Conformal invariance and microscopic sensitivity in cosmic inflation

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Conformal universe

Our studies into the nature of supergravity inflation and worldsheet inflation point towards an intrinsic difficulty in describing microscopic theories for inflation. In both cases, the sensitivity of inflation to the underlying details of the theory demands a level of understanding that is presently unobtainable and may remain unobtainable for the (near) future. For this reason, we are led to consider different approaches to understand the structure of inflation. Conventionally we can rely on the symmetries of the theory to better understand its behavior. For inflation, which is a quasi-de Sitter evolution, the late-time geometry has a conformal isometry group. Hence, the restrictions and structure of conformal field theory are expected to be imprinted in the late-time phenomena of the theory.

With this observation in mind, string theory provides a new approach in the manner in which these symmetry considerations can be written down. The techniques in the context of holography make full use of all of our knowledge of microscopic physics. As we have seen in section 3.2.3, in the case of the known holographic realization of anti-de Sitter spacetimes, the gravity theory can be understood from a field theory perspective, interchanging strongly coupled and weakly coupled regimes. Hopefully, knowing that anti-de Sitter and de Sitter spacetimes are closely related mathematically, a holographic study of (quasi-)de Sitter spacetimes yields similar results, although the subject still needs to be shaped and molded before its full power can be used.

In this chapter we present a small but important step towards a better understanding of a cosmological holographic duality, at the level of the constraining symmetries of inflation. We study the correlation functions of primordial curvature perturbations generated during inflation, specifically the power spectrum and the bispectrum, from a purely conformal point of view. At a technical level, many techniques that are developed in chapter 3 and put to use in chapter 5 will again prove to be useful, because again, conformal invariance and the deviation away from pure conformality are central to the analysis. This chapter can therefore be seen as yet another example displaying the breadth of the applicability of conformal invariance within physics. The chapter is based on [253].

6.1 Introduction

Phenomenologically, the inflationary paradigm provides a satisfying explanation for the initial value problems of the standard big bang model. Over the last decades, we have gathered increasing evidence for the existence of an epoch of primordial inflation, most importantly through the appearance of acoustic oscillations in the temperature anisotropies of the cosmic microwave background radiation [10]. Via a careful study of the relation between theory and observation, inflation enables us to open a new window towards the study of the structure of our universe at very high energies. As explained in chapter 2, the most accurate mapping we possess between theory and observation is that of the n-point functions of curvature perturbations. The observation of primordial gravitational waves, of features in the power spectrum of primordial density fluctuations [51, 52] or the observation of any type of non-Gaussianities [45, 46, 54] would all pave the way for a leap in our understanding of the primordial phase of the universe. It is for this reason that there is much research devoted to the structure of the two- and three-point functions of primordial density perturbations. A true understanding of the structure of the power spectrum and bispectrum may be a direct probe of new physics, once the required sensitivity is obtained observationally. To satisfy this need, different theoretical techniques have been developed in the literature for calculating the three-point function [57, 59, 62, 76, 254].

Direct calculation of these correlation functions, however, can be rather involved [56, 57, 61, 62], as the organization imposed by the slow-roll expansion does not necessarily ensure that the expressions remain tractable at intermediate steps. As such, the underlying structure behind the final result is obscured. It would certainly be welcoming to have alternative ways to derive these non-Gaussian correlation functions which emphasize strongly the symmetries of inflation. In particular, slow-roll inflation is a quasi-de Sitter expansion and as such, it is expected that the correlation functions of inflationary curvature perturbations inherit constraints from the (remnants of the) isometry group of the de Sitter phase. At late times, the isometry group of the de Sitter phase asymptotically reduces to three-dimensional Euclidian conformal symmetry, which suggests that the late-time correlation functions generated during inflation are naturally constrained by this (broken) conformal symmetry [57, 255–257]. In this chapter, we investigate the constraints of late-time de Sitter symmetry and the effect of its breaking on the bispectrum of primordial density fluctuations in single field slow-roll inflation.

