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Chapter 5

Sporadic simple groups

This chapter is devoted for a proof of **Theorem C** as the following.

Theorem C. *If G is a finitely generated profinite group such that almost every nonabelian composition factor is isomorphic to a sporadic simple group and $P_G(s)$ is rational, then $G/\mathrm{Frat}(G)$ is a finite group.*

The list of sporadic simple groups with their orders and automorphism groups is described in Table 5.1.

Note that if X is an almost simple group whose socle S is a sporadic group, then either $X = S$ or $X = S \cdot 2$. If $X = S$ then $P_{X,S}(s) = P_S(s)$. If $X = S \cdot 2$ then $X/S \cong C_2$, and so

$$P_X(s) = P_{X,S}(s)P_{X/S}(s) = P_{X,S}(s)\left(1 - \frac{1}{2^s}\right)$$

which implies that $P^{(2)}_{X,S}(s) = P^{(2)}_X(s)$. This helps us analyse $P^{(2)}_{X,S}(s)$ easily from the knowledge of X. Therefore, we will focus only on odd indices of subgroups of X. Let Γ_X be the set of useful subgroups of X of odd index, and let $m_X = \min\{|X : H| : H \in \Gamma_X\}$. Then m_X is the index of a maximal subgroup of X. In order to apply our machinery, we need to look for a prime divisor p_X of m_X such that if p_X divides the index $|X : H|$ of some subgroup $H \in \Gamma_X$, then $v_{p_X}(m_X) = v_{p_X}(|X : H|)$. Notice also that for any prime $p \in \pi(m_X)$, if there is some $H \in \Gamma_X$ such that $p \mid |X : H|$, then $m_X \leq |X : H|$. The list of m_X and all potential p_X for all sporadic groups is included in Table 5.3.

Proof of Theorem C. Since there are only 26 sporadic simple groups, by Step 1 in Section 2.6, the set $\pi(G)$ is finite. We have that

$$P_G(s) = \prod_{i \in J} P_i(s)$$

where J is the set of indices i with G_i/G_{i+1} non-Frattini. Let $\mathcal{T}$ be the set of almost simple groups X such that $\mathrm{soc}(X)$ is a sporadic simple group and there exist infinitely many $i \in J$ with $X_i \cong X$, and let $I = \{i \in J \mid X_i \in \mathcal{T}\}$. By Proposition 2.6.3, the set $J \setminus I$ is finite. We have to prove that J is finite; this is equivalent to showing that $I = \emptyset$. For any almost simple group X, let $\Omega(X)$ be the set of the odd integers $m \in \mathbb{N}$ such that:

- X contains at least one subgroup Y such that $X = Y \cdot \mathrm{soc}(X)$ and $|X : Y| = m$.
- if $X = Y \cdot \mathrm{soc}(X)$ and $|X : Y| = m$, then Y is a maximal subgroup of X.

Note that if $m \in \Omega(X)$, $X = Y \cdot \mathrm{soc}(X)$ and $|X : Y| = m$, then $\mu_X(Y) = -1$; in particular, $b_m(X) < 0$. This implies that if $m \in \Omega(X_i)$ then $b_{i,m^{r_i}} < 0$. Certainly $\Omega(X)$ is not empty and its smallest element is the smallest index m_X of a supplement of $\mathrm{soc}(X)$ in X (see Table 5.3). In a few cases, we need to know another integer n_X in $\Omega(X)$, given in Table 5.4.

Since $J \setminus I$ is finite and $P_G(s) = \prod_{i \in J} P_i(s)$ is rational, also $\prod_{i \in I} P_i(s)$ is rational. This implies from Lemma 1.2.5 that the product $Q(s) = \prod_{i \in I} P_i^{(2)}(s)$ is also rational. For a fixed prime p, let $\Lambda_p = \{i \in I \mid p \in \pi(P_i^{\{2\}}(s))\}$.

