Z} \text{ and } \bar{b} \in \mathbb{Z}/d\mathbb{Z} \right\}, \quad d \in \mathbb{N},$$

and note the isomorphism $\mathcal{Q}_\theta \cong C^*(\mathbb{H}/Z(d))$. The nilpotent group $\mathbb{H}/Z(d)$ is of Type I since $\mathbb{H}/Z(d)$ has normal abelian subgroups of finite index, e.g.,

$$\left\{ \begin{pmatrix} 1 & ad & \bar{b} \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}, \quad a, c \in \mathbb{Z} \text{ and } \bar{b} \in \mathbb{Z}/d\mathbb{Z} \right\}.$$

Hence, all irreducible representations are monomial by Theorem 3.2.5.

A Wiener Lemma can now be deduced from Corollary 3.3.4.

Note that in the general study of invertibility in this example we have not used Allan's local principle or any results from Section 3.4 explicitly.

We give another example of a group where Theorem 3.5.3 simplifies the analysis.

Example 3.5.2. Let us denote by $\mathfrak{D}$ the group

$$\left\{ \begin{pmatrix} 1 & a & c & f \\ 0 & 1 & b & e \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix} : a, b, c, d, e, f \in \mathbb{Z} \right\}.$$

One can easily verify that the center of this group is given by

$$\left\{ \begin{pmatrix} 1 & 0 & 0 & f \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : f \in \mathbb{Z} \right\} \simeq \mathbb{Z}$$

and that we have a localisation like in the Heisenberg case.

Let us choose a monomial representation. First note that $\mathcal{D}$ can be identified with a semi-direct product $D_1 D_2$ of the normal abelian subgroup

$$D_1 := \left\{ \begin{pmatrix} 1 & 0 & c & f \\ 0 & 1 & b & e \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : b, c, e, f \in \mathbb{Z} \right\} \simeq \mathbb{Z}^4$$

and the closed abelian subgroup D_2 which is given by

$$\left\{ \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \;:\; a, d \in \mathbb{Z} \right\} \simeq \mathbb{Z}^2 \,.$$

In such a situation the construction of induced representation becomes very easy. We refer to [18, Section 2.4] for all the details. Now let $\sigma_{\theta_b,\theta_c,\theta_e,\theta_f}$ be the one-dimensional representation of D_1 given by

$$\sigma_{\theta_b,\theta_c,\theta_e,\theta_f}\left(\begin{pmatrix} 1 & 0 & c & f \\ 0 & 1 & b & e \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) = e^{2\pi i \theta_b b} e^{2\pi i \theta_c c} e^{2\pi i \theta_e e} e^{2\pi i \theta_f f} \,.$$

The inclusion map from D_2 to $\mathcal{D}$ will serve as a cross-section. The induced representation $\mathrm{Ind}_{D_1}^{\mathcal{D}}(\sigma_{\theta_b,\theta_c,\theta_e,\theta_f})$ (is unitary equivalent to a representation which) acts on the Hilbert space $\ell^2(\mathbb{Z}^2, \mathbb{C})$ and is given by

$$\begin{aligned} &\left(\mathrm{Ind}_{D_1}^{\mathcal{D}}(\sigma_{\theta_b,\theta_c,\theta_e,\theta_f}) \left(\begin{pmatrix} 1 & 0 & c & f \\ 0 & 1 & b & e \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & a & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{pmatrix} \right) F \right)(k,l) \\ &= e^{2\pi i \theta_b b} e^{2\pi i \theta_c (c-kb)} e^{2\pi i \theta_e (e+lb)} e^{2\pi i \theta_f (f+lc-ke-klb)} F(k-a, l-d)\,, \end{aligned} \tag{3.5.2}$$

for every $a, b, c, d, e, f, k, l \in \mathbb{Z}$ and $F \in \ell^2(\mathbb{Z}^2, \mathbb{C})$.

The localisation fibres are indexed by θ_f. It is clear that for every irrational θ_f and arbitrary $\theta_b, \theta_c, \theta_e$,

$$\ker\left(\mathrm{Ind}_{D_1}^{\mathcal{D}}(\sigma_{\theta_b,\theta_c,\theta_e,\theta_f}) \right) = \mathcal{J}_{\theta_f} \,.$$

In the case of irrational θ_f, the commutant of $\mathrm{Ind}_{D_1}^{\mathcal{D}}(\sigma_{\theta_b,\theta_c,\theta_e,\theta_f})(\mathcal{D})$ in $\mathcal{B}(\ell^2(\mathbb{Z}^2,\mathbb{C}))$ is trivial which is equivalent to irreducibility by Schur's Lemma. Hence, for irrational θ_f the two-sided closed ideal $\mathcal{J}_{\theta_f}$ is maximal by Theorem 3.5.3 and one has to consider only a single representation, e.g., the one given in (3.5.2) for fixed parameters $\theta_b, \theta_c, \theta_e$, to study invertibility on these fibres.

3.6 Finite dimensional approximation

The following proposition follows from Theorem 3.2.11 and might be useful for checking invertibility of $f \in \mathbb{Z}[\mathbb{H}]$ via numerical simulations.

Proposition 3.6.1. *Let $f \in \mathbb{Z}[\mathbb{H}]$. Then α_f is expansive if and only if there exists a constant $C > 0$ such that $\pi(f)$ is invertible and $\|\pi(f)^{-1}\| \le C$ for every finite-dimensional irreducible representation π of $\mathbb{H}$.*

For the proof of the Proposition we work with the representations $\pi_\theta^{(1,1)}$ in (3.2.14). For irrational θ,

$$(\pi_\theta^{(1,1)}(x)h)(t) = h(t+\theta), \quad (\pi_\theta^{(1,1)}(y)h)(t) = e^{2\pi i t}h(t)\,, \tag{3.6.1}$$

for every $h \in L^2(\mathbb{T}, \lambda_{\mathbb{T}})$ and $t \in \mathbb{T}$. For rational θ of the form $\theta = p/q$ with $(p, q) = 1$, and for $s, t \in \mathbb{T}$, we replace the Lebesgue measure $\lambda = \lambda_{\mathbb{T}}$ in (3.6.1) by the uniform probability measure ν_q concentrated on the cyclic group $\{1/q, \dots, (q-1)/q, 1\} \subset \mathbb{T}$.

Proof. One direction is obvious. For the converse, assume that α_f is non-expansive, but that there exists a constant $C > 0$ such that $\pi(f)$ is invertible and $\|\pi(f)^{-1}\| \le C$ for every finite-dimensional irreducible representation π of $\mathbb{H}$.

Since α_f is non-expansive, there exists an irrational θ (by our assumption) such that the operator $\pi_\theta^{(1,1)}(f)$ has no bounded inverse due to Theorem 3.2.11 and its proof. Therefore, $\pi_\theta^{(1,1)}(f)$ is either not bounded from below or its range is not dense in the representation space or both.

We consider first the case where $\pi_\theta^{(1,1)}(f)$ is not bounded from below. Then there exists, for every $\varepsilon > 0$, an element $h_\varepsilon \in L^2(\mathbb{T}, \lambda_{\mathbb{T}})$ with $\|h_\varepsilon\|_2 = 1$ and $\|\pi_\theta^{(1,1)}(f)h_\varepsilon\|_2 < \varepsilon$. By approximating the h_ε by continuous functions we may obviously assume that each h_ε is continuous.

