Random walks in dynamic random environments

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Chapter 5

Law of large numbers for one-dimensional transient RW on the exclusion process

5.1 Introduction and result

In this chapter we present some results from an ongoing project with R.S. dos Santos and F. Völlering.

5.1.1 Slow-mixing REs and the exclusion process

In Chapter 2 we derived a LLN for the RW in (2.6) when the dynamic RE has the cone-mixing property in Definition 2.1. In particular, Theorem 2.2 holds for the more general model in Section 1.3.2 in which the RW X has two different (not only opposite) drifts $\alpha_0 - \beta_0$ and $\alpha_1 - \beta_1$ on top of holes and particles, respectively. The weak point of Theorem 2.2 is that many natural and interesting examples of dynamic REs are not cone-mixing, e.g., conservative dynamics like the exclusion process or, more generally, Kawasaki dynamics.

It is worthwhile to investigate examples of slow-mixing REs, because significantly different behavior may occur compared to fast-mixing REs, such as cone-mixing REs. Indeed, in Chapter 4 we have already met the case of a RW X on the one-dimensional simple symmetric exclusion (SSE) with opposite drifts on top of particles and holes (i.e., $\alpha_1 - \beta_1 = \beta_0 - \alpha_0$). In particular, in Section 4.1.4 we presented the results of some simulations for the asymptotic speed of X, which suggest that X is recurrent if and only if $\rho = \frac{1}{2}$, and that X is ballistic as soon as it is transient. Thus, the transient regime with zero speed, which is known to occur for static REs (see Section 1.1.1.1), does not survive in the dynamic setup, because even a 'slow' motion of the particles in the RE makes it hard for a 'trap' to survive. Nevertheless, similarly to the one-dimensional static RE and in contrast to the fast-mixing dynamic RE, Proposition 4.4 shows that when we look at large deviation estimates for the empirical speed of X, the slow-mixing properties of the exclusion process allow for a 'trap' to persist up to time t with a probability that is decaying sub-exponentially in t. Furthermore, similarly to the static RE (see Section 1.1.1.2), we may expect a sub-diffusive scaling limit for X to occur at least in the recurrent case, i.e., for $\rho = \frac{1}{2}$.

These results and observations motivate the interest in slow-mixing REs. In this chapter we prove a LLN under a somewhat strong drift condition, which represents a small step forward. At the end we mention some further extensions that are still part of a work in progress.

5.1.2 Model and main theorem

Consider a dynamic RE ξ constituted by a SSE (see Section 4.1.3) starting from a Bernoulli product measure ν_ρ of density ρ. Let

$$X = (X)_{t\geq 0} \tag{5.1}$$

be the RW in dynamic RE defined in Section 1.3.2, under the the following drift conditions:

$$\alpha_1 > \alpha_0 > \beta_0 > \beta_1 > 0, \quad \alpha_1 + \beta_1 = \alpha_0 + \beta_0, \quad \alpha_0 - \beta_0 > 1. \tag{5.2}$$

Note that the jump rate of the SSE equals 1 and that the latter condition implies

$$\liminf_{t\to\infty} X_t/t \geq \alpha_0 - \beta_0 > 1 \qquad \mathbb{P}_{\nu_\rho,0} - a.s. \tag{5.3}$$

Theorem 5.1. *Assume (5.2). Then, for any $\rho \in (0,1)$, there exists a constant $v > 1$ such that*

$$\lim_{t\to\infty} X_t/t = v \qquad \mathbb{P}_{\nu_\rho,0} - a.s. \tag{5.4}$$

5.2 Proof of Theorem 5.1

The main idea in the proof is that, under the third condition in (5.2), X travels to the right faster than the 'information' in the RE. As a consequence, it is possible to construct certain regeneration times at which the RE to the right of X is *freshly* sampled from its equilibrium distribution.

5.2.1 Coupling and minimal walker

In this section we show that the RW X defined in (5.1) can be constructed from an independent homogeneous RW and the RE. In particular, the following construction is valid for any general dynamic RE constituted by an IPS $\xi = (\xi_t)_{t \geq 0}$.

