Random walks in dynamic random environments
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Chapter 3

Annealed central limit theorem for RW in mixing dynamic RE

3.1 Introduction and main result

In this chapter we continue to investigate the model in Section 2.1.1. We show that under a certain strong-mixing assumption on the RE ξ, called *n-cone-mixing* (see Definition 3.1), the RW X satisfies an annealed invariance principle with a Brownian motion as scaling limit. The proof of this functional CLT relies on a direct adaptation of a technique used in [36] for static REs. The *n-cone-mixing* property is a technical assumption directly connected with the machinery used in the proof. In Section 3.2.4 we will exhibit examples of dynamic REs satisfying this assumption.

We first need some definitions. Recall (2.20), and let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^2$. Put $\ell = (0,1)$. For $x = (z,m) \in \mathbb{H} = \mathbb{Z} \times \mathbb{N}_0$, let

$$C_N(x) = \Big\{ u \in \mathbb{R} \times [0,\infty) \colon \sqrt{2}/2 \|u - x\| \leq (u - x) \cdot \ell \leq N \Big\} \tag{3.1}$$

be the cone of angle $\pi/2$ with tip in (z,m) truncated at time $N+m$.

For fixed $L \geq 0$, let $\{C_{N_i}(x_i) \colon x_i = (z_i, m_i) \in \mathbb{H}\}_{i=1}^n$ be a set of n truncated cones such that, for $1 \leq i < n$,

$$m_1 \geq L, \quad m_{i+1} = N_i + m_i + L, \quad | z_{i+1} - z_i | \leq N_i. \tag{3.2}$$

We call these *nested-cones*. In words, we are considering n space-time truncated cones separated in time by a distance L such that the $(i+1)$-st cone is contained in the i-th extended cone.

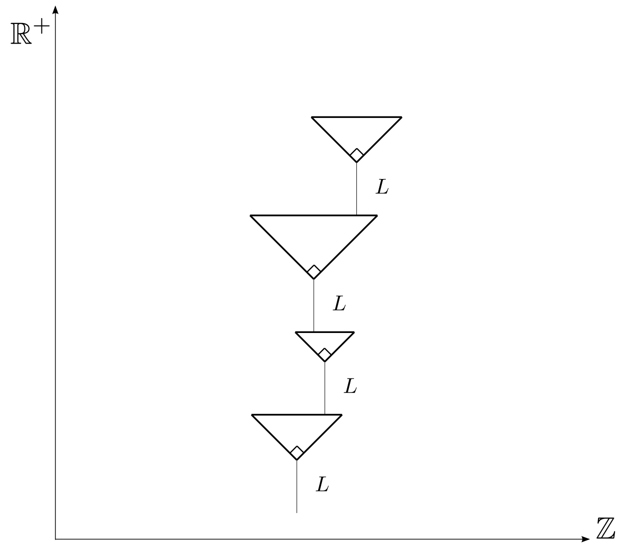

FIGURE 3.1: Example of 4 nested-cones.

Definition 3.1. Fix $L \geq 0$ and any set of n *nested-cones* $\{C_{N_i}(x_i)\colon\ x_i = (z_i, m_i) \in \mathbb{H}\}_{i=1}^n$. A dynamic RE ξ on $\Omega = \{0,1\}^{\mathbb{Z}}$ is said to be *n-cone-mixing* if for any $n \in \mathbb{N}$ there exists a function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$, with

$$\int_0^\infty \Psi(t)\mathrm{d}t < \infty, \tag{3.3}$$

such that

$$\sup_{\substack{A\in\mathcal{G}_n, B\in\mathcal{G}^{<n}\\ \eta,\eta'\in\Omega}} \Big| P_\eta(A \mid B) - P_{\eta'}(A \mid B) \Big| \leq \Psi(nL), \tag{3.4}$$

where

$$\begin{aligned} \mathcal{G}_n &= \sigma\Big\{\xi_s(z)\colon\ (z,s) \in C_{N_n}(x_n)\Big\}, \\ \mathcal{G}^{<n} &= \sigma\left\{\xi_s(z)\colon\ (z,s) \in \bigcup_{i=1}^{n-1} C_{N_i}(x_i)\right\}. \end{aligned} \tag{3.5}$$

Note that if a dynamic RE is *n-cone-mixing*, then the associated path measure P^μ in (1.23) satisfies the *cone-mixing* property in Definition 2.1. Indeed, (2.11) follows easily from (3.4) with $n = 1$. Therefore, by Theorem 2.2, X satisfies a strong LLN with asymptotic speed v. We are now ready to state the main result of this chapter.

Theorem 3.2. *Assume* (2.3) *and suppose that ξ is n-cone-mixing. Then there exists a deterministic $\sigma^2 \in (0,\infty)$ such that, under the annealed measure $\mathbb{P}_{\mu,0}$, the path $(S_t(s))_{s\geq 0}$, with*

$$S_t(s) = \frac{X_{ts} - vts}{\sqrt{t}} \tag{3.6}$$

and taking values in the space of right-continuous functions with left limits, converges weakly to a Brownian motion with variance σ^2 as $t \to \infty$.

The proof of Theorem 3.2 will be given in Section 3.2. In Section 3.3 we give an alternative proof in the context of the perturbative regime introduced in Section 2.3. Indeed, in the latter regime, the strong control on the environment process allows for a much simpler proof than in the general case, and the claim can be easily obtained via a martingale approximation in the spirit of Kipnis-Varhadan [61].

