Random walks in dynamic random environments

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Chapter 2

Law of large numbers for a class of RW in dynamic RE

This chapter is based on a paper with Frank den Hollander and Frank Redig that has been submitted to Electronic Journal of Probability.

Abstract

In this paper we consider a class of one-dimensional interacting particle systems in equilibrium, constituting a dynamic random environment, together with a nearest-neighbor random walk that on occupied/vacant sites has a local drift to the right/left. We adapt a regeneration-time argument originally developed by Comets and Zeitouni [35] for static random environments to prove that, under a space-time mixing property for the dynamic random environment called cone-mixing, the random walk has an a.s. constant global speed. In addition, we show that if the dynamic random environment is exponentially mixing in space-time and the local drifts are small, then the global speed can be written as a power series in the size of the local drifts. From the first term in this series the sign of the global speed can be read off.

The results can be easily extended to higher dimensions.

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2.1 Introduction and main result

In Section 2.1 we define the random walk in dynamic random environment, introduce a space-time mixing property for the random environment called cone-mixing, and state our law of large numbers for the random walk subject to cone-mixing. In Section 2.2 we give the proof of the law of large numbers with the help of a space-time regeneration-time argument. In Section 2.3 we assume a stronger space-time mixing property, namely, exponential mixing, and derive a series expansion for the global speed of the random walk in powers of the size of the local drifts. This series expansion converges for small enough local drifts and its first term allows us to determine the sign of the global speed. (The perturbation argument underlying the series expansion provides an alternative proof of the law of large numbers.) In Section 2.4 we give examples of random environments that are cone-mixing. In Section 2.5 we compute the first three terms in the expansion for an independent spin-flip dynamics.

2.1.1 Model

Let $\Omega = \{0,1\}^{\mathbb{Z}}$. Let $C(\Omega)$ be the set of continuous functions on Ω taking values in $\mathbb{R}$, $\mathcal{P}(\Omega)$ the set of probability measures on Ω, and $D_\Omega[0,\infty)$ the path space, i.e., the set of càdlàg functions on $[0,\infty)$ taking values in Ω. In what follows,

$$\xi = (\xi_t)_{t\geq 0} \quad \text{with} \quad \xi_t = \{\xi_t(x)\colon\, x \in \mathbb{Z}\} \tag{2.1}$$

is an interacting particle system taking values in Ω, with $\xi_t(x) = 0$ meaning that site x is vacant at time t and $\xi_t(x) = 1$ that it is occupied. The paths of ξ take values in $D_\Omega[0,\infty)$. The law of ξ starting from $\xi_0 = \eta$ is denoted by P^η. The law of ξ when ξ_0 is drawn from $\mu \in \mathcal{P}(\Omega)$ is denoted by P^μ, and is given by

$$P^\mu(\cdot) = \int_\Omega P^\eta(\cdot)\,\mu(\mathrm{d}\eta). \tag{2.2}$$

Through the sequel we will assume that

$$P^\mu \text{ is stationary and ergodic under space-time shifts.} \tag{2.3}$$

Thus, in particular, μ is a homogeneous extremal equilibrium for ξ. The Markov semigroup associated with ξ is denoted by $S_{\mathrm{IPS}} = (S_{\mathrm{IPS}}(t))_{t\geq 0}$. This semigroup acts from the left on $C(\Omega)$ as

$$\big(S_{\mathrm{IPS}}(t)f\big)(\cdot) = E^{(\cdot)}[f(\xi_t)], \qquad f \in C(\Omega), \tag{2.4}$$

and acts from the right on $\mathcal{P}(\Omega)$ as

$$\big(\nu S_{\mathrm{IPS}}(t)\big)(\cdot) = P^{\nu}(\xi_t \in \cdot\,), \qquad \nu \in \mathcal{P}(\Omega). \tag{2.5}$$

See Liggett [63], Chapter I, for a formal construction.

Conditional on ξ, let

$$X = (X_t)_{t\geq 0} \tag{2.6}$$

be the random walk with local transition rates

$$\begin{aligned} x \to x+1 \quad &\text{at rate} \quad \alpha\, \xi_t(x) + \beta\,[1-\xi_t(x)], \\ x \to x-1 \quad &\text{at rate} \quad \beta\, \xi_t(x) + \alpha\,[1-\xi_t(x)], \end{aligned} \tag{2.7}$$

where w.l.o.g.

$$0 < \beta < \alpha < \infty. \tag{2.8}$$

Thus, on occupied sites the random walk has a local drift to the right while on vacant sites it has a local drift to the left, of the same size. Note that the sum of the jump rates $\alpha+\beta$ is independent of ξ. Let P_0^{ξ} denote the law of X starting from $X_0 = 0$ conditional on ξ, which is the *quenched* law of X. The *annealed* law of X is

$$\mathbb{P}_{\mu,0}(\cdot) = \int_{D_{\Omega}[0,\infty)} P_0^{\xi}(\cdot)\, P^{\mu}(\mathrm{d}\xi). \tag{2.9}$$

2.1.2 Cone-mixing and law of large numbers

In what follows we will need a *mixing property* for the law P^{μ} of ξ. Let $\cdot$ and $\|\cdot\|$ denote the inner product, respectively, the Euclidean norm on $\mathbb{R}^2$. Put $\ell = (0,1)$. For $\theta \in (0, \frac{1}{2}\pi)$ and $t \geq 0$, let

$$C_t^{\theta} = \big\{u \in \mathbb{Z} \times [0,\infty)\colon\ (u - t\ell)\cdot \ell \geq \|u - t\ell\| \cos\theta\big\} \tag{2.10}$$

be the cone whose tip is at $t\ell = (0,t)$ and whose wedge opens up in the direction ℓ with an angle θ on either side (see Figure 2.1). Note that if $\theta = \frac{1}{2}\pi$ ($\theta = \frac{1}{4}\pi$), then the cone is the half-plane (quarter-plane) above $t\ell$.

Definition 2.1. A probability measure P^{μ} on $D_{\Omega}[0,\infty)$ satisfying (2.3) is said to be cone-mixing if, for all $\theta \in (0, \frac{1}{2}\pi)$,

$$\lim_{t\to\infty} \sup_{\substack{A\in\mathcal{F}_0,\, B\in\mathcal{F}_t^{\theta} \\ P^{\mu}(A)>0}} \Big| P^{\mu}(B \mid A) - P^{\mu}(B)\Big| = 0, \tag{2.11}$$

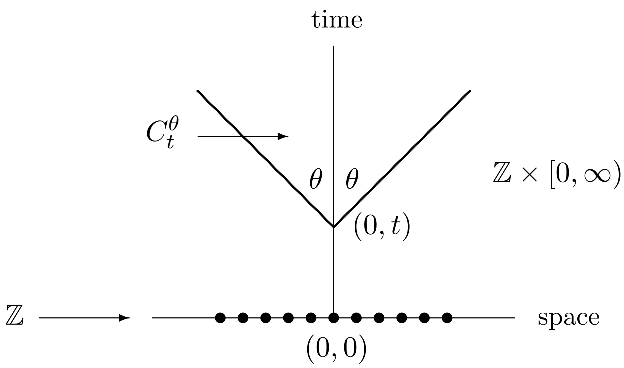

FIGURE 2.1: The cone C_t^θ.

where

$$\begin{aligned} \mathcal{F}_0 &= \sigma\big\{\xi_0(x)\colon\ x \in \mathbb{Z}\big\}, \\ \mathcal{F}_t^\theta &= \sigma\big\{\xi_s(x)\colon\ (x,s) \in C_t^\theta\big\}. \end{aligned} \tag{2.12}$$

In Section 2.4 we give examples of interacting particle systems that are cone-mixing.

We are now ready to formulate our law of large numbers (LLN).

Theorem 2.2. *Assume* (2.3). *If P^μ is cone-mixing, then there exists a $v \in \mathbb{R}$ such that*

$$\lim_{t\to\infty} X_t/t = v \qquad \mathbb{P}_{\mu,0} - a.s. \tag{2.13}$$

The proof of Theorem 2.2 is given in Section 2.2, and is based on a *regeneration-time argument* originally developed by Comets and Zeitouni [35] for static random environments (based on earlier work by Sznitman and Zerner [92]).

We have no criterion for when $v < 0$, $v = 0$ or $v > 0$. In view of (2.8), a naive guess would be that these regimes correspond to $\rho < \frac{1}{2}$, $\rho = \frac{1}{2}$ and $\rho > \frac{1}{2}$, respectively, with $\rho = P^\mu(\xi_0(0) = 1)$ the density of occupied sites. However, $v = (2\tilde{\rho} - 1)(\alpha - \beta)$, with $\tilde{\rho}$ the asymptotic fraction of time spent by the walk on occupied sites, and the latter is a non-trivial function of P^μ, α and β. We do not (!) expect that $\tilde{\rho} = \frac{1}{2}$ when $\rho = \frac{1}{2}$ in general. Clearly, if P^μ is invariant under swapping the states 0 and 1, then $v = 0$.

2.1.3 Global speed for small local drifts

For small $\alpha - \beta$, X is a perturbation of simple symmetric random walk. In that case it is possible to derive an expansion of v in powers of $\alpha - \beta$, provided P^μ satisfies an exponential space-time mixing property referred to as $M < \epsilon$ (Liggett [63], Section I.3). Under this mixing property, μ is even uniquely ergodic.

Suppose that ξ has shift-invariant local transition rates

$$c(A,\eta), \qquad A \subset \mathbb{Z} \text{ finite}, \, \eta \in \Omega, \tag{2.14}$$

i.e., $c(A,\eta)$ is the rate in the configuration η to change the states at the sites in A, and $c(A,\eta) = c(A+x, \tau_x \eta)$ for all $x \in \mathbb{Z}$ with τ_x the shift of space over x. Define

$$\begin{aligned} M &= \sum_{A \ni 0} \sum_{x \neq 0} \sup_{\eta \in \Omega} |c(A,\eta) - c(A,\eta^x)|, \\ \epsilon &= \inf_{\eta \in \Omega} \sum_{A \ni 0} |c(A,\eta) + c(A,\eta^0)|, \end{aligned} \tag{2.15}$$

where η^x is the configuration obtained from x by changing the state at site x. The interpretation of (2.15) is that M is a measure for the *maximal* dependence of the transition rates on the states of single sites, while ϵ is a measure for the *minimal* rate at which the states of single sites change. See Liggett [63], Section I.4, for examples.

Theorem 2.3. *Assume* (2.3) *and suppose that* $M < \epsilon$. *If* $\alpha - \beta < \frac{1}{2}(\epsilon - M)$, *then*

$$v = \sum_{n \in \mathbb{N}} c_n \, (\alpha - \beta)^n \in \mathbb{R} \quad \text{with} \quad c_n = c_n(\alpha + \beta; P^\mu), \tag{2.16}$$

where $c_1 = 2\rho - 1$ *and* $c_n \in \mathbb{R}$, $n \in \mathbb{N}\backslash\{1\}$, *are given by a recursive formula (see Section* 2.3.3*).*

The proof of Theorem 2.3 is given in Section 2.3, and is based on an analysis of the semigroup associated with the environment process, i.e., the environment as seen relative to the random walk. The generator of this process turns out to be a sum of a large part and a small part, which allows for a perturbation argument. In Section 2.4 we show that $M < \epsilon$ implies cone-mixing for spin-flip systems, i.e., systems for which $c(A,\eta) = 0$ when $|A| \geq 2$.

It follows from Theorem 2.3 that for $\alpha - \beta$ small enough the global speed v changes sign at $\rho = \frac{1}{2}$:

$$v = (2\rho - 1)(\alpha - \beta) + O\big((\alpha - \beta)^2\big) \text{ as } \alpha \downarrow \beta \text{ for } \rho \text{ fixed.} \tag{2.17}$$

We will see in Section 2.3.3 that $c_2 = 0$ when μ is a *reversible* equilibrium, in which case the error term in (2.17) is $O((\alpha - \beta)^3)$.