If $i \in \Lambda_{31}$, then 31 divides $|S_i|$ and $S_i \in \{\mathrm{J}_4, \mathrm{Ly}, \mathrm{O'N}, \mathrm{BM}, \mathrm{M}, \mathrm{Th}\}$. Moreover 31^2 does not divide $|S_i|$ so if n is odd, divisible by 31 and $b_{i,n} \neq 0$ then $n = x^{r_i}$ and $v_{31}(x) = 1$. Let $m_i = n_{X_i}$ if $S_i \cong \mathrm{Th}$, and $m_i = m_{X_i}$ otherwise. Since m_i is the smallest odd number divisible by 31 and equal to the index in X_i of a supplement of S_i we get:

$$\begin{aligned} w &= \min\{x \in \mathbb{N} \mid x \text{ is odd }, v_{31}(x) = 1 \text{ and } b_{i,x^{r_i}} \neq 0 \text{ for some } i \in I\} \\ &= \min\{x \in \mathbb{N} \mid x \text{ is odd }, v_{31}(x) = 1 \text{ and } b_{i,x^{r_i}} \neq 0 \text{ for some } i \in \Lambda_{31}\} \\ &= \min\{m_i \mid i \in \Lambda_{31}\}. \end{aligned}$$

But then by Proposition 2.4.3, the following Dirichlet series is rational:

$$\prod_{i \in \Lambda_{31}} \left(1 + \frac{b_{i,w^{r_i}}}{(w^{r_i})^s}\right).$$

We have $b_{i,w^{r_i}} < 0$ if $m_i = w$, $b_{i,w^{r_i}} = 0$ otherwise. By applying Corollary 2.5.5, we get that $\{i \in \Lambda_{31} \mid m_i = w\}$ is a finite set, and this implies $\Lambda_{31} = \emptyset$.

Now consider Λ_{23}. Since $\Lambda_{31} = \emptyset$, if $i \in \Lambda_{23}$ then $S_i \in \{\mathrm{M}_{23}, \mathrm{M}_{24}, \mathrm{Co}_1, \mathrm{Co}_2, \mathrm{Co}_3, \mathrm{Fi}_{23}, \mathrm{Fi}_{24}{}'\}$. We can repeat the argument used to proved that $\Lambda_{31} = \emptyset$. Let $m_i = n_{X_i}$ if $S_i \cong \mathrm{Co}_1$, $m_i = m_{X_i}$ otherwise and let $w = \min\{m_i \mid i \in \Lambda_{23}\}$. By applying Corollary 2.5.5, we get that $\{i \in \Lambda_{23} \mid m_i = w\}$ is a finite set, and this implies $\Lambda_{23} = \emptyset$.

New we consider Λ_{11}. Since $\Lambda_{31} \cup \Lambda_{23} = \emptyset$, if $i \in \Lambda_{11}$, then $S_i \in \{\mathrm{M}_{11}, \mathrm{M}_{12}, \mathrm{M}_{22}, \mathrm{J}_1,$ HS, Suz, McL, HN, $\mathrm{Fi}_{22}\}$. Let $m_i = n_{X_i}$ if $S_i \cong \mathrm{Fi}_{22}$ or $S_i \cong \mathrm{Fi}_{24}'$, $m_i = m_{X_i}$ otherwise and let $w = \min\{m_i \mid i \in \Lambda_{11}\}$. As before, by applying Corollary 2.5.5, we get that $\{i \in \Lambda_{23} \mid m_i = w\}$ is a finite set, and this implies $\Lambda_{11} = \emptyset$. Continuing our procedure, we consider Λ_{17}: if $i \in \Lambda_{17}$, then $S_i \in \{\mathrm{J}_3, \mathrm{He}\}$ and we can take $w = \min\{m_{X_i} \mid i \in \Lambda_{17}\}$ and deduce that $\Lambda_{17} = \emptyset$. Next we take $w = m_X$ with $\mathrm{soc}(X) = \mathrm{Ru}$ to prove $\Lambda_{29} = \emptyset$ and finally we take $w = m_X$ with $\mathrm{soc}(X) = \mathrm{J}_2$ to prove $\Lambda_7 = \emptyset$. In conclusion, the set

$$I = \Lambda_{31} \cup \Lambda_{23} \cup \Lambda_{11} \cup \Lambda_{29} \cup \Lambda_{17} \cup \Lambda_7$$