Let $M = (M_t)_{t \geq 0}$ be a homogeneous continuous-time RW with jump rates α_0 and β_0, to the right and to the left, respectively. Let $(b_n)_{n \in \mathbb{N}}$ an i.i.d. sequence of Bernoulli random variables with parameter $(\alpha_1 - \alpha_0)/\beta_0$. The path of the RW X in (5.1) can be constructed as a function of

$$(M, (b_n)_{n \in \mathbb{N}}, \xi) \tag{5.5}$$

by using the following rules:

1. $M_0 = X_0 = 0$.
2. X jumps only when M jumps.
3. If M_t jumps to the right at time t, then so does X_t.
4. If M_t jumps to the left at time t and X_t is on top of a hole, i.e., $\xi_t(X_t) = 0$, then X_t jumps to the left too.
5. If M_t jumps to the left at time t and X is on top of a particle, i.e., $\xi_t(X_t) = 1$, then X_t jumps to the right when an independent Bernoulli trial with parameter $(\alpha_1 - \alpha_0)/\beta_0$ succeeds, and jumps to the left otherwise.

Denote by

$$\left(\widetilde{P}, \Gamma, \mathcal{F}_t\right) \tag{5.6}$$

the probability space associated with (5.5), with

$$\mathcal{F}_t = \sigma\left(\{M_s\}_{s \leq t}, \{b_n\}_{n \leq m_t}, \{\xi_s\}_{s \leq t}\right), \tag{5.7}$$

where m_t is the number of jumps of M up to time t, which is distributed according to a Poisson random variable with parameter $(\alpha_0 + \beta_0)t$.

By construction, for any $t \geq 0$,

$$M_t \leq X_t \qquad \widetilde{P} - a.s. \tag{5.8}$$

We are therefore justified to call M the *minimal walker*.

5.2.2 Graphical representation: symmetric exclusion as an interchange process

The interchange process $\gamma = (\gamma_t)_{t\geq 0}$ on $\mathbb{Z}$ is a process, taking values on the permutations of $\mathbb{Z}$, that can be defined through a graphical representation as follows. Start with a permutation γ_0. We call the state of the coordinates of γ 'agents'. We take γ_0 to be the identity, i.e., the agents are $(\ldots, -2, -1, 0, 1, 2, \ldots)$. Associate to each non-directed nearest-neighbor edge $(x, x+1)$ in $\mathbb{Z}$ an independent Poisson clock $I^{x,x+1} = (I_t^{x,x+1})_{t\geq 0}$ ticking at rate 1. Denote by

$$I = \{I^{x,x+1} : \, x \in \mathbb{Z}\} \tag{5.9}$$

the set of all those clocks. Then γ_t is obtained from γ_0 by exchanging the labels of x and $x+1$ each time the Poisson clock $I^{x,x+1}$ rings. In particular, $\gamma_t(x) \in \mathbb{Z}$ represents the starting position of the agent who at time t is at site x.

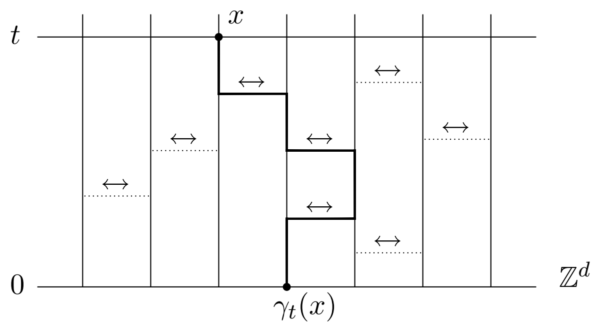

FIGURE 5.1: Graphical representation. The dashed lines are the links given by the realization of I. The thick line represents the path of the agent $\gamma_t(x)$.

Given the interchange process γ, the simple symmetric exclusion process (SSE) (see Section 4.1.3) $\xi = (\xi_t)_{t\geq 0}$ on $\mathbb{Z}$ starting from a configuration $\eta \in \Omega = \{0,1\}^{\mathbb{Z}}$ can be obtained from γ by putting $\xi_t(x) = \eta(\gamma_t(x))$.

The interpretation is that, in the interchange process, the agents move around in the lattice by exchanging their places with their nearest neighbors. For exclusion, we choose one of two states for these agents at the start (1 or 0, which we refer to as 'particle' and 'hole') and assign the state of a site at a later time as the initial state of the agent who is there at this time.

