Random walks in dynamic random environments

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Chapter 1

Introduction: Random walks in random environments (RWRE)

In Sections 1.1 and 1.2 we introduce RWs in *static* and *dynamic* REs and we present a brief overview of the known results relevant for our discussion. In Section 1.3 we define the class of models that are the core of this thesis, i.e., RWs on interacting particle systems. In Section 1.4 we briefly mention other topics related to RWRE that are not covered in this introduction.

1.1 Static RE

The first model of a static RWRE appeared in the biophysics literature (Chernov [33], Temkin [93]) as a toy model for replication of DNA chains. In the early 70' Solomon [80] began a rigorous mathematical analysis of such models by considering a RW in a static RE on the one-dimensional integer lattice. Nowadays the behavior of this random process is completely understood. An overview of the results relevant for our discussion will be presented in Section 1.1.1. In Section 1.1.2 we describe the multi-dimensional case. Most of the techniques used in one dimension cannot be applied in the multi-dimensional setting, due to a more complicated structure of hitting times. Although powerful tools have been developed in the last twenty years and many important results have been achieved, several problems are still open. We will briefly describe what is known in the literature. For detailed statements, proofs and methods we refer the reader to [89, 99].

A formal definition of RW in *static* RE on $\mathbb{Z}^d$ is as follows.

Definition 1.1. (RW in static RE)
For each site $x \in \mathbb{Z}^d$, consider a $2d$-dimensional vector $\xi(x,\cdot) = \{\xi(x,e) \in [0,1] : e \in \mathbb{Z}^d, |e| = 1\}$ such that $\sum_{e:|e|=1} \xi(x,e) = 1$. Let S be the set of all possible values of these vectors, and let $\Omega = S^{\mathbb{Z}^d}$. Given a probability measure μ on Ω, we call a *random environment* an element $\xi \in \Omega$ distributed according to μ. For each realization of $\xi \in \Omega$, we define the RW X in the environment ξ as the Markov chain $X = (X_n)_{n\in\mathbb{N}}$ with state space $\mathbb{Z}^d$ and transition probabilities

$$P^{\xi}(X_{n+1} = x + e | X_n = x) = \xi(x,e), \quad e \in \mathbb{Z}^d, |e| = 1. \tag{1.1}$$

We write P^{ξ}_z to denote the *quenched* law of the RW in the environment ξ starting from position z. We write $\mathbb{P}_z$ to denote the *annealed* law starting from z, i.e.,

$$\mathbb{P}_z(X \in \cdot) = \int_{\Omega} P^{\xi}_z(X \in \cdot)\mu(\mathrm{d}\xi). \tag{1.2}$$

We write $E^{\xi}_z, \mathbb{E}_z$ and E_μ, respectively, for expectation with respect to the laws $P^{\xi}_z, \mathbb{P}_z$ and μ.

Henceforth we say that a statement involving the RW X holds $\mathbb{P}_z$-a.s., if for μ-almost every ξ the statement holds P^{ξ}_z-a.s. Note that under the *quenched* law X is a space-inhomogeneous Markov chain, whereas under the *annealed* law X is space-homogeneous but not Markovian. The definition above could have been stated without the nearest-neighbor restriction. This choice was made to avoid cumbersome notations and further technicalities. In the sequel we will sometimes point out when results hold without this restriction.

1.1.1 One dimension

1.1.1.1 Ergodic behavior

The first natural problem is to determine when X is transient or recurrent, whether it admits an asymptotic deterministic speed (under the quenched and the annealed law), i.e., a Law of Large Numbers (LLN), and what can be said about this speed. The next theorem answers these questions.

Theorem 1.2. (Transience, recurrence, LLN)
Let $\xi_x = \xi(x,1)$ and $\rho_x = (1-\xi_x)/\xi_x$. Assume that

$$\mu(\xi_x \in (0,1)) = 1, \tag{1.3}$$

and that μ is stationary and ergodic under translations. Then

1. $\mathbb{P}_0$*-a.s., X is recurrent if* $\mathbb{E}_\mu[\log \rho_0] = 0$*, transient to the left if* $\mathbb{E}_\mu[\log \rho_0] > 0$*, and transient to the right if* $\mathbb{E}_\mu[\log \rho_0] < 0$.

2. $\mathbb{P}_0$*-a.s., there exists a deterministic* $v \in (-1, 1)$ *such that*

$$\lim_{n\to\infty} \frac{X_n}{n} = v \begin{cases} > 0, & \text{if } \sum_{i=1}^{\infty} \mathbb{E}_\mu\Big[\prod_{j=0}^{i} \rho_{-j}\Big] < \infty, \\ < 0, & \text{if } \sum_{i=1}^{\infty} \mathbb{E}_\mu\Big[\prod_{j=0}^{i} \rho_{-j}^{-1}\Big] < \infty, \\ = 0, & \text{if both these conditions fail.} \end{cases} \tag{1.4}$$

3. *If μ is a product measure, then*

$$v = \begin{cases} (1 - \mathbb{E}_\mu[\rho_0])/(1 + \mathbb{E}_\mu[\rho_0]), & \text{if } \mathbb{E}_\mu[\rho_0] < 1, \\ -(1 - \mathbb{E}_\mu[\rho_0^{-1}])/(1 + \mathbb{E}_\mu[\rho_0^{-1}]), & \text{if } \mathbb{E}_\mu[\rho_0^{-1}] < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{1.5}$$

This result is mainly due to Solomon [80]. The original paper only deals with the case in which μ is a product measure. The generalization to the ergodic setup was proven later in [1].