3.2 Proof of Theorem 3.2

In this section we prove Theorem 3.2 by adapting the proof of the CLT for random walks in *static* random environments developed by Comets and Zeitouni [36]. The proof heavily uses the regeneration scheme introduced in Section 2.2.3 and is based on the following steps. In Section 3.2.1 we show that the path of the RW Z in (2.29), together with the evolution of the RE ξ between regeneration times, can be encoded into a *chain with complete connections* for which the dependence of the future on the past can be controlled by the *n-cone-mixing* condition. Chains with complete connections are natural extensions of Markov chains when the transitions of the associated stochastic process depend on its full past. For details we refer the reader to [45, 57]. In Section 3.2.2, using standard results from the theory of such chains, we prove an invariance principle. In Section 3.2.3, we show how Theorem 3.2 follows from the latter.

3.2.1 A chain with complete connections

We construct a chain with complete connections that carries the necessary information relative to the evolution of the path of the RW Z in (2.29), together with the states of the RE ξ inside the truncated cones visited by the path between regeneration times. Lemma 3.3 below uses the *n-cone-mixing* property to control the dependence of the future evolution of the chain on its past. In particular, we will see that the influence of the past decays as fast as the correlations in the RE.

We start by defining the relevant state space. Recall (3.1) and for $N \in \mathbb{N}$ let

$$\mathcal{P}_N = \left\{ \begin{array}{c} \underline{x} = (x(0), x(1), \ldots, x(N)) \in C_N(0)^N : \\ x(0) = 0,\ x(i+1) \sim x(i),\ i = 0, 1, \ldots, N-1 \end{array} \right\} \tag{3.7}$$

be the set of possible paths of the process Z within the truncated cone $C_N(0)$, where $x(i+1) \sim x(i)$ stands for $|x_1(i+1) - x_1(i)| = 1, x_2(i+1) - x_2(i) = 1$. Define

$$\mathcal{T} = \bigcup_{N \in \mathbb{N}} \{N\} \times \mathcal{P}_N \times \mathcal{E}_N, \tag{3.8}$$

where $\mathcal{E}_N = \{\xi_t(z)\colon\ (z,t) \in C_N(0)\}$ is the set of possible values of the environment ξ in the cone $C_N(0)$. Let

$$\mathcal{W} = \big\{\mathcal{T} \cup \{s\}\big\}^{\mathbb{N}} \tag{3.9}$$

be the set of infinite vectors with components either in $\mathcal{T}$ or equal to the stopping symbol s, with the restriction that if $w_k = s$ then $w_i = s$ for $i \geq k$. Note that, for fixed $L \in \mathbb{N}$, the sequence of regeneration times $\left(\tau_k^{(L)}\right)_{k\in\mathbb{N}}$ in (2.35), together with the path Z, determine an infinite sequence $\underline{r} = (r_1, r_2, \dots) \in \mathcal{W}$ given by

$$r_k = \left(S_{k+1,L},\ \left(Z_{\tau_k^{(L)}+j}\right)_{j=1}^{S_{k+1,L}},\ \left\{\xi_t(z)\colon\ (z,t) \in C_{S_{k+1,L}}\left(Z_{\tau_k^{(L)}}\right)\right\}\right) \in \mathcal{T}, \quad k \in \mathbb{N}, \tag{3.10}$$

with

$$S_{k,L} = \tau_k^{(L)} - \tau_{k-1}^{(L)} - L, \quad k \in \mathbb{N}. \tag{3.11}$$

Observe that the sequence $\underline{r} = (r_1, r_2, \dots) \in \mathcal{W}$ encodes the information relative to the environment and the path of the walker just after time $S_{1,L} = \tau_1^{(L)} - L$.

Next, we define a set in which we can gather the information prior to time $S_{1,L}$, i.e.,

$$\mathcal{U} = \left\{ \begin{array}{c} u = \big(M, y(1), y(2), \dots, y(M), \underline{\xi}(u)\big) : \\ M \in \mathbb{N},\ y(i) \in \mathbb{H},\ y(i+1) \sim y(i),\ i = 0, 1, \dots, M-1 \end{array} \right\} \tag{3.12}$$

with $\underline{\xi}(u) = \{\xi_t : t \leq M\}$.

Recall the sigma-fields in (2.38). For $A \in \mathcal{H}_1$, write

$$A = \bigcup_{(z,n)\in\mathbb{H}} A_{z,n}, \quad A_{z,n} = A \cap \big\{S_{1,L} = n, Z_{S_{1,L}} = (z,n)\big\} \in \mathcal{U}. \tag{3.13}$$

Then the law $\bar{\mathbb{P}}_{\mu,(0,0)}$ induces a probability measure $\mathbb{Q}$ on $\mathcal{U}$ such that

$$\mathbb{Q}(A_{z,n}) = \bar{\mathbb{P}}_{\mu,(0,0)}\Big(\big(S_{1,L}, Z_1, \dots, Z_n, \{\xi_t\colon\ t \leq n\}\big) \in A_{z,n}\Big), \quad (z,n) \in \mathbb{H}. \tag{3.14}$$