In Section 2.5 we consider an independent spin-flip dynamics such that 0 changes to 1 at rate γ and 1 changes to 0 at rate δ, where $0 < \gamma, \delta < \infty$. By reversibility, $c_2 = 0$. We show that

$$c_3 = \frac{4}{U^2} \, \rho(1-\rho)(2\rho - 1) \, f(U,V), \quad f(U,V) = \frac{2U + V}{\sqrt{V^2 + 2UV}} - \frac{2U + 2V}{\sqrt{V^2 + UV}} + 1, \tag{2.18}$$

with $U = \alpha + \beta$, $V = \gamma + \delta$ and $\rho = \gamma/(\gamma + \delta)$. Note that $f(U,V) < 0$ for all U,V and $\lim_{V\to\infty} f(U,V) = 0$ for all U. Therefore (2.18) shows that

$$\begin{aligned} &(1) \quad c_3 > 0 \text{ for } \rho < \tfrac{1}{2},\ c_3 = 0 \text{ for } \rho = \tfrac{1}{2},\ c_3 < 0 \text{ for } \rho > \tfrac{1}{2}, \\ &(2) \quad c_3 \to 0 \text{ as } \gamma + \delta \to \infty \text{ for fixed } \rho \neq \tfrac{1}{2} \text{ and fixed } \alpha + \beta. \end{aligned} \tag{2.19}$$

If $\rho = \frac{1}{2}$, then the dynamics is invariant under swapping the states 0 and 1, so that $v = 0$. If $\rho > \frac{1}{2}$, then $v > 0$ for $\alpha - \beta > 0$ small enough, but v is smaller in the random environment than in the average environment, for which $v = (2\rho - 1)(\alpha - \beta)$ ("slow-down phenomenon"). In the limit $\gamma + \delta \to \infty$ the walk sees the average environment.

2.1.4 Discussion and outline

Three classes of dynamic random environments have been studied in the literature so far:

(1) *Independent in time*: globally updated at each unit of time ;

(2) *Independent in space*: locally updated according to independent single-site Markov chains;

(3) *Dependent in space and time.*

Our models fit into class (3), which is the most challenging and still is far from being understood. For an extended list of references we refer the reader to [4].

Many results, like a LLN, annealed and quenched invariance principles or decay of correlations, have been obtained for the above three classes under suitable extra assumptions. In particular, it is assumed either that the random enviornment has a strong space-time mixing property and/or that the transition probabilities of the walks are close to constant, i.e., small perturbation of a homogeneous random walk.

The LLN in Theorem 2.2 is a successful attempt to move away from the restrictions. Cone mixing is one of the weakest mixing conditions under which we may expect to be able to derive a LLN via regeneration times: no rate of mixing is imposed in (2.11). Still, (2.11) is not optimal because it is a *uniform* mixing condition. For instance, the simple symmetric exclusion process, which has a one-parameter family of equilibria parameterized by the particle density, is not cone-mixing.

Our expansion of the global speed in Theorem 2.3 which is a perturbation of a homogeneous random walk falls in class (3), but unlike what was done in previous works, it offers an explicit control on the coefficients and on the domain of convergence of the expansion.

Both Theorem 2.2 and 2.3 are easily extended to higher dimensions (with the obvious generalization of cone-mixing), and to random walks whose step rates are local functions of the environment, i.e., in (2.7) replace $\xi_t(x)$ by $R(\tau_x \xi_t)$, with τ_x the shift over x and R any cylinder function on Ω. It is even possible to allow for steps with a finite range. All that is needed is that the total jump rate is independent of the random environment. The reader is invited to take a look at the proofs in Sections 2.2 and 2.3 to see why.

In the context of Theorem 2.3, the LLN can be extended to a central limit theorem (CLT) with somewhat strong mixing assumptions and to a large deviation principle (LDP), issues which we plan to address in future work.

2.2 Proof of Theorem 2.2

In this section we prove Theorem 2.2 by adapting the proof of the LLN for random walks in *static* random environments developed by Comets and Zeitouni [35]. The proof proceeds in seven steps. In Section 2.2.1 we look at a discrete-time random walk X on $\mathbb{Z}$ in a *dynamic* random environment and show that it is equivalent to a discrete-time random walk Y on

$$\mathbb{H} = \mathbb{Z} \times \mathbb{N}_0 \tag{2.20}$$

in a *static* random environment that is *directed* in the vertical direction. In Section 2.2.2 we show that Y in turn is equivalent to a discrete-time random walk Z on $\mathbb{H}$ that suffers *time lapses*, i.e., random times intervals during which it does not observe the random environment and does not move in the horizontal direction. Because of the cone-mixing property of the random environment, these time lapses have the effect of *wiping out the memory.* In Section 2.2.3 we introduce *regeneration times* at which, roughly speaking, the future of Z becomes independent of its past. Because Z is directed, these regeneration times are stopping times. In Section 2.2.4 we derive a bound on the moments of the gaps between the regeneration times. In Section 2.2.5 we recall a basic coupling property for sequences of random variables that are weakly dependent. In Section 2.2.6, we collect the various ingredients and prove the LLN for Z, which will immediately imply the LLN for X. In Section 2.2.7, finally, we show how the LLN for X can be extended from *discrete* time to *continuous* time.

The main ideas in the proof all come from [35]. In fact, by exploiting the directedness we are able to simplify the argument in [35] considerably.

2.2.1 Space-time embedding

Conditional on ξ, we define a *discrete-time* random walk on $\mathbb{Z}$

$$X = (X_n)_{n\in\mathbb{N}_0} \tag{2.21}$$

with transition probabilities

$$P_0^\xi\big(X_{n+1} = x+i \mid X_n = x\big) = \begin{cases} p\,\xi_{n+1}(x) + q\,[1-\xi_{n+1}(x)] & \text{if } i=1,\\ q\,\xi_{n+1}(x) + p\,[1-\xi_{n+1}(x)] & \text{if } i=-1,\\ 0 & \text{otherwise,}\end{cases} \tag{2.22}$$

where $x\in\mathbb{Z}$, $p\in(\frac{1}{2},1)$, $q=1-p$, and P_0^ξ denotes the law of X starting from $X_0=0$ conditional on ξ. This is the discrete-time version of the random walk defined in (2.6–2.7), with p and q taking over the role of $\alpha/(\alpha+\beta)$ and $\beta/(\alpha+\beta)$. As in Section 2.1.1, we write P_0^ξ to denote the *quenched* law of X and $\mathbb{P}_{\mu,0}$ to denote the *annealed* law of X.

Our interacting particle system ξ is assumed to start from an equilibrium measure μ such that the path measure P^μ is stationary and ergodic under space-time shifts and is cone-mixing. Given a realization of ξ, we observe the values of ξ at integer times $n\in\mathbb{Z}$, and introduce a random walk on $\mathbb{H}$

$$Y = (Y_n)_{n\in\mathbb{N}_0} \tag{2.23}$$

with transition probabilities

$$P_{(0,0)}^\xi\big(Y_{n+1} = x+e \mid Y_n = x\big) = \begin{cases} p\,\xi_{x_2+1}(x_1) + q\,[1-\xi_{x_2+1}(x_1)] & \text{if } e=\ell^+,\\ q\,\xi_{x_2+1}(x_1) + p\,[1-\xi_{x_2+1}(x_1)] & \text{if } e=\ell^-,\\ 0 & \text{otherwise,}\end{cases} \tag{2.24}$$

where $x=(x_1,x_2)\in\mathbb{H}$, $\ell^+=(1,1)$, $\ell^-=(-1,1)$, and $P_{(0,0)}^\xi$ denotes the law of Y given $Y_0=(0,0)$ conditional on ξ. By construction, Y is the random walk on $\mathbb{H}$ that moves inside the cone with tip at $(0,0)$ and angle $\frac{1}{4}\pi$, and jumps in the directions either l^+ or l^-, such that

$$Y_n = (X_n, n), \qquad n\in\mathbb{N}_0. \tag{2.25}$$

We refer to $P_{(0,0)}^\xi$ as the quenched law of Y and to

$$\mathbb{P}_{\mu,(0,0)}(\cdot) = \int_{D_\Omega[0,\infty)} P_{(0,0)}^\xi(\cdot)\,P^\mu(\mathrm{d}\xi) \tag{2.26}$$

as the annealed law of Y. If we manage to prove that there exists a $u = (u_1, u_2) \in \mathbb{R}^2$ such that

$$\lim_{n\to\infty} Y_n/n = u \qquad \mathbb{P}_{\mu,(0,0)} - a.s., \tag{2.27}$$

then, by (2.25), $u_2 = 1$, and the LLN in Theorem 2.2 holds with $v = u_1$.

2.2.2 Adding time lapses

Put $\Lambda = \{0, \ell^+, \ell^-\}$. Let $\epsilon = (\epsilon_i)_{i\in\mathbb{N}}$ be an i.i.d. sequence of random variables taking values in Λ according to the product law $W = w^{\otimes\mathbb{N}}$ with marginal

$$w(\epsilon_1 = e) = \begin{cases} r & \text{if } e \in \{\ell^+, \ell^-\}, \\ p & \text{if } e = 0, \end{cases} \tag{2.28}$$

with $r = \frac{1}{2}q$. For fixed ξ and ϵ, introduce a second random walk on $\mathbb{H}$

$$Z = (Z_n)_{n\in\mathbb{N}_0} \tag{2.29}$$

with transition probabilities

$$\begin{aligned} &\bar{P}^{\xi,\epsilon}_{(0,0)}\big(Z_{n+1} = x + e \mid Z_n = x\big) \\ &\qquad = 1_{\{\epsilon_{n+1}=e\}} + \frac{1}{p}\, 1_{\{\epsilon_{n+1}=0\}} \left[P^{\xi}_{(0,0)}\big(Y_{n+1} = x + e \mid Y_n = x\big) - r\right], \end{aligned} \tag{2.30}$$

where $x \in \mathbb{H}$ and $e \in \{\ell^+, \ell^-\}$, and $\bar{P}^{\xi,\epsilon}_{(0,0)}$ denotes the law of Z given $Z_0 = (0,0)$ conditional on ξ, ϵ. In words, if $\epsilon_{n+1} \in \{\ell^+, \ell^-\}$, then Z takes step ϵ_{n+1} at time $n+1$, while if $\epsilon_{n+1} = 0$, then Z copies the step of Y.

The quenched and annealed laws of Z defined by

$$\bar{P}^{\xi}_{(0,0)}(\cdot) = \int_{\Lambda^{\mathbb{N}}} \bar{P}^{\xi,\epsilon}_{(0,0)}(\cdot)\, W(\mathrm{d}\epsilon), \qquad \bar{\mathbb{P}}_{\mu,(0,0)}(\cdot) = \int_{D_\Omega[0,\infty)} \bar{P}^{\xi}_{(0,0)}(\cdot)\, P^{\mu}(\mathrm{d}\xi), \tag{2.31}$$

coincide with those of Y, i.e.,

$$\bar{P}^{\xi}_{(0,0)}(Z \in \cdot\,) = P^{\xi}_{(0,0)}(Y \in \cdot\,), \qquad \bar{\mathbb{P}}_{\mu,(0,0)}(Z \in \cdot\,) = \mathbb{P}_{\mu,(0,0)}(Y \in \cdot\,). \tag{2.32}$$

In words, Z becomes Y when the average over ϵ is taken. The importance of (2.32) is two-fold. First, to prove the LLN for Y in (2.27) it suffices to prove the LLN for Z. Second, Z suffers time lapses during which its transitions are dictated by ϵ rather than ξ. By the cone-mixing property of ξ, these time lapses will allow ξ to steadily loose memory, which will be a crucial element in the proof of the LLN for Z.

2.2.3 Regeneration times

Fix $L \in 2\mathbb{N}$ and define the L-vector

$$\epsilon^{(L)} = (\ell^+, \ell^-, \dots, \ell^+, \ell^-), \tag{2.33}$$

where the pair ℓ^+, ℓ^- is alternated $\frac{1}{2}L$ times. Given $n \in \mathbb{N}_0$ and $\epsilon \in \Lambda^{\mathbb{N}}$ with $(\epsilon_{n+1}, \dots, \epsilon_{n+L}) = \epsilon^{(L)}$, we see from (2.30) that (because $\ell^+ + \ell^- = (0,2) = 2\ell$)

$$\bar{P}^{\xi,\epsilon}_{(0,0)}\big(Z_{n+L} = x + L\ell \mid Z_n = x\big) = 1, \qquad x \in \mathbb{H}, \tag{2.34}$$

which means that the stretch of walk $Z_n, \dots, Z_{n+L}$ travels in the vertical direction ℓ irrespective of ξ.

Define *regeneration times*

$$\tau_0^{(L)} = 0, \qquad \tau_{k+1}^{(L)} = \inf\big\{n > \tau_k^{(L)} + L\colon\, (\epsilon_{n-L}, \dots, \epsilon_{n-1}) = \epsilon^{(L)}\big\}, \qquad k \in \mathbb{N}. \tag{2.35}$$

Note that these are stopping times w.r.t. the filtration $\mathcal{G} = (\mathcal{G}_n)_{n \in \mathbb{N}}$ given by

$$\mathcal{G}_n = \sigma\{\epsilon_i\colon\, 1 \le i \le n\}, \qquad n \in \mathbb{N}. \tag{2.36}$$

Also note that, by the product structure of $W = w^{\otimes \mathbb{N}}$ defined in (2.28), we have $\tau_k^{(L)} < \infty$ $\bar{\mathbb{P}}_0$-a.s. for all $k \in \mathbb{N}$.