When μ is a product measure, we can already appreciate some surprising features. For instance, if $\mathbb{E}_\mu[\log \rho_0] < 0$, then μ-a.s. $\lim_{n\to\infty} X_n = +\infty$. However, by Jensen's inequality, $\mathbb{E}_\mu[\log \rho_0] \leq \log \mathbb{E}_\mu[\rho_0]$, and if $\mathbb{E}_\mu[\rho_0] > 1$, then $v = 0$, in which case X is transient with zero speed. In other words, the RWRE moves to infinity in a sub-ballistic manner, a phenomenon that never happens for a homogeneous RW. This behavior is due to the presence of 'traps' in the environment: localized pockets in which the walk spends a long time because the transition probabilities push it towards the center of the pocket. In particular, it can be shown that if $v \geq 0$, then $v < 2\mathbb{E}_\mu[\xi_0] - 1 = \bar{v}$. By interpreting $\bar{v}$ as the speed of a homogeneous nearest-neighbor RW jumping to the right with probability $\mathbb{E}_\mu[\xi_0]$ and to the left with probability $1 - \mathbb{E}_\mu[\xi_0]$ ('average medium RW'), we see that in general the RE causes a *slow-down* with respect to the average environment.

1.1.1.2 Scaling limits

Next, we may ask whether X when properly scaled admits a limiting law. Results in this direction have been derived in a number of papers. The invariance principles are typically different under the quenched and the annealed law, and several types of scaling laws occur depending on μ. For example, in the recurrent case ($\mathbb{E}_\mu[\log \rho_0] = 0$), Sinai [79] proved that extreme sub-diffusive behavior holds, i.e. ,

$$\frac{\sigma^2 X_n}{(\log n)^2} \xrightarrow[n\to\infty]{(\mathbb{P}_0)} Z, \qquad \sigma^2 = \mathbb{E}_\mu[(\log \rho_0)^2] \in (0, \infty), \tag{1.6}$$

where Z is a functional of a standard Wiener process (independent of μ) with a non-trivial law that was later identified by Kesten [59].

1.1.1.3 Large deviations

The last item of interest for our introduction is the analysis of the large deviation behavior of the empirical speed of X. We briefly recall that a family of probability measures $(P_n)_{n\in\mathbb{N}}$ satisfies a Large Deviation Principle (LDP) with rate a_n and with rate function I if, for any measurable set A,

$$-\inf_{\theta\in \mathrm{int}(A)} I(\theta) \le \liminf_{n\to\infty} \frac{1}{a_n} \log P_n(A) \le \limsup_{n\to\infty} \frac{1}{a_n} \log P_n(A) \le -\inf_{\theta\in\bar{A}} I(\theta), \tag{1.7}$$

where $\bar{A}$ and $\mathrm{int}(A)$, are, respectively, the closure and the interior of A. If we consider the family of probability measures associated with the empirical speed of X, i.e., $P_n(\cdot) = P(X_n/n \in \ \cdot), n \in \mathbb{N}$, then a LDP for this family tells us how unlikely it is to observe the walk travelling at any given speed we may be interested in. The most general large deviation result for the one-dimensional RWRE is the following.

Theorem 1.3. (Quenched and Annealed LDP)
Assume that μ is stationary and ergodic. Then, for μ-a.e. realization of ξ, the family of probability measures $P_0^\xi(X_n/n \in \ \cdot), n \in \mathbb{N}$, satisfies a LDP with rate n and with convex deterministic rate function I_μ^{que}. Moreover, the family of probability measures $\mathbb{P}_0(X_n/n \in \ \cdot), n \in \mathbb{N}$, satisfies a LDP with rate n and with convex rate function

$$I_\mu^{\mathrm{ann}}(\theta) = \inf_{\nu\in\mathcal{M}_e} [h(\nu|\mu) + I_\nu^{\mathrm{que}}(\theta)], \tag{1.8}$$

where $\mathcal{M}_e$ denotes the set of stationary and ergodic measures on Ω, and $h(\nu|\mu)$ is the relative entropy of ν with respect to μ. In particular, $I_\mu^{\mathrm{ann}}(\theta) \le I_\mu^{\mathrm{que}}(\theta)$. Furthermore in some cases both rate functions are not strictly convex, and are zero in the interval $[0, v]$ (and only in this interval).

From this general statement, we can already appreciate two interesting and unusual features: the rate functions need not be strictly convex and they may vanish on $[0, v]$, indicating sub-exponential decay for the probability of slow-down. In contrast, we recall that for homogeneous RWs the corresponding rate function is strictly convex and vanishes only at the typical speed v (see e.g. [39, 52]). The quenched LDP when $(\xi_x)_{x\in\mathbb{Z}}$ is an i.i.d. sequence was derived in [49], while the annealed LDP, refined quenched estimates and the generalization to ergodic REs were obtained later in [34, 38, 46, 70, 71, 99]. In the next section we give an explicit example.