Many of the techniques we use, have come of age in the context of holography and as such, our presentation and analysis have a distinct holographic flavor. Holographic duality between gravity theories and gauge theories [129, 130] is arguably the most profound and deep achievement of string theory in the last fifteen years, with the realization of the AdS/CFT-correspondence [90, 136, 147, 148]. The duality enables us to understand a physical system from a different perspective, thereby emphasizing aspects that had gone unnoticed before in the original description. For this reason, holography can provide fundamental new insights into the structure of the phenomena. When applied to critical phenomena in condensed matter systems, a holographic understanding already seems to bear fruit [258–262].

Given the close relationship to anti-de Sitter spaces, our cosmic evolution might also be described by some conformal field theory. Indeed, after the proposed AdS/CFT-correspondence, the related dS/CFT-correspondence was quickly formulated [144–146] and further investigated in the context of inflation [263–265]. However, no concrete proposal for a dS/CFT-correspondence exists and there are fundamental objections against a dS/CFT-correspondence [266–268]. Taking this into consideration, we emphasize that our viewpoint is more modest, and depends only on the *symmetries*. In our considerations, the late-time de Sitter symmetry will lead the way to a different perspective, in terms of terminology *inspired* by (A)dS/CFT [57, 263, 265]. Ultimately it is the symmetries, or the approximate lack thereof, of the late-time behavior of the observed perturbations that constrain the form of the n-point correlation functions.

In [264, 265] it was shown that the nearly scale invariant power spectrum of curvature density perturbations can be fully understood from the constrained form of two-point functions in a conformal field theory. This means that the universal behavior of the inflationary power spectrum can be explained as a critical phenomenon, suggesting that there should not be any finetuning problems. The main motivation for this chapter is to study to what extent this can be generalized to the three-point function. Since slow-roll inflation is a quasi-de Sitter evolution, the exact conformal symmetry is broken. This is understood holographically in terms of a renormalization group flow, which has been extensively studied in the context of AdS/CFT [138–142] as explained in chapter 3. The underlying symmetry imposes Ward identities on the correlation function that restrict the form of the stress-energy tensor, i.e. the holographic dual of the curvature perturbations, in terms of the correlation functions of the nearly marginal operator driving the renormalization group flow. The final result should then be obtained by finding the solution to the renormalization group equations.

The scale invariance of the power spectrum and the conformal symmetries of a de Sitter spacetime are a striking feature of the inflationary epoch. It is therefore not surprising that in recent literature active investigations are undertaken to understand the structure of the power spectrum, bispectrum and trispectrum in terms of conformal symmetries, for both scalar as well as tensor perturbations [57, 255–257, 269–278]. These studies recognize that a pure de Sitter phase is the zeroth order result in a slow-roll inflation calculation and hence, the observed correlation functions are constrained by conformal symmetry to leading order [255–257, 269]. To further understand the connection with inflation, a departure from conformal symmetry is necessary, which can be studied through consistency relations between the n-point function and the squeezed limit of $n+1$-correlation functions [57, 270–274, 279, 280] or in terms of spontaneously broken symmetries [275–278]. We provide a supplementary view by studying the departure from conformal symmetry as a renormalization group flow.

Other studies employing the strengths of holographic renormalization to the inflationary bispectrum exist [254, 281–283]. The approach undertaken in [254] provides an alternative method for calculating the three-point correlation function, which provides a valuable consistency check and a clear insight in the dS/CFT-correspondence. The techniques from AdS/CFT used by [254] are to regulate divergences in a calculation that is in essence a bulk calculation. As such it is not clear to us how the three-point function that they obtain could be found from a conformal field theory. The purpose of our study is to supplement their analysis, fully from the perspective of a boundary conformal field theory.