Next, recall (5.5). By the coupling with the *minimal walker* M of the previous section, we have that, for any starting configuration $\eta \in \Omega$, X is a function of

$$(M, (b_n)_{n\in\mathbb{N}}, I) \text{ and } \eta, \tag{5.10}$$

where, in the coupling space (5.6), $\mathcal{F}_t$ is given by

$$\mathcal{F}_t = \sigma\left(\{M_s\}_{s\le t}, \{b_n\}_{n\le m_t}, \{I_s\}_{s\le t}\right). \tag{5.11}$$

In particular, if we consider $\zeta, \eta \in \Omega$ such that $\zeta \succeq \eta$ (where $\succeq$ denotes the partial order on Ω), then for any $t \ge 0$ we have by construction

$$M_t \le X_t(\eta) \le X_t(\zeta) \qquad \widetilde{P} - a.s., \tag{5.12}$$

where $X_t(\eta)$ and $X_t(\zeta)$ represent the RW starting from η and ζ, respectively.

5.2.3 Marked agents set

As the RW X moves, it will meet the agents of the interchange process. Sometimes, due to the coupling with the *minimal walker*, it will not need to know their state in order to proceed, i.e., when the *minimal walker* M goes to the right. If M goes to the left, then X will have to 'ask' the agent at its current position what is its state to know how to move. We say that at this time X and the agent 'meet', and we call an agent *marked* at time t if it has met X at some time $s \le t$. For any $t \ge 0$, we can define A_t to be the set of *marked* agents up to time t. For reasons that will become clear at the end of this section, we add to this *marked* agents set all the agents $x \le 0$.

Formally, define $A_0 = \{x \in \mathbb{Z}\colon\ x \le 0\}$, let t be a time at which M_t jumps to the left, and put

$$A_t = A_{t^-} \cup \{\gamma_t(X_t)\}. \tag{5.13}$$

Next, let

$$U_1 = \inf\{t > 0\colon\ M_t \ne 0\} = \inf\{t > 0\colon\ X_t \ne 0\} \tag{5.14}$$

and define

$$\tau_0 = \inf\Big\{t \ge U_1\colon\ X_t > \max\{x \in \mathbb{Z}\colon\ \gamma_t(x) \in A_t\}\Big\}, \tag{5.15}$$

i.e., the first time such that all the sites with *marked* agents are to the left of X_t.

Lemma 5.2. *Let τ_0 be as in (5.15) and denote by $\widetilde{E}$ the expectation w.r.t. $\widetilde{P}$. Then*

$$\widetilde{E}[\tau_0^2] < \infty.$$

Proof. Let

$$Y = (Y_t)_{t\ge 0} \tag{5.16}$$

be a path starting from 0 that jumps to the right according to the realization of the process I in (5.9), see Figure 5.2.

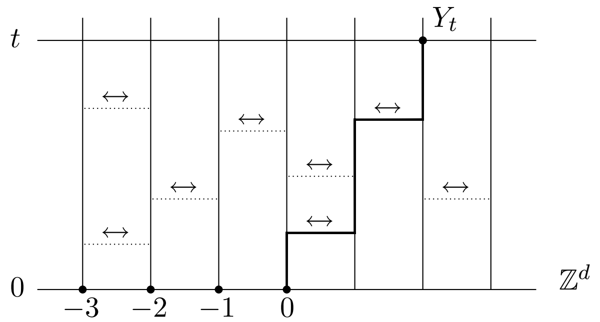

FIGURE 5.2: As in Figure 5.1, the dashed lines are links given by the realization of I. The path Y starts at the origin and goes only to the right following the links determined by I.

Then Y is distributed according to a Poisson process with rate 1.

Denote by $\gamma^{-1}(x) = \big(\gamma_t^{-1}(x)\big)_{t\geq 0}$ the path of the agent x. By construction, for any $x \leq 0$ and $t \geq 0$, $\gamma_t^{-1}(x) \leq Y_t$. Furthermore, let $S_1 = \inf\{t > 0\colon\ M_t - Y_t > 0\}$, and note that

$$\tau_0 \leq S_1. \tag{5.17}$$

Recalling that the *minimal walker* M is independent of the RE, while Y is a function of the RE, we have that $Z = (Z_t)_{t\geq 0}$ with $Z_t = M_t - Y_t$ is a continuous-time homogeneous RW, starting from the origin, that jumps to the right at rate α_0 and to the left at rate $\beta_0 + 1$. Since $\alpha_0 - \beta_0 > 1$ by the third condition in (5.2), Z is transient to the right with positive speed $\alpha_0 - \beta_0 - 1 > 0$. Thus, $\mathbb{E}[S_1^2] < \infty$, and the claim follows from (5.17). ∎