Let q be a rational prime, and let p satisfy $|\theta - p/q| < 1/q$. Then

$$\int |h_\varepsilon|^2 d\nu_q \quad \text{and} \quad \int |\pi_\theta^{(1,1)}(f)h_\varepsilon|^2 d\nu_q$$

are Riemann approximations to the corresponding integrals with respect to λ. Hence,

$$\lim_{q\to\infty} \int |h_\varepsilon|^2 d\nu_q = 1 \quad \text{and} \quad \lim_{q\to\infty} \int |\pi_\theta^{(1,1)}(f)h_\varepsilon|^2 d\nu_q \le \varepsilon^2\,.$$

Furthermore, as $q \to \infty$, $\pi_{p/q}^{(1,1)}(f)h_\varepsilon$ converges uniformly to $\pi_\theta^{(1,1)}(f)h_\varepsilon$. From this we deduce that

$$\limsup_{q\to\infty} \int |\pi_{p/q}^{(1,1)}(f)h_\varepsilon|^2 d\nu_q \le \varepsilon^2\,.$$

This clearly violates the hypothesis that $\pi_{p/q}^{(1,1)}(f)$, q prime, have uniformly bounded inverses.

Finally, assume that $\pi_\theta^{(1,1)}(f)$ has no dense image in $L^2(\mathbb{T}, \lambda)$. In that case the adjoint operator $(\pi_\theta^{(1,1)}(f))^* = \pi_\theta^{(1,1)}(f^*)$ is not injective [4]. Furthermore, by our assumptions, $\|\pi(f^*)^{-1}\| \le C$ for every finite-dimensional irreducible representation π of $\mathbb{H}$. The same arguments as in the first part of the proof lead to a contradiction. □

3.7 A connection to Time-Frequency-Analysis

In this section we explore a connection to Time-Frequency Analysis.

3.7.1 The twisted convolution algebra

In [12] we determined the explicit form of the ideals $\mathcal{J}_m$ for the group algebra $\ell^1(\mathbb{H},\mathbb{C})$ of the discrete Heisenberg group. Let us recall this result. We write a typical element in $\ell^1(\mathbb{H},\mathbb{C})$ in the normal form:

$$\sum_{(k,l,m)\in\mathbb{Z}^3} f_{(k,l,m)} x^k y^l z^m \,,$$

$f_{(k,l,m)} \in \mathbb{C}$ and $\sum_{(k,l,m)\in\mathbb{Z}^3} |f_{(k,l,m)}| < \infty$. We identify the center of $\ell^1(\mathbb{H},\mathbb{C})$ with $\ell^1(\mathbb{Z},\mathbb{C})$ since the center of the group $\mathbb{H}$ is generated by powers of z. The maximal ideal space $\mathrm{Max}(\ell^1(\mathbb{Z},\mathbb{C}))$ is canonically homeomorphic to $\widehat{\mathbb{Z}} \cong \mathbb{T}$. The smallest closed two-sided ideal in $\ell^1(\mathbb{H},\mathbb{C})$ containing $m_\theta \in \mathrm{Max}(\ell^1(\mathbb{Z},\mathbb{C}))$, $\theta \in \mathbb{T}$, is given by the subset $\mathcal{J}_\theta \subset \ell^1(\mathbb{H},\mathbb{C})$ which consists of all elements $f \in \ell^1(\mathbb{H},\mathbb{C})$ such that

$$f^\theta := \sum_{(k,l,m)\in\mathbb{Z}^3} f_{(k,l,m)} x^k y^l e^{2\pi i m\theta} = 0_{\ell^1(\mathbb{H},\mathbb{C})} \,.$$

The next definition plays an important role in the field of Time-Frequency-Analysis. Fix $\theta \in \mathbb{T}$. The *twisted convolution* $\natural_\theta$ on $\ell^1(\mathbb{Z}^2,\mathbb{C})$ is defined as follows. Let $a, b \in \ell^1(\mathbb{Z}^2,\mathbb{C})$, then

$$(a \natural_\theta b)_{m,n} = \sum_{k,l\in\mathbb{Z}} a_{k,l} b_{m-k,n-l} e^{2\pi i (m-k) l\theta} \,.$$

Moreover, define the involution $a^*_{k,l} = \overline{a_{-k,-l}} e^{2\pi i k l \theta}$ for every $a \in \ell^1(\mathbb{Z}^2,\mathbb{C})$. The triple $(\ell^1(\mathbb{Z}^2,\mathbb{C}), \natural_\theta, {}^*)$ forms a Banach-*-algebra – the so called *twisted convolution algebra*.

The Banach-algebras $\mathcal{Q}_\theta = \ell^1(\mathbb{H},\mathbb{C})/\mathcal{J}_\theta$ and $(\ell^1(\mathbb{Z}^2,\mathbb{C}), \natural_\theta, {}^*)$ are connected by the *-isomorphism $\kappa\colon \mathcal{Q}_\theta \longrightarrow (\ell^1(\mathbb{Z}^2,\mathbb{C}), \natural_\theta, {}^*)$ defined by

$$\kappa(\Phi_\theta(f)) = f^\theta \,.$$

[4] For an operator A acting on a Hilbert space $\mathcal{H}$ one has $(\ker A)^\perp = \overline{\mathrm{im} A^*}$.

3.7.2 Wiener's Lemma for twisted convolution algebras

Principal results were obtained by Janssen [17] and Gröchenig and Leinert [13]. Let α, β be strictly positive real parameters and let $\theta = \alpha\beta$. On the Hilbert space $L^2(\mathbb{R}, \mathbb{C})$ define the translation operator T_α and the modulation operator M_β as follows:

$$(T_\alpha F)(t) = F(t+\alpha) \quad \text{and} \quad (M_\beta F)(t) = e^{2\pi i \beta t} F(t) \tag{3.7.1}$$

where $F \in L^2(\mathbb{R}, \mathbb{C})$ and $t \in \mathbb{R}$. The representation $\pi_{\alpha,\beta}$ of $(\ell^1(\mathbb{Z}^2, \mathbb{C}), \natural_\theta, ^*)$ on $L^2(\mathbb{R}, \mathbb{C})$ is defined as follows: for each $a \in \ell^1(\mathbb{Z}^2, \mathbb{C})$, let

$$\pi_{\alpha,\beta}(a) = \sum_{k,l \in \mathbb{Z}} a_{k,l} T_\alpha^k M_\beta^l .$$

Gröchenig and Leinert established the following Wiener Lemma for twisted convolution algebras.

Theorem 3.7.1 ([13, Lemma 3.3]). *Suppose that $\theta \in \mathbb{T}$, $\alpha\beta = \theta \mod 1$ and that $a \in \ell^1(\mathbb{Z}^2, \mathbb{C})$ and $\pi_{\alpha,\beta}(a)$ is invertible on $L^2(\mathbb{R}, \mathbb{C})$. Then a is invertible in $(\ell^1(\mathbb{Z}^2, \mathbb{C}), \natural_\theta, ^*)$.*

The representation $\pi_{\alpha,\beta}$ of $(\ell^1(\mathbb{Z}^2, \mathbb{C}), \natural_\theta, ^*)$ induces a representation of $\mathbb{H}$, $\ell^1(\mathbb{H}, \mathbb{C})$ and $\mathcal{Q}_\theta$ on $L^2(\mathbb{R}, \mathbb{C})$ in a canonical way:

$$\pi_{\alpha,\beta}(x) = T_\alpha, \quad \pi_{\alpha,\beta}(y) = M_\beta, \quad \text{and} \quad \pi_{\alpha,\beta}(z) = e^{2\pi i \theta} .$$

The representations $\pi_{\alpha,\beta}$ appear in the literature under various names: Stone-von Neumann, Weyl-Heisenberg or Schrödinger representations of $\mathbb{H}_\mathbb{R}$.