Furthermore, the law $\bar{\mathbb{P}}_{\mu,(0,0)}(\cdot \mid \mathcal{H}_1)$ induces a probability distribution on the sequence $\underline{r} = (r_1, r_2, \dots) \in \mathcal{W}$ in (3.10). Indeed, for fixed $k \in \mathbb{N}$, note that $\bar{\mathbb{P}}_{\mu,(0,0)}(r_k \in \cdot \mid \mathcal{H}_k)$ defines a measurable function $h_k(\cdot \mid w_{k-1}, \dots, w_1, u)$ on $\mathcal{U} \times \mathcal{T}^{k-1}$ such that

$$\begin{aligned} \bar{\mathbb{E}}_{\mu,(0,0)}[\mathbb{1}_A \mathbb{1}_B] &= \bar{\mathbb{E}}_{\mu,(0,0)}[\mathbb{1}_A \bar{\mathbb{P}}_{\mu,(0,0)}((r_1, \dots, r_k) \in B \mid \mathcal{H}_1)] \\ &= \int_A \mathbb{Q}(\mathrm{d}u) \int_{\mathcal{T}} \cdots \int_{\mathcal{T}} \mathbb{1}_B \prod_{i=1}^{k} h_i(\mathrm{d}w_i \mid w_{i-1}, \dots, w_1, u), \end{aligned} \tag{3.15}$$

with $B \subset \mathcal{T}^k$.

In the following lemma we provide an estimate to control the dependence on the past in the sequence $\underline{r}$ whose law is governed by the random kernels $(h_k)_{k\in\mathbb{N}}$. In particular, we show that the influence of the past decays as fast as the correlations in the environment controlled by the Ψ-function in Definition 3.1.

Lemma 3.3. *Let $j \geq i \geq k$, $\underline{w}^{(i)} = (w_i, \dots, w_1)$ and $\underline{w'}^{(j)} = (w'_j, \dots, w'_1)$ be such that $w_{i-l} = w'_{j-l}$ for $l = 0, 1, \dots, k$. Then*

$$\sup_{u,u'\in\mathcal{U}} \left\| h_{i+1}(\cdot \mid \underline{w}^{(i)}, u) - h_{j+1}(\cdot \mid \underline{w'}^{(j)}, u') \right\|_{\mathrm{tv}} \leq \Psi(kL). \tag{3.16}$$

Proof. Observe that the maximum in the left-hand side of (3.16) is attained for $i = j = k$. Therefore, we restrict the proof to this case.

For $u = (M, y(1), y(2), \dots, y(M), \underline{\xi}(u)) \in \mathcal{U}$ and $w_i = (N_i, x(1), x(2), \dots, x(N_i), \underline{\xi}(C_i)) \in \mathcal{T}$, where $\underline{\xi}(C_i)$ denotes the state of the environment in a certain truncated cone C_i, let π be the projection on $\mathcal{U}$ and $\mathcal{T}$, given by, respectively, $\pi(u) = (M, y(1), y(2), \dots, y(M))$ and $\pi(w_i) = (N_i, x(1), x(2), \dots, x(N_i))$. Thus, the first i regeneration points and regeneration times can be reconstructed from $u, \underline{w}^{(i)}$ as follows:

$$\overline{x}_0 = Z_{\tau_1^{(L)}} = y(M) + (0, L), \quad \overline{x}_i = Z_{\tau_{i+1}^{(L)}} = \overline{x}_{i-1} + x(N_i) + (0, L), \tag{3.17}$$

$$\overline{t}_i = \tau_{i+1}^{(L)} = M + L + \sum_{j=1}^{i} (N_j + L). \tag{3.18}$$

Note that the entire path of Z up to time $\overline{t}_i$ is also encoded in $(\pi(u), \pi(w_1), \dots, \pi(w_i))$. Hereafter we denote this path by $\tilde{x} = \tilde{x}\big(\pi(u), \pi(w_1), \dots, \pi(w_i)\big)$, and its k-th component by $\tilde{x}[k]$. In particular, $\tilde{x}[\overline{t}_j] = \overline{x}_j$.

Next, consider a non-negative bounded random variable F measurable w.r.t. $\mathcal{H}_1$. For any given $\pi_0 \in \pi(\mathcal{T})$, there exists a non-negative bounded random variable F_{π_0}, measurable w.r.t. $\sigma\left(\underline{\xi}(u), \{\epsilon_k \colon k = 1, \dots, M\}\right)$, such that $F = F_{\pi_0}$ on the event $\{\pi(r_0) = \pi_0\}$.

Similarly, let G be a non-negative bounded random variable measurable w.r.t. $\sigma(r_1, \dots, r_i)$. For all $\underline{\pi}(i) \in \pi(\mathcal{T})^i$, there exists a random variable $G_{\underline{\pi}(i)}$ measurable w.r.t. $\sigma(\Lambda_\xi(i))$, with

$$\Lambda_\xi(i) = \left\{ \xi_t(z) \colon (z, t) \in \bigcup_{j=1}^{i} (C_j + \overline{x}_{j-1}) \right\}, \tag{3.19}$$

such that $G = G_{\underline{\pi}(i)}$ on the event $\{\pi(r_k) = \pi_k \colon k = 1, \dots, i\}$.