The study of [281–283] is a far-reaching, technically advanced understanding of a proposed dS/CFT-correspondence. The authors apply the correspondence to a free conformal field theory, thereby calculating the bispectrum of a strongly coupled gravitational theory. Our investigations are concerned with ordinary slow-roll inflation, which is already a solution in *classical* general relativity. Hence, we are forced to consider an arbitrary (strongly coupled) conformal field theory. The reason we are still capable of considering interactions between operators is that the perturbations are dictated by the renormalization group flow and conformal symmetry alone, allowing us to circumvent any expected problems regarding the strong coupling of the field theory. On the other hand, as far as our analysis goes, symmetry may not completely specify the full structure of the bispectrum, whereas [281–283] find explicit predictions.

This chapter is organized as follows. In section 6.2 we review the relation between time evolution during inflation and the energy scale of the boundary field theory, i.e. holographic renormalization in the dS/CFT-correspondence. Then, in section 6.3 we relate the power spectrum and bispectrum of primordial density fluctuations to Ward identities between the trace of the field theory stress-energy tensor and the operator dual to the inflaton field. We will use this relation in 6.4 to investigate the structure of the inflationary two- and three-point functions, focussing on the consistency condition between them and on their behavior under a renormalization of the field theory. Technical aspects concerning the Ward identities and the Fourier transform of the three-point function are summarized in appendices.

6.2 Cosmology and the dS/CFT-correspondence

6.2.1 Renormalization group flow and cosmic evolution

The discovery of an explicit realization of the holographic principle [129, 130] in anti-de Sitter geometry [136] immediately sparked the question whether other spacetime geometries could be seen to have a holographic dual as well. The holographic principle itself does not rely on the precise structure of the spacetime geometry and it would be rather unsatisfactory if no other realizations could be found. Since anti-de Sitter geometry is mathematically very similar to de Sitter space, a natural candidate for an extension of the AdS/CFT-duality is de Sitter geometry [145]. A realization of the dS/CFT-correspondence would phenomenologically be very interesting, as our own universe is observed to currently resemble de Sitter geometry [19, 20]. In theory, we could therefore enlarge our understanding of our own spacetime geometry through holographic means, by borrowing results from the mathematically related and much better understood AdS/CFT-correspondence [144, 145, 284, 285].

Not only for the present de Sitter geometry would the existence of a dS/CFT-correspondence be very interesting, but also for the primordial inflationary epoch, which follows a quasi-de Sitter evolution [146, 263]. In this chapter we will continue our study of inflation from a holographic point of view, but we only consider holography at its minimum (necessary) level, viz. that of the symmetries between the theories. We will consider what structure the asymptotic de Sitter symmetries impose on the late-time two- and three-point correlation functions. Investigations of a correspondence between other correlation functions and thereby a first indication of a more complete dS/CFT-correspondence is left for future research.

The dS/CFT-correspondence predicts a relation between cosmic evolution of the de Sitter spacetime and scale invariance in the (boundary) field theory, similar to the relation in AdS/CFT between radial coordinates in the bulk and renormalization group flow on the boundary field theory [138–142]. This relation follows from one of the isometries of the pure de Sitter geometry,

$$ds^2 = -dt^2 + a(t)^2 d\boldsymbol{x}^2,$$

in flat spatial slicing with $a(t) = e^{Ht}$. It is invariant under the combined transformation [146]

$$t \to t + \Delta t, \qquad \boldsymbol{x} \to e^{-H\Delta t}\boldsymbol{x}. \tag{6.1}$$

The parameter $\mu = Ha$ related to this time translation has dimensions of energy and is therefore taken to be the typical energy scale of the boundary theory. With this, the cosmic evolution can be seen as a reversed renormalization group flow in the field theory [146]. One identifies primordial stages of the cosmic evolution with the IR fixed point of the field theory and to study the late-time behavior of the gravity theory, one can consider the field theory around the UV fixed point.