1.1.1.4 An example

Let $\xi = (\xi_x)_{x\in\mathbb{Z}} \in \{0,1\}^{\mathbb{Z}}$ be a random sequence distributed according to a Bernoulli product measure ν_ρ with parameter $\rho \in (0,1)$. When $\xi_x = 1$ we say that site x is occupied, while when $\xi_x = 0$ we say that it is vacant. In particular, ρ represents the density of the occupied sites. Conditional on ξ, let $X = (X_n)_{n\in\mathbb{N}_0}$ be the RW with local transition probabilities

$$P^{\xi}(X_{n+1} = x + e \mid X_n = x) = \begin{cases} p\xi_x + q(1-\xi_x), & \text{if } e = +1, \\ q\xi_x + p(1-\xi_x), & \text{if } e = -1, \end{cases} \tag{1.9}$$

where w.l.o.g. we assume that $p = 1 - q \in \left(\frac{1}{2}, 1\right)$. Note that the formulation of the model in this example is consistent with (1.1). Thus, on occupied sites the RW has a local drift to the right while on vacant sites it has a local drift to the left, of the same size. Note that for $p = \frac{1}{2}$ the model reduces to a simple RW and for $\rho = 1$(respectively 0) to a RW with drift $2p - 1$(respectively $1 - 2p$). From Theorem 1.2, we have that X is recurrent if $\rho = \frac{1}{2}$, transient to the right(left) if $\rho > \frac{1}{2}$ $\left(< \frac{1}{2}\right)$. Moreover, μ-a.s.,

$$\lim_{n\to\infty} X_n/n = v \begin{cases} = 0, & \text{if } \rho \in [q,p], \\ > 0, & \text{if } \rho \in (p,1], \\ < 0, & \text{if } \rho \in [0,q). \end{cases} \tag{1.10}$$

We thus see that if $\rho \in \left(\frac{1}{2}, p\right]$, then the walk will eventually go to the right but at zero speed. This effect is due to the presence of 'traps' in the environment. Indeed, even though occupied sites are more frequent than vacant sites, on its way to $+\infty$, X will cross arbitrarily long intervals in which the local drift is pointing to the left, which results in a displacement of X that grows sub-linearly.

When we look at the large deviations of the empirical speed of X, we see that 'trapping effects' play an important role even in the transient regime with non-zero speed. Without loss of generality we will restrict to the case $\rho \in \left[\frac{1}{2}, 1\right)$.

Theorem 1.4. (Quenched LDP)
For μ-a.e. ξ, the family of probability measures $P_0^{\xi}(X_n/n \in \cdot), n \in \mathbb{N}$, satisfies a LDP

with rate n and with deterministic rate function I^{que} that can be computed in terms of a variational problem and that has the following properties:

1. *I^{que} is continuous and convex on $[-1,1]$ and infinite elsewhere.*
2. *$I^{\mathrm{que}}(-\theta) = I^{\mathrm{que}}(\theta) + \theta(2\rho - 1)\log\left(\frac{p}{q}\right)$ for $\theta \in (0,1]$.*
3. *I^{que} is zero on $[0,v]$ and strictly positive on $(v,1]$.*
4. *I^{que} is strictly convex and analytic on $(v,1)$.*

Here are qualitative pictures of $\theta \mapsto I^{\mathrm{que}}(\theta)$ on $[-1,1]$ in the three respective cases:

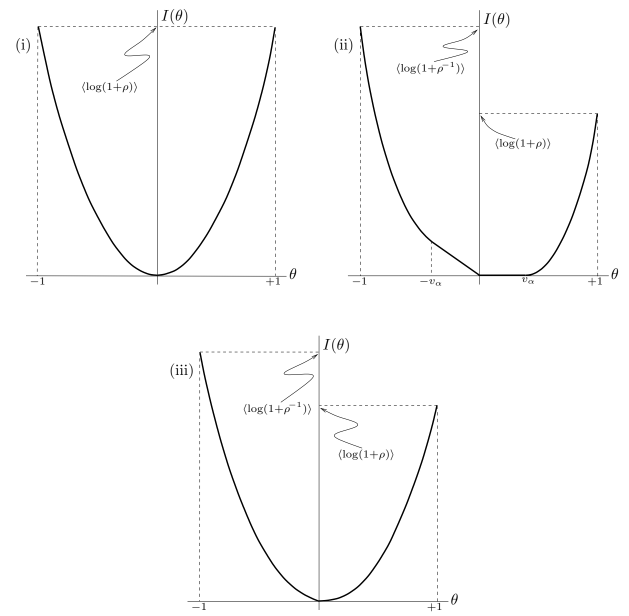

FIGURE 1.1: (i) recurrent; (ii) transient: positive speed; (iii) transient: zero speed. Permission to use the picture has been kindly granted by F. den Hollander [52]. The notations $I(\theta)$, ρ and $< \cdot >$, stand for I^{que}, $\frac{q}{p}\xi_0 + \frac{p}{q}[1-\xi_0]$ and E^{ξ}, respectively.

From Theorem 1.4 (see [34, 49]) we see that both in the recurrent case and in the transient case with zero speed the rate function has a unique zero at $\theta = 0$ and is strictly convex everywhere, while in the transient case with positive speed the rate function has two linear pieces: one horizontal piece for $\theta \in [0,v]$ and one tilted piece for $\theta \in [-v,0]$.

The flat piece means that speeds smaller than the typical speed v are not exponentially costly. This is again because the RE contains long stretches of sites where the local drifts point to the left. Between 0 and θn the longest stretches have a length of order $\log n$, and for the walker to lose a time of order n in these stretches has a cost that is sub-exponential in n.

Under the annealed measure, as stated in Theorem 1.3, an LDP is satisfied as well. The corresponding rate function I^{ann} is given by (1.8). A symmetry relation as in part 2 of Theorem 1.4 does not hold. In particular, I^{ann} and I^{que} coincide on $[0, v]$. Moreover a small linear piece can be present in the annealed rate function for some choice of the parameters, see [34].