Recall Definition 2.1 and put

$$\Phi(t) = \sup_{\substack{A \in \mathcal{F}_0,\, B \in \mathcal{F}_t^\theta \\ P^\mu(A) > 0}} \Big| P^\mu(B \mid A) - P^\mu(B) \Big|. \tag{2.37}$$

Cone-mixing is the property that $\lim_{t \to \infty} \Phi(t) = 0$ (for all cone angles $\theta \in (0, \frac{1}{2}\pi)$, in particular, for $\theta = \frac{1}{4}\pi$ needed here). Let

$$\mathcal{H}_k = \sigma\Big((\tau_i^{(L)})_{i=0}^k,\, (Z_i)_{i=0}^{\tau_k^{(L)}},\, (\epsilon_i)_{i=0}^{\tau_k^{(L)}-1},\, \{\xi_t\colon\, 0 \le t \le \tau_k^{(L)} - L\}\Big), \qquad k \in \mathbb{N}. \tag{2.38}$$

This sequence of sigma-fields allows us to keep track of the walk, the time lapses and the environment up to each regeneration time. Our main result in the section is the following.

Lemma 2.4. *For all $L \in 2\mathbb{N}$ and $k \in \mathbb{N}$,*

$$\Big\| \bar{\mathbb{P}}_{\mu,(0,0)}\Big(Z^{[k]} \in \cdot \mid \mathcal{H}_k\Big) - \bar{\mathbb{P}}_{\mu,(0,0)}\big(Z \in \cdot\,\big) \Big\|_{\mathrm{tv}} \le \Phi(L), \tag{2.39}$$

where

$$Z^{[k]} = \left(Z_{\tau_k^{(L)}+n} - Z_{\tau_k^{(L)}} \right)_{n\in\mathbb{N}_0} \tag{2.40}$$

and $\|\cdot\|_{\rm tv}$ *is the total variation norm.*

Proof. We give the proof for $k = 1$. Let $A \in \sigma(\mathbb{H}^{\mathbb{N}_0})$ be arbitrary, and abbreviate $1_A = 1_{\{Z\in A\}}$. Let h be any $\mathcal{H}_1$-measurable non-negative random variable. Then, for all $x \in \mathbb{H}$ and $n \in \mathbb{N}$, there exists a random variable $h_{x,n}$, measurable w.r.t. the sigma-field

$$\sigma\left((Z_i)_{i=0}^n,\, (\epsilon_i)_{i=0}^{n-1},\, \{\xi_t\colon\ 0 \le t < n-L\}\right), \tag{2.41}$$

such that $h = h_{x,n}$ on the event $\{Z_n = x, \tau_1^{(L)} = n\}$. Let $E_{P^\mu\otimes W}$ and $\mathrm{Cov}_{P^\mu\otimes W}$ denote expectation and covariance w.r.t. $P^\mu \otimes W$, and write θ_n to denote the shift of time over n. Then

$$\begin{aligned}
\bar{\mathbb{E}}_{\mu,(0,0)}\left(h\left[1_A\circ\theta_{\tau_1^{(L)}}\right]\right) &= \sum_{x\in\mathbb{H},n\in\mathbb{N}} E_{P^\mu\otimes W}\left(\bar{E}_0^{\xi,\epsilon}\left(h_{x,n}\left[1_A\circ\theta_n\right]1_{\left\{Z_n=x,\tau_1^{(L)}=n\right\}}\right)\right)\\
&= \sum_{x\in\mathbb{H},n\in\mathbb{N}} E_{P^\mu\otimes W}\big(f_{x,n}(\xi,\epsilon)\,g_{x,n}(\xi,\epsilon)\big)\\
&= \bar{\mathbb{E}}_{\mu,(0,0)}(h)\,\bar{\mathbb{P}}_{\mu,(0,0)}(A) + \rho_A,
\end{aligned} \tag{2.42}$$

where

$$f_{x,n}(\xi,\epsilon) = \bar{E}_{(0,0)}^{\xi,\epsilon}\left(h_{x,n}\,1_{\left\{Z_n=x,\tau_1^{(L)}=n\right\}}\right), \quad g_{x,n}(\xi,\epsilon) = \bar{P}_x^{\theta_n\xi,\theta_n\epsilon}(A), \tag{2.43}$$

and

$$\rho_A = \sum_{x\in\mathbb{H},n\in\mathbb{N}} \mathrm{Cov}_{P^\mu\otimes W}\big(f_{x,n}(\xi,\epsilon), g_{x,n}(\xi,\epsilon)\big). \tag{2.44}$$

By (2.11), we have

$$\begin{aligned}
|\rho_A| &\le \sum_{x\in\mathbb{H},n\in\mathbb{N}} \left|\mathrm{Cov}_{P^\mu\otimes W}\big(f_{x,n}(\xi,\epsilon), g_{x,n}(\xi,\epsilon)\big)\right|\\
&\le \sum_{x\in\mathbb{H},n\in\mathbb{N}} \Phi(L)\,E_{P^\mu\otimes W}\big(f_{x,n}(\xi,\epsilon)\big)\,\sup_{\xi,\epsilon} g_{x,n}(\xi,\epsilon)\\
&\le \Phi(L) \sum_{x\in\mathbb{H},n\in\mathbb{N}} E_{P^\mu\otimes W}\big(f_{x,n}(\xi,\epsilon)\big) = \Phi(L)\,\bar{\mathbb{E}}_{\mu,(0,0)}(h).
\end{aligned} \tag{2.45}$$

Combining (2.42) and (2.45), we get

$$\left|\bar{\mathbb{E}}_{\mu,(0,0)}\left(h\left[1_A\circ\theta_{\tau_1^{(L)}}\right]\right) - \bar{\mathbb{E}}_{\mu,(0,0)}(h)\,\bar{\mathbb{P}}_{\mu,(0,0)}(A)\right| \le \Phi(L)\,\bar{\mathbb{E}}_{\mu,(0,0)}(h). \tag{2.46}$$

Now pick $h = 1_B$ with $B \in \mathcal{H}_1$ arbitrary. Then (2.46) yields

$$\left|\bar{\mathbb{P}}_{\mu,(0,0)}\left(Z^{[k]} \in A \mid B\right) - \bar{\mathbb{P}}_{\mu,(0,0)}\left(Z \in A\right)\right| \leq \Phi(L) \text{ for all } A \in \sigma(\mathbb{H}^{\mathbb{N}_0}),\ B \in \mathcal{H}_1. \quad (2.47)$$

There are only countably many cylinders in $\mathbb{H}^{\mathbb{N}_0}$, and so there is a subset of $\mathcal{H}_1$ with P^μ-measure 1 such that, for all B in this set, the above inequality holds simultaneously for all A. Take the supremum over A to get the claim for $k = 1$.

The extension to $k \in \mathbb{N}$ is straightforward. ∎

2.2.4 Gaps between regeneration times

Define (recall (2.35))

$$T_k^{(L)} = r^L\left(\tau_k^{(L)} - \tau_{k-1}^{(L)}\right), \qquad k \in \mathbb{N}. \quad (2.48)$$

Note that $T_k^{(L)}$, $k \in \mathbb{N}$, are i.i.d. In this section we prove two lemmas that control the moments of these increments.

Lemma 2.5. *For every $\alpha > 1$ there exists an $M(\alpha) < \infty$ such that*

$$\sup_{L \in 2\mathbb{N}} \bar{\mathbb{E}}_{\mu,(0,0)}\left([T_1^{(L)}]^\alpha\right) \leq M(\alpha). \quad (2.49)$$

Proof. Fix $\alpha > 1$. Since $T_1^{(L)}$ is independent of ξ, we have

$$\bar{\mathbb{E}}_{\mu,(0,0)}\left([T_1^{(L)}]^\alpha\right) = E_W\left([T_1^{(L)}]^\alpha\right) \leq \sup_{L \in 2\mathbb{N}} E_W\left([T_1^{(L)}]^\alpha\right), \quad (2.50)$$

where E_W is expectation w.r.t. W. Moreover, for all $a > 0$, there exists a constant $C = C(\alpha, a)$ such that

$$[aT_1^{(L)}]^\alpha \leq C\, e^{aT_1^{(L)}}, \quad (2.51)$$

and hence

$$\bar{\mathbb{E}}_{\mu,(0,0)}\left([T_1^{(L)}]^\alpha\right) \leq \frac{C}{a^\alpha} \sup_{L \in 2\mathbb{N}} E_W\left(e^{aT_1^{(L)}}\right). \quad (2.52)$$

Thus, to get the claim it suffices to show that, for a small enough,

$$\sup_{L \in 2\mathbb{N}} E_W\left(\mathrm{e}^{aT_1^{(L)}}\right) < \infty. \quad (2.53)$$

To prove (2.53), let

$$I = \inf\left\{m \in \mathbb{N}\colon\ (\epsilon_{mL}, \ldots, \epsilon_{(m+1)L-1}) = \epsilon^{(L)}\right\}. \quad (2.54)$$

By (2.28), I is geometrically distributed with parameter r^L. Moreover, $\tau_1^{(L)} \leq (I+1)L$. Therefore

$$
\begin{aligned}
E_W\left(\mathrm{e}^{aT_1^{(L)}}\right) &= E_W\left(\mathrm{e}^{ar^L\tau_1^{(L)}}\right) \leq \mathrm{e}^{ar^L L}\, E_W\left(\mathrm{e}^{ar^L IL}\right) \\
&= \mathrm{e}^{ar^L L} \sum_{j\in\mathbb{N}} (\mathrm{e}^{ar^L L})^j\, (1-r^L)^{j-1}\, r^L = \frac{r^L \mathrm{e}^{2ar^L L}}{\mathrm{e}^{ar^L L}(1-r^L)},
\end{aligned} \tag{2.55}
$$

with the sum convergent for $0 < a < (1/r^L L)\log[1/(1-r^L)]$ and tending to zero as $L \to \infty$ (because $r < 1$). Hence we can choose a small enough so that (2.53) holds. ∎

Lemma 2.6. $\liminf_{L\to\infty} \bar{\mathbb{E}}_{\mu,(0,0)}(T_1^{(L)}) > 0$.

Proof. Note that $\bar{\mathbb{E}}_{\mu,(0,0)}(T_1^{(L)}) < \infty$ by Lemma 2.5. Let $N = (N_n)_{n\in\mathbb{N}_0}$ be the Markov chain with state space $S = \{0,1,\ldots,L\}$, starting from $N_0 = 0$, such that $N_n = s$ when

$$
s = 0 \,\vee\, \max\left\{k \in \mathbb{N}\colon\, (\epsilon_{n-k},\ldots,\epsilon_{n-1}) = (\epsilon_1^{(L)},\ldots,\epsilon_k^{(L)})\right\} \tag{2.56}
$$

(with $\max\emptyset = 0$). This Markov chain moves up one unit with probability r, drops to 0 with probability $p+r$ when it is even, and drops to 0 or 1 with probability p, respectively, r when it is odd. Since $\tau_1^{(L)} = \min\{n \in \mathbb{N}_0\colon\, N_n = L\}$, it follows that $\tau_1^{(L)}$ is bounded from below by a sum of independent random variables, each bounded from below by 1, whose number is geometrically distributed with parameter r^{L-1}. Hence

$$
\bar{\mathbb{P}}_{\mu,(0,0)}\left(\tau_1^{(L)} \geq c\,r^{-L}\right) \geq (1-r^{L-1})^{\lfloor cr^{-L}\rfloor}. \tag{2.57}
$$

Since

$$
\begin{aligned}
\bar{\mathbb{E}}_{\mu,(0,0)}(T_1^{(L)}) &= r^L\, \bar{\mathbb{E}}_{\mu,(0,0)}(\tau_1^{(L)}) \\
&\geq r^L\, \bar{\mathbb{E}}_{\mu,(0,0)}\left(\tau_1^{(L)}\, 1_{\{\tau_1^{(L)}\geq cr^{-L}\}}\right) \geq c\,\bar{\mathbb{P}}_{\mu,(0,0)}\left(\tau_1^{(L)} \geq cr^{-L}\right),
\end{aligned} \tag{2.58}
$$

it follows that

$$
\liminf_{L\to\infty} \bar{\mathbb{E}}_{\mu,(0,0)}(\tau_1^{(L)}) \geq c\,\mathrm{e}^{-c/r}. \tag{2.59}
$$

This proves the claim. ∎

2.2.5 A coupling property for random sequences

In this section we recall a technical lemma that will be needed in Section 2.2.6. The proof of this lemma is a standard coupling argument (see e.g. Berbee [7], Lemma 2.1).