In the context of inflation, this observation suggests a natural description for the inflationary dynamics in terms of the renormalization group flow. Inflation occurred at early times in the cosmic evolution, right after the field theory IR fixed point. It is described by the inflaton scalar field $\phi(t, \boldsymbol{x})$ coupled gravitationally to a background FLRW spacetime that is spatially flat, in accordance with the Friedmann equations. The asymptotic value $\phi_0(\boldsymbol{x})$ of the inflaton scalar field $\phi(t, \boldsymbol{x})$, for $t \to \infty$, acts in the dual conformal field theory as the coupling $u = \phi_0$ to an operator O. As a consequence of this coupling, the conformal field theory S_0, which describes the asymptotic symmetry of pure de Sitter spacetime, is perturbed

$$S_u = S_0 + \int d^3\boldsymbol{x}\, uO. \tag{6.2}$$

When the operator is non-marginal, it will induce a renormalization group flow, which in the case of the cosmic evolution is reversed and ends asymptotically in the UV fixed point of the theory.

While one can consider the asymptotic behavior of inflation from the point of view of the field theory IR fixed point [263], from the UV fixed point [265] or from the bulk gravitational IR point of view [264], it is important to realize that inflation itself actually is an epoch *along* the renormalization group flow, cf. figure 6.1. The essence of slow-roll inflation is that at every point along the inflationary flow the spacetime can be approximated by a de Sitter phase. Typically, a particular de Sitter phase is chosen as the pivot point around which the slow-roll expansion is defined [28, 30]. Similarly, at any intermediate point along the renormalization group flow,

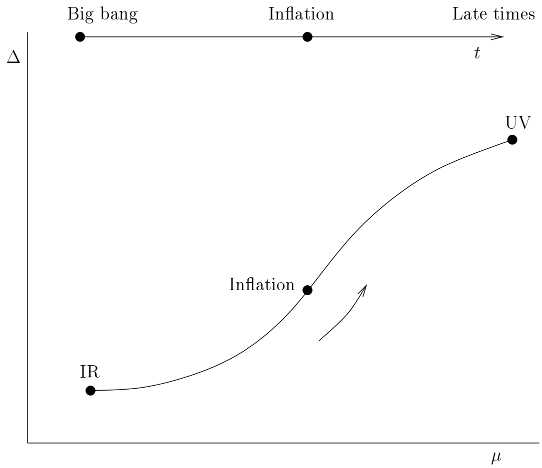

Figure 6.1: *The cosmic evolution can be seen as a reversed renormalization group flow, from the IR fixed point of the dual theory to the UV fixed point of the dual theory. Inflation occurs at a certain intermediate stage during the renormalization group flow. As is usual for inflation, a pivot point along the flow is chosen around which the slow-roll expansion can be studied. We observe the effects of inflation at late times, corresponding to the UV fixed point of the renormalization group flow.*

the dual description is approximately a conformal field theory itself, about which the effects of the flow can be expanded. When considering the correlation functions at late times, it is important to realize that the result has to be related to this intermediate renormalization group point, rather than the IR or UV fixed points.

6.2.2 Holographic slow-roll parameters

The close relation between the renormalization group flow induced by the non-marginal operator O and the inflationary solution for the bulk field $\phi(t, \boldsymbol{x})$ can be made technically more precise [263]. For a massive scalar field and taking $a(t) \sim e^{Ht}$ in the asymptotic limit $t \to \infty$, the equations of motion determine the asymptotic solution as $\phi = \phi_0(\boldsymbol{x}) e^{\lambda_\pm H t}$, where

$$\lambda_\pm = -\frac{3}{2} \pm \sqrt{\left(\frac{3}{2}\right)^2 - \frac{m^2}{H^2}}. \tag{6.3}$$

With the identification $\mu \propto e^{Ht}$, the invariance of the full asymptotic solution under the transformation (6.1) dictates ϕ_0 to transform as

$$\phi_0(\boldsymbol{x}) \to \phi_0(\boldsymbol{x}e^{-H\Delta t})e^{-\lambda_\pm H\Delta t} = \left(\frac{\mu}{\mu_0}\right)^{-\lambda_\pm} \phi_0\left(\frac{\mu_0}{\mu}\boldsymbol{x}\right).$$

Hence, identifying $\phi_0(\boldsymbol{x})$