1.1.2 Higher dimensions

In the multi-dimensional setup, even fundamental questions like recurrence vs. transience and the existence of a limiting speed remain partially open. We give here a brief summary of the main results and formulate unsolved conjectures. For a more detailed overview we refer the reader to [89, 99]. In what follows we restrict to the case where $\big(\xi(x,\cdot)\big)_{x\in\mathbb{Z}^d}$ is an i.i.d. sequence satisfying the so-called *ellipticity* condition

$$\mu\left(\inf_{|e|=1} \xi(x,e) > 0\right) = 1. \tag{1.11}$$

Let $\mathbb{S}^{d-1}$ be the unit sphere. Given a vector $l \in \mathbb{S}^{d-1}$, consider the event

$$A_l = \left\{\lim_{n\to\infty} X_n \cdot l = \infty\right\}, \tag{1.12}$$

where $\cdot$ denotes the vector inner product. In 1981 [58] Kalikow proved that $\mathbb{P}_0(A_l \cup A_{-l}) \in \{0,1\}$ for all $l \in \mathbb{S}^{d-1}$ and $d \geq 1$, and he conjectured that if μ is *uniformly elliptic*, i.e., there exists a constant $\delta > 0$ such that

$$\mu\left(\inf_{|e|=1} \xi(x,e) > \delta\right) = 1, \tag{1.13}$$

then

$$\mathbb{P}_0(A_l) \in \{0,1\} \quad \forall l \in \mathbb{S}^{d-1}. \tag{1.14}$$

Furthermore, he formulated a technical condition (known as Kalikow's condition; see [58]) that ensures a strong bias in the direction l and implies (1.14).

In $d = 1$, the 0-1 law in (1.14) is a simple consequence of Theorem 1.2. For $d = 2$, (1.14) has been proven in [92] under the ellipticity condition in (1.11). For $d \geq 3$, (1.14) is still open and is a cornerstone to prove a LLN, as shown by the following theorem.

Theorem 1.5. (LLN)
Assume that μ is uniformly elliptic. Fix $l \in \mathbb{S}^{d-1}$. Then there exist $v^+, v^- \in [0,1]$ such that

$$\lim_{n\to\infty} \frac{X_n \cdot l}{n} = v^+ 1_{A_l} - v^- 1_{A_{-l}} \quad \mathbb{P}_0 - a.s. \tag{1.15}$$

In particular, for $d = 2$ the LLN holds.

The proof of this theorem [92, 101], and many other results in the multi-dimensional setting, are based on a construction of *regeneration times* introduced by Sznitman and Zerner [92]. Roughly speaking, a random time $\tau \in \mathbb{N}$ is a regeneration time in direction l if $X_\tau \cdot l \geq X_n \cdot l$ for all $n \leq \tau$ and $X_\tau \cdot l < X_n \cdot l$ for all $n > \tau$, i.e., $X_n \cdot l$ achieves a record at time τ and never moves backward from that record. Once these times are constructed, it is possible to show that the sequences of space and time increments between regeneration times form i.i.d. sequences, from which the LLN and the CLT can be derived.

Further results when μ is i.i.d. and uniformly elliptic were obtained in [8, 95]. In these papers it is shown that there are at most two deterministic limit points for the sequence $(X_n/n)_{n\in\mathbb{N}}$, say, v_1 and v_2. If $v_1 \neq v_2$ then there exists a constant $a \geq 0$ such that $v_2 = -av_1$. For $d \geq 5$, [8] proves that if $v_1 \neq v_2$, then at least one of them is zero.

There is no general criterion to establish when RWRE in $d \geq 2$ is transient or recurrent, although one expects transience as soon as $d \geq 3$. Moreover, when the LLN holds, no explicit formula for the limiting speed v is known. A natural question is to at least understand under which condition RWRE is *ballistic*, i.e., $v \neq 0$. Some results have been obtained in this direction in the last decade. This problem is related to the properties of the RE and the possible presence of 'traps' (i.e., regions where the walk may spend a long time with a high probability). In [100], the author considered the drift at the origin

$$d_0 = \sum_{i=1}^{d} [\xi(0, e_i) - \xi(0, -e_i)] \cdot e_i, \tag{1.16}$$

with $\{e_i\}_{i=1}^d$ the canonical basis of $\mathbb{Z}^d$, and showed that if, for some $l \in \mathbb{S}^{d-1}$, $d_0 \cdot l > 0$ for μ-a.e. environment, then X_n/n converges to a deterministic v with $v \cdot l > 0$. Such REs are called *non-nestling*. The interest is in understanding the so-called *nestling* REs, i.e., when the origin belongs to the closed convex hull of the support of d_0, for which a non-ballistic regime might be possible . Some progress has been achieved by Sznitman in [88, 89], who formulated the following conditions that guarantee ballisticity even for the nestling case. Given a direction $l \in \mathbb{S}^{d-1}$ and $b, L > 0$, define the slab $U_{b,l,L} = \{x \in \mathbb{Z}^d : -bL < x \cdot l < L\}$ and the exit time $\tau_{b,l,L} = \inf\{n \in \mathbb{N} : X_n \notin U_{b,l,L}\}$. Let $\gamma \in (0, 1]$. Condition $(T)_\gamma|l$ is said to hold relative to $l \in \mathbb{S}^{d-1}$ if, for all $l' \in \mathbb{S}^{d-1}$ in a neighborhood of l and all $b > 0$,

$$\limsup_{L\to\infty} L^{-\gamma} \log \mathbb{P}_0(X_{\tau_{b,l',L}} \cdot l' < 0) < 0. \tag{1.17}$$

In words, consider a slab in $\mathbb{Z}^d$ contained between the hyperplanes normal to l at distance L and $-bL$ in direction l. When $\gamma = 1$, the condition $(T)_1|l$ holds if the probability of exit from this slab in direction $-l$ is exponentially small in L. Condition $(T')|l$ is said to hold if condition $(T)_\gamma|l$ holds for all $\gamma \in (0,1)$. Clearly

$$(T)_1|l \Longrightarrow (T')|l \Longrightarrow (T)_\gamma|l \text{ for } \gamma \in (0,1), \tag{1.18}$$

and it is believed that $(T)_1|l, (T')|l$ and $(T)_\gamma|l$ are equivalent. This equivalence is still open and some recent progress can be found in [43, 44, 78]. The importance of these conditions is given by the following theorem due to Sznitman [88].