As an immediate corollary of Theorem 3.7.1 one obtains the following Wiener Lemma for the discrete Heisenberg group.

Theorem 3.7.2. *Let $f \in \ell^1(\mathbb{H}, \mathbb{C})$, then f is invertible if and only if $\pi_{\alpha,\beta}(f)$ is invertible for each non-zero pair $\alpha, \beta \in \mathbb{R}$.*

Proof. The result follows by combining Allan's local principle with Wiener's Lemma for twisted convolution algebras. □

Finally, we give an alternative proof of Wiener's Lemma for the twisted convolution algebra. We need first the following lemmas.

Lemma 3.7.1. *The twisted convolution algebra $(\ell^1(\mathbb{Z}^2, \mathbb{C}), \natural_\theta, ^*)$ is symmetric.*

Proof. First, recall that the Banach algebras $(\ell^1(\mathbb{Z}^2, \mathbb{C}), \natural_\theta, ^*)$ and $\mathcal{Q}_\theta$ are *-isomorphic. For every $a \in \ell^1(\mathbb{H}, \mathbb{C})$ the following holds: if $\Phi_\theta(a) \in \mathcal{Q}_\theta$ is not invertible, then a is not invertible in $\ell^1(\mathbb{H},\mathbb{C})$ by Allan's local principle. Hence, $\sigma_{\mathcal{Q}_\theta}(\Phi_\theta(a)) \subseteq \sigma_{\ell^1(\mathbb{H},\mathbb{C})}(a)$ for every $a \in \ell^1(\mathbb{H},\mathbb{C})$. In particular, for every $a \in \ell^1(\mathbb{H},\mathbb{C})$,

$$\sigma_{\mathcal{Q}_\theta}(\Phi_\theta(a^*a)) \subseteq \sigma_{\ell^1(\mathbb{H},\mathbb{C})}(a^*a) \subseteq [0,\infty)$$

by the symmetry of $\ell^1(\mathbb{H},\mathbb{C})$. □

Lemma 3.7.2. *Consider irrational $\theta \in \mathbb{T}$. Then $a \in (\ell^1(\mathbb{Z}^2,\mathbb{C}), \natural_\theta, ^*)$ is invertible if and only if $\pi_{\alpha,\beta}$ has a bounded inverse, where $\alpha\beta = \theta \mod 1$.*

Proof. We just have to show that the non-invertibility of the element $a \in \mathcal{Q}_\theta$ (note $(\ell^1(\mathbb{Z}^2,\mathbb{C}), \natural_\theta, ^*) \simeq \mathcal{Q}_\theta$), for irrational θ, implies that a is not invertible in the irrational rotation algebra $\mathcal{R}_\theta$. Since $\mathcal{Q}_\theta$ is symmetric (cf. Lemma 3.7.1), there exists an irreducible unitary representation π of $\mathbb{H}$ such that π vanishes on $\mathcal{J}_\theta$ and $\pi(\Phi_\theta(a))v = 0$ for some non-zero vector $v \in \mathcal{H}_\pi$. This implies that a is not invertible in $\mathcal{R}_\theta$ and, in particular, not in its realisation $\pi_{\alpha,\beta}(C^*(\mathbb{H}))$ with $\alpha\beta = \theta \mod 1$. □

The proof of Lemma 3.7.2 basically says that for irrational θ the Banach algebra $\mathcal{Q}_\theta$ is inversed closed in $\mathcal{R}_\theta$.

The representation $\pi_{\alpha,\beta}$ of $\mathbb{H}$ can be decomposed in the following way (cf. [2]). Let ν be the Haar measure on $(0,\theta]$, where $\theta \in \mathbb{T}$ with $\theta = \alpha\beta \mod 1$. There is a unitary operator

$$U : L^2(\mathbb{R}) \longrightarrow \int_{(0,\theta]}^{\oplus} [\ell^2(\mathbb{Z},\mathbb{C})]_t \, d\nu(t)$$

and a family of representations $\{\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,s}) \,:\, s \in (0,\theta]\}$ such that $\pi_{\alpha,\beta}$ is unitary equivalent via U to the direct integral

$$\int_{(0,\theta]}^{\oplus} \mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,t}) \, d\nu(t) \,.$$

This decomposition tells us immediately that

$$\ker(\pi_{\alpha,\beta}) = \bigcap_{t\in(0,\theta]} \ker\left(\mathrm{Ind}_N^{\mathbb{H}}(\sigma_{\theta,t})\right) = \mathcal{J}_\theta$$

and hence that $\pi_{\alpha,\beta}(\ell^1(\mathbb{H},\mathbb{C})) \simeq \ell^1(\mathbb{H},\mathbb{C})/\mathcal{J}_\theta = \mathcal{Q}_\theta$. From this observation we get the following lemma.

Lemma 3.7.3. *Let $\theta \in \mathbb{T}$ be rational. Then $a \in \mathcal{Q}_\theta$ is invertible if and only if $\pi_{\alpha,\beta}(a)$ is invertible in $\mathcal{B}(L^2(\mathbb{R},\mathbb{C}))$.*

Proof of Theorem 3.7.1. Combine Lemma 3.7.2 and Lemma 3.7.3. □

Remark 3.7.4. The decomposition (3.7.2) of $\pi_{\alpha,\beta}$ depends only on the product $\alpha\beta = \theta \mod 1$ and hence is independent of the particular choice of α and β. Hence, in Theorem 3.7.1 and Theorem 3.7.2 one has to consider, e.g., $\alpha = \theta$ and $\beta = 1$ only.

3.7.3 An application to algebraic dynamical systems

As already mentioned in the Introduction the problem of deciding on the invertibility in $\ell^1(\mathbb{H}, \mathbb{C})$ has an application in algebraic dynamics. The following result is important to check invertibility for $f \in \mathbb{Z}[\mathbb{H}]$ in the group algebra $\ell^1(\mathbb{H}, \mathbb{C})$ because it tells us that $\pi_{\alpha,\beta}(f)$ has a trivial kernel in $L^2(\mathbb{R}, \mathbb{C})$ for $\alpha, \beta \neq 0$.

Theorem 3.7.3 ([22]). *Let G be a non-zero element in $L^2(\mathbb{R}, \mathbb{C})$, then for every finite set $A \subseteq \mathbb{Z}^2$ the set $\{T_\alpha^k M_\beta^l G \,:\, (k,l) \in A\}$ is linear independent over $\mathbb{C}$.*

The following result is a reformulation of Theorem 3.7.3 and gives a complete description of the spectrum of an operator $\pi_{\alpha,\beta}(f)$, for $\alpha, \beta \in \mathbb{R} \smallsetminus \{0\}$ and $f \in \mathbb{C}[\mathbb{H}]$, where $\mathbb{C}[\mathbb{H}]$ is the ring of functions $\mathbb{H} \longrightarrow \mathbb{C}$ with finite support.