Next, define the events

$$B(\pi_0) = \{Z_k = \tilde{x}[k]\colon\ k = 0, \dots, \bar{t}_0\}, \tag{3.20}$$

and

$$B(\underline{\pi}(i)) = \{Z_{k+\bar{t}_0} - Z_{\bar{t}_0} = \tilde{x}[k+\bar{t}_0] - \tilde{x}[\bar{t}_0]\colon k = 0, \dots, \bar{t}_i - \bar{t}_0\}, \tag{3.21}$$

and the random variable

$$G'_{\pi_0,\underline{\pi}(i)} = G_{\underline{\pi}(i)} \bar{P}_0^{\xi,\epsilon}\left(B\left(\underline{\pi}\left(i\right)\right) \mid Z_n, n \le \bar{t}_0, Y_{\bar{t}_0} = \overline{x}_0\right), \tag{3.22}$$

which is measurable w.r.t. the σ-algebra generated by $\Lambda_\xi(i)$. Abbreviate $\mathbb{1}_A = \mathbb{1}_{\{r_0 \in A\}}$ for a measurable subset $A \subset \mathcal{T}$, and write θ_n to denote the shift of time over n.

By using the above notations and the Markov property, we can write

$$\begin{aligned}
&\bar{\mathbb{E}}_{\mu,(0,0)}\left(FG\left[\mathbb{1}_A \circ \theta_{\tau^{(L)}_{i+1}}\right]\right) \\
&= \sum_{\pi_0,\underline{\pi}(i)} E_{P^\mu \otimes W}\left(\bar{E}_0^{\xi,\epsilon}\left(F_{\pi_0}\, \mathbb{1}_{B(\pi_0)}\, G_{\underline{\pi}(i)} \mathbb{1}_{B(\underline{\pi}(i))}\left[1_A \circ \theta_{\bar{t}_i}\right]\right)\right) \\
&= \sum_{\pi_0,\underline{\pi}(i)} E_{P^\mu \otimes W}\left(\bar{E}_0^{\xi,\epsilon}\left(F_{\pi_0}\, \mathbb{1}_{B(\pi_0)}\, G_{\underline{\pi}(i)} \mathbb{1}_{B(\underline{\pi}(i))}\right) \bar{P}_{\overline{x}_i}^{\theta_{\bar{t}_i}(\xi,\epsilon)}(A)\right) \\
&= \sum_{\pi_0,\underline{\pi}(i)} E_{P^\mu \otimes W}\left(E_{P^\mu \otimes W}\left(\bar{E}_0^{\xi,\epsilon}\left(F_{\pi_0}\, \mathbb{1}_{B(\pi_0)}\, G_{\underline{\pi}(i)} \mathbb{1}_{B(\underline{\pi}(i))}\right) \bar{P}_{\overline{x}_i}^{\theta_{\bar{t}_i}(\xi,\epsilon)}(A) \,\Big|\, \Lambda_\xi(i)\right)\right) \\
&= \sum_{\pi_0,\underline{\pi}(i)} E_{P^\mu \otimes W}\left(G'_{\pi_0,\underline{\pi}(i)} E_{P^\mu \otimes W}\left(F_{\pi_0} \bar{P}_0^{\xi,\epsilon}\left(B(\pi_0)\right) \bar{P}_{\overline{x}_i}^{\theta_{\bar{t}_i}(\xi,\epsilon)}(A) \,\Big|\, \Lambda_\xi(i)\right)\right),
\end{aligned} \tag{3.23}$$

where the sum on $\pi_0, \underline{\pi}(i)$ runs over $\pi(\mathcal{T})^{i+1}$. Define

$$\rho_A = \mathrm{Cov}_{P^\mu \otimes W(\cdot|\Lambda_\xi(i))}\left[\bar{P}_{\overline{x}_i}^{\theta_{\bar{t}_i}(\xi,\epsilon)}(A); F_{\pi_0} \bar{P}_0^{\xi,\epsilon}\left(B(\pi_0)\right)\right], \tag{3.24}$$

and

$$\widetilde{\rho_A} = \sum_{\pi_0,\underline{\pi}(i)} E_{P^\mu \otimes W}\left(G'_{\pi_0,\underline{\pi}(i)} \rho_A\right). \tag{3.25}$$

Write $\widetilde{h_{i+1}}(\cdot \mid \underline{w}^{(i)})$ for the conditional law of r_{i+1} given $r^{(i)} = (r_1, \dots, r_i)$, and note that

$$\widetilde{h_{i+1}}(A \mid \underline{w}^{(i)}) = E_{P^\mu \otimes W}\left(\bar{P}_{\overline{x}_i}^{\theta_{\bar{t}_i}(\xi,\epsilon)}(A) \,\Big|\, \Lambda_\xi(i)\right) \tag{3.26}$$

on the event $B(\underline{\pi}(i)) \cap B(\pi_0)$. Combining (3.23), (3.25) and (3.26), we have