Theorem 1.6. (Ballisticity and CLT under Sznitman's $(T')|l$ condition)
Assume that μ is i.i.d. and uniformly elliptic and that condition $(T')|l$ holds relative to $l \in \mathbb{S}^{d-1}$. Then X satisfies a LLN with a deterministic limiting speed v such that $v \cdot l > 0$. Moreover, there exists a deterministic $\sigma > 0$ such that, under the annealed measure $\mathbb{P}_0$, $(X_n - nv)/\sigma\sqrt{n}$ converges in distribution to a standard Gaussian random variable.

Other recent results in the i.i.d. setting can be found in [27, 28].

When we drop the i.i.d. assumption on μ further complications arise. If the environment has a finite-range dependence, then a slight modification of the arguments for the i.i.d. situation, developed in [76, 99], shows that the LLN and the CLT carry over. If the space correlations are long-range but strong mixing in some appropriate sense, then only few results have been obtained. In this context, [35, 36] derived a LLN and a CLT via a regeneration-time argument under a uniform mixing condition. In [72] a LLN was derived by analyzing the *environment process*, i.e., the environment as seen from the point of view of the walker. [29] developed a renormalization scheme to prove a CLT when the transition probabilities of the RW are sufficiently close to those of a simple RW.

Large deviations for the empirical speed X_n/n have been studied only recently. The main result is stated in the next theorem due to Varadhan [94].

Theorem 1.7. (Quenched and Annealed LDP)
Let $d \geq 2$. Assume that μ is uniformly elliptic and ergodic. Then, for μ-a.e. realization of ξ, the family of probability measures $P_0^\xi(X_n/n \in \cdot\,), n \in \mathbb{N}$, satisfies a LDP with rate n and with convex deterministic rate function I^{que}. If μ is i.i.d., then also an annealed LDP is satisfied with rate n and with convex rate function I^{ann}. Furthermore, in the latter case I^{que} and I^{ann} have the same zero set, and this set is convex and consists of either a single point or a line segment.

The proof of Theorem 1.7 is based on a subadditivity argument. As for $d = 1$, the rate functions are in general not strictly convex. A relation like (1.8) is not available. It is not known under which conditions and in which region the two rate functions coincide. Partial progress and relalted results can be found in [73, 86, 87, 97].

1.2 Dynamic RE

In this section we introduce RWs in dynamic REs, which will be the main topic of this thesis. This is a variant of the problem in the previous section (see Definition 1.1) in which the environment ξ evolves in time according to a given autonomous dynamics. In other words, ξ is given by a collection of random vectors $\{\xi_n(x,\cdot) : x \in \mathbb{Z}^d, n \in \mathbb{N}_0\}$ with a prescribed joint law, and X is a RW with space-time dependent transition probabilities given by

$$P^\xi(X_{n+1} = x + e | X_n = x) = \xi_n(x, e), \quad e \in \mathbb{Z}^d, |e| = 1, n \in \mathbb{N}_0. \tag{1.19}$$

Under the quenched law P_0^ξ, X is now a space-time inhomogeneous Markov chain. Due to the dynamics of the environment, we expect different behavior than in the static situation. In particular, trapping phenomena, which played a chief role in static models, may not survive. The next sections are devoted to a brief exposition of the different types of problems that have been studied so far in the literature. We will first describe the easiest models and then move on to the more challenging ones.

1.2.1 Early work

In 1986 [64] Madras studied a one-dimensional RW that is a deterministic functional of a randomly fluctuating environment, which can be considered as a degenerate case of a RW in a dynamic RE. The model is defined as follows. For each $x \in \mathbb{Z}$, let $\xi = (\xi_t(x))_{t\geq 0}$ be an independent stationary continuous-time Markov process with state space $\{+1, -1\}$ and transition probability matrix

$$\begin{pmatrix} p_{-1,-1}(t) & p_{-1,+1}(t) \\ p_{+1,-1}(t) & p_{+1,+1}(t) \end{pmatrix} = \begin{pmatrix} q + pr^t & p - pr^t \\ q - qr^t & p + qr^t \end{pmatrix},$$

where $p = \alpha/(\alpha + \beta), q = 1 - p, r = e^{-(\alpha+\beta)}$, and $\alpha, \beta \in (0, \infty)$. Let $X = (X_t)_{t\geq 0}$ be a RW, starting from the origin ($X_0 = 0$) and moving on $\mathbb{R}$ as follows. For $i \in \mathbb{N}_0$, define

$$X_t = X_i + (t - i)\xi_i(X_i), \quad i < t \leq i + 1. \tag{1.20}$$

Thus, X represents the motion of a particle, travelling on $\mathbb{R}$ at unit speed, that at each unit of time chooses its direction according to the state of the local environment. By using the ergodic properties of ξ it can be shown that the process X has a stationary and exponentially mixing measure, which can be used to derive a recurrence criterion, a strong LLN and a CLT. In particular, X is recurrent if and only if $\alpha = \beta$. This model does not exhibit surprising behavior and, in contrast to RWRE models, has just one level of randomness. Nevertheless, the above results were obtained with the help of highly non-trivial methods and were the first in the dynamic setting.