is empty, the Theorem follows. □

Tables

Table 5.1: Sporadic simple groups

S	$\lvert\mathrm{Out}(S)\rvert$	$\lvert S\rvert$
M_{11}	1	$2^4 \cdot 3^2 \cdot 5 \cdot 11$
M_{12}	2	$2^6 \cdot 3^3 \cdot 5 \cdot 11$
M_{22}	2	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
M_{23}	1	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
M_{24}	1	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
J_1	1	$2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 19$
J_2	2	$2^7 \cdot 3^3 \cdot 5^2 \cdot 7$
J_3	2	$2^7 \cdot 3^5 \cdot 5 \cdot 17 \cdot 19$
J_4	1	$2^{21} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11^3 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$
HS	2	$2^9 \cdot 3^2 \cdot 5^3 \cdot 7 \cdot 11$
Suz	2	$2^{13} \cdot 3^7 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
McL	2	$2^7 \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11$
Ru	1	$2^{14} \cdot 3^3 \cdot 5^3 \cdot 7 \cdot 13 \cdot 29$

He	2	$2^{10} \cdot 3^3 \cdot 5^2 \cdot 7^3 \cdot 17$
Ly	1	$2^8 \cdot 3^7 \cdot 5^6 \cdot 7 \cdot 11 \cdot 31 \cdot 37 \cdot 67$
O'N	2	$2^9 \cdot 3^4 \cdot 5 \cdot 7^3 \cdot 11 \cdot 19 \cdot 31$
Co_1	1	$2^{21} \cdot 3^9 \cdot 5^4 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$
Co_2	1	$2^{18} \cdot 3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Co_3	1	$2^{10} \cdot 3^7 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Fi_{22}	2	$2^{17} \cdot 3^9 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13$
Fi_{23}	1	$2^{18} \cdot 3^{13} \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
Fi'_{24}	2	$2^{21} \cdot 3^{16} \cdot 5^2 \cdot 7^3 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
HN	2	$2^{14} \cdot 3^6 \cdot 5^6 \cdot 7 \cdot 11 \cdot 19$
Th	1	$2^{15} \cdot 3^{10} \cdot 5^3 \cdot 7^2 \cdot 13 \cdot 19 \cdot 31$
BM	1	$2^{41} \cdot 3^{13} \cdot 5^6 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 47$
M	1	$2^{46} \cdot 3^{20} \cdot 5^9 \cdot 7^6 \cdot 11^2 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 59 \cdot 71$

Table 5.2: Maximal subgroups of odd index of sporadic groups and their automorphism groups

Group	Maximal subgroup	Index
M_{11}	$M_{10} \cong A_6 \cdot 2$	11
	$M_9 : 2 \cong 3^2 : Q_8 \cdot 2$	$5 \cdot 11$
	$M_8 : S_3 \cong 2 \cdot S_4$	$3 \cdot 5 \cdot 11$
M_{12}	$M_8 \cdot S_4 \cong 2^{1+4} \cdot S_3$	$3^2 \cdot 5 \cdot 11$
	$4^2 : D_{12}$	$3^2 \cdot 5 \cdot 11$
$M_{12} \cdot 2$	$2^{1+4} \cdot S_3 : 2$	$3^2 \cdot 5 \cdot 11$
	$4^2 : D_{12} \cdot 2$	$3^2 \cdot 5 \cdot 11$
M_{22}	$2^4 : A_6$	$7 \cdot 11$
	$2^4 : S_5$	$3 \cdot 7 \cdot 11$
$M_{22} \cdot 2$	$2^4 : A_6 : 2$	$7 \cdot 11$
	$2^5 : S_5$	$3 \cdot 7 \cdot 11$
M_{23}	M_{22}	23
	$PSL(3,4) : 2$	$11 \cdot 23$
	$2^4 : A_7$	$11 \cdot 23$
	$2^4 : (3 \times A_5) : 2$	$7 \cdot 11 \cdot 23$
M_{24}	$2^4 \cdot A_8$	$3 \cdot 11 \cdot 23$
	$(2^6 \cdot 3) \cdot S_6$	$7 \cdot 11 \cdot 23$
	$2^6 \cdot (S_3 \times PSL(2,7))$	$3 \cdot 5 \cdot 11 \cdot 23$
J_1	$2^3 : 7 : 3$	$5 \cdot 11 \cdot 19$
	$2 \times A_5$	$7 \cdot 11 \cdot 19$