$$
\begin{aligned}
&\bar{\mathbb{E}}_{\mu,(0,0)}\left(FG\left[\mathbb{1}_A\circ\theta_{\tau^{(L)}_{i+1}}\right]\right)=\widetilde{\rho_A}+\\
&\sum_{\pi_0,\underline{\pi}(i)} E_{P^\mu\otimes W}\left(G'_{\pi_0,\underline{\pi}(i)}E_{P^\mu\otimes W}\left(F_{\pi_0}\bar{P}_0^{\xi,\epsilon}\left(B(\pi_0)\right)\,\Big|\,\Lambda_\xi(i)\right)E_{P^\mu\otimes W}\left(\bar{P}_{\overline{x}_i}^{\theta_{\bar{t}_i}(\xi,\epsilon)}(A)\,\Big|\,\Lambda_\xi(i)\right)\right)\\
&=\widetilde{\rho_A}+\sum_{\pi_0,\underline{\pi}(i)} E_{P^\mu\otimes W}\left(G'_{\pi_0,\underline{\pi}(i)}E_{P^\mu\otimes W}\left(F_{\pi_0}\bar{P}_0^{\xi,\epsilon}\left(B(\pi_0)\right)\,\Big|\,\Lambda_\xi(i)\right)\widetilde{h_{i+1}}(A\mid \underline{w}^{(i)})\right)\\
&=\widetilde{\rho_A}+\sum_{\pi_0,\underline{\pi}(i)}\bar{\mathbb{E}}_{\mu,(0,0)}\left(F_{\pi_0}\,\mathbb{1}_{B(\pi_0)}\,G_{\underline{\pi}(i)}\mathbb{1}_{B(\underline{\pi}(i))}\widetilde{h_{i+1}}(A\mid \underline{w}^{(i)})\right)\\
&=\widetilde{\rho_A}+\bar{\mathbb{E}}_{\mu,(0,0)}\left(FG\,\widetilde{h_{i+1}}(A\mid r^{(i)})\right).
\end{aligned}
\tag{3.27}
$$

Observe at this point that, for g measurable w.r.t. $\sigma\big(\xi_t(z)\colon (z,t)\in C_{i+1}+\overline{x}_i\big)$, the n-cone-mixing in (3.4), together with the Markovian nature of the RE ξ, imply that, $\bar{\mathbb{P}}_{\mu,(0,0)}$-a.s.

$$
\big|E_\mu\left[g\mid \underline{\xi}(u)\cup\Lambda_\xi(i)\right]-E_\mu\left[g\mid\Lambda_\xi(i)\right]\big|\le\Psi(iL)\|g\|_\infty. \tag{3.28}
$$

Consequently, for f measurable w.r.t. $\sigma\big(\underline{\xi}(u)\big)$, we have

$$
\begin{aligned}
&\big|E_\mu\left[fg\mid\Lambda_\xi(i)\right]-E_\mu\left[f\mid\Lambda_\xi(i)\right]E_\mu\left[g\mid\Lambda_\xi(i)\right]\big|\\
&=\big|E_\mu\left[fE_\mu\left[g\,|\underline{\xi}(u)\cup\Lambda_\xi(i)\right]\mid\Lambda_\xi(i)\right]-E_\mu\left[f\mid\Lambda_\xi(i)\right]E_\mu\left[g\mid\Lambda_\xi(i)\right]\big|\\
&\le\Psi(iL)\|g\|_\infty E_\mu\left[|f|\mid\Lambda_\xi(i)\right].
\end{aligned}
\tag{3.29}
$$

By estimating (3.24) with the help of (3.29), we obtain from (3.25) that

$$
|\widetilde{\rho_A}|\le\Psi(iL)\bar{\mathbb{E}}_{\mu,(0,0)}\left(FG\right). \tag{3.30}
$$

Finally, combining (3.27) and (3.30), we get

$$
\left|\bar{\mathbb{E}}_{\mu,(0,0)}\left(FG\left[\mathbb{1}_A\circ\theta_{\tau^{(L)}_{i+1}}\right]\right)-\bar{\mathbb{E}}_{\mu,(0,0)}\left(FG\,\widetilde{h_{i+1}}(A\mid r^{(i)})\right)\right|\le\Psi(iL)\bar{\mathbb{E}}_{\mu,(0,0)}\left(FG\right), \tag{3.31}
$$

which, in view of (3.15), implies(3.16). ∎

With the help of Lemma 3.3, we show in the following lemma that the kernel h_k converges as $k\to\infty$ to a kernel h that is independent of $u\in\mathcal{U}$.

Lemma 3.4. *Let $d(w,w') = 2^{-\min\{i\in\mathbb{N}:\ w_i\neq w_i'\}}$ be the lexicographic distance on the space $\mathcal{W}$ defined in (3.9), and let $M(\mathcal{T})$ be the set of probability measures on $\mathcal{T}$. For $\underline{w}^{(k)} = (w_k, w_{k-1}, \dots, w_1) \in \mathcal{T}^k$, define $\underline{w} = (w_k, w_{k-1}, \dots, w_1, s, s, \dots) \in \mathcal{W}$. Then, there exists a measurable kernel*

$$h : \mathcal{W} \longrightarrow M(\mathcal{T}) \tag{3.32}$$

such that

$$\sup_{\substack{k\geq i,\, u\in\mathcal{U},\, \underline{w}^{(k-1)}\in\mathcal{T}^{k-1},\\ w'\in\mathcal{W}:\, d(\underline{w},w')<2^{-i}}} \Big\| h_k(\cdot \mid \underline{w}^{(k-1)}, u) - h(\cdot \mid w') \Big\|_{\mathrm{tv}} \leq \Psi(iL) \tag{3.33}$$

and

$$\sup_{w,w'\in\mathcal{W}:\, d(w,w')<2^{-k}} \big\| h(\cdot \mid w) - h(\cdot \mid w') \big\|_{\mathrm{tv}} \leq 2\Psi(kL). \tag{3.34}$$

