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Stellingen

behorende bij het proefschrift

Profinite groups with a rational probabilistic zeta function

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In all propositions, G denotes a finitely generated profinite group.
We define the Dirichlet series

$$P_G(s) := \sum_{n\in\mathbb{N}} \frac{a_n(G)}{n^s}, \text{ with } a_n(G) := \sum_{|G:H|=n} \mu_G(H)$$

where μ_G is the Möbius function defined on the lattice of open subgroups of G recursively by $\mu_G(G) = 1$ and $\mu_G(H) = -\sum_{H<K\leq G} \mu_G(K)$ if $H < G$.

1. Let p be a fixed prime. Assume that almost every nonabelian composition factor of G is a simple group of Lie type over a field of characteristic p. If $P_G(s)$ is rational then $G/\mathrm{Frat}(G)$ is finite. [Theorem A, Chapter 3]

2. Assume that almost every nonabelian composition factor of G is isomorphic to $\mathrm{PSL}(2,p)$ for some prime $p \geq 5$. If $P_G(s)$ is rational then $G/\mathrm{Frat}(G)$ is finite. [Theorem B, Chapter 4]

3. Assume that almost every nonabelian composition factor of G is a sporadic simple group. If $P_G(s)$ is rational then $G/\mathrm{Frat}(G)$ is finite. [Theorem C, Chapter 5]

4. Assume that almost every nonabelian composition factor of G is isomorphic to either $\mathrm{PSL}(2,p)$ for some prime $p \geq 5$, or a sporadic simple group, or an alternating group $\mathrm{Alt}(n)$ with n either a prime or a 2-power. If $P_G(s)$ is rational then $G/\mathrm{Frat}(G)$ is finite. [Theorem D, Chapter 6]

5. Assume that G has linear subgroup growth, i.e., there exists a non-negative real number c such that for all n, the number of index n subgroups in G is at most cn. If $P_G(s)$ is rational then $G/\mathrm{Frat}(G)$ is finite.

The following two propositions are concerned with the convergence of the series $P_G(s)$. Let $\Lambda(G)$ be the set of finite monolithic groups L such that $\mathrm{soc}(L) = S_1 \times \cdots \times S_r$ is nonabelian and L is an epimorphic image of G. For each $L \in \Lambda(G)$, let X_L be the subgroup of $\mathrm{Aut}(S_1)$ induced by the conjugation action of $N_L(S_1)$ on S_1. Then X_L is a finite almost simple group with socle S_1 (see Section 2, Chapter 2).
Fix two non-negative real numbers c_1 and c_2. A finite almost simple group X is said to be (c_1, c_2)-bounded if the following hold.

- $b_n^*(X) \leq n^{c_1}$ for all n, where $b_n^*(X)$ denotes the number of index n subgroups Y of X with $X = Y \cdot \mathrm{soc}(X)$;
- $|\mu_X(Y)| \leq |X : Y|^{c_2}$ for each $Y \leq X$ with $X = Y \cdot \mathrm{soc}(X)$.

It is known that if X_L is (c_1, c_2)-bounded for all $L \in \Lambda(G)$ then $P_G(s)$ converges absolutely on some half-plane (see [Luc11a]).

6. If X is a finite almost simple group with socle $\mathrm{PSL}(2,q)$, where $q \geq 4$ is a prime power, then X is $(2,1)$-bounded.

7. If almost every nonabelian composition factor of G is isomorphic to $\mathrm{PSL}(2,q)$, where $q \geq 4$ is a prime power, then $P_G(s)$ converges absolutely in some half-plane.

For each $n \in \mathbb{N}$, let $b_n(G)$ be the number of subgroups H of index n in G with $\mu_G(H) \neq 0$ and $c_n(G)$ the number of subgroups of index n that can be written as an intersection of maximal subgroups of G. Since each H with $\mu_G(H) \neq 0$ is an intersection of maximal subgroups (see [Hal36]), we have that $b_n(G) \leq c_n(G)$ for each $n \in \mathbb{N}$. It is conjectured that there exists a non-negative real number α such that for all n, one has $b_n(G) \leq n^{\alpha}$.

8. If G is pro-supersoluble then there exists a non-negative real number β such that for all n, one has $c_n(G) \leq n^{\beta}$.

9. Not every direct product of finite simple groups is the profinite completion of an abstract group.