Theorem 3.7.4. *Let $f \in \mathbb{C}[\mathbb{H}]$ with $f^\theta \neq 0$ for $\theta = \alpha\beta \neq 0$, $\alpha, \beta \in \mathbb{R}$, then $\lambda - \pi_{\alpha,\beta}(f)$ is injective and has dense range in $L^2(\mathbb{R}, \mathbb{C})$ but is not bounded from below for all $\lambda \in \sigma(\pi_{\alpha,\beta}(f))$.*

Proof. Suppose $f \in \mathbb{C}[\mathbb{H}]$ is such that $\pi_{\alpha,\beta}(f) \neq 0$ and $\lambda \in \sigma(\pi_{\alpha,\beta}(f))$. By Theorem 3.7.3, for every non-zero $G \in L^2(\mathbb{R}, \mathbb{C})$ the finite linear combination

$$(\lambda - \pi_{\alpha,\beta}(f))G = \left(\lambda - \sum_{(k,l,m)\in\mathbb{Z}^3} f_{(k,l,m)} T_\alpha^k M_\beta^l e^{2\pi i\theta m}\right) G \neq 0 \,.$$

This is equivalent to the injectivity of $\lambda - (\pi_{\alpha,\beta}(f))$.

Suppose the range of $\lambda - \pi_{\alpha,\beta}(f)$ is not dense in $L^2(\mathbb{R}, \mathbb{C})$. Then

$$(\pi_{\alpha,\beta}(\lambda - f))^* = \pi_{\alpha,\beta}((\lambda - f)^*)$$

is not injective (cf. the footnote on page 111) which is a contradiction to Theorem 3.7.3 because $(\lambda - f)^* \in \mathbb{C}[\mathbb{H}]$. Hence, $\lambda - \pi_{\alpha,\beta}(f)$ not being invertible on $L^2(\mathbb{R}, \mathbb{C})$ is equivalent to $\lambda - \pi_{\alpha,\beta}(f)$ not being bounded from below. □

Therefore, non-expansiveness of α_f can be checked via two different approaches:

- The dual of $\mathbb{H}$: Does there exists an irreducible representation π of $\mathbb{H}$ such that 0 is an eigenvalue of $\pi(f)$.

- Stone-von Neumann representations: For all Stone-von Neumann representations $\pi_{\alpha,\beta}$, 0 is an eigenvalue of $\pi_{\alpha,\beta}(f)$ if and only if $\pi_{\alpha,\beta}(f) = 0$; and $\pi_{\alpha,\beta}(f)$ is not invertible if and only if $\pi_{\alpha,\beta}(f)$ is not bounded from below.

Remark 3.7.5. The authors are not aware whether the approach based on Theorem 3.2.7 and the construction of the dual of $\mathbb{H}$ via ergodic quasi-invariant measures are well-known results in the field of Time-Frequency Analysis. It would be interesting to investigate whether this eigenvalue approach would simplify the problem of deciding on invertibility – at least – for some examples $f \in \ell^1(\mathbb{H}, \mathbb{C}) \smallsetminus \mathbb{C}[\mathbb{H}]$.

3.8 Examples

We now demonstrate how to apply Wiener's Lemma to obtain easily verifiable sufficient conditions for non-expansivity of a principal algebraic action.

Let $f \in \mathbb{Z}[\mathbb{H}]$ be of the form

$$f = g_1(y,z)x - g_0(y,z) \tag{3.8.1}$$

with $g_1(y,z), g_0(y,z) \in \mathbb{Z}[y,z] \simeq \mathbb{Z}[\mathbb{Z}^2]$.

We set

$$\mathsf{U}(g_i) = \{(\zeta,\chi) \in \mathbb{S}^2 \,:\, g_i(\zeta,\chi) = 0\},\ i = 0,1\,.$$

In [21] (cf. [12, Theorem 2.6] for a proof) the following result was established. For linear $f \in \mathbb{Z}[\mathbb{H}]$ of the form (3.8.1) with $\mathsf{U}(g_i) = \varnothing$ for $i = 0,1$, the action α_f is expansive if and only if

$$\int_{\mathbb{S}}\int_{\mathbb{S}} (\log|g_1(\zeta,\chi)| - \log|g_0(\zeta,\chi)|)\, \lambda_{\mathbb{S}}(d\zeta)\lambda_{\mathbb{S}}(d\chi) \neq 0\,,$$

where $\lambda_{\mathbb{S}}$ is the Lebesgue measure on $\mathbb{S}$. Equivalently, α_f is expansive if and only if

$$\mathfrak{m}(g_0) \neq \mathfrak{m}(g_1)\,,$$

where $\mathfrak{m}(h)$ is the so-called logarithmic Mahler measure of a polynomial $h \in \mathbb{C}[\mathbb{Z}^d]$ defined by

$$\mathfrak{m}(h) = \int_{\mathbb{T}^d} \log|h(e^{2\pi i\theta_1},\ldots,e^{2\pi i\theta_d})|\, d\theta_1\cdots d\theta_d\,.$$

In this section we use results on invertibility to derive criteria for non-expansiveness of principal actions of elements f in $\mathbb{Z}[\mathbb{H}]$ of the form $f = g_1(y,z)x - g_0(y,z)$ in cases when the unitary varieties $\mathsf{U}(g_0)$ and $\mathsf{U}(g_1)$ are not necessarily empty.

For every $\chi \in \mathbb{S}$, consider the rational function ψ_χ on $\mathbb{S}$:

$$\psi_\chi(\zeta) = \frac{g_0(\zeta, \chi)}{g_1(\zeta\chi^{-1}, \chi)}$$

and consider the map $\psi\colon \mathbb{N} \times \mathbb{S} \longrightarrow \mathbb{C}$ given by

$$\psi_\chi(n, \zeta) = \begin{cases} 1 & \text{if } n = 0 \\ \prod_{j=0}^{n-1} \psi_\chi(\zeta\chi^{-j}) & \text{if } n \geq 1. \end{cases}$$

3.8.1 Either $\mathsf{U}(g_0)$ or $\mathsf{U}(g_1)$ is a non-empty set

We fix the following notation. For every $\chi \in \mathbb{S}$ and $i = 0, 1$, put

$$\mathsf{U}_\chi(g_i) = \{\zeta \in \mathbb{S} \,:\, g_i(\zeta, \chi) = 0\},$$

and

$$g_{i,\chi}(y) = g_i(y, \chi),$$

which we will view as a Laurent polynomial in y with complex coefficients, i.e., $g_{i,\chi} \in \mathbb{C}[\mathbb{Z}]$ for every χ and $i = 0, 1$. Note also that the set $\mathsf{U}_\chi(g_i)$ is infinite if and only if $g_{i,\chi}$ is the zero polynomial.

For notational convenience we put

$$\phi_\chi(\zeta) = \log|\psi_\chi(\zeta)| \quad \text{and} \quad \phi_\chi(n, \zeta) = \log|\psi_\chi(n, \zeta)|,$$

for every $\zeta \in \mathbb{S}$ and $n \geq 0$.