Proof. Fix $u \in \mathcal{U}$ and $w = (w_1, w_2, \dots) \in \mathcal{W}$, and put $w^{(k)} = (w_1, \dots, w_k) \in \mathcal{T}^k$. By Lemma 3.3, we have that

$$\sup_{u,u'\in\mathcal{U},\, w\in\mathcal{W}} \Big\| h_k(\cdot \mid w^{(k-1)}, u) - h_{k'}(\cdot \mid w^{(k'-1)}, u') \Big\|_{\mathrm{tv}} \leq \Psi((k\wedge k')L). \tag{3.35}$$

Therefore the sequence $\big(h_k(\cdot \mid w^{(k-1)}, u)\big)_{k\in\mathbb{N}}$ of kernels in $M(\mathcal{T})$ forms a Cauchy sequence w.r.t. the total variation distance, and the completeness of $M(\mathcal{T})$ ensures the existence of a limit $h(\cdot \mid w, u)$. Furthermore, from (3.35) we have that

$$\sup_{u,u'\in\mathcal{U},\, w\in\mathcal{W}} \Big\| h_k(\cdot \mid w^{(k-1)}, u') - h(\cdot \mid w, u) \Big\|_{\mathrm{tv}} \leq \sum_{i\geq k} \Psi(iL),$$

which, in view of (3.3), implies that $h(\cdot \mid w, u) = h(\cdot \mid w)$ does not depend on $u \in \mathcal{U}$. In particular, the estimates in (3.33) and (3.34) follow easily from (3.35).

∎

3.2.2 Invariance principle for the chain with complete connections

In Section 3.2.1 we constructed a chain with complete connections on $\mathcal{W}$ defined via the kernel h. From the latter we next construct a Markov chain $(w(n))_{n\in\mathbb{N}}$ with state space $\mathcal{W}$ for which we can use standard results from the theory of chains with complete connections.

Let $w(n) = (w_1(n), w_2(n), \dots) \in \mathcal{W}$, and let $y(n+1) \in \mathcal{T}$ be a random variable distributed according to $h(\cdot \mid w(n))$. The next state of the chain, $w(n+1)$, is obtained by setting

$$w_1(n+1) = y(n+1), \quad w_i(n+1) = w_{i-1}(n), \quad i \geq 2. \tag{3.36}$$

In particular, Lemma 3.4 implies that the chain $(w(n))_{n\in\mathbb{N}}$ satisfies conditions $FLS(\mathcal{T},1)$ and $M(1)$ in [57], pages 47 and 51. Thus, by Theorem 2.2.7 in [57], it is uniformly ergodic with a unique invariant measure P^w. Next, given $y = (N, x(1), x(2), \dots, x(N), \underline{\xi}(C)) \in \mathcal{T}$, set $f(y) = x(N)$ and $g(y) = N$. The integrability condition (2.49) in Lemma 2.5 implies that

$$\sup_{w\in\mathcal{W}} \int_{\mathcal{T}} |f(y)|^\alpha h(dy|w) < \infty, \quad \alpha > 1. \tag{3.37}$$

Therefore, by Proposition 4.1.1 and Theorem 4.1.2 in [57], we have that, P^w-a.s.,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} g(w_1(i)) = E_{P^w}[g(w_1)] = C_1, \quad \lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(w_1(i)) = E_{P^w}[f(w_1)] = C_2. \tag{3.38}$$

Furthermore, by the ϕ mixing property (see [57]) of $f(w_1(i))$ given by Theorem 2.1.5 in [57], together with (3.37) and Theorem 4.1.5 in [57], the following invariance principle holds. Let $c = C_2/C_1$, and

$$\Upsilon_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{\lfloor nt \rfloor} \big[f(w_1(i)) - c g(w_1(i)) \big], \quad n \in \mathbb{N}, t \geq 0. \tag{3.39}$$

Then, under P^w, the path $\Upsilon_n(t)$, converges weakly to a Brownian motion with a non-degenerate deterministic variance that is independent of the initial condition w.

3.2.3 Invariance principle for the random walk

It remains to show that the invariance principle in Section 3.2.2 for the chain $(w(n))_{n\in\mathbb{N}_0}$ implies the invariance principle of Theorem 3.2. To this aim, consider the random process

$$\left(\widetilde{S}_n(k)\right)_{k\in\mathbb{N}} \quad \text{with} \quad \widetilde{S}_n(k) = \frac{Z_{\tau_k^{(L)}} - c\tau_k^{(L)}}{\sqrt{n}}. \tag{3.40}$$

We first construct a coupling that allows us to compare $\widetilde{S}_n$ with Υ_n. After that we pass from $\widetilde{S}_n$ to S_t defined in (3.6).

Fix $w \in \mathcal{W}$ and $\epsilon \in (0,1)$. Consider an enlarged probability space, with law $P_{\epsilon,w}$, on which there exist a sequence $(r_k)_{k\in\mathbb{N}}$ distributed according to $\bar{\mathbb{P}}_{\mu,(0,0)}(\underline{r} \in \cdot \mid \mathcal{H}_1)$, with $\underline{r}$ as in (3.10), and a sequence $(w(k))_{k\in\mathbb{N}}$ distributed according to P^w. On this enlarged probability space, by using (3.33), we can couple $(r_k)_{k\in\mathbb{N}}$ and $(w(k))_{k\in\mathbb{N}}$ in a recursive manner such that

$$P_{\epsilon,w}\big(r_{i+1} = w_1(i+1) \mid r_1,\dots,r_i, w_1(1),\dots,w_1(i)\big) \geq 1-\Psi(kL) \tag{3.41}$$

on the event $\{r_l = w_1(l), i-k+1 \leq l \leq i\}$ for any $k \in \{1,\dots,i\}$. Hence, by (3.41) and the fact that $\sum_{k\in\mathbb{N}}\Psi(kL) < \infty$, we have a sequence $k_0(\epsilon) < \infty$, with $k_0(\epsilon) \to \infty$ as $\epsilon \to 0$, such that

$$P_{\epsilon,w}\big(\exists k \geq k_0(\epsilon)\colon\ r_k \neq w_1(k)\big) \leq \epsilon. \tag{3.42}$$