Lemma 2.7. *Let $(U_i)_{i\in\mathbb{N}}$ be a sequence of random variables whose joint probability law P is such that, for some marginal probability law μ and $a \in [0,1]$,*

$$\Big\| P\big(U_i \in \cdot \mid \sigma\{U_j\colon 1 \le j < i\}\big) - \mu(\cdot)\Big\|_{\mathrm{tv}} \le a \quad a.s. \qquad \forall\, i \in \mathbb{N}. \tag{2.60}$$

Then there exists a sequence of triples of random variables $(\widetilde{U}_i, \Delta_i, \widehat{U}_i)_{i\in\mathbb{N}}$ satisfying

- *(a)* $(\widetilde{U}_i, \Delta_i)_{i\in\mathbb{N}}$ *are i.i.d.,*
- *(b)* $\widetilde{U}_i$ *has probability law* μ*,*
- *(c)* $P(\Delta_i = 0) = 1 - a$, $P(\Delta_i = 1) = a$*,*
- *(d)* Δ_i *is independent of* $(\widetilde{U}_j, \Delta_j)_{1\le j<i}$ *and* $\widehat{U}_i$*,*

such that for all $i \in \mathbb{N}$

$$U_i = (1-\Delta_i)\widetilde{U}_i + \Delta_i \widehat{U}_i \quad \text{in distribution.} \tag{2.61}$$

2.2.6 LLN for Y

Similarly as in (2.48), define

$$Z_k^{(L)} = r^L \left(Z_{\tau_k^{(L)}} - Z_{\tau_{k-1}^{(L)}} \right), \qquad k \in \mathbb{N}. \tag{2.62}$$

In this section we prove the LLN for these increments and this will imply the LLN in (2.27).

Proof. By Lemma 2.4, we have

$$\Big\| \bar{\mathbb{P}}_{\mu,(0,0)}\big((T_k^{(L)}, Z_k^{(L)}) \in \cdot \mid \mathcal{H}_{k-1}\big) - \mu^{(L)}(\cdot)\Big\|_{\mathrm{tv}} \le \Phi(L) \quad a.s. \qquad \forall\, k \in \mathbb{N}, \tag{2.63}$$

where

$$\mu^{(L)}(A \times B) = \bar{\mathbb{P}}_{\mu,(0,0)}\big(T_1^{(L)} \in A, Z_1^{(L)} \in B\big) \qquad \forall\, A \subset r^L\mathbb{N},\, B \subset r^L\mathbb{H}. \tag{2.64}$$

Therefore, by Lemma 2.7, there exists an i.i.d. sequence of random variables

$$(\widetilde{T}_k^{(L)}, \widetilde{Z}_k^{(L)}, \Delta_k^{(L)})_{k\in\mathbb{N}} \tag{2.65}$$

on $r^L\mathbb{N} \times r^L\mathbb{H} \times \{0,1\}$, where $(\widetilde{T}_k^{(L)}, \widetilde{Z}_k^{(L)})$ is distributed according to $\mu^{(L)}$ and $\Delta_k^{(L)}$ is Bernoulli distributed with parameter $\Phi(L)$, and also a sequence of random variables

$$(\widehat{Z}_k^{(L)}, \widehat{Z}_k^{(L)})_{k\in\mathbb{N}}, \tag{2.66}$$

where $\Delta_k^{(L)}$ is independent of $(\widehat{Z}_k^{(L)}, \widehat{Z}_k^{(L)})$ and of

$$\widetilde{\mathcal{G}}_k = \sigma\{(\widetilde{T}_l^{(L)}, \widetilde{Z}_l^{(L)}, \Delta_l^{(L)}) \colon\ 1 \le l < k\}, \tag{2.67}$$

such that

$$(T_k^{(L)}, Z_k^{(L)}) = (1 - \Delta_k^{(L)})\,(\widetilde{T}_k^{(L)}, \widetilde{Z}_k^{(L)}) + \Delta_k^{(L)}\,(\widehat{Z}_k^{(L)}, \widehat{Z}_k^{(L)}). \tag{2.68}$$

Let

$$z_L = \bar{\mathbb{E}}_{\mu,(0,0)}(Z_1^{(L)}), \tag{2.69}$$

which is finite by Lemma 2.5 because $|Z_1^{(L)}| \le T_1^{(L)}$.

Lemma 2.8. *There exists a sequence of numbers* $(\delta_L)_{L\in\mathbb{N}_0}$*, satisfying* $\lim_{L\to\infty}\delta_L = 0$*, such that*

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{k=1}^{n} Z_k^{(L)} - z_L\right| < \delta_L \quad \bar{\mathbb{P}}_{\mu,(0,0)} - a.s. \tag{2.70}$$

Proof. With the help of (2.68) we can write

$$\frac{1}{n}\sum_{k=1}^{n} Z_k^{(L)} = \frac{1}{n}\sum_{k=1}^{n} \widetilde{Z}_k^{(L)} - \frac{1}{n}\sum_{k=1}^{n} \Delta_k^{(L)}\widetilde{Z}_k^{(L)} + \frac{1}{n}\sum_{k=1}^{n} \Delta_k^{(L)}\widehat{Z}_k^{(L)}. \tag{2.71}$$

By independence, the first term in the r.h.s. of (2.71) converges $\bar{\mathbb{P}}_{\mu,(0,0)}$-a.s. to z_L as $L \to \infty$. Hölder's inequality applied to the second term gives, for $\alpha, \alpha' > 1$ with $\alpha^{-1} + \alpha'^{-1} = 1$,

$$\left|\frac{1}{n}\sum_{k=1}^{n} \Delta_k^{(L)}\widetilde{Z}_k^{(L)}\right| \le \left(\frac{1}{n}\sum_{k=1}^{n}\left|\Delta_k^{(L)}\right|^{\alpha'}\right)^{\frac{1}{\alpha'}} \left(\frac{1}{n}\sum_{k=1}^{n}\left|\widetilde{Z}_k^{(L)}\right|^{\alpha}\right)^{\frac{1}{\alpha}}. \tag{2.72}$$

Hence, by Lemma 2.5 and the inequality $|\widetilde{Z}_k^{(L)}| \le \widetilde{T}_k^{(L)}$ (compare (2.48) and (2.62)), we have

$$\limsup_{n\to\infty}\left|\frac{1}{n}\sum_{k=1}^{n} \Delta_k^{(L)}\widetilde{Z}_k^{(L)}\right| \le \Phi(L)^{\frac{1}{\alpha'}}\, M(\alpha)^{\frac{1}{\alpha}} \quad \bar{\mathbb{P}}_{\mu,(0,0)} - a.s. \tag{2.73}$$

It remains to analyze the third term in the r.h.s. of (2.71). Since $|\Delta_k^{(L)}\widehat{Z}_k^{(L)}| \le Z_k^{(L)}$, it follows from Lemma 2.5 that

$$\begin{aligned} M(\alpha) &\ge \bar{\mathbb{E}}_{\mu,(0,0)}\big(|Z_k^{(L)}|^{\alpha}\big) \\ &\ge \bar{\mathbb{E}}_{\mu,(0,0)}\Big(|\Delta_k^{(L)}\widehat{Z}_k^{(L)}|^{\alpha} \mid \widetilde{\mathcal{G}}_k\Big) = \Phi(L)\ \bar{\mathbb{E}}_{\mu,(0,0)}\Big(|\widehat{Z}_k^{(L)}|^{\alpha} \mid \widetilde{\mathcal{G}}_k\Big) \quad a.s. \end{aligned} \tag{2.74}$$

Next, put $\widehat{Z}_k^{*(L)} = \bar{\mathbb{E}}_{\mu,(0,0)}(\widehat{Z}_k^{(L)} \mid \widetilde{\mathcal{G}}_k)$ and note that

$$M_n = \frac{1}{n}\sum_{k=1}^{n} \Delta_k^{(L)} \left(\widehat{Z}_k^{(L)} - \widehat{Z}_k^{*(L)}\right) \tag{2.75}$$

is a mean-zero martingale w.r.t. the filtration $\widetilde{\mathcal{G}} = (\widetilde{\mathcal{G}}_k)_{k\in\mathbb{N}}$. By the Burkholder-Gundy maximal inequality (Williams [96], (14.18)), it follows that, for $\beta = \alpha \wedge 2$,

$$\begin{aligned} &\bar{\mathbb{E}}_{\mu,(0,0)}\left(\left|\sup_{n\in\mathbb{N}} M_n\right|^\beta\right) \leq C(\beta)\,\bar{\mathbb{E}}_{\mu,(0,0)}\Big(\sum_{k\in\mathbb{N}} \frac{[\Delta_k^{(L)}(\widehat{Z}_k^{(L)} - \widehat{Z}_k^{*(L)})]^2}{k^2}\Big)^{\beta/2} \\ &\leq C(\beta)\sum_{k\in\mathbb{N}} \bar{\mathbb{E}}_{\mu,(0,0)}\left(\frac{|\Delta_k^{(L)}(\widehat{Z}_k^{(L)} - \widehat{Z}_k^{*(L)})|^\beta}{k^\beta}\right) \leq C'(\beta), \end{aligned} \tag{2.76}$$

for some constants $C(\beta), C'(\beta) < \infty$. Hence M_n a.s. converges to an integrable random variable as $n \to \infty$, and by Kronecker's lemma $\lim_{n\to\infty} M_n = 0$ a.s. Moreover, if $\Phi(L) > 0$, then by Jensen's inequality and (2.74) we have

$$|\widehat{Z}_k^{*(L)}| \leq \left[\bar{\mathbb{E}}_{\mu,(0,0)}\left(|\widehat{Z}_k^{(L)}|^\alpha \mid \widetilde{\mathcal{G}}_k\right)\right]^{\frac{1}{\alpha}} \leq \left(\frac{M(\alpha)}{\Phi(L)}\right)^{\frac{1}{\alpha}} \quad \bar{\mathbb{P}}_{\mu,(0,0)} - a.s. \tag{2.77}$$

Hence

$$\left|\frac{1}{n}\sum_{k=1}^{n} \Delta_k^{(L)} \widehat{Z}_k^{*(L)}\right| \leq \left(\frac{M(\alpha)}{\Phi(L)}\right)^{\frac{1}{\alpha}} \frac{1}{n}\sum_{k=1}^{n} \Delta_k^{(L)}. \tag{2.78}$$

As $n \to \infty$, the r.h.s. converges $\bar{\mathbb{P}}_{\mu,(0,0)}$-a.s. to $M(\alpha)^{\frac{1}{\alpha}}\Phi(L)^{\frac{1}{\alpha'}}$. Therefore, recalling (2.78) and choosing $\delta_L = 2M(\alpha)^{\frac{1}{\alpha}}\Phi(L)^{\frac{1}{\alpha'}}$, we get the claim. ∎

Finally, since $\widetilde{Z}_k^{(L)} \geq r^L$ and

$$\frac{1}{n}\sum_{k=1}^{n} T_k^{(L)} = t_L = \bar{\mathbb{E}}_{\mu,(0,0)}(T_1^{(L)}) > 0 \quad \bar{\mathbb{P}}_{\mu,(0,0)} - a.s., \tag{2.79}$$

Lemma 2.8 yields

$$\limsup_{n\to\infty}\left|\frac{\frac{1}{n}\sum_{k=1}^{n} Z_k^{(L)}}{\frac{1}{n}\sum_{k=1}^{n} T_k^{(L)}} - \frac{z_L}{t_L}\right| < C_1\,\delta_L \quad \bar{\mathbb{P}}_{\mu,(0,0)} - a.s. \tag{2.80}$$

for some constant $C_1 < \infty$ and L large enough. By (2.48) and (2.62), the quotient of sums in the l.h.s. equals $Z_{\tau_n^{(L)}}/\tau_n^{(L)}$. It therefore follows from a standard interpolation argument that

$$\limsup_{n\to\infty}\left|\frac{Z_n}{n} - \frac{z_L}{t_L}\right| < C_2\,\delta_L \quad \bar{\mathbb{P}}_{\mu,(0,0)} - a.s. \tag{2.81}$$

for some constant $C_2 < \infty$ and L large enough. This implies the existence of the limit $\lim_{L\to\infty} z_L/t_L$, as well as the fact that $\lim_{n\to\infty} Z_n/n = u$ $\bar{\mathbb{P}}_{\mu,(0,0)}$-a.s., which in view of (2.32) is equivalent to the statement in (2.27) with $u = (v, 1)$. ∎

2.2.7 From discrete to continuous time

It remains to show that the LLN derived in Sections 2.2.1–2.2.6 for the discrete-time random walk defined in (2.21–2.22) can be extended to the continuous-time random walk defined in (2.6–2.7).