J_2	$2^{1+4} : A_5$ $2^{2+4} : (3 \times S_3)$	$3^2 \cdot 5 \cdot 7$ $3 \cdot 5^2 \cdot 7$
$J_2 \cdot 2$	$2^{1+4} \cdot A_5 \cdot 2$ $2^{2+4} : (3 \times S_3) \cdot 2$	$3^2 \cdot 5 \cdot 7$ $3 \cdot 5^2 \cdot 7$
J_3	$2^{1+4} : A_5$ $2^{2+4} : (3 \times S_3)$	$3^4 \cdot 17 \cdot 19$ $3^3 \cdot 5 \cdot 17 \cdot 19$
$J_3 \cdot 2$	$2^{1+4} \cdot S_5$ $2^{2+4} : (S_3 \times S_3)$	$3^4 \cdot 17 \cdot 19$ $3^3 \cdot 5 \cdot 17 \cdot 19$
J_4	$2^{11} : M_{24}$ $2^{1+12} \cdot 3M_{22} : 2$ $2^{3+12} \cdot (S_5 \times PSL(3,2))$	$11^2 \cdot 29 \cdot 31 \cdot 37 \cdot 43$ $11^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$ $3 \cdot 11^2 \cdot 23 \cdot 29 \cdot 31 \cdot 37 \cdot 43$
Co_1	$2^{11} : M_{24}$ $2^{1+8} \cdot O_8^+(2)$ $2^{2+12} : (A_8 \times S_3)$ $2^{4+12} \cdot (S_3 \times 3S_6)$	$3^6 \cdot 5^3 \cdot 7 \cdot 13$ $3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 23$ $3^6 \cdot 5^3 \cdot 7 \cdot 11 \cdot 13 \cdot 23$ $3^5 \cdot 5^3 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23$
Co_2	$2^{10} : M_{22} : 2$ $2^{1+8} : S_6(2)$ $2^{4+10} \cdot (S_5 \times S_3)$	$3^4 \cdot 5^2 \cdot 23$ $3^2 \cdot 5^2 \cdot 11 \cdot 23$ $3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 23$
Co_3	$2 \cdot S_6(2)$ $2^4 \cdot A_8$ $2^2 \cdot [2^7 \cdot 3^2] \cdot S_3$	$3^3 \cdot 5^2 \cdot 11 \cdot 23$ $3^5 \cdot 5^2 \cdot 11 \cdot 23$ $3^4 \cdot 5^3 \cdot 7 \cdot 11 \cdot 23$
Ly	$3 \cdot \mathrm{McL} : 2$ $2 \cdot A_{11}$	$5^3 \cdot 31 \cdot 37 \cdot 67$ $3^3 \cdot 5^4 \cdot 31 \cdot 37 \cdot 67$
Th	$2^5 \cdot PSL(5,2)$ $2^{1+8} \cdot A_9$	$3^8 \cdot 5^2 \cdot 7 \cdot 13 \cdot 19$ $3^6 \cdot 5^2 \cdot 7 \cdot 13 \cdot 19 \cdot 31$
Suz	$2^{1+6} \cdot U_4(2)$ $3^5 : M_{11}$ $2^{4+6} : 3A_6$ $2^{2+8} : (A_5 \times S_3)$	$3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ $3^9 \cdot 5 \cdot 7 \cdot 13$ $3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ $3^5 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
$\mathrm{Suz} \cdot 2$	$2^{1+6} \cdot U_4(2) \cdot 2$ $3^5(M_{11} \times 2)$ $2^{4+6} : 3S_6$ $2^{2+8} : (S_5 \times S_3)$	$3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ $3^9 \cdot 5 \cdot 7 \cdot 13$ $3^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ $3^5 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
Fi_{22}	$2^{10} : M_{12}$ $(2 \times 2^{1+8} \cdot U_4(2)) : 2$ $2^{5+8} : (S_3 \times A_6)$	$3^7 \cdot 5 \cdot 13$ $3^5 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ $3^6 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
$\mathrm{Fi}_{22} \cdot 2$	$2^{10} : M_{12} : 2$ $(2 \times 2^{1+8} \cdot U_4(2) : 2) : 2$ $2^{5+8} : (S_3 \times S_6)$	$3^7 \cdot 5 \cdot 13$ $3^5 \cdot 5 \cdot 7 \cdot 11 \cdot 13$ $3^6 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
	$2 \cdot \mathrm{Fi}_{22}$	$3^4 \cdot 17 \cdot 23$