The crucial point, which we state in the next proposition, is that if we start from a configuration $\eta \in \Omega$ sampled from ν_ρ to the right of the origin, then, no matter what is η to the left of the origin, the RW X at time τ_0 will still see to its right a configuration that is *freshly* sampled from ν_ρ. Such a fact is related to the nature of the SSE and its construction from the interchange process, and it is the main ingredient for the proof of the LLN.

Let $\mathbb{Z}_{>0} = \{x \in \mathbb{Z}\colon\ x > 0\}$, and put $\mathbb{Z}_{\leq 0} = \mathbb{Z} \setminus \mathbb{Z}_{>0}$. Given $\zeta \in \{0,1\}^{\mathbb{Z}_{\leq 0}}$, let $\nu_\rho^{(\zeta)}$ be the product measure of single site measures on Ω given by

$$\begin{aligned} &\nu_\rho^{(\zeta)}\left(\eta(x) = \zeta(x)\right) = 1,\ \text{ if } x \in \mathbb{Z}_{\leq 0},\\ &\nu_\rho^{(\zeta)}\left(\eta(x) = 1\right) = \rho,\ \text{ otherwise,} \end{aligned} \tag{5.18}$$

i.e., $\nu_\rho^{(\zeta)}$ coincides with ν_ρ on $\{0,1\}^{\mathbb{Z}_{>0}}$, and is the delta measure δ_ζ on $\{0,1\}^{\mathbb{Z}_{\leq 0}}$.

Proposition 5.3. *For any $\zeta \in \{0,1\}^{\mathbb{Z}_{\leq 0}}$, let ξ be the SSE starting from $\nu_\rho^{(\zeta)}$, and denote by $\mathbb{P}_{\nu_\rho^{(\zeta)},0}$ the law $\widetilde{P}$ when the starting configuration η is sampled from $\nu_\rho^{(\zeta)}$. Then, for any finite $B \subset \{0,1\}^{\mathbb{Z}_{>0}}$,*

$$\mathbb{P}_{\nu_\rho^{(\zeta)},0}\Big(\xi_{\tau_0}(X_{\tau_0} + \cdot) \in B \mid (X_t)_{t\leq\tau_0}\Big) = \nu_\rho(B), \tag{5.19}$$

i.e., the SSE at time τ_0 to the right of X_{τ_0} is independent of $(X_t)_{t\leq\tau_0}$, and is distributed according to ν_ρ.

Proof. $(X_t)_{t\leq\tau_0}$ is a function of $(M_t)_{t\leq\tau_0}$, $\{b_n\}_{n\leq m_{\tau_0}}$, $(I_t)_{t\leq\tau_0}$ (see (5.10)), and the state of the agents belonging to A_{τ_0}. Therefore $(X_t)_{t\leq\tau_0}$ is independent of $\{\xi_0(x)\colon\ x \in \mathbb{Z}\backslash A_{\tau_0}\}$. By the definition of τ_0, $\gamma_{\tau_0}(x) \in \mathbb{Z}_{>0}\backslash A_{\tau_0}$, for all $x > X_{\tau_0}$. Therefore, since $\nu_\rho^{(\zeta)}$ coincides with ν_ρ on $\{0,1\}^{\mathbb{Z}_{>0}}$, it follows that $\xi_{\tau_0}(x)$ is a Bernoulli random variable with parameter ρ for $x > X_{\tau_0}$. ∎

5.2.4 Right walker and a sub-additivity argument

Denote by $\underline{1} \in \{0,1\}^{\mathbb{Z}_{\leq 0}}$ the configuration with all coordinates equal to 1. Let

$$R = (R_t)_{t\geq 0} \tag{5.20}$$

be the RW X starting from $\nu_\rho^{(\underline{1})}$. For any $\zeta \in \{0,1\}^{\mathbb{Z}_{\leq 0}}$, if we denote by $X(\zeta)$ a RW starting from $\nu_\rho^{(\zeta)}$, then, as a consequence of (5.12), for any $t \geq 0$ we have that

$$M_t \leq X_t(\zeta) \leq R_t \qquad \widetilde{P} - a.s. \tag{5.21}$$

We call R the *right walker*. We anticipate that in the sequel we first prove that R satisfies a LLN, and then Theorem 5.1 follows by showing that the limiting speed of the *right walker* does not depend on the configuration $\underline{1}$.