Let $\chi = (\chi_n)_{n\in\mathbb{N}_0}$ denote the jump times of the continuous-time random walk $X = (X_t)_{t\geq 0}$ (with $\chi_0 = 0$). Let Q denote the law of χ. The increments of χ are i.i.d. random variables, independent of ξ, whose distribution is exponential with mean $1/(\alpha+\beta)$. Define

$$\begin{aligned} \xi^* &= (\xi^*_n)_{n\in\mathbb{N}_0} \quad \text{with} \quad \xi^*_n = \xi_{\chi_n}, \\ X^* &= (X^*_n)_{n\in\mathbb{N}_0} \quad \text{with} \quad X^*_n = X_{\chi_n}. \end{aligned} \tag{2.82}$$

Then X^* is a discrete-time random walk in a discrete-time random environment of the type considered in Sections 2.2.1–2.2.6, with $p = \alpha/(\alpha+\beta)$ and $q = \beta/(\alpha+\beta)$. Lemma 2.9 below shows that the cone-mixing property of ξ carries over to ξ^* under the joint law $P^\mu \times Q$. Therefore we have (recall (2.9))

$$\lim_{n\to\infty} X^*_n/n = v^* \quad \text{exists } (\mathbb{P}_{\mu,0}\times Q) - a.s. \tag{2.83}$$

Since $\lim_{n\to\infty} \chi_n/n = 1/(\alpha+\beta)$ Q-a.s., it follows that

$$\lim_{n\to\infty} X_{\chi_n}/\chi_n = (\alpha+\beta)v^* \quad \text{exists } (\mathbb{P}_{\mu,0}\times Q) - a.s. \tag{2.84}$$

A standard interpolation argument now yields (2.13) with $v = (\alpha+\beta)v^*$.

Lemma 2.9. *If ξ is cone-mixing with angle $\theta > \arctan(\alpha+\beta)$, then ξ^* is cone-mixing with angle $\frac{1}{4}\pi$.*

Proof. Fix $\theta > \arctan(\alpha+\beta)$, and put $c = c(\theta) = \cot\theta < 1/(\alpha+\beta)$. Recall from (2.10) that C^θ_t is the cone with angle θ whose tip is at $(0, t)$. For $M \in \mathbb{N}$, let $C^\theta_{t,M}$ be the cone obtained from C^θ_t by extending the tip to a rectangle with base M, i.e.,

$$C^\theta_{t,M} = C^\theta_t \cup \{([-M, M]\cap\mathbb{Z})\times[t,\infty)\}. \tag{2.85}$$

Because ξ is cone-mixing with angle θ, and

$$C^\theta_{t,M} \subset C^\theta_{t-cM}, \qquad M\in\mathbb{N}, \tag{2.86}$$

ξ is cone-mixing with angle θ and base M, i.e., (2.11) holds with C_t^θ replaced by $C_{t,M}^\theta$. This is true for every $M \in \mathbb{N}$.

Define, for $t \geq 0$ and $M \in \mathbb{N}$,

$$\begin{aligned} \mathcal{F}_t^\theta &= \sigma\{\xi_s(x)\colon (x,s) \in C_t^\theta\}, \\ \mathcal{F}_{t,M}^\theta &= \sigma\{\xi_s(x)\colon (x,s) \in C_{t,M}^\theta\}, \end{aligned} \tag{2.87}$$

and, for $n \in \mathbb{N}$,

$$\begin{aligned} \mathcal{F}_n^* &= \sigma\{\xi_m^*(x)\colon (x,m) \in C_n^{\frac{1}{4}\pi}\}, \\ \mathcal{G}_n &= \sigma\{\chi_m\colon m \geq n\}, \end{aligned} \tag{2.88}$$

where $C_n^{\frac{1}{4}\pi}$ is the discrete-time cone with tip $(0,n)$ and angle $\frac{1}{4}\pi$.

Fix $\delta > 0$. Then there exists an $M = M(\delta) \in \mathbb{N}$ such that $Q(D[M]) \geq 1 - \delta$ with $D[M] = \{\chi_n/n \geq c \;\forall\, n \geq M\}$. For $n \in \mathbb{N}$, define

$$D_n = \{\chi_n/n \geq c\} \cap \sigma^n D[M], \tag{2.89}$$

where σ is the left-shift acting on χ. Since $c < 1/(\alpha+\beta)$, we have $P(\chi_n/n \geq c) \geq 1-\delta$ for $n \geq N = N(\delta)$, and hence $P(D_n) \geq (1-\delta)^2 \geq 1 - 2\delta$ for $n \geq N = N(\delta)$. Next, observe that

$$B \in \mathcal{F}_n^* \Longrightarrow B \cap D_n \in \mathcal{F}_{cn,M}^\theta \otimes \mathcal{G}_n \tag{2.90}$$

(the r.h.s. is the product sigma-algebra). Indeed, on the event D_n we have $\chi_m \geq cm$ for $m \geq n + M$, which implies that, for $m \geq M$,

$$(x,m) \in C_n^{\frac{1}{4}\pi} \Longrightarrow |x| + m \geq n \Longrightarrow c|x| + \chi_n \geq cn \Longrightarrow (x, \chi_m) \in C_{cn,M}^\theta. \tag{2.91}$$

Now put $\bar{P}^\mu = P^\mu \otimes Q$ and, for $A \in \mathcal{F}_0$ with $P^\mu(A) > 0$ and $B \in \mathcal{F}_n^*$ estimate

$$|\bar{P}^\mu(B \mid A) - \bar{P}^\mu(B)| \leq I + II + III \tag{2.92}$$

with

$$\begin{aligned} I &= |\bar{P}^\mu(B \mid A) - \bar{P}^\mu(B \cap D_n \mid A)|, \\ II &= |\bar{P}^\mu(B \cap D_n \mid A) - \bar{P}^\mu(B \cap D_n)|, \\ III &= |\bar{P}^\mu(B \cap D_n) - \bar{P}^\mu(B)|. \end{aligned} \tag{2.93}$$

Since D_n is independent of A, B and $P(D_n) \geq 1-2\delta$, it follows that $I \leq 2\delta$ and $III \leq 2\delta$ uniformly in A and B. To bound II, we use (2.90) to estimate

$$II \leq \sup_{\substack{A \in \mathcal{F}_0,\, B' \in \mathcal{F}^{\theta}_{cn,M} \otimes \mathcal{G}_n \\ P^{\mu}(A)>0}} |\bar{P}^{\mu}(B' \mid A) - \bar{P}^{\mu}(B')|. \tag{2.94}$$

But the r.h.s. is bounded from above by

$$\sup_{\substack{A \in \mathcal{F}_0,\, B'' \in \mathcal{F}^{\theta}_{cn,M} \\ P^{\mu}(A)>0}} |P^{\mu}(B'' \mid A) - P^{\mu}(B'')| \tag{2.95}$$

because, for every $B'' \in \mathcal{F}^{\theta}_{cn,M}$ and $C \in \mathcal{G}_n$,

$$|\bar{P}^{\mu}(B'' \times C \mid A) - \bar{P}^{\mu}(B'' \times C)| = |[P^{\mu}(B'' \mid A) - P^{\mu}(B'')]\, Q(C)| \leq |P^{\mu}(B'' \mid A) - P^{\mu}(B'')|, \tag{2.96}$$

where we use that C is independent of A, B''.

Finally, because ξ is cone-mixing with angle θ and base M, (2.95) tends to zero as $n \to \infty$, and so by combining (2.92–2.95) we get

$$\limsup_{n\to\infty} \sup_{\substack{A \in \mathcal{F}_0,\, B \in \mathcal{F}^{*}_n \\ P^{\mu}(A)>0}} |\bar{P}^{\mu}(B \mid A) - \bar{P}^{\mu}(B)| \leq 4\delta. \tag{2.97}$$

Now let $\delta \downarrow 0$ to obtain that ξ^* is cone mixing with angle $\frac{1}{4}\pi$. ∎

2.2.8 Remarks on the cone-mixing assumption

By using the cone-mixing assumption and the auxiliary process Z introduced in Section 2.2.2, we could have followed a shorter approach to derive the strong LLN in Theorem 2.2, avoiding the technicalities of Sections 2.2.5 and 2.2.6. Indeed, it is possible to deduce that the process of the environment as seen from the walk admits a mixing equilibrium measure μ_e. Consequently, a weak law of large numbers, L^2 convergence, and an almost sure convergence with respect to μ_e can be inferred. If we could subsequently show that the equilibrium measure μ is absolutely continuous with respect to μ_e (which is not trivial in the present generality), then Theorem 2.2 would follow.

As pointed out in Section 2.1.4, cone-mixing is one of the weakest assumptions under which we may expect to get the strong LLN, since no rate of mixing is imposed in (2.11). If we strengthen (2.11) to an exponential decay of the function in (2.37), then it seems possible to adapt the proof in [36] to derive an annealed CLT in the present context.

2.3 Series expansion for $M < \epsilon$

Throughout this section we assume that the dynamic random environment ξ falls in the regime for which $M < \epsilon$ (recall (2.14)). In Section 2.3.1 we define the *environment process*, i.e., the environment as seen relative to the position of the random walk. In Section 2.3.2 we prove that this environment process has a *unique ergodic equilibrium* μ_e, and we derive a series expansion for μ_e in powers of $\alpha - \beta$ that converges when $\alpha - \beta < \frac{1}{2}(\epsilon - M)$. In Section 2.3.3 we use the latter to derive a series expansion for the global speed v of the random walk.

2.3.1 Definition of the environment process

Let $X = (X_t)_{t\geq 0}$ be the random walk defined in (2.6–2.7). For $x \in \mathbb{Z}$, let τ_x denote the shift of space over x.

Definition 2.10. The environment process is the Markov process $\zeta = (\zeta_t)_{t\geq 0}$ with state space Ω given by

$$\zeta_t = \tau_{X_t}\xi_t, \qquad t \geq 0, \tag{2.98}$$

where

$$(\tau_{X_t}\xi_t)(x) = \xi_t(x + X_t), \qquad x \in \mathbb{Z},\, t \geq 0. \tag{2.99}$$

Equivalently, if ξ has generator L_{IPS}, then ζ has generator L given by

$$(Lf)(\eta) = c^+(\eta)\big[f(\tau_1\eta) - f(\eta)\big] + c^-(\eta)\big[f(\tau_{-1}\eta) - f(\eta)\big] + (L_{\mathrm{IPS}}f)(\eta), \quad \eta \in \Omega, \tag{2.100}$$

where f is an arbitrary cylinder function on Ω and

$$\begin{aligned} c^+(\eta) &= \alpha\,\eta(0) + \beta\,[1 - \eta(0)], \\ c^-(\eta) &= \beta\,\eta(0) + \alpha\,[1 - \eta(0)]. \end{aligned} \tag{2.101}$$

Let $S = (S(t))_{t\geq 0}$ be the semigroup associated with the generator L. Suppose that we manage to prove that ζ is ergodic, i.e., there exists a unique probability measure μ_e on Ω such that, for any cylinder function f on Ω,

$$\lim_{t\to\infty} (S(t)f)(\eta) = \langle f\rangle_{\mu_e} \qquad \forall\, \eta \in \Omega, \tag{2.102}$$

where $\langle\cdot\rangle_{\mu_e}$ denotes expectation w.r.t. μ_e. Then, picking $f = \phi_0$ with $\phi_0(\eta) = \eta(0)$, $\eta \in \Omega$, we have

$$\lim_{t\to\infty} (S(t)\phi_0)(\eta) = \langle\phi_0\rangle_{\mu_e} = \widetilde{\rho} \qquad \forall\, \eta \in \Omega \tag{2.103}$$

for some $\widetilde{\rho} \in [0,1]$, which represents the limiting probability that X is on an occupied site given that $\xi_0 = \zeta_0 = \eta$ (note that $(S(t)\phi_0)(\eta) = E^\eta(\zeta_t(0)) = E^\eta(\xi_t(X_t))$).

Next, let N_t^+ and N_t^- be the number of shifts to the right, respectively, left up to time t in the environment process. Then $X_t = N_t^+ - N_t^-$. Since $M_t^j = N_t^j - \int_0^t c^j(\eta_s)\,\mathrm{d}s$, $j \in \{+,-\}$, are martingales with stationary and ergodic increments, we have

$$X_t = M_t + (\alpha - \beta)\int_0^t \big(2\eta_s(0) - 1\big)\,\mathrm{d}s \tag{2.104}$$

with $M_t = M_t^+ - M_t^-$ a martingale with stationary and ergodic increments. It follows from (2.103–2.104) that

$$\lim_{t\to\infty} X_t/t = (2\widetilde{\rho} - 1)(\alpha - \beta) \qquad \mu - a.s. \tag{2.105}$$

In Section 2.3.2 we prove the existence of μ_e, and show that it can be expanded in powers of $\alpha - \beta$ when $\alpha - \beta < \frac{1}{2}(\epsilon - M)$. In Section 2.3.3 we use this expansion to obtain an expansion of $\widetilde{\rho}$.

2.3.2 Unique ergodic equilibrium measure for the environment process

In Section 2.3.2.1 we prove four lemmas controlling the evolution of ζ. In Section 2.3.2.2 we use these lemmas to show that ζ has a unique ergodic equilibrium measure μ_e that can be expanded in powers of $\alpha - \beta$, provided $\alpha - \beta < \frac{1}{2}(\epsilon - M)$.