Next, recall Lemma 2.6, fix $T > 0$, and let

$$I_T = 2(T+1)/(Jr^{-L}) \quad \text{with} \quad J = \liminf_{L\to\infty} \bar{\mathbb{E}}_{\mu,(0,0)}(T_1^{(L)}). \tag{3.43}$$

From (3.42), we have that

$$P_{\epsilon,w}\left(\sup_{k_0(\epsilon)\leq k\leq nI_T} \|\widetilde{S}_n(k) - \widetilde{S}_n(k_0(\epsilon)) - \Upsilon_n(k/n) - \Upsilon_n(k_0(\epsilon)/n)\|_1 > 0\right) \leq \epsilon. \tag{3.44}$$

Moreover, for any $\delta > 0$,

$$\lim_{n\to\infty} \bar{\mathbb{P}}_{\mu,(0,0)}\left(\sup_{t\leq\tau_1^{(L)}} \|Z_t\|_1 > \delta\sqrt{n}\right) \leq \lim_{n\to\infty} \bar{\mathbb{P}}_{\mu,(0,0)}\left(\tau_1^{(L)} > \delta\sqrt{n}\right) = 0, \tag{3.45}$$

and, by using (2.49) with $\alpha > 1$, we get

$$\begin{aligned}
&\bar{\mathbb{P}}_{\mu,(0,0)}\left(\sup_{1\leq k\leq n}\left\{\sup_{\tau_k^{(L)}\leq t\leq\tau_{k+1}^{(L)}}\left\{\|Z_t - Z_{\tau_k^{(L)}}\|_1 + (t-\tau_k^{(L)})\right\}\right\} > 3\delta\sqrt{n} \mid \mathcal{H}_1\right)\\
&\leq \bar{\mathbb{P}}_{\mu,(0,0)}\left(\sup_{1\leq k\leq n}\left\{\tau_{k+1}^{(L)} - \tau_k^{(L)}\right\} > \delta\sqrt{n}\right) = 1 - \left[1 - \bar{\mathbb{P}}_{\mu,(0,0)}\left(\tau_1^{(L)} > \delta\sqrt{n}\right)\right]^n. \qquad (3.46)\\
&\leq 1 - \left[1 - \frac{\bar{\mathbb{E}}_{\mu,(0,0)}\left[\left(\tau_1^{(L)}\right)^\alpha\right]}{(\delta\sqrt{n})^\alpha}\right]^n \leq 1 - \left[1 - \frac{M(\alpha)r^{-\alpha L}}{(\delta\sqrt{n})^\alpha}\right]^n,
\end{aligned}$$

The r.h.s. of (3.46) tends to zero as $n \to \infty$. Therefore, in view of (3.45) and (3.46), taking first $n \to \infty$ and then $\epsilon \to 0$ in (3.44), we see that the invariance principle for Υ_n in (3.39) can be transferred to an invariance principle for $\widetilde{S}_n(\lfloor tn \rfloor)$ under $\bar{\mathbb{P}}_{\mu,(0,0)}$, on the interval $[0, I_T]$, with the same covariance.

To return to the original process Z, note that by (3.38) and (3.44) we have that

$$
\begin{aligned}
&\limsup_{n\to\infty} \bar{\mathbb{P}}_{\mu,(0,0)}\left(\sup_{k\le nI_T} \mid \frac{\tau_k^{(L)}}{n} - C_1\frac{k}{n} \mid > \delta\right) \\
&\le \limsup_{\epsilon\to 0}\limsup_{n\to\infty} P_{\epsilon,w}\left(\sup_{k\le nI_T} |\frac{\tau_k^{(L)}}{n} - C_1\frac{k}{n}| > \delta\right) = 0
\end{aligned}
. \tag{3.47}
$$

On the other hand, by (3.42), we have that

$$
\limsup_{n\to\infty} \bar{\mathbb{P}}_{\mu,(0,0)}\left(\tau_{nI_T}^{(L)} < Tn\right) \le \limsup_{\epsilon\to 0}\limsup_{n\to\infty} P_{\epsilon,w}\left(\tau_{nI_T}^{(L)} < Tn\right) = 0. \tag{3.48}
$$

Thus, by (3.44) and the stability of the invariance principle under random time changes (see [14]) we obtain the invariance principle under $\bar{\mathbb{P}}_{\mu,(0,0)}$, for

$$
\left(\frac{Z_{\lfloor nt\rfloor} - vnt}{\sqrt{n}}\right)_{n\in\mathbb{N}},
$$

which due to (2.32) carries over to Y, and in particular to its first component (see (2.25)). To pass to continuous time, note that the jump times of X in (2.6) are distributed according to a Poisson process with parameter $\alpha+\beta$ independently of the environment. Therefore, again by the stability of the invariance principle under random time changes, Theorem 3.2 holds.