We next construct a renewal structure in the coupling space (5.6). The idea of this construction is that, starting from R and from the τ_0 associated to R, we have that, by Proposition 5.3, at time τ_0 the states of the SSE ξ to the right of $R_{\tau_0}^{(0)}$ are distributed according to ν_ρ. At time τ_0 we define a new configuration $\eta^{(1)}$ of the SSE from ξ_{τ_0}, by replacing all its states to the left of R_{τ_0} by 1 (i.e., put $\xi_{\tau_0}(x) = 1$ for $x \leq R_{\tau_0}$), and we define $R^{(1)}$ to be the RW evolving as X in Section 5.2.1 starting at time τ_0 at position R_{τ_0} from this new configuration of the SSE. In particular, such $R^{(1)}$ has the following properties:

1. $R^{(1)}$ is a function of $\big(\{M_t\}_{t\geq\tau_0}, \{b_n\}_{n\geq m_{\tau_0}}, \{I\}_{t\geq\tau_0}\big)$ and $\eta^{(1)}$.

2. By (5.12), $R^{(1)}$ is coupled to R in such a way that $R_{t+\tau_0} - R_{\tau_0} \leq R_t^{(1)}$ for $t \geq 0$.
3. R and $\left(R_t^{(1)} - R_0^{(1)}\right)_{t\geq 0}$ have the same distribution.

We can then repeat the same argument to construct a new RW $R^{(n)}$ from $R^{(n-1)}$ for any $n \in \mathbb{N}$.

More precisely, let $\eta^{(0)} \in \Omega$ be a configuration sampled from $\nu_\rho^{(1)}$, set $R^{(0)} = R$, and construct inductively the random vector-sequence

$$\left\{\left(\eta^{(n)}, R^{(n)}, \tau_n\right)\right\}_{n\in\mathbb{N}}, \quad R^{(n)} = \left(R_t^{(n)}\right)_{t\geq 0}, \tag{5.22}$$

as follows. For $n \in \mathbb{N}$, let $\eta^{(n)} \in \Omega$ given by

$$\eta^{(n)}(x) = \begin{cases} \eta^{(n-1)}\left(\gamma_{\tau_{n-1}}(x)\right), & \text{if } x > R^{(n-1)}_{\tau_{n-1}}, \\ 1, & \text{otherwise.} \end{cases} \tag{5.23}$$

For $t \geq \tau_{n-1}$, let $R^{(n)} = \left(R_t^{(n)}\right)_{t\geq 0}$ be the RW evolving according to the rules given for X in Section 5.2.1, starting from $R^{(n-1)}_{\tau_{n-1}}$ with initial states of the RE given by $\eta^{(n)}$.

Let $A_t^{(n)}$ be the marked agents set constructed from $R_t^{(n)}$ as in (5.13), namely, set $A_0^{(n)} = \left\{x \in \mathbb{Z}\colon\ x \leq R_0^{(n)} = R^{(n-1)}_{\tau_{n-1}}\right\}$, let t be a time at which $M_{\tau_{n-1}+t}$ jumps to the left, and put

$$A_t^{(n)} = A_{t^-}^{(n)} \cup \left\{\gamma_t\left(R_t^{(n)}\right)\right\}. \tag{5.24}$$

Define

$$\tau_n = \inf\left\{t \geq U_1\colon\ R_t^{(n)} > \max\left\{x \in \mathbb{Z}\colon\ \gamma_t(x) \in A_t^{(n)}\right\}\right\}. \tag{5.25}$$

As a consequence of this construction, it follows from (5.12) that

$$R^{(n)}_{t+\tau_n} - R^{(n)}_{\tau_n} \leq R_t^{(n+1)} \qquad \widetilde{P} - a.s. \tag{5.26}$$

The main advantage is now that, by Proposition 5.3, $\left\{\left(\eta^{(n)}, R^{(n)}, \tau_n\right)\right\}_{n\in\mathbb{N}}$ is a stationary sequence.