We need some notation. Let $\|\cdot\|_\infty$ be the sup-norm on $C(\Omega)$. Let $|||\cdot|||$ be the triple norm on Ω defined as follows. For $x \in \mathbb{Z}$ and a cylinder function f on Ω, let

$$\Delta_f(x) = \sup_{\eta\in\Omega} |f(\eta^x) - f(\eta)| \tag{2.106}$$

be the maximum variation of f at x, where η^x is the configuration obtained from η by flipping the state at site x, and put

$$|||f||| = \sum_{x\in\mathbb{Z}} \Delta_f(x). \tag{2.107}$$

It is easy to check that, for arbitrary cylinder functions f and g on Ω,

$$|||fg||| \le \|f\|_\infty\, |||g||| + \|g\|_\infty\, |||f|||. \tag{2.108}$$

2.3.2.1 Decomposition of the generator of the environment process

Lemma 2.11. *Assume* (2.3) *and suppose that* $M < \epsilon$. *Write the generator of the environment process* ζ *defined in* (2.100) *as*

$$L = L_0 + L_* = (L_{\text{SRW}} + L_{\text{IPS}}) + L_*, \tag{2.109}$$

where

$$\begin{aligned} (L_{\text{SRW}} f)(\eta) &= \tfrac{1}{2}(\alpha+\beta)\Big[f(\tau_1\eta) + f(\tau_{-1}\eta) - 2f(\eta)\Big], \\ (L_* f)(\eta) &= \tfrac{1}{2}(\alpha-\beta)\Big[f(\tau_1\eta) - f(\tau_{-1}\eta)\Big]\big(2\eta(0)-1\big). \end{aligned} \tag{2.110}$$

Then L_0 *is the generator of a Markov process that still has* μ *as an equilibrium, and that satisfies*

$$|||S_0(t)f||| \le \mathrm{e}^{-ct}\, |||f||| \tag{2.111}$$

and

$$\|S_0(t)f - \langle f\rangle_\mu\|_\infty \le C\,\mathrm{e}^{-ct}\, |||f|||, \tag{2.112}$$

where $S_0 = (S_0(t))_{t\ge 0}$ *is the semigroup associated with the generator* L_0, $c = \epsilon - M$, *and* $C < \infty$ *is a positive constant.*

Proof. Note that L_{SRW} and L_{IPS} commute. Therefore, for an arbitrary cylinder function f on Ω, we have

$$|||S_0(t)f||| = |||\mathrm{e}^{tL_{\text{SRW}}}\big(\mathrm{e}^{tL_{\text{IPS}}}f\big)||| \le |||\mathrm{e}^{tL_{\text{IPS}}}f||| \le \mathrm{e}^{-ct}\, |||f|||, \tag{2.113}$$

where the first inequality uses that $\mathrm{e}^{tL_{\text{SRW}}}$ is a contraction semigroup, and the second inequality follows from the fact that ξ falls in the regime $M < \epsilon$ (see Liggett [63], Theorem I.3.9). The inequality in (2.112) follows by a similar argument. Indeed,

$$\|S_0(t)f - \langle f\rangle_\mu\|_\infty = \|\mathrm{e}^{tL_{\text{SRW}}}\big(\mathrm{e}^{tL_{\text{IPS}}}f\big) - \langle f\rangle_\mu\|_\infty \le \|\mathrm{e}^{tL_{\text{IPS}}}f - \langle f\rangle_\mu\|_\infty \le C\,\mathrm{e}^{-ct}\, |||f|||, \tag{2.114}$$

where the last inequality again uses that ξ falls in the regime $M < \epsilon$ (see Liggett [63], Theorem I.4.1). The fact that μ is an equilibrium measure is trivial, since L_{SRW} only acts on η by shifting it. ∎

Note that L_{SRW} is the generator of simple random walk on $\mathbb{Z}$ jumping at rate $\alpha+\beta$. We view L_0 as the generator of an unperturbed Markov process and L_* as a perturbation of L_0. The following lemma gives us control of the latter.

Lemma 2.12. *For any cylinder function f on Ω,*

$$\|L_* f\|_\infty \le (\alpha-\beta)\|f\|_\infty \tag{2.115}$$

and

$$|||L_* f||| \le 2(\alpha-\beta)\, |||f||| \qquad \text{if } \langle f\rangle_\mu = 0. \tag{2.116}$$

Proof. To prove (2.115), estimate

$$\begin{aligned} \|L_* f\|_\infty &= \tfrac{1}{2}(\alpha-\beta)\, \|\left[f(\tau_1\,\cdot) - f(\tau_{-1}\,\cdot)\right]\big(2\phi_0(\cdot)-1\big)\|_\infty \\ &\le \tfrac{1}{2}(\alpha-\beta)\, \|f(\tau_1\,\cdot) + f(\tau_{-1}\,\cdot)\|_\infty \le (\alpha-\beta)\,\|f\|_\infty. \end{aligned} \tag{2.117}$$

To prove (2.116), recall (2.110) and estimate

$$\begin{aligned} |||L_* f||| &= \tfrac{1}{2}(\alpha-\beta)\, |||\left[f(\tau_1\,\cdot) - f(\tau_{-1}\,\cdot)\right]\big(2\phi_0(\cdot)-1\big)||| \\ &\le \tfrac{1}{2}(\alpha-\beta)\Big(|||f(\tau_1\cdot)(2\phi_0(\cdot)-1)||| + |||f(\tau_{-1}\cdot)(2\phi_0(\cdot)-1)|||\Big) \\ &\le (\alpha-\beta)\Big(\|f\|_\infty\, |||\,(2\phi_0-1)||| + |||f|||\, \|(2\phi_0-1)\|_\infty\Big) \\ &= (\alpha-\beta)\Big(\|f\|_\infty + |||f|||\Big) \le 2(\alpha-\beta)|||f|||, \end{aligned} \tag{2.118}$$

where the second inequality uses (2.108) and the third inequality follows from the fact that $\|f\|_\infty \le |||f|||$ for any f such that $\langle f\rangle_\mu = 0$. ∎

We are now ready to expand the semigroup S of ζ. Henceforth abbreviate

$$c = \epsilon - M. \tag{2.119}$$

Lemma 2.13. *Let $S_0 = (S_0(t))_{t\ge 0}$ be the semigroup associated with the generator L_0 defined in* (2.110). *Then, for any $t \ge 0$ and any cylinder function f on Ω,*

$$S(t)f = \sum_{n\in\mathbb{N}} g_n(t,f), \tag{2.120}$$

where

$$g_1(t,f) = S_0(t)f \quad \text{and} \quad g_{n+1}(t,f) = \int_0^t S_0(t-s)\, L_*\, g_n(s,f)\,\mathrm{d}s, \qquad n\in\mathbb{N}. \tag{2.121}$$

Moreover, for all $n \in \mathbb{N}$,

$$\|g_n(t,f)\|_\infty \le |||f|||\, \left(\frac{2(\alpha-\beta)}{c}\right)^{n-1} \tag{2.122}$$

and

$$|||g_n(t,f)||| \leq \mathrm{e}^{-ct}\,\frac{[2(\alpha-\beta)t]^{n-1}}{(n-1)!}\,|||f|||, \tag{2.123}$$

where $0! = 1$. *In particular, for all* $t > 0$ *and* $\alpha - \beta < \frac{1}{2}c$ *the series in* (2.120) *converges uniformly in* η.

Proof. Since $L = L_0 + L_*$, Dyson's formula gives

$$\mathrm{e}^{tL}\,f = \mathrm{e}^{tL_0} f + \int_0^t \mathrm{e}^{(t-s)L_0}\,L_*\,\mathrm{e}^{sL}\,f\,\mathrm{d}s, \tag{2.124}$$

which, in terms of semigroups, reads

$$S(t)f = S_0(t)f + \int_0^t S_0(t-s)L_*\,S(s)f\,\mathrm{d}s. \tag{2.125}$$

The expansion in (2.120–2.121) follows from (2.125) by induction on n.

We next prove (2.123) by induction on n. For $n = 1$ the claim is immediate. Indeed, by Lemma 2.11 we have the exponential bound

$$|||g_1(t,f)||| = |||S_0(t)f||| \leq \mathrm{e}^{-ct}\,|||f|||. \tag{2.126}$$

Suppose that the statement in (2.123) is true up to n. Then

$$\begin{aligned}
|||g_{n+1}(t,f)||| &= |||\int_0^t S_0(t-s)\,L_*\,g_n(s,f)\,\mathrm{d}s\,||| \\
&\leq \int_0^t |||S_0(t-s)\,L_*\,g_n(s,f)|||\,\mathrm{d}s \\
&\leq \int_0^t \mathrm{e}^{-c(t-s)}\,|||L_*\,g_n(s,f)|||\,\mathrm{d}s \\
&= \int_0^t \mathrm{e}^{-c(t-s)}\,|||L_*\left(g_n(s,f) - \langle g_n(s,f)\rangle_\mu\right)|||\,\mathrm{d}s \\
&\leq 2(\alpha-\beta)\int_0^t \mathrm{e}^{-c(t-s)}\,|||g_n(s,f)|||\,\mathrm{d}s, \\
&\leq |||f|||\,\mathrm{e}^{-ct}\,[2(\alpha-\beta)]^n \int_0^t \frac{s^{n-1}}{(n-1)!}\,\mathrm{d}s \\
&= |||f|||\,\mathrm{e}^{-ct}\,\frac{[2(\alpha-\beta)t]^n}{n!},
\end{aligned} \tag{2.127}$$

where the third inequality uses (2.116), and the fourth inequality relies on the induction hypothesis.

Using (2.123), we can now prove (2.122). Estimate

$$\begin{aligned}
\|g_{n+1}(t,f)\|_\infty &= \left\| \int_0^t S_0(t-s)\, L_*\, g_n(s,f) \mathrm{d}s \right\|_\infty \\
&\le \int_0^t \|L_*\, g_n(s,f)\|_\infty \, \mathrm{d}s \\
&= \int_0^t \left\| L_* \left(g_n(s,f) - \langle g_n(s,f)\rangle_\mu \right) \right\|_\infty \mathrm{d}s \\
&\le (\alpha-\beta) \int_0^t \left\| g_n(s,f) - \langle g_n(s,f)\rangle_\mu \right\|_\infty \mathrm{d}s \qquad (2.128)\\
&\le (\alpha-\beta) \int_0^t |||g_n(s,f)||| \, \mathrm{d}s \\
&\le (\alpha-\beta) |||f||| \int_0^t \mathrm{e}^{-cs} \frac{[2(\alpha-\beta)s]^{n-1}}{(n-1)!} \mathrm{d}s \\
&\le |||f||| \left(\frac{2(\alpha-\beta)}{c} \right)^n ,
\end{aligned}$$

where the first inequality uses that $S_0(t)$ is a contraction semigroup, while the second and fourth inequality rely on (2.115) and (2.123). ∎

We next show that the functions in (2.120) are uniformly close to their average value.

Lemma 2.14. *Let*

$$h_n(t,f) = g_n(t,f) - \langle g_n(t,f)\rangle_\mu, \quad t \ge 0,\ n \in \mathbb{N}. \qquad (2.129)$$

Then

$$\|h_n(t,f)\|_\infty \le C\, \mathrm{e}^{-ct} \frac{[2(\alpha-\beta)t]^{n-1}}{(n-1)!} |||f|||, \qquad (2.130)$$

for some $C < \infty$.

Proof. Note that $|||h_n(t,f)||| = |||g_n(t,f)|||$ for $t \ge 0$ and $n \in \mathbb{N}$, and estimate

$$\begin{aligned}
\|h_{n+1}(t,f)\|_\infty &= \left\| \int_0^t \Big(S_0(t-s)\, L_*\, g_n(s,f) - \langle L_*\, g_n(s,f)\rangle_\mu \Big) \mathrm{d}s \right\|_\infty \\
&\le C \int_0^t \mathrm{e}^{-c(t-s)} \, |||L_*\, g_n(s,f)||| \, \mathrm{d}s \\
&= C \int_0^t \mathrm{e}^{-c(t-s)} \, |||L_*\, h_n(s,f)||| \, \mathrm{d}s \\
&\le C\, 2(\alpha-\beta) \int_0^t \mathrm{e}^{-c(t-s)} \, |||h_n(s,f)||| \, \mathrm{d}s \qquad (2.131)\\
&\le C\, |||f|||\, \mathrm{e}^{-ct} \, [2(\alpha-\beta)]^n \int_0^t \frac{s^{n-1}}{(n-1)!} \mathrm{d}s \\
&= C\, |||f|||\, \mathrm{e}^{-ct} \, \frac{[2(\alpha-\beta)t]^n}{n!},
\end{aligned}$$

where the first inequality uses (2.112), while the second and third inequality rely on (2.116) and (2.123). ∎

2.3.2.2 Expansion of the equilibrium measure of the environment process

We are finally ready to state the main result of this section.