Fi_{23}

	$2^2 \cdot U_6(2) \cdot 2$ $2^{11} \cdot M_{23}$ $(2^2 \times 2^{1+8}) \cdot (3 \times U_4(2)) \cdot 2$ $2^{6+8} : (A_7 \times S_3)$	$3^7 \cdot 5 \cdot 13 \cdot 17 \cdot 23$ $3^{11} \cdot 5 \cdot 13 \cdot 17$ $3^8 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17 \cdot 23$ $3^{10} \cdot 5 \cdot 11 \cdot 13 \cdot 17 \cdot 23$
Fi'_{24}	$2^{11} \cdot M_{24}$ $2^{1+12} \cdot 3U_4(2) \cdot 2$ $2^{3+12} \cdot (PSL(3,2) \times A_6)$ $2^{6+8} \cdot (S_3 \times A_8)$	$3^{13} \cdot 5 \cdot 7^2 \cdot 13 \cdot 17 \cdot 23 \cdot 29$ $3^9 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$ $3^{13} \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$ $3^{13} \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
$\mathrm{Fi}'_{24} \cdot 2$	$2^{12} \cdot M_{24}$ $2^{1+12} \cdot 3U_4(2) \cdot 2^2$ $2^{3+12} \cdot (PSL(3,2) \times S_6)$ $2^{7+8} \cdot (S_3 \times A_8)$	$3^{13} \cdot 5 \cdot 7^2 \cdot 13 \cdot 17 \cdot 23 \cdot 29$ $3^9 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$ $3^{13} \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$ $3^{13} \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
O'N	$4 \cdot PSL(3,4) : 2$ $4^3 \cdot PSL(2,31)$	$3^2 \cdot 7^2 \cdot 11 \cdot 19 \cdot 31$ $3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 19 \cdot 31$
O'N.2	$4 \cdot PSL(3,4) : 2^2$ $4^3 \cdot (PSL(2,31) \times 2)$	$3^2 \cdot 7^2 \cdot 11 \cdot 19 \cdot 31$ $3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 19 \cdot 31$
Ru	$2^{3+8} : PSL(3,2)$ $2 \cdot 2^{4+6} : S_5$	$3^2 \cdot 5^3 \cdot 13 \cdot 29$ $3^2 \cdot 5^2 \cdot 7 \cdot 13 \cdot 29$
He	$2^6 : 3 \cdot S_6$ $2^{1+6} \cdot PSL(3,2)$	$5 \cdot 7^3 \cdot 17$ $3^2 \cdot 5^2 \cdot 7^2 \cdot 17$
$\mathrm{He} \cdot 2$	$2^{4+4} \cdot (S_3 \times S_3) \cdot 2$ $2^{1+6} \cdot PSL(3,2) \cdot 2$	$3 \cdot 5^2 \cdot 7^3 \cdot 17$ $3^2 \cdot 5^2 \cdot 7^2 \cdot 17$
HS	$4^3 : PSL(3,2)$ $4 \cdot 2^4 : S_5$	$3 \cdot 5^3 \cdot 11$ $3 \cdot 5^2 \cdot 7 \cdot 11$
$\mathrm{HS} \cdot 2$	$4^3 : (PSL(3,2) \times 2)$ $2^{1+6} : S_5$	$3 \cdot 5^3 \cdot 11$ $3 \cdot 5^2 \cdot 7 \cdot 11$
McL	$U_4(3)$ M_{22} $PSL(3,4) : 2$ $2 \cdot A_8$ $2^4 : A_7$	$5^2 \cdot 11$ $3^4 \cdot 5^2$ $3^4 \cdot 5^2 \cdot 11$ $3^4 \cdot 5^2 \cdot 11$ $3^4 \cdot 5^2 \cdot 11$
$\mathrm{McL} \cdot 2$	$U_4(3) : 2$ $PSL(3,4) : 2^2$ $2 \cdot S_8$ $2^{2+4} : (S_3 \times S_3)$	$5^2 \cdot 11$ $3^4 \cdot 5^2 \cdot 11$ $3^4 \cdot 5^2 \cdot 11$ $3^4 \cdot 5^3 \cdot 7 \cdot 11$
HN	$2^{1+8} \cdot (A_5 \times A_5) \cdot 2$ $2^3 \cdot 2^2 \cdot 2^6 \cdot (3 \times PSL(3,2))$	$3^4 \cdot 5^4 \cdot 7 \cdot 11 \cdot 19$ $3^4 \cdot 5^6 \cdot 11 \cdot 19$
$\mathrm{HN} \cdot 2$	$2^{1+8} \cdot (A_5 \times A_5) \cdot 2^2$ $2^3 \cdot 2^2 \cdot 2^6 \cdot (3 \times PSL(3,2)) \cdot 2$	$3^4 \cdot 5^4 \cdot 7 \cdot 11 \cdot 19$ $3^4 \cdot 5^6 \cdot 11 \cdot 19$
BM	$2^{1+22} \cdot \mathrm{Co}_2$ $2^{9+16} \cdot S_8(2)$	$3^7 \cdot 5^3 \cdot 7 \cdot 13 \cdot 19 \cdot 31 \cdot 47$ $3^8 \cdot 5^4 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 23 \cdot 31 \cdot 47$