1.2.2 Space-time i.i.d. RE

In 1992, Boldrighini et al. [16] introduced the first model of a RW in a dynamic RE. Since then this model has been studied intensively under several assumptions and using different tools. Though results like LLNs and CLTs have been derived, the general picture is far from being understood. The simplest setting is when the environment $\xi = \{\xi_n(x, \cdot) : x \in \mathbb{Z}^d, n \in \mathbb{N}\}$ is a collection of i.i.d. random variables, which we call i.i.d. space-time RE. Note that this is equivalent to a $(d+1)$-dimensional RW in a static i.i.d. RE in which, at each time step, one coordinate of the walk increases deterministically by one unit. Under the annealed measure, this RWRE becomes a simple RW in an averaged environment. Thus, the interest is in studying the quenched properties. The most general result has been derived in [74]. With the help of a martingale approach for additive functionals of a Markov chain, they obtained a quenched CLT in arbitrary dimension. In particular, they showed that the displacement of the RW in the i.i.d. space-time RE always has diffusive behavior with deterministic parameters. Similar results under a somewhat stronger condition on the RE were already found in [11], [22–24], via a cluster-expansion technique together with a small-noise assumption (see (1.21)), and in [6], with the help of generating functions.

A variant of the i.i.d. setting has been considered in [26]. Here, the environment is independent in time but has spatial correlation, i.e., at each time unit a new RE is sampled from a given distribution with dependence in space. The RW X is taken to be a perturbation of a homogeneous RW with transition kernel $p(e)$, namely,

$$P^{\xi}(X_{n+1} = x + e | X_n = x) = p(e) + \epsilon c(e; \xi_n(x)), \quad e \in \mathbb{Z}^d, |e| = 1, \tag{1.21}$$

where ϵ is a small positive parameter. The function c is such that (1.21) are transition probabilities, and represents the influence of ξ on the evolution of X. For ϵ small ('small noise regime'), a quenched CLT was proved with Brownian motion as scaling limit.

An analysis of the large deviations for RW in a space-time i.i.d. RE is presented in [98]. Except for our result in Chapter 4, this is the only paper dealing with LDPs in the dynamic setup. Indeed, as we pointed out in Section 1.1.2, even for static RE in $d \geq 2$ the large deviation analysis is difficult and is still far from being understood. Under the annealed law, RW in a space-time i.i.d. RE behaves like a homogeneous RW, for which the LDP for the empirical speed is given by Cramer's theorem [39, 52]. [98] shows that also under the quenched law a LDP for the empirical speed holds when $d \geq 3$. In particular, for speeds that are sufficiently close to the typical speed v, the quenched and annealed rate functions coincide. Furthermore, conditioned on any rare event (i.e., the empirical speed being any value different from v), the empirical process associated with *the environment process*, i.e., the environment as seen from the walker, converges to a certain stationary process, both under the quenched and the annealed law.

1.2.3 Time-dependent RE

Further complications arise when considering dynamic RE ξ in which the collection $\xi = \{\xi_n(x, \cdot) : x \in \mathbb{Z}^d, n \in \mathbb{N}\}$ is i.i.d. in space but Markovian in time, i.e., at each site x there is an independent copy of the same ergodic Markov chain. Note that, in this setup, the loss of time-independence makes even the annealed properties of X non-trivial. Such problems have been investigated in [5], [13], [41]. [13] considers the case in which the transition probabilities of the RW depend weakly on the environment ('small noise regime'; see (1.21)). By means of a cluster-expansion technique, it is proved that a quenched CLT holds a.s. for any $d \geq 3$. Similar results have been obtained in [5]. Via a probabilistic argument based on regeneration times, and under ellipticity conditions weaker than in [13], a strong LLN and a CLT were derived under the annealed law for any $d \geq 1$, and a quenched invariance principle only in high dimensions, namely, $d > 7$. Further progress was achieved with the help of an analytical approach to analyze *the environment process* in two recent papers [41, 42]. [41] deals with the case in which the transition probabilities of the RW are again weakly dependent on the environment, while the environment has a deterministic but strongly chaotic evolution. In [42], the authors consider a RW that is strongly dependent on a dynamic RE that again is assumed to be independent in space and Markovian in time. In both papers, a strong LLN and a CLT have been proven, under both the annealed and the quenched law.

1.2.4 Space-time mixing RE

A major challenge is to consider more general REs in which correlations in both space and time are allowed. Results in this direction have been obtained recently in [30, 40].

Both papers deal with finite-range RWs whose transition probabilities depend weakly on a RE whose space-time correlations decay exponentially. By means of a renormalization group technique [30], respectively, by analyzing *the environment process* via a martingale approximation [40], they proved a LLN and showed that in the scaling limit the behavior is diffusive for any $d \geq 1$. In particular, [30] does not assume a Markovian structure of the RE.

1.3 RW on an Interacting Particle System (IPS)

We are finally ready to introduce a class of RW in dynamic RE that will be the main subject of this thesis, namely, our RE will evolve as an Interacting Particle System (IPS). The main reason for this choice is that IPSs constitute a well-established research area and are natural examples of dynamic RE with space-time correlations. In the next sections we first define the class of IPSs we are interested in, providing some explicit examples, and then introduce our class of RWs. To avoid heavy notation, the definitions are stated for $d = 1$ and for nearest-neighbor RW, even though they easily extend to $d \geq 2$ and/or to more general step distributions. Such possible extensions will be pointed out in the next chapters.