Theorem 3.8.1. *Let $f \in \mathbb{Z}[\mathbb{H}]$ be of the form $f = g_1(y, z)x - g_0(y, z)$. Suppose there exists an element $\chi \in \mathbb{S}$ of infinite order which satisfies either of the following conditions.*

(i) $\mathsf{U}_\chi(g_0) = \varnothing$, $\mathsf{U}_\chi(g_1) \neq \varnothing$ *and* $\int \phi_\chi d\lambda_{\mathbb{S}} < 0$.

(ii) $\mathsf{U}_\chi(g_0) \neq \varnothing$, $\mathsf{U}_\chi(g_1) = \varnothing$ *and* $\int \phi_\chi d\lambda_{\mathbb{S}} > 0$.

Then α_f is non-expansive.

Proof. We will prove only the first case, the second case can be proven similarly.

Suppose f is such that (X_f, α_f) is expansive and the conditions in (i) are satisfied. We will now show that certain consequences of expansivity of α_f are inconsistent with the conditions in (i). Hence, by arriving to a contradiction, we will prove that under (i) α_f is not expansive.

We know that (X_f, α_f) is expansive if and only if f is invertible in $\ell^1(\mathbb{H}, \mathbb{C})$. Hence (X_f, α_f) is expansive if and only if there exists a $w \in \ell^1(\mathbb{H}, \mathbb{C})$,

$$w = \sum_{k,l,m} w_{k,l,m} y^l x^k z^m \,,$$

such that

$$f \cdot w = w \cdot f = 1_{\ell^1(\mathbb{H},\mathbb{C})} \,.$$

Suppose $\theta \in (0, 1]$ is irrational and that $\chi = e^{2\pi i\theta} \in \mathbb{S}$ satisfies condition (i). Consider the following representation $\pi_{1,\theta}$ of $\ell^1(\mathbb{H}, \mathbb{C})$ on $L^2(\mathbb{R}, \mathbb{C})$, defined by

$$(\pi_{1,\theta}(x)F)(t) = T_1 F(t) = F(t+1), \quad (\pi_{1,\theta}(y)F)(t) = M_\theta F(t) = e^{2\pi i\theta t} F(t),$$
$$\text{and} \quad (\pi_{1,\theta}(z)F)(t) = e^{2\pi i\theta} F(t) \,.$$

If

$$f = g_1(y, z)x - g_0(y, z) \quad \text{and} \quad w = f^{-1} = \sum_{k,l,m} w_{k,l,m} y^k x^l z^m \,,$$

then

$$\pi_{1,\theta}(f) = g_1(e^{2\pi i\theta t}, e^{2\pi i\theta})T_1 - g_0(e^{2\pi i\theta t}, e^{2\pi i\theta})$$

and

$$\begin{aligned} \pi_{1,\theta}(w) &= \sum_{(k,l,m)\in\mathbb{Z}^3} w_{k,l,m} M_\theta^k T_1^l \chi^m \\ &= \sum_{l\in\mathbb{Z}} \Big[\sum_{(k,m)\in\mathbb{Z}^2} w_{(k,l,m)} e^{2\pi i\theta tk} e^{2\pi i\theta m} \Big] T_1^l \,. \end{aligned}$$

Set

$$P_{l,\theta}(t) \coloneqq \sum_{(k,m)\in\mathbb{Z}^2} w_{(k,l,m)} e^{2\pi i\theta tk} e^{2\pi i\theta m} \,,$$

then $\pi_{1,\theta}(w) = \sum_{l\in\mathbb{Z}} P_{l,\theta}(t) T_1^l$.

The functions $P_{l,\theta}(\cdot)\colon \mathbb{R} \longrightarrow \mathbb{C}$, $l \in \mathbb{Z}$, are bounded and continuous. Indeed, for any $l \in \mathbb{Z}$

$$P_{l,\theta}(t) = \sum_{(k,m)\in\mathbb{Z}^2} w_{(k,l,m)} e^{2\pi i\theta tk} e^{2\pi i\theta m}$$

is a Fourier series with absolutely convergent coefficients:

$$\begin{aligned} \sum_{k\in\mathbb{Z}} \Big| \sum_{m\in\mathbb{Z}} w_{(k,l,m)} e^{2\pi i\theta m} \Big| &\le \sum_{k\in\mathbb{Z}} \sum_{m\in\mathbb{Z}} |w_{(k,l,m)}| \\ &\le \|w\|_{\ell^1(\mathbb{H},\mathbb{C})} < \infty \,. \end{aligned}$$

For similar reasons,

$$\sum_{l\in\mathbb{Z}} \sup_{t\in\mathbb{R}} |P_{l,\theta}(t)| \leq \sum_{l\in\mathbb{Z}} \Big[\sum_{k,m} |w_{k,l,m}|\Big] \tag{3.8.2}$$

$$= \|w\|_{\ell^1(\mathbb{H},\mathbb{C})} < \infty\,. \tag{3.8.3}$$

Since $w \cdot f = 1_{\ell^1(\mathbb{H},\mathbb{C})}$ and $\pi_{1,\theta}(1_{\ell^1(\mathbb{H},\mathbb{C})}) = 1_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}))}$ – the identity operator on $L^2(\mathbb{R},\mathbb{C})$, one has

$$\begin{aligned}
1_{\mathcal{B}(L^2(\mathbb{R},\mathbb{C}))} &= \pi_{1,\theta}(w)\pi_{1,\theta}(f) \\
&= \Big[\sum_{l\in\mathbb{Z}} P_{l,\theta}(t)T_1^l\Big] \cdot \Big[g_1(e^{2\pi i\theta t}, e^{2\pi i\theta})T_1 - g_0(e^{2\pi i\theta t}, e^{2\pi i\theta})\Big] \\
&= \Big[\sum_{l\in\mathbb{Z}} P_{l,\theta}(t)T_1^l\Big] \cdot \Big[g_{1,\chi}(e^{2\pi i\theta t})T_1 - g_{0,\chi}(e^{2\pi i\theta t})\Big] \\
&= \sum_{l\in\mathbb{Z}} \Big[P_{l-1,\theta}(t)g_{1,\chi}(e^{2\pi i\theta(t+l-1)}) - P_{l,\theta}(t)g_{0,\chi}(e^{2\pi i\theta(t+l)})\Big]T_1^l\,.
\end{aligned}$$

Set

$$Q_{l,\theta}(t) = P_{l-1,\theta}(t)g_{1,\chi}(e^{2\pi i\theta(t+l-1)}) - P_{l,\theta}(t)g_{0,\chi}(e^{2\pi i\theta(t+l)})\,. \tag{3.8.4}$$

Since $\{Q_{l,\theta}(\cdot)|\, l \in \mathbb{Z}\}$ are again bounded continuous functions, one concludes that

$$Q_{0,\theta}(t) \equiv 1 \quad \text{and} \quad Q_{l,\theta}(t) \equiv 0, \quad \text{for every } l \neq 0.$$

Hence, for every $t \in \mathbb{R}$, one has

$$Q_{0,\theta}(t) = P_{-1,\theta}(t)g_{1,\chi}(e^{2\pi i\theta(t-1)}) - P_{0,\theta}(t)g_{0,\chi}(e^{2\pi i\theta t}) = 1\,, \tag{3.8.5}$$

and for every $l \geq 1$

$$Q_{l,\theta}(t) = P_{l-1,\theta}(t)g_{1,\chi}(e^{2\pi i\theta(t+l-1)}) - P_{l,\theta}(t)g_{0,\chi}(e^{2\pi i\theta(t+l)}) = 0\,. \tag{3.8.6}$$