Theorem 2.15. *For $\alpha - \beta < \frac{1}{2}c$, the environment process ζ has a unique invariant measure μ_e. In particular, for any cylinder function f on Ω,*

$$\langle f \rangle_{\mu_e} = \lim_{t\to\infty} \langle S(t)f \rangle_{\mu} = \sum_{n\in\mathbb{N}} \lim_{t\to\infty} \langle g_n(t,f) \rangle_{\mu}. \tag{2.132}$$

Proof. By Lemma 2.14, we have

$$\begin{aligned}\Big\| S(t)f - \langle S(t)f \rangle_{\mu} \Big\|_{\infty} &= \Big\| \sum_{n\in\mathbb{N}} g_n(t,f) - \langle \sum_{n\in\mathbb{N}} g_n(t,f) \rangle_{\mu} \Big\|_{\infty} = \Big\| \sum_{n\in\mathbb{N}} h_n(t,f) \Big\|_{\infty} \\ &\le \sum_{n\in\mathbb{N}} \|h_n(t,f)\|_{\infty} \le C\,\mathrm{e}^{-ct}\,|||f|||\sum_{n\in\mathbb{N}} \frac{[2(\alpha-\beta)t]^n}{n!} \\ &= C|||f|||\,\mathrm{e}^{-t[c-2(\alpha-\beta)]}.\end{aligned} \tag{2.133}$$

Since $\alpha-\beta < \frac{1}{2}c$, we see that the r.h.s. of (2.133) tends to zero as $t \to \infty$. Consequently, the l.h.s. tends to zero uniformly in η, and this is sufficient to conclude that the set $\mathcal{I}$ of equilibrium measures of the environment process is a singleton, i.e., $\mathcal{I} = \{\mu_e\}$. Indeed, suppose that there are two equilibrium measures $\nu, \nu' \in \mathcal{I}$. Then

$$\begin{aligned}|\langle f\rangle_{\nu} - \langle f\rangle_{\nu'}| &= |\langle S(t)f\rangle_{\nu} - \langle S(t)f\rangle_{\nu'}| \\ &\le |\langle S(t)f\rangle_{\nu} - \langle S(t)f\rangle_{\mu}| + |\langle S(t)f\rangle_{\nu'} - \langle S(t)f\rangle_{\mu}| \\ &= |\langle [S(t)f - \langle S(t)f\rangle_{\mu}]\rangle_{\nu}| + |\langle [S(t)f - \langle S(t)f\rangle_{\mu}]\rangle_{\nu'}| \\ &\le 2\,\|S(t)f - \langle S(t)f\rangle_{\mu}\|_{\infty}.\end{aligned} \tag{2.134}$$

Since the l.h.s. of (2.134) does not depend on t, and the r.h.s. tends to zero as $t \to \infty$, we have $\nu = \nu' = \mu_e$. Next, μ_e is uniquely ergodic, meaning that the environment process converges to μ_e as $t \to \infty$ no matter what its starting distribution is. Indeed, for any μ',

$$|\langle S(t)f\rangle_{\mu'} - \langle S(t)f\rangle_{\mu}| = |\langle [S(t)f - \langle S(t)f\rangle_{\mu}]\rangle_{\mu'}| \le \|S(t)f - \langle S(t)f\rangle_{\mu}\|_{\infty}, \tag{2.135}$$

and therefore

$$\begin{aligned}\langle f\rangle_{\mu_e} &= \lim_{t\to\infty} S(t)f = \lim_{t\to\infty}\langle S(t)f\rangle_\mu = \lim_{t\to\infty}\left\langle \sum_{n\in\mathbb{N}} g_n(t,f)\right\rangle_\mu \\ &= \lim_{t\to\infty}\sum_{n\in\mathbb{N}}\langle g_n(t,f)\rangle_\mu = \sum_{n\in\mathbb{N}}\lim_{t\to\infty}\langle g_n(t,f)\rangle_\mu,\end{aligned} \tag{2.136}$$

where the last equality is justified by the bound in (2.122) in combination with the dominated convergence theorem. ∎

We close this section by giving a more transparent description of μ_e, more suitable for explicit computation.

Theorem 2.16. *For $\alpha - \beta < \frac{1}{2}c$,*

$$\langle f\rangle_{\mu_e} = \sum_{n\in\mathbb{N}}\langle \Psi_n\rangle_\mu \tag{2.137}$$

with

$$\Psi_1 = f \quad \textit{and} \quad \Psi_{n+1} = L_* L_0^{-1}(\Psi_n - \langle\Psi_n\rangle_\mu), \qquad n\in\mathbb{N}, \tag{2.138}$$

where $L_0^{-1} = \int_0^\infty S_0(t)\,\mathrm{d}t$ (whose domain is the set of all $f\in C(\Omega)$ with $\langle f\rangle_\mu = 0$).

Proof. By (2.136), the claim is equivalent to showing that for all $n \geq 1$

$$\lim_{t\to\infty}\langle g_n(t,f)\rangle_\mu = \langle\Psi_n\rangle_\mu. \tag{2.139}$$

First consider the case $n = 2$. Then

$$\begin{aligned}\lim_{t\to\infty}\langle g_2(t,f)\rangle_\mu &= \lim_{t\to\infty}\left\langle \int_0^t \mathrm{d}s\, S_0(t-s)\, L_*\, g_1(s,f)\right\rangle_\mu \\ &= \lim_{t\to\infty}\left\langle \int_0^t \mathrm{d}s\ L_*\, g_1(s,f)\right\rangle_\mu \\ &= \lim_{t\to\infty}\left\langle \int_0^t \mathrm{d}s\ L_* S_0(s) f\right\rangle_\mu \\ &= \lim_{t\to\infty}\left\langle \int_0^t \mathrm{d}s\ L_*\big[S_0(s)(f-\langle f\rangle_\mu)\big]\right\rangle_\mu \\ &= \left\langle \lim_{t\to\infty} L_* \int_0^t \mathrm{d}s\ S_0(s)(f-\langle f\rangle_\mu)\right\rangle_\mu = \langle L_* L_0^{-1}(f-\langle f\rangle_\mu)\rangle_\mu,\end{aligned} \tag{2.140}$$

where the second equality uses that μ is invariant w.r.t. S_0, while the fifth equality uses the linearity and continuity of L_* in combination with the bound in (2.122).

For general n, the argument runs as follows. First write

$$
\begin{aligned}
&\langle g_n(t,f)\rangle_\mu \\
&= \left\langle \int_0^t \mathrm{d}s\, S_0(t-t_1)\, L_*\, g_{n-1}(t_1,f) \right\rangle_\mu \\
&= \left\langle \int_0^t \mathrm{d}t_1\;\, L_*\, g_{n-1}(t_1,f) \right\rangle_\mu \\
&= \left\langle \int_0^t \mathrm{d}t_1 \int_0^{t_1} \mathrm{d}t_2 \cdots \int_0^{t_{n-1}} \mathrm{d}t_n \left[L_* S_0(t_1-t_2)\cdots L_* S_0(t_{n-1}-t_n) L_* S_0(t_n)\right] f \right\rangle_\mu \\
&= \left\langle \int_0^t \mathrm{d}t_n \int_0^{t-t_n} \mathrm{d}t_{n-1} \cdots \int_0^{t-t_2} \mathrm{d}t_1 \left[L_* S_0(t_1) L_* S_0(t_2)\cdots L_* S_0(t_{n-1}) L_* S_0(t_n)\right] f \right\rangle_\mu .
\end{aligned}
\tag{2.141}
$$

Next let $t \to \infty$ to obtain

$$
\begin{aligned}
&\lim_{t\to\infty} \langle g_n(t,f)\rangle_\mu \\
&= \left\langle \int_0^\infty \mathrm{d}t_n \int_0^\infty \mathrm{d}t_{n-1} \cdots \int_0^\infty \mathrm{d}t_1 \left[L_* S_0(t_1) L_* S_0(t_2)\cdots L_* S_0(t_{n-1}) L_* S_0(t_n)\right] f \right\rangle_\mu \\
&= \left\langle L_* \int_0^\infty \mathrm{d}t_1\, S_0(t_1)\, L_* \int_0^\infty \mathrm{d}t_2\, S_0(t_2) \cdots L_* \int_0^\infty \mathrm{d}t_n\, S_0(t_n)\, (f - \langle f\rangle_\mu) \right\rangle_\mu \\
&= \left\langle L_* \int_0^\infty \mathrm{d}t_1\, S_0(t_1)\, L_* \int_0^\infty \mathrm{d}t_2\, S_0(t_2) \cdots L_* L_0^{-1} (f - \langle f\rangle_\mu) \right\rangle_\mu \\
&= \left\langle L_* \int_0^\infty \mathrm{d}t_1\, S_0(t_1)\, L_* \int_0^\infty \mathrm{d}t_2\, S_0(t_2) \cdots L_* \int_0^\infty \mathrm{d}t_{n-1}\, S_0(t_{n-1}) \Psi_2 \right\rangle_\mu ,
\end{aligned}
\tag{2.142}
$$

where we insert $L_* L_0^{-1}(f - \langle f\rangle_\mu) = \Psi_2$. Iteration shows that the latter expression is equal to

$$
\begin{aligned}
\left\langle L_* \int_0^\infty \mathrm{d}t_1\, S_0(t_1) \Psi_{n-1} \right\rangle_\mu &= \left\langle L_* \int_0^\infty \mathrm{d}t_1\, S_0(t_1) (\Psi_{n-1} - \langle \Psi_{n-1}\rangle_\mu) \right\rangle_\mu \\
&= \left\langle L_* L_0^{-1} (\Psi_{n-1} - \langle \Psi_{n-1}\rangle_\mu) \right\rangle_\mu = \langle \Psi_n\rangle_\mu .
\end{aligned}
\tag{2.143}
$$

∎

2.3.3 Expansion of the global speed

As we argued in (2.105), the global speed of X is given by

$$
v = (2\widetilde{\rho} - 1)(\alpha - \beta) \tag{2.144}
$$

with $\widetilde{\rho} = \langle \phi_0 \rangle_{\mu_e}$. By using Theorem 2.16, we can now expand $\widetilde{\rho}$.

First, if $\langle \phi_0 \rangle_\mu = \rho$ is the particle density, then

$$\widetilde{\rho} = \langle \phi_0 \rangle_{\mu_e} = \rho + \sum_{n=2}^{\infty} \langle \Psi_n \rangle_\mu, \tag{2.145}$$

where Ψ_n is constructed recursively via (2.138) with $f = \phi_0$. We have

$$\langle \Psi_n \rangle_\mu = d_n \, (\alpha - \beta)^{n-1}, \quad n \in \mathbb{N}, \tag{2.146}$$

where $d_n = d_n(\alpha + \beta; P^\mu)$, and the factor $(\alpha - \beta)^{n-1}$ comes from the fact that the operator L_* is applied $n-1$ times to compute Ψ_n, as is seen from (2.138). Recall that, in (2.110), L_{SRW} caries the prefactor $\alpha + \beta$, while L_* carries the prefactor $\alpha - \beta$. Combining (2.144–2.145), we have

$$v = \sum_{n \in \mathbb{N}} c_n \, (\alpha - \beta)^n, \tag{2.147}$$

with $c_1 = 2\rho - 1$ and $c_n = 2d_n$, $n \in \mathbb{N} \backslash \{1\}$.

For $n = 2, 3$ we have

$$\begin{aligned} c_2 &= 2 \big\langle \phi_0 L_0^{-1} \big(\phi_1 - \phi_{-1} \big) \big\rangle_\mu \\ c_3 &= \tfrac{1}{2} \left\langle \psi_0 L_0^{-1} \big[\psi_{-1} L_0^{-1} \bar{\phi}_{-2} - \psi_1 L_0^{-1} \bar{\phi}_0 - \psi_{-1} L_0^{-1} \bar{\phi}_0 + \psi_1 L_0^{-1} \bar{\phi}_2 \big] \right\rangle_\mu , \end{aligned} \tag{2.148}$$

where $\phi_i(\eta) = \eta(i)$, $\eta \in \Omega$, $\bar{\phi}_i = \phi_i - \langle \phi_i \rangle_\mu$ and $\psi_i = 2\phi_i - 1$. It is possible to compute c_2 and c_3 for appropriate choices of ξ.

If the law of ξ is invariant under reflection w.r.t. the origin, then ξ has the same distribution as ξ' defined by $\xi'(x) = \xi(-x)$, $x \in \mathbb{Z}$. In that case $c_2 = 0$, and consequently $v = (2\rho - 1)(\alpha - \beta) + O((\alpha - \beta)^3)$. For examples of interacting particle systems with $M < \epsilon$, see Liggett [63], Section I.4. Some of these examples have the reflection symmetry property.