Lemma 5.4. *Let $T_n = \sum_{i=1}^n \tau_n$. For integers $0 \leq m < n$, define the double indexed random variables*

$$\bar{R}_{m,n} = R^{(m)}_{T_n - T_m}. \tag{5.27}$$

Then there exists a constant $c(R) \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \frac{\bar{R}_{0,n}}{n} = \lim_{n\to\infty} \frac{R_{T_n}}{n} = c(R) \qquad \widetilde{P} - a.s. \tag{5.28}$$

Proof. The proof relies on the subadditive ergodic theorem of Liggett (see Theorem 1.10 in [63]). By (5.26), for any $0 \leq m < n$, we have

$$\bar{R}_{0,n} = \bar{R}_{0,m} + (\bar{R}_{0,n} - \bar{R}_{0,m}) \leq \bar{R}_{0,m} + \bar{R}_{m,n} \qquad \widetilde{P} - a.s. \tag{5.29}$$

Moreover, by construction and since $\{(R^{(n)}, \tau_n)\}_{n\in\mathbb{N}}$ is a stationary sequence, for every $n > m$, $\{\bar{R}_{m+k,n+k}\}_{k\in\mathbb{N}_0}$ is a sequence of i.i.d. random variables. Therefore, for each $m \in \mathbb{N}_0$, the joint distribution of $\{\bar{R}_{m+1,m+k+1}\}_{k\in\mathbb{N}}$ is the same as that of $\{\bar{R}_{m,m+k}\}_{k\in\mathbb{N}}$. Furthermore, for each $k \in \mathbb{N}$, we have that $\{\bar{R}_{nk,(n+1)k}\}_{n\in\mathbb{N}}$ is a stationary and ergodic process. Therefore the assumptions of Theorem 1.10 in [63] are satisfied, and the claim follows. ∎

5.2.5 LLN

Lemma 5.5. *There exists a constant $v(R) > 1$ such that*

$$\lim_{t\to\infty} \frac{R_t}{t} = v(R) \qquad \mathbb{P}_{\nu_\rho^{(1)},0} - a.s. \tag{5.30}$$

Proof. For $t \geq 0$, let $n(t)$ be such that

$$T_{n(t)} \leq t < T_{n(t)+1}. \tag{5.31}$$

Denote by $\mathbb{E}_{\nu_\rho^{(1)},0}$ the expectation associated to $\mathbb{P}_{\nu_\rho^{(1)},0}$. By Lemma 5.2, $\mathbb{E}_{\nu_\rho^{(1)},0}[\tau_0] < \infty$. Since $T_{n(t)}/n(t) \to \mathbb{E}_{\nu_\rho^{(1)},0}[\tau_0]$ as $n \to \infty$, dividing by $n(t)$ and taking $t \to \infty$ in (5.31), we have

$$\lim_{t\to\infty} \frac{n(t)}{t} = \frac{1}{\mathbb{E}_{\nu_\rho^{(1)},0}[\tau_0]} \qquad \mathbb{P}_{\nu_\rho^{(1)},0} - a.s. \tag{5.32}$$

By Lemma 5.4 and (5.32), we get

$$\lim_{t\to\infty} \frac{R^{(0)}_{T_{n(t)}}}{t} = \lim_{t\to\infty} \frac{R_{T_{n(t)}}}{n(t)} \frac{n(t)}{t} = \frac{c(R)}{\mathbb{E}_{\nu_\rho^{(1)},0}[\tau_0]} =: v(R). \tag{5.33}$$

Since

$$\frac{R_t}{t} = \frac{|R_t - R_{T_{n(t)}}|}{t} + \frac{R_{T_{n(t)}}}{t}, \tag{5.34}$$

the claim follows by combining (5.33) and (5.34), and observing that

$$\limsup_{t\to\infty} \frac{|R_t - R_{T_{n(t)}}|}{t} = 0 \qquad \mathbb{P}_{\nu_\rho^{(1)},0} - a.s. \tag{5.35}$$

To show (5.35) we argue as follows. Note first that R can be coupled with a Poisson process $N = (N_t)_{t\geq 0}$ of rate $\alpha_0 + \beta_0$ such that

$$R_t \leq N_t \quad \text{for any } t \geq 0. \tag{5.36}$$

In particular, it follows from Lemma 5.2 that $\mathbb{E}_{\nu_\rho^{(1)},0}\left[N_{\tau_0}^2\right] < \infty$, which together with (5.36) ensures that there exists a constant $C \in (0,\infty)$ such that

$$\mathbb{E}_{\nu_\rho^{(1)},0}\left[\max_{t\leq\tau_0} |R_t|^2\right] \leq C. \tag{5.37}$$