Since $\mathsf{U}_\chi(g_0) = \varnothing$, equations (3.8.6) imply that

$$P_{l,\theta}(t) = P_{l-1,\theta}(t) \cdot \frac{g_{1,\chi}(e^{2\pi i\theta(t+l-1)})}{g_{0,\chi}(e^{2\pi i\theta(t+l)})}$$

for every t, and hence for each $l \geq 1$ and every $t \in \mathbb{R}$, one has

$$P_{l,\theta}(t) = P_{0,\theta}(t) \cdot \frac{g_{1,\chi}(e^{2\pi i\theta t})}{g_{0,\chi}(e^{2\pi i\theta(t+1)})} \cdots \frac{g_{1,\chi}(e^{2\pi i\theta(t+l-1)})}{g_{0,\chi}(e^{2\pi i\theta(t+l)})}\,,$$

or

$$P_{l,\theta}(t) = P_{0,\theta}(t)\frac{1}{\psi_{\chi^{-1}}(l, \zeta_t\chi)}$$

where $\zeta_t = e^{2\pi i\theta t}$.

Then since $\mathsf{U}_\chi(g_0) = \varnothing$, the logarithmic Mahler measure $\mathfrak{m}(g_{0,\chi})$ is finite, and hence $\mathfrak{m}(g_{1,\chi}) > -\infty$. Therefore, $g_{1,\chi}(\eta)$ is not identically 0 on $\mathbb{S}^1$, and since $g_{1,\chi}$ is a polynomial, we can conclude that $\mathsf{U}_\chi(g_1)$ is finite. Therefore, the set of points

$$\mathcal{B}_1 = \left\{\zeta \in \mathbb{S}^1 : \zeta e^{2\pi i\theta k} \in \mathsf{U}_\chi(g_1) \quad \text{for some } k \in \mathbb{Z}\right\} = \bigcup_{k\in\mathbb{Z}} R_\theta^k(\mathsf{U}_\chi(g_1))$$

is at most countable, and hence has Lebesgue measure 0.

Both functions $\log|g_{0,\chi}(\cdot)|$ and $\log|g_{1,\chi}(\cdot)|$ are integrable. Moreover, the irrational rotation $R_\theta\colon \mathbb{T} \longrightarrow \mathbb{T}$ is an ergodic transformation. By Birkhoff's ergodic theorem there exists a set $\mathcal{B}_2 \subset \mathbb{S}^1$ of full Lebesgue measure such that for any $\zeta \in \mathcal{B}_2$

$$\frac{1}{n}\sum_{k=1}^{n} \log|g_{0,\chi}(\zeta e^{2\pi i\theta k})| \to \mathfrak{m}(g_{0,\chi})\,,$$
$$\frac{1}{n}\sum_{k=0}^{n-1} \log|g_{1,\chi}(\zeta e^{2\pi i\theta k})| \to \mathfrak{m}(g_{1,\chi})\,.$$

Therefore, since $\mathfrak{m}(g_{1,\chi}) > \mathfrak{m}(g_{0,\chi})$, on the set of full measure $\mathcal{B}_1^c \cap \mathcal{B}_2$

$$\Psi_n(\zeta) = \frac{1}{\psi_{\chi^{-1}}(n, \zeta\chi)} \neq 0 \quad \forall n \geq 1, \tag{3.8.7}$$

and

$$\lim_{n\to\infty} \Psi_n(\zeta) = +\infty\,. \tag{3.8.8}$$

Since $\theta \neq 0$, the set of points

$$\mathcal{C} = \left\{t \in \mathbb{R} \,:\, e^{2\pi i t\theta} \notin \mathcal{B}_1^c \cap \mathcal{B}_2\right\}$$

has full measure, and for every $t \in \mathcal{C}$, one has that the sum

$$\sum_{l\geq 0} |P_{l,\theta}(t)| = |P_{0,\theta}(t)| + |P_{0,\theta}(t)|\Psi_1(e^{2\pi i t\theta}) + \ldots + |P_{0,\theta}(t)|\Psi_l(e^{2\pi i t\theta}) + \ldots \tag{3.8.9}$$

is finite if and only if $|P_{0,\theta}(t)| = 0$. Combining this fact with the uniform bound (3.8.2) – (3.8.3), we are able to conclude that

$$P_{0,\theta}(t) = 0 \tag{3.8.10}$$

on a set of full measure in $\mathbb{R}$. The function $P_{0,\theta}(t)$ is continuous and therefore, $P_{0,\theta}$ must be the identically zero function on $\mathbb{R}$.

Finally, consider the remaining equation (3.8.5) for $Q_{0,\theta}(t)$. Since $P_{0,\theta}(t) \equiv 0$, one has that there exists a continuous bounded function $P_{-1,\theta}$ such that

$$P_{-1,\theta}(t) g_{1,\chi}(e^{2\pi i\theta(t-1)}) = 1 \,, \tag{3.8.11}$$

for every $t \in \mathbb{R}$. However, since the unitary variety $\mathsf{U}_\chi(g_1)$ is not empty, one can find $t \in \mathbb{R}$ such that

$$g_{1,\chi}(e^{2\pi i\theta(t-1)}) = 0 \,,$$

and hence, (3.8.11) cannot be satisfied. Therefore, we arrived to a contradiction with the earlier assumption that α_f is expansive. $\square$

The assumption that $\int \phi_\chi d\lambda_{\mathbb{S}} < 0$ cannot be dropped in (i) of Theorem 3.8.1 as the following simple minded example shows.

Example 3.8.1. Suppose χ is not a root of unity and $\mathsf{U}_\chi(g_1) \neq \varnothing$. Set $g_0(y,z) \equiv K$, where we pick $K \in \mathbb{N}$ such that

1. $K > \|g_1(y,z)\|_{\ell^1(\mathbb{H},\mathbb{C})}$;
2. $\int \phi_\chi d\lambda_{\mathbb{S}} > 0$.

Then $f = g_1(y,z)x - K$ is invertible since

$$f = K\left(\frac{g_1(y,z)x}{K} - 1\right) \quad \text{and} \quad \left\|\frac{g_1(y,z)x}{K}\right\|_{\ell^1(\mathbb{H},\mathbb{C})} < 1 \,.$$

3.8.2 The sets $\mathsf{U}(g_0)$ and $\mathsf{U}(g_1)$ are both non-empty

Let us denote by

$$\mathrm{Orb}_\chi(\zeta) = \{\zeta\chi^n \,:\, n \in \mathbb{Z}\}$$

the orbit of ζ under the circle rotation $R_\chi\colon \mathbb{S} \longrightarrow \mathbb{S}$ with $\zeta \mapsto \zeta\chi$, for every $\zeta \in \mathbb{S}$. We consider first linear elements $f = g_1(y,z)x - g_0(y,z)$ for which

1. $\mathsf{U}(g_0) \neq \varnothing$ and $\mathsf{U}(g_1) \neq \varnothing$;
2. and there exists an element $(\zeta,\chi) \in \mathsf{U}(g_0)$ with

$$\{(\eta,\chi) \in \mathbb{S}^2 \,:\, \eta \in \mathrm{Orb}_\chi(\zeta)\} \cap \mathsf{U}(g_1) \neq \varnothing \,.$$

Theorem 3.8.2. *If f is of the form* (3.8.1) *and for some $m \in \mathbb{Z}$*

$$\mathsf{U}(g_1) \cap \{(\xi\chi^m,\chi) \in \mathbb{S}^2 \,:\, (\xi,\chi) \in \mathsf{U}(g_0)\} \neq \varnothing \,,$$

then α_f is non-expansive.