An alternative formula for c_2 is (recall (2.110))

$$c_2 = 2 \int_0^\infty \mathrm{d}t \, \Big(E_{\text{SRW},1}[K(Y_t, t)] - E_{\text{SRW},-1}[K(Y_t, t)] \Big), \tag{2.149}$$

where

$$K(i, t) = E_{P^\mu}[\xi_0(0) \xi_t(i)] = \langle \phi_0 \, (S_{\text{IPS}}(t) \phi_i) \rangle_\mu, \qquad i \in \mathbb{Z}, \, t \geq 0, \tag{2.150}$$

is the space-time correlation function of the interacting particle system (with generator L_{IPS}), and $E_{\text{SRW},i}$ is the expectation over simple random walk $Y = (Y_t)_{t \geq 0}$ jumping at rate $\alpha + \beta$ (with generator L_{SRW}) starting from i. If μ is a *reversible* equilibrium, then (recall (2.3))

$$K(i,t) = \langle \phi_0 \, (S_{\mathrm{IPS}}(t)\phi_i) \rangle_\mu = \langle (S_{\mathrm{IPS}}(t)\,\phi_0)\phi_i \rangle_\mu = \langle (S_{\mathrm{IPS}}(t)\phi_{-i})\,\phi_0 \rangle_\mu = K(-i,t), \tag{2.151}$$

implying that $c_2 = 0$.

In Section 2.5 we compute c_3 for the independent spin-flip dynamics, for which $c_2 = 0$.

2.4 Examples of cone-mixing

2.4.1 Spin-flip systems in the regime $M < \epsilon$

Let ξ be a spin-flip system for which $M < \epsilon$. We recall that in a spin-flip system only one coordinate changes in a single transition. The rate to flip the spin at site $x \in \mathbb{Z}$ in configuration $\eta \in \Omega$ is $c(x,\eta)$. As shown in Steif [85] and in Maes and Shlosman [65], two copies ξ, ξ' of the spin-flip system starting from configurations η, η' can be coupled such that, uniformly in t and η, η',

$$\widehat{P}_{\eta,\eta'}\big(\exists\, s \geq t\colon\ \xi_s(x) \neq \xi'_s(x)\big) \leq \sum_{\substack{y \in \mathbb{Z}:\\ \eta(y) \neq \eta'(y)}} \mathrm{e}^{-\epsilon t}\left(\mathrm{e}^{\Gamma t}\right)(y,x) \leq \mathrm{e}^{-(\epsilon - M)t}, \tag{2.152}$$

where $\widehat{P}_{\eta,\eta'}$ is the Vasershtein coupling (or basic coupling), and Γ is the matrix $\Gamma = (\gamma(u,v))_{u,v\in\mathbb{Z}}$ with elements

$$\gamma(u,v) = \sup_{\eta\in\Omega} |c(u,\eta) - c(u,\eta^v)|. \tag{2.153}$$

Recall (2.15) to see that Γ is a bounded operator on $\ell_1(\mathbb{Z})$ with norm M (see also Liggett [63], Section I.3).

Define

$$\rho(t) = \sup_{\eta,\eta'\in\Omega} \widehat{P}_{\eta,\eta'}\big(\exists\, s \geq t\colon\ \xi_s(0) \neq \xi'_s(0)\big), \qquad t \geq 0. \tag{2.154}$$

Recall Definition 2.1, fix $\theta \in (0, \frac{1}{2}\pi)$ and put $c = c(\theta) = \cot\theta$. For $B \in \mathcal{F}_t^\theta$, estimate

$$\begin{aligned} |P_\eta(B) - P_{\eta'}(B)| &\le \widehat{P}_{\eta,\eta'}\big(\exists\, x \in \mathbb{Z}\ \exists\, s \ge t + c|x|\colon\ \xi_s(x) \neq \xi'_s(x)\big) \\ &\le \sum_{x\in\mathbb{Z}} \widehat{P}_{\eta,\eta'}\big(\exists\, s \ge t + c|x|\colon\ \xi_s(x) \neq \xi'_s(x)\big) \\ &\le \sum_{x\in\mathbb{Z}} \rho(t + c|x|) \\ &\le \rho(t) + 2\int_0^\infty \rho(t+cu)\,\mathrm{d}u \\ &= \rho(t) + \frac{2}{c}\int_0^\infty \rho(t+v)\,\mathrm{d}v. \end{aligned} \tag{2.155}$$

Since this estimate is uniform in B and η, η', it follows that for the cone mixing property to hold it suffices that

$$\int_0^\infty \rho(v)\,\mathrm{d}v < \infty. \tag{2.156}$$

It follows from (2.152) that $\rho(t) \le \mathrm{e}^{-(\epsilon - M)t}$, which indeed is integrable.

Note that if the supremum in (2.154) is attained at the same pair of starting configurations η, η' for all $t \ge 0$, then (2.156) amounts to the condition that the average coupling time at the origin for this pair is finite.

2.4.2 Attractive spin-flip dynamics

An attractive spin-flip system ξ has rates $c(x, \eta)$ satisfying

$$\begin{aligned} c(x,\eta) &\le c(x,\eta') \quad \text{if } \eta(x) = \eta'(x) = 0, \\ c(x,\eta) &\ge c(x,\eta') \quad \text{if } \eta(x) = \eta'(x) = 1, \end{aligned} \tag{2.157}$$

whenever $\eta \le \eta'$ (see Liggett [63], Chapter III). If $c(x, \eta) = c(x + y, \tau_y\eta)$ for all $y \in \mathbb{Z}$, then attractivity implies that, for any pair of configurations η, η',

$$\widehat{P}_{\eta,\eta'}\big(\exists\, s \ge t\colon\ \xi_s(x) \neq \xi'_s(x)\big) \le \widehat{P}_{[0],[1]}\big(\exists\, s \ge t\colon\ \xi_s(0) \neq \xi'_s(0)\big), \tag{2.158}$$

where [0] and [1] are the configurations with all 0's and all 1's, respectively. Proceeding as in (2.155), we find that for the cone-mixing property to hold it suffices that

$$\int_0^\infty \rho^*(v)\,\mathrm{d}v < \infty, \qquad \rho^*(t) = \widehat{P}_{[0],[1]}\big(\exists\, s \ge t\colon\ \xi_s(0) \neq \xi'_s(0)\big). \tag{2.159}$$

Examples of attractive spin-flip systems are the (ferromagnetic) Stochastic Ising Model, the Contact Process, the Voter Model, and the Majority Vote Process (see Liggett [63], Chapter III). For the one-dimensional Stochastic Ising Model, $t \mapsto \rho^*(t)$ decays exponentially fast at any temperature (see Holley [53]). The same is true for the one-dimensional Majority Vote Process (Liggett [63], Example III.2.12). Hence both are cone-mixing. The one-dimensional Voter Model has equilibria $p\delta_{[0]} + (1-p)\delta_{[1]}$, $p \in [0,1]$, and therefore is not interesting for us. The Contact Process has equilibria $p\delta_{[0]} + (1-p)\nu$, $p \in [0,1]$, but ν is not cone-mixing.

In view of the remark made at the end of Section 2.1.4, we note the following. For the Stochastic Ising Model in dimensions $d \geq 2$ exponentially fast decay occurs only at high enough temperature (Martinelli [66], Theorem 4.1). The Voter Model in dimensions $d \geq 3$ has non-trivial ergodic equilibria, but none of these is cone-mixing. The same is true for the Contact Process in dimensions $d \geq 2$.

2.4.3 Space-time Gibbs measures

We next give an example of a discrete-time dynamic random environment that is cone-mixing but not Markovian. Accordingly, in (2.12) we must replace $\mathcal{F}_0$ by $\mathcal{F}_{-\mathbb{N}_0} = \{\xi_t(x)\colon\, x \in \mathbb{Z},\, t \in (-\mathbb{N}_0)\}$. Let $\sigma = \{\sigma(x,y)\colon\, (x,y) \in \mathbb{Z}^2\}$ be a two-dimensional Gibbsian random field in the Dobrushin regime (see Georgii [47], Section 8.2). We can define a discrete-time dynamic random environment ξ on Ω by putting

$$\xi_t(x) = \sigma(x,t) \qquad (x,t) \in \mathbb{Z}^2. \tag{2.160}$$

The cone-mixing condition for ξ follows from the mixing condition of σ in the Dobrushin regime. In particular, the decay of the mixing function Φ in (2.37) is like the decay of the Dobrushin matrix, which can be polynomial.

2.5 Independent spin-flips

Let ξ be the Markov process with generator L_{ISF} given by

$$(L_{\mathrm{ISF}} f)(\eta) = \sum_{x \in \mathbb{Z}} c(x,\eta)\, \big[f(\eta^x) - f(\eta)\big], \qquad \eta \in \Omega, \tag{2.161}$$

where

$$c(x,\eta) = \gamma[1 - \eta(x)] + \delta\eta(x), \tag{2.162}$$

i.e., 0's flip to 1's at rate γ and 1's flip to 0's at rate δ, independently of each other. Such a ξ is an example of a dynamics with $M < \epsilon$, for which Theorem 2.16 holds. From the expansion of the global speed in (2.147) we see that $c_2 = 0$, because the dynamics is invariant under reflection in the origin. We explain the main ingredients that are needed to compute c_3 in (2.18).

The equilibrium measure of ξ is the Bernoulli product measure ν_ρ with parameter $\rho = \gamma/(\gamma+\delta)$. We therefore see from (2.148) that we must compute expressions of the form

$$I(j,i) = \big\langle (2\eta(0)-1)L_0^{-1}\big[(2\eta(j)-1)L_0^{-1}(\eta(i)-\rho)\big]\big\rangle_{\nu_\rho}, \tag{2.163}$$

where η is a typical configuration of the environment process $\zeta = (\zeta_t)_{t\geq 0} = (\tau_{X_t}\xi_t)_{t\geq 0}$ (recall Definition 2.10), and

$$(j,i) \in A = \{(-1,-2),(-1,0),(1,0),(1,2)\}. \tag{2.164}$$

By Lemma 2.11 we have $L_0 = L_{\mathrm{SRW}} + L_{\mathrm{ISF}}$, with L_{SRW} the generator of simple random walk on $\mathbb{Z}$ jumping at rate $U = \alpha+\beta$. Hence

$$(S_0(t)\eta)(i) = E_R^\eta[\eta_t(i)] = \sum_{y\in\mathbb{Z}} p_{Ut}(0,y)\, E_{\mathrm{ISF}}^{\tau_y\eta}[\eta_t(i)] = \sum_{y\in\mathbb{Z}} p_{Ut}(0,y)\, E_{\mathrm{ISF}}^{\eta}[\eta_t(i-y)], \tag{2.165}$$

where τ_y is the shift of space over y, and

$$E_{\mathrm{ISF}}^{\eta}[\eta_t(i)] = \eta(i)\,\mathrm{e}^{-Vt} + \rho(1-\mathrm{e}^{-Vt}) \tag{2.166}$$

with $V = \gamma+\delta$, and $p_t(0,y)$ is the transition kernel of simple random walk on $\mathbb{Z}$ jumping at rate 1. Therefore, by (2.165–2.166), we have

$$L_0^{-1}(\eta(i)-\rho) = \int_0^\infty S_0(t)(\eta(i)-\rho)\,\mathrm{d}t = \sum_{y\in\mathbb{Z}} \eta(i-y)\,G_V(y) - \rho\,\frac{1}{V} \tag{2.167}$$

with

$$G_V(y) = \int_0^\infty \mathrm{e}^{-Vt}\,p_{Ut}(0,y)\,\mathrm{d}t. \tag{2.168}$$

With these ingredients we can compute (2.163), ending up with

$$c_3 = \sum_{(j,i)\in A} I(j,i) = \frac{4}{U}\,\rho(2\rho-1)(1-\rho)\left[\frac{2U+V}{U}\,G_V(0) - \frac{3U+2V}{U}\,G_{2V}(0) - G_{2V}(1)\right]. \tag{2.169}$$

The expression between square brackets can be worked out, because

$$G_V(0) = \int_0^\infty \mathrm{e}^{-Vt}\,p_{Ut}(0,0)\,\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{\mathrm{d}\theta}{(U+V)-U\cos\theta} = \frac{1}{\sqrt{(U+V)^2-U^2}} \tag{2.170}$$

and

$$G_V(1) = \frac{U+V}{U}\,G_V(0) - \frac{1}{U}, \tag{2.171}$$

where the latter is derived by using that

$$\frac{\partial}{\partial t}\, p_{Ut}(0,0) = \tfrac{1}{2}U\left[p_{Ut}(0,1) + p_{Ut}(0,-1) - 2p_{Ut}(0,0)\right] \tag{2.172}$$

and $p_{Ut}(0,1) = p_{Ut}(0,-1)$. This leads to (2.18).