	$2^{2+10+20}:(M_{22:2\times S_3})$ $2^{3+(1\cdot 4)+1\cdot 2+(12\cdot 3)}\cdot(S_5\times PSL(3,2))$	$3^{10}\cdot 5^5\cdot 7\cdot 13\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47$ $3^{11}\cdot 5^5\cdot 7\cdot 11\cdot 13\cdot 17\cdot 19\cdot 23\cdot 31\cdot 47$
M	$2^{1+24}\cdot \mathrm{Co}_1$	$3^{11}\cdot 5^5\cdot 7^4\cdot 11\cdot 13^2\cdot 17\cdot 19\cdot 29$ $\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71$
	$2^{10+16}\cdot O_{10}^{+}(2)$	$3^{15}\cdot 5^7\cdot 7^5\cdot 11^2\cdot 13^2\cdot 13^3\cdot 19\cdot 23$ $\cdot 29\cdot 41\cdot 47\cdot 59\cdot 71$
	$2^{2+11+22}\cdot(M_{24}\times S_3)$	$3^{16}\cdot 5^8\cdot 7^5\cdot 11\cdot 13^2\cdot 17\cdot 19\cdot 29$ $\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71$
	$2^{5+10+20}\cdot(S_3\times PSL(5,2))$	$3^{17}\cdot 5^8\cdot 7^5\cdot 11^2\cdot 13^3\cdot 17\cdot 19\cdot 23$ $\cdot 29\cdot 41\cdot 47\cdot 59\cdot 71$
	$2^{3+6+12+18}(PSL(3,2)\times 3S_6)$	$3^{17}\cdot 5^8\cdot 7^5\cdot 11^2\cdot 13^3\cdot 17\cdot 19\cdot 23$ $\cdot 29\cdot 31\cdot 41\cdot 47\cdot 59\cdot 71$