By the Markov inequality and (5.37), for any $\epsilon > 0$ we have

$$\begin{aligned}\mathbb{P}_{\nu_\rho^{(1)},0}\left(|R_t - R_{T_{n(t)}}| \geq \epsilon t\right) &\leq \mathbb{P}_{\nu_\rho^{(1)},0}\left(\max_{T_{n(t)}\leq t\leq T_{n(t)+1}} |R_t - R_{T_{n(t)}}| \geq \epsilon t\right)\\ &= \mathbb{P}_{\nu_\rho^{(1)},0}\left(\max_{t\leq\tau_0} |R_t| \geq \epsilon t\right) \leq \frac{\mathbb{E}_{\nu_\rho^{(1)},0}\left[\max_{t\leq\tau_0}|R_t|^2\right]}{(\epsilon t)^2}\\ &\leq C(\epsilon t)^{-2}.\end{aligned} \tag{5.38}$$

Finally, (5.35) follows from (5.38) and the Borel-Cantelli lemma. ∎

Next, let $\underline{0} \in \{0,1\}^{\mathbb{Z}_{\leq 0}}$ be the configuration with all coordinates equal to 0. Let

$$L = (L_t)_{t\geq 0} \tag{5.39}$$

be the RW X starting from $\nu_\rho^{(0)}$.

For any $\zeta \in \{0,1\}^{\mathbb{Z}_{\leq 0}}$, if we denote by $X(\zeta)$ the RW starting from $\nu_\rho^{(\zeta)}$, then, as a consequence of (5.12), for any $t \geq 0$ we have that

$$M_t \leq L_t \leq X_t(\zeta) \leq R_t \qquad \widetilde{P} - a.s. \tag{5.40}$$

Note that by repeating the same argument as in Section 5.2.4 and in the proof of Lemma 5.5 for the *left walker* L, we get that there exists a constant $v(L) > 1$ such that

$$\lim_{t\to\infty} \frac{L_t}{t} = v(L) \qquad \mathbb{P}_{\nu_\rho^{(0)},0} - a.s.$$

The only difference is that in Lemma 5.5 we obtain super-additivity instead of sub-additivity. Finally, by observing that $v(R) = v(L)$, Theorem 5.1 follows from (5.40).

To see this latter observation we argue as follows. As both L and R are identical if they do not encounter an agent from the left of the origin, the associated speeds are the same on this event. Thus, if this event is not a null-set, then, as the speeds are a.s. constants on the whole space, we obtain $v(R) = v(L)$. To show that the latter event has positive probability, recall the RW Y in (5.16) and observe that the event that L and R do not encounter an agent from the left of the origin includes the event $\{M_t > Y_t, \quad \forall t \geq 0\}$, which has positive probability due to the third drift condition in (5.2).

5.3 Concluding remarks

The assumption that the total jump rates of X are the same on top of particles or holes (i.e., $\alpha_0 + \beta_0 = \alpha_1 + \beta_1$ in (5.2)) is not relevant for the proof and can be easily dropped by constructing a different coupling with the minimal walker in Section 5.2.1, and essentially keeping the rest of the proof unchanged. We made this assumption just to avoid cumbersome notations.

The proof of Theorem 5.1 is simple and uses the specific nature of the SSE. Indeed, we exploited the graphical representation of the SSE, in particular, its construction from the interchange process, to ensure the integrability of the time τ_0 in (5.15) and to ensure that the sequence in (5.22) is stationary.

We are currently working on extensions of Theorem 2.2 for a larger class of dynamic RE under strong drift assumptions as in (5.3), namely, for dynamic RE in which, intuitively, the 'information' travels to the right slower than the minimal drift of X. If we consider other dynamic RE, like e.g. an asymmetric exclusion process or a Poissonian field of independent RWs, then we cannot a priori be sure of the existence of a non-degenerate integrable time at which X observes to its right a RE in equilibrium. A heavier regeneration scheme in the spirit of Chapter 2 seems to be needed. We plan to treat such cases in future works.