1.3.1 IPS

1.3.1.1 Definition

Let $\Omega = \{0,1\}^{\mathbb{Z}^d}$. Denote by $D_\Omega[0,\infty)$ the set of paths in Ω that are right-continuous and have left limits. Let $\{P^\eta, \eta \in \Omega\}$ be a collection of probability measures on $D_\Omega[0,\infty)$ satisfying the Markov property. An IPS

$$\xi = (\xi_t)_{t\geq 0} \quad \text{with} \quad \xi_t = \{\xi_t(x)\colon\, x \in \mathbb{Z}^d\}, \tag{1.22}$$

is a Markov process on Ω with law P^η, when $\xi_0 = \eta \in \Omega$ is the starting configuration. We say that site x at time t is vacant or occupied when $\xi_t(x) = 0$ or 1.

Let $\mathcal{P}(\Omega)$ be the set of probability measures on Ω. Given $\mu \in \mathcal{P}(\Omega)$, we denote by P^μ the law of ξ when ξ_0 is drawn from $\mu \in \mathcal{P}(\Omega)$, i.e.,

$$P^\mu(\cdot) = \int_\Omega P^\eta(\cdot)\, \mu(\mathrm{d}\eta). \tag{1.23}$$

Throughout the sequel we will assume that

$$P^{\mu} \text{ is stationary and ergodic under space-time shifts.} \tag{1.24}$$

Thus, in particular, μ is a homogeneous extremal equilibrium for ξ.

Let $C(\Omega)$ be the set of continuous functions on Ω taking values in $\mathbb{R}$, viewed as a Banach space with norm

$$\|f\|_{\infty} = \sup_{\eta \in \Omega} |f(\eta)|. \tag{1.25}$$

The Markov semigroup associated with ξ is denoted by $S_{\mathrm{IPS}} = (S_{\mathrm{IPS}}(t))_{t \geq 0}$. This semigroup acts from the left on $C(\Omega)$ as

$$\big(S_{\mathrm{IPS}}(t) f\big)(\cdot) = E^{(\cdot)}[f(\xi_t)], \qquad f \in C(\Omega), \tag{1.26}$$

and acts from the right on $\mathcal{P}(\Omega)$ as

$$\big(\nu S_{\mathrm{IPS}}(t)\big)(\cdot) = P^{\nu}(\xi_t \in \cdot\,), \qquad \nu \in \mathcal{P}(\Omega). \tag{1.27}$$

In particular, we assume that ξ is a *Feller* process, i.e., $S_{\mathrm{IPS}}(t) f \in C(\Omega)$ for every $t \geq 0$ and $f \in C(\Omega)$.

Informally, an IPS is a collection of particles on the integer lattice evolving in a Markovian way. Depending on the specific transition rates between the different configurations, we obtain several types of IPS. Each particle may interact with the others: the evolution of each particle is defined in terms of local transition rates that may depend on the state of the system in a neighborhood of the particle. For a formal construction, we refer the reader to Liggett [63], Chapter I. Some explicit examples will be given below, and in the next chapters whenever needed.

1.3.1.2 Examples

(1) **Stochastic Ising Model (SIM)**

This model goes back to Glauber [48] and was introduced as a model for magnetism. The SIM is a Markov process on $\Omega' = \{+1, -1\}^{\mathbb{Z}^d}$, where each site represents an iron atom whose spin can be either up $(+1)$ or down (-1). In the original and easiest formulation, the dynamics can be described as follows. Let $\beta = T^{-1} \geq 0$ represent the inverse of the temperature T of the system. Given a starting configuration of spins $\eta \in \Omega'$, the spin

$\eta(x)$ at site x flips to $-\eta(x)$ at rate

$$c(x,\eta) = \exp\Big\{ -\beta \sum_{y:|y-x|=1} \eta(x)\eta(y) \Big\}, x \in \mathbb{Z}^d, \eta \in \Omega'. \tag{1.28}$$

With this choice of the rates we see that each spin tends to be aligned with its neighborhood. Indeed, the flip rate in (1.28) is higher when the spin at x differs from most of its neighbors than when it agrees with most of them. Such a monotonicity property is called *attractiveness* (see Section 2.4.2). In the language of statistical mechanics it is referred to as *ferromagnetism.* Note that, replacing the state space Ω' by Ω, we can pass from the 'spin interpretation' of the system to an interpretation of an IPS in which particles/holes flip into holes/particles.

Depending on the temperature and the dimension, the SIM shows interesting behavior. For example, when $d = 1$ it admits a unique ergodic measure for any $\beta \in \mathbb{R}^+$, while for $d \geq 2$ there exists a critical $\beta_c(d)$ such that for $\beta > \beta_c(d)$ there are at least two extremal invariant measures (which means that the system has a phase transition).

For $d \geq 1$, if $\beta = 0$, then the SIM is an example of an independent spin-flip dynamics (see Section 2.5), namely, the coordinates $\eta_t(x)$ become independent two-state Markov chains and the system has a unique ergodic measure given by the Bernoulli product measure with density $\frac{1}{2}$. The dynamics defined by the rates in (1.28) is only an example of a SIM. It is possible to also consider flip rates that depend not only on nearest-neighbor sites. For a general definition of the stochastic Ising model, see Liggett [63] Chapter 4.

(2) **Exclusion Process (EP)**

Let $p(x,y), x,y \in \mathbb{Z}^d$, be a transition kernel of a finite-range homogeneous RW on $\mathbb{Z}^d$. Given a configuration $\eta \in \Omega = \{0,1\}^{\mathbb{Z}^d}$, let $\{x \in \mathbb{Z}^d : \eta(x) = 1\}$ be the set of locations of the particles at time 0. The exclusion process is the IPS in which particles move according to the following rules:

- A particle at site x waits an exponential time with mean 1 and then chooses a site y with probability $p(x,y)$.
- The particle jumps to site y if this site is vacant, but does not jump when it is occupied.