Although, this result could be proven with the help of Stone-von Neumann representations as well, it is more suitable to use monomial representations. For the following discussion it is convenient to work with slightly modified versions of the monomial representations defined in (3.5.1). For every $\zeta, \chi \in \mathbb{S}$ let $\pi^{(\zeta,\chi)}$ be the representation of $\mathbb{H}$ acting on $\ell^2(\mathbb{Z},\mathbb{C})$ which fulfils

$$(\pi^{(\zeta,\chi)}(x)F)(n) = F(n+1), \quad (\pi^{(\zeta,\chi)}(y)F)(n) = \zeta\chi^n F(n) \quad \text{and} \tag{3.8.12}$$

$$(\pi^{(\zeta,\chi)}(z)F)(n) = \chi F(n) \tag{3.8.13}$$

for each $F \in \ell^2(\mathbb{Z},\mathbb{C})$ and $n \in \mathbb{Z}$.

Proof. Consider the case $m \geq 0$ first. Suppose f is invertible and hence $\pi(f)$ is invertible for every unitary representations of $\mathbb{H}$ and in particular, $\pi^{(\zeta,\chi)}(f)$ is invertible for every pair $(\zeta,\chi) \in \mathbb{S}^2$. By the assumptions of theorem there exists a pair $(\xi,\chi) \in \mathbb{S}^2$ such that

$$g_1(\xi\chi^m,\chi) = 0 \quad \text{and} \quad g_0(\xi,\chi) = 0\,. \tag{3.8.14}$$

Without loss of generality we may assume that m is the minimal power such that (3.8.14) is satisfied, i.e., $g_1(\xi\chi^l,\chi) \neq 0$ for each $l \in \mathbb{Z}$ with $0 \leq l \leq m-1$.

Suppose $G \in \ell^2(\mathbb{Z},\mathbb{C})$ is in the image of $\pi^{(\xi,\chi)}(f)$, then there exists an $F \in \ell^2(\mathbb{Z},\mathbb{C})$ such that

$$G(n) = g_1(\xi\chi^n,\chi)F(n+1) - g_0(\xi\chi^n,\chi)F(n) \tag{3.8.15}$$

holds for every $n \in \mathbb{Z}$. Given the choice of (ξ,χ) (cf. (3.8.14)), one immediately concludes that

$$G(0) = g_1(\xi,\chi)F(1)\,.$$

If $m = 0$, then $G(0) = 0$, and we arrive to a contradiction with the assumption that $\pi^{(\xi,\chi)}(f)$ is invertible, and hence has a dense range in $\ell^2(\mathbb{Z},\mathbb{C})$: Indeed, for every $F \in \ell^2(\mathbb{Z},\mathbb{C})$ one has that

$$(\pi^{(\xi,\chi)}(f)F)(0) = 0$$

and hence, the range of $\pi^{(\xi,\chi)}(f)$ is not dense.

If $m > 0$, then F must satisfy the following system of linear equations

$$G(0) = g_1(\xi,\chi)F(1) \tag{3.8.16}$$

$$G(l) = g_1(\xi\chi^l,\chi)F(l+1) - g_0(\xi\chi^l,\chi)F(l), \quad 1 \leq l \leq m-1 \tag{3.8.17}$$

$$G(m) = -g_0(\xi\chi^m,\chi)F(m)\,. \tag{3.8.18}$$

We can eliminate $F(1), F(2), \ldots, F(m)$ in (3.8.16) – (3.8.18) to obtain an expression for $G(m)$ in terms of $G(0), G(1), \ldots, G(m-1)$. Indeed, one can easily verify that, for each $0 \le l \le m-1$, $F(l+1)$ can be written as

$$F(l+1) = \frac{G(l)}{g_1(\xi\chi^l,\chi)} + \frac{g_0(\xi\chi^l,\chi)}{g_1(\xi\chi^l,\chi)} F(l) = \dots \tag{3.8.19}$$

$$= \sum_{i=0}^{l} \frac{G(l-i)}{g_1(\xi\chi^l,\chi)} \prod_{n=0}^{i-1} \frac{g_0(\xi\chi^{l-n},\chi)}{g_1(\xi\chi^{l-n-1},\chi)} \tag{3.8.20}$$

$$= \sum_{i=0}^{l} \frac{G(l-i)}{g_1(\xi\chi^l,\chi)} \psi_\chi(i,\xi\chi^l) \tag{3.8.21}$$

(we use the convention that the empty product $\prod_\varnothing$ is equal to 1). Due to our choice of m, $F(l+1)$ in (3.8.19) is well-defined. Moreover, since $G(m) = g_0(\xi\chi^m,\chi)F(m)$, one gets

$$\begin{aligned} G(m) &= -g_0(\xi\chi^m,\chi)F(m) \\ &= -\sum_{i=0}^{m-1} G(m-1-i)\psi_\chi(i+1,\xi\chi^{m-1}) . \end{aligned}$$

Hence, $G(m)$ depends continuously on the values $G(0), G(1), \dots, G(m-1)$. Again, this is a contradiction with our hypothesis that $\pi^{(\xi,\chi)}(f)$ has dense range in $\ell^2(\mathbb{Z},\mathbb{C})$.

If $m < 0$, then we choose π such that

$$(\pi(x)F)(n) = F(n-1) \quad \text{and} \quad (\pi(y)F)(n) = \xi\chi^{-n}F(n)$$

for each $F \in \ell^2(\mathbb{Z},\mathbb{C})$ and $n \in \mathbb{Z}$. Exactly the same arguments as in the case $m \ge 0$ can be used to get a contradiction to the invertibility of $\pi(f)$. $\square$

Corollary 3.8.2. *Let f be of the form* (3.8.1) *which satisfies the conditions of Theorem* 3.8.2. *Then $\alpha_{f^\diamond}$ defined by $f^\diamond = g_0(y,z)x - g_1(y,z)$ is non-expansive.*

Proof. Since

$$\mathsf{U}(g_1) \cap \{(\xi\chi^m,\chi) \in \mathbb{S}^2 \,:\, (\xi,\chi) \in \mathsf{U}(g_0)\} \neq \varnothing ,$$

there exists a $k \in \mathbb{Z}$ such that

$$\mathsf{U}(g_0) \cap \left\{(\xi\chi^k,\chi) \in \mathbb{S}^2 \,:\, (\xi,\chi) \in \mathsf{U}(g_1)\right\}$$

is non-empty as well. Hence, Theorem 3.8.2 guarantees the non-expansiveness of $\alpha_{f^\diamond}$. $\square$

Example 3.8.3. Consider

$$g_1(y,z) = 1 - y - y^{-1} - z - z^{-1} \quad \text{and} \quad g_0(y,z) = 3 - y - y^{-1} - z - z^{-1} .$$