3.2.4 Examples of mixing dynamic RE

We give here some example of *n-cone-mixing* dynamic RE according to Definition 3.1.
(1) **Independent spin-flip dynamics**
Let $\xi = (\xi_t)_{t\ge 0}$ be an independent spin-flip dynamics (see Section 2.5). Recall the notations of Section 3.1. Fix a set of n *nested-cones* $\{C_{N_i}(x_i)\colon\, x_i = (z_i, m_i) \in \mathbb{H}\}_{i=1}^n$. Define

$$
R_n = \{y \in \mathbb{Z}\colon\ \mid y - z_n \mid \le N_n\}
$$

to be the set of sites in $\mathbb{Z}$ belonging to the n-th cone, and

$$
R^{<n} = \{y \in \mathbb{Z}\colon\ (y,s) \in C_{N_i}(x_i) \text{ for some } i \le n-1\}
$$

to be the set of sites belonging to the first $n-1$ cones. For any subsets $A \in \mathcal{G}_n, B \in \mathcal{G}^{<n}$, and any two starting configurations $\eta, \eta' \in \Omega$, in the spirit of Section 2.4.1, estimate

$$\begin{aligned} \Big| P_\eta(A \mid B) - P_{\eta'}(A \mid B) \Big| &\leq \widehat{P}_{\eta,\eta'}\big(\exists\, (z,s) \in C_{N_n}(x_n):\ \xi_s(z) \neq \xi'_s(z) \mid B\big) \\ &\leq \sum_{z \in R_n \setminus R^{<n}} \widehat{P}\big(\exists\, s \geq m_n + |z_n - z|:\ \xi_s(0) \neq \xi'_s(0)\big) \\ &\leq c_1 \mathrm{e}^{-c_2 m_n} \leq c_1 \mathrm{e}^{-c_2 nL}, \end{aligned} \tag{3.49}$$

for some constants $c_1, c_2 > 0$. In the second inequality we have used the independence in space and $\widehat{P}$ stands for the single-site basic coupling measure. In the third inequality we used the exponential convergence to equilibrium and in the fourth inequality that $m_n \geq nL$.

(2) **Space-time strong-mixing Gibbsian field and IPS in the regime** $M < \epsilon$ Consider a dynamic RE ξ constituted by a space-time Gibbsian field as in the example of Section 2.4.3. As shown in [36] (see just after Eq. (2.7) therein), by requiring that the Gibbsian field ξ is strong-mixing in the sense of Definition 1.7 in [36] (see Eq. (1.9) therein), it follows that ξ is an *n-cone-mixing* dynamic RE.

If ξ is a spin-flip system in the regime $M < \epsilon$ (see Section 2.4.1), then due to spatial correlations the argument used in (3.49) does not hold. Nevertheless, such systems are equivalent, in terms of mixing properties, to a Gibbsian field in the uniqueness regime at high temperature (see e.g. [65, 66]), and are therefore expected to satisfy the *n-cone-mixing* property in Definition 3.1. We plan to settle this technical issue in the future.

3.3 CLT in the perturbative regime

In the context of Section 2.3, the proof of Theorem 3.2 does not need the machinery of the previous section. Indeed, as we pointed out in (2.104), $X_t - vt$ can be decomposed as a sum of a martingale $(M_t)_{t\geq 0}$ and an additive functional of the environment process $(\eta_t)_{t\geq 0}$, i.e.,

$$X_t - vt = M_t + (\alpha - \beta) \int_0^t \big(2\eta_s(0) - 1\big)\, \mathrm{d}s - vt = M_t + \int_0^t f(\eta_s)\mathrm{d}s. \tag{3.50}$$

In the spirit of Kipnis-Varadhan [61], we would like to write the additive functional in (3.50) as the sum of a martingale $(M'_t)_{t\geq 0}$ plus a term $(\epsilon_t)_{t\geq 0}$ that is negligible when we divide by $\sqrt{t}$, i.e.,

$$\int_0^t f(\eta_s)\mathrm{d}s = M'_t + \epsilon_t, \quad \epsilon_t = o(\sqrt{t}). \tag{3.51}$$

Since the environment process is not in general a reversible Markov process in $L_2(\mu_e)$, we cannot directly apply the theorem stated in [61]. Nevertheless, several refinements of the Kipnis-Varadhan approach have been obtained for non-reversible Markov processes, e.g. [54], Corollary 3.2, gives a sufficient condition for a martingale approximation, namely,

$$\int_1^\infty t^{-1/2} \left\| S(t)f \right\|_2 \mathrm{d}t < \infty, \tag{3.52}$$

where $(S(t))_{t\geq 0}$ is the semigroup associated with $(\eta_t)_{t\geq 0}$ and $\|\cdot\|_2$ denote the $L_2(\mu_e)$-norm. From (2.133) we easily see that (3.52) holds. Indeed,

$$\left\| S(t)f \right\|_2 \leq \left\| S(t)f \right\|_\infty \leq Ce^{-[c-2(\alpha-\beta)]t}. \tag{3.53}$$

Hence, (3.50) holds, and we can write

$$X_t - vt = M_t + \int_0^t f(\eta_s)\mathrm{d}s = M_t + M'_t + \epsilon_t = M''_t + \epsilon_t. \tag{3.54}$$

The invariance principle for $(X_t)_{t\geq 0}$ then follows from the standard invariance principle for martingales (see e.g. [14]).