Table 5.3: Odd useful indices and potential primes

X	m_X	p_X
M_{11}	11	11
M_{12}	$3^2\cdot 5\cdot 11$	$5,11$
$M_{12}\cdot 2$	$3^2\cdot 5\cdot 11$	$5,11$
M_{22}	$7\cdot 11$	$7,11$
$M_{22}\cdot 2$	$7\cdot 11$	$7,11$
M_{23}	23	23
M_{24}	$3\cdot 11\cdot 23$	$11,23$
J_1	$5\cdot 11\cdot 19$	$5,11,19$
J_2	$3^2\cdot 5\cdot 7$	7
$J_2\cdot 2$	$3^2\cdot 5\cdot 7$	7
J_3	$3^4\cdot 17\cdot 19$	$17,19$
$J_3\cdot 2$	$3^4\cdot 17\cdot 19$	$17,19$
J_4	$11^2\cdot 29\cdot 31\cdot 37\cdot 43$	$29,31,37,43$
HS	$3\cdot 5^3\cdot 11$	11
HS $\cdot 2$	$3\cdot 5^3\cdot 11$	11
Suz	$3^3\cdot 5\cdot 7\cdot 11\cdot 13$	$7,11,13$
Suz $\cdot 2$	$3^3\cdot 5\cdot 7\cdot 11\cdot 13$	$7,11,13$
McL	$5^2\cdot 11$	11
McL $\cdot 2$	$5^2\cdot 11$	11
Ru	$3^2\cdot 5^3\cdot 13\cdot 29$	$13,29$
He	$5\cdot 7^3\cdot 17$	17
He $\cdot 2$	$3^2\cdot 5^2\cdot 7^2\cdot 17$	17
Ly	$5^3\cdot 31\cdot 37\cdot 67$	$31,37,67$
O'N	$3^2\cdot 7^2\cdot 11\cdot 19\cdot 31$	$11,19,31$

$\mathrm{O'N} \cdot 2$	$3^2 \cdot 7^2 \cdot 11 \cdot 19 \cdot 31$	$11, 19, 31$
Co_1	$3^6 \cdot 5^3 \cdot 7 \cdot 13$	13
Co_2	$3^4 \cdot 5^2 \cdot 23$	23
Co_3	$3^3 \cdot 5^2 \cdot 11 \cdot 23$	$11, 23$
Fi_{22}	$3^7 \cdot 5 \cdot 13$	13
$\mathrm{Fi}_{22} \cdot 2$	$3^7 \cdot 5 \cdot 13$	13
Fi_{23}	$3^4 \cdot 17 \cdot 23$	$17, 23$
Fi'_{24}	$3^{13} \cdot 5 \cdot 7^2 \cdot 13 \cdot 17 \cdot 29$	$11, 13, 17, 23, 29$
$\mathrm{Fi}'_{24} \cdot 2$	$3^{13} \cdot 5 \cdot 7^2 \cdot 13 \cdot 17 \cdot 29$	$11, 13, 17, 23, 29$
HN	$3^4 \cdot 5^4 \cdot 7 \cdot 11 \cdot 19$	$7, 11, 19$
$\mathrm{HN} \cdot 2$	$3^4 \cdot 5^4 \cdot 7 \cdot 11 \cdot 19$	$7, 11, 19$
Th	$3^8 \cdot 5^2 \cdot 7 \cdot 13 \cdot 19$	$13, 19$
BM	$3^7 \cdot 5^3 \cdot 7 \cdot 13 \cdot 19 \cdot 31 \cdot 47$	$13, 19, 31, 47$
M	$3^{11} \cdot 5^5 \cdot 7^4 \cdot 11 \cdot 13^2 \cdot 17 \cdot 19 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$	$17, 19, 29, 31, 41, 47, 59, 71$

Table 5.4: n_X

X	$n(X)$
Co_1	$3^4 \cdot 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 23$
Fi_{22}	$3^5 \cdot 5 \cdot 7 \cdot 11 \cdot 13$
Fi'_{24}	$3^9 \cdot 5 \cdot 11 \cdot 7^2 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
$\mathrm{Aut}(\mathrm{Fi}'_{24})$	$3^9 \cdot 5 \cdot 11 \cdot 7^2 \cdot 13 \cdot 17 \cdot 23 \cdot 29$
Th	$3^8 \cdot 5^2 \cdot 7 \cdot 13 \cdot 19 \cdot 31$