The exclusion process is an example of a conservative IPS (i.e., the number of particles is preserved by the evolution) in which at each transition two coordinates of the system may change. It was originally introduced by Spitzer [81] as a model for a lattice gas at infinite temperature.

(3) **Contact Process (CP)**

The contact process (introduced by Harris [50]) is a toy model for the spread of an infection in a large population of individuals. Let $\Omega = \{0,1\}^{\mathbb{Z}^d}$. Each site $x \in \mathbb{Z}^d$ represents an individual. Given $\eta \in \Omega$, we say that the individual x is infected or healthy if $\eta(x)$ equals 1 respectively 0. The evolution of the system makes a healthy individual infected at rate λ times the number of infected neighbors, while infected individuals recover independently at rate 1. In other words, for each $x \in \mathbb{Z}^d$, if $\eta(x) = 1$, then $\eta(x)$ flips to 0 at rate 1, while if $\eta(x) = 0$, then it flips to 1 at rate $\lambda \sum_{y:|y-x|=1} \eta(y)$, where $\lambda \geq 0$ is a parameter representing the intensity of the infection spread. It is easy to see that the pointmass concentrated at the configuration with all 0's is a trivial invariant measure. It is possible to prove that for any $d \geq 1$ there exists a critical value $\lambda_c(d) \in (0,\infty)$ such that for $\lambda > \lambda_c(d)$ the system has at least one non-trivial invariant measure.

1.3.2 RW on IPS

Conditional on a realization of an IPS ξ, let

$$X = (X_t)_{t\geq 0} \tag{1.29}$$

be the RW with local transition rates

$$\begin{aligned} x \to x+1 \quad &\text{at rate} \quad \alpha_1\, \xi_t(x) + \alpha_0\, [1-\xi_t(x)], \\ x \to x-1 \quad &\text{at rate} \quad \beta_1\, \xi_t(x) + \beta_0\, [1-\xi_t(x)], \end{aligned} \tag{1.30}$$

where $\alpha_1, \beta_1, \alpha_0, \beta_0 \in (0,\infty)$ with $\alpha_1 + \beta_1 = \alpha_0 + \beta_0$. Thus, on occupied sites the RW has a local drift $\alpha_1 - \beta_1$, while on vacant sites it has a local drift $\alpha_0 - \beta_0$. Note that the sum of the jump rates is independent of ξ. Let P_0^ξ denote the law of X starting from $X_0 = 0$ conditional on ξ, which is the *quenched* law of X. The *annealed* law of X is

$$\mathbb{P}_{\mu,0}(\cdot) = \int_{D_\Omega[0,\infty)} P_0^\xi(\cdot)\, P^\mu(\mathrm{d}\xi). \tag{1.31}$$

Note that X is a continuous-time variant of the RW in a dynamic RE defined in (1.19).

By choosing $\alpha_1 = \beta_0, \alpha_0 = \beta_1$ with $\alpha_1 > \beta_1$, we obtain the continuous-time dynamic analogue of the static model given in Section 1.1.1.4, where $\alpha_1/(\alpha_1+\beta_1)$ takes over the role of p.

1.4 Related models

We close by listing some topics which are closely related to RWRE but not covered in this introduction.

- *Non-nearest-neighbor RWRE*: When dealing with non-nearest-neighbor finite-range RW in static and dynamic RE, some results and techniques we discussed in this chapter can be easily extended; see e.g. [40, 41, 73, 95]. Nevertheless, in dropping the nearest-neighbor assumption extra difficulties may arise and tools to analyze Lyapunov exponents associated with certain random matrices are needed (see e.g. [31, 60]).

- *RW in dynamic RE with mutual interaction*: These are models in which the dynamics of the RE is locally affected by the evolution of the RW (recall that in Section 1.2 we dealt with situations where the RE is completely independent of the RW). Under certain assumptions on the mutual interaction, LLNs, CLTs and LDPs have been obtained for such models in [11, 16, 19–21, 56].

- *RW on random graphs*: Several papers in the literature have been focusing on the asymptotic properties of RWs that evolves on a realization of a random graph. Two main classes concern random subgraphs of $\mathbb{Z}^d$ like percolation clusters (see e.g. [9, 10, 68, 90]), and random trees like Galton-Watson branching processes (see e.g. [37, 69]).

- *Random conductance model*: In these models, with each bond (x, y) of the integer lattice $\mathbb{Z}^d$ is associated a random variable $C_{x,y} \geq 0$ representing a conductance, with $C = \{C_{x,y}\}_{x,y\in\mathbb{Z}^d}$ i.i.d. Given a realization of C, the aim is to study the behavior of the RW whose transition probabilities from site x to site y are given by $C_{x,y}/\sum_{z:|z-x|=1} C_{x,z}$. Such a model is closely related to RWs on supercritical percolation clusters. Annealed and quenched CLTs for this RW were derived in [15, 62, 77].

- *Diffusion with random potential*: These models represent the natural analogue of RWRE in the theory of diffusion processes. Informally speaking, the idea is to find a 'solution' to the stochastic differential equation $\mathrm{d}X_t = -\frac{1}{2}\nabla V(X_t)\mathrm{d}t + \mathrm{d}W_t$, $X_0 = 0$, where the function $V = F + B$ is a sum of a deterministic function $F : \mathbb{R}^d \to \mathbb{R}$ plus a random field B indexed by $\mathbb{R}^d$, $B = (B(x))_{x\in\mathbb{R}^d}$, and W is a d-dimensional Brownian motion independent of V. For results on this topic we refer the reader to [32, 67, 91] and the references therein.