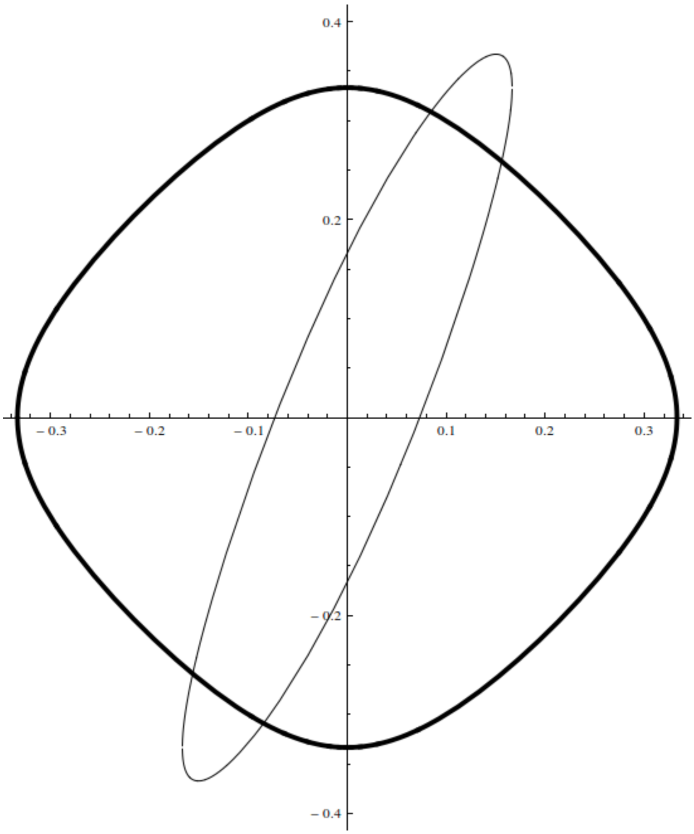

Figure 3.1: In this Figure the curves corresponding to the solution sets K (thick line) and $K[2]$ (thin line) are plotted.

We will show that the dynamical systems (X_f, α_f) and $(X_{f^\diamond}, \alpha_{f^\diamond})$, with $f = g_1(y,z)x - g_0(y,z)$ and $f^\diamond = g_0(y,z)x - g_1(y,z)$, are non-expansive.

For this purpose we introduce, for $m \in \mathbb{Z}$, the 'm-sheared version' of $g_0(y,z)$ given by

$$g_0^{(m)}(y,z) = g_0(yz^m, z) = 3 - yz^m - y^{-1}z^{-m} - z - z^{-1}\,;$$

and note that

$$\mathsf{U}(g_1) \cap \{(\xi\chi^m, \chi) \in \mathbb{S}^2 \,:\, (\xi,\chi) \in \mathsf{U}(g_0)\} \neq \varnothing \iff \mathsf{U}(g_1) \cap \mathsf{U}(g_0^{(m)}) \neq \varnothing\,.$$

The Fourier transforms of $g_1(y,z)$ and $g_0^{(m)}(y,z)$ are given by the functions

$$\begin{aligned}(\mathcal{F}g_1)(s,t) &= 1 - 2\cos(2\pi s) - 2\cos(2\pi t) \quad \text{and}\\ \left(\mathcal{F}g_0^{(m)}\right)(s,t) &= 3 - 2\cos(2\pi(s+mt)) - 2\cos(2\pi t)\,,\end{aligned}$$

respectively. Let

$$\begin{aligned}K &= \{(s,t) \in \mathbb{T}^2 \,:\, (\mathcal{F}g_1)(s,t) = 0\} \quad \text{and}\\ K[m] &= \left\{(s,t) \in \mathbb{T}^2 \,:\, \left(\mathcal{F}g_0^{(m)}\right)(s,t) = 0\right\}\,.\end{aligned}$$

Fix $m \in \mathbb{Z}$. By solving the equations

$$(\mathcal{F}g_1)(s,t) = 0 \quad \text{and} \quad \left(\mathcal{F}g_0^{(m)}\right)(s',t') = 0$$

for s and s' we get curves $s(t)$ and $s'(t')$ corresponding to the solution sets K and $K[m]$. If these curves intersect, then K and $K[m]$ have a non-empty intersection. It is clear that $(s,t) \in K$ if and only if $(e^{2\pi is}, e^{2\pi it}) \in \mathsf{U}(g_1)$. For every $m \in \mathbb{Z}$ the sets $K[m]$ and $\mathsf{U}(g_0^{(m)})$ are related in the same way.

The sets K and $K[2]$ have a non-empty intersection as Figure 3.1 shows; while $K \cap K[0] = \varnothing$ and $K \cap K[1] = \varnothing$.

Since the conditions of Theorem 3.8.2 and Corollary 3.8.2 are satisfied, f and $f^\diamond$ are not invertible.

The next result can be easily deduced from the proof of Theorem 3.8.1.

Theorem 3.8.3. *Let $f \in \mathbb{Z}[\mathbb{H}]$ be of the form* (3.8.1). *Suppose there exists an element $\chi \in \mathbb{S}$ of infinite order such that the following conditions are satisfied*

$$\mathsf{U}_\chi(g_0) \neq \varnothing \quad \textit{and} \quad \mathsf{U}_\chi(g_1) \neq \varnothing\,, \tag{3.8.22}$$

and

$$\mathfrak{m}(g_{0,\chi}) \neq \mathfrak{m}(g_{1,\chi})\,.$$

Then α_f is non-expansive.

Proof. Suppose (3.8.22) is satisfied. Let us first treat the trivial cases.

If $g_{0,\chi}(y)$ is the zero-element in $\mathbb{C}[\mathbb{Z}]$, then for every $\zeta \in \mathbb{S}$

$$\pi^{(\zeta,\chi)}(f) = \pi^{(\zeta,\chi)}(g_1(y,z)x)\,.$$

Fix $\xi \in \mathsf{U}_\chi(g_1)$, which is a non-empty set by the assumptions of the theorem. Since one has $(\pi^{(\xi,\chi)}(f)F)(0)$ is equal to 0, for every $F \in \ell^2(\mathbb{Z},\mathbb{C})$, 0 is an element of $\sigma(\pi^{(\xi,\chi)}(f))$ and hence f is not invertible. The same conclusions can be drawn for the cases $g_{1,\chi} = 0_{\mathbb{C}[\mathbb{Z}]}$ and $g_{0,\chi} = g_{1,\chi} = 0_{\mathbb{C}[\mathbb{Z}]}$.

Next consider the case where $g_{0,\chi}$ and $g_{1,\chi}$ are not the zero elements in $\mathbb{C}[\mathbb{Z}]$, which implies that $\mathfrak{m}(g_{0,\chi})$ and $\mathfrak{m}(g_{1,\chi})$ are finite and moreover $\mathsf{U}_\chi(g_0)$ and $\mathsf{U}_\chi(g_1)$ are finite sets. Suppose that $\mathfrak{m}(g_{0,\chi}) < \mathfrak{m}(g_{1,\chi})$. We follow the line of arguments in the proof of Theorem 3.8.1. The only adaption one has to make is to take the countable set

$$\mathcal{B} = \left\{ t \in \mathbb{R} \,:\, e^{2\pi i t}\chi^k \in \mathsf{U}_\chi(g_0) \text{ for some } k \in \mathbb{Z} \right\}$$

into consideration, i.e., to exclude points in $\mathcal{B}$ in the equations (3.8.7) – (3.8.10).

The case $\mathfrak{m}(g_{0,\chi}) > \mathfrak{m}(g_{1,\chi})$ can be proven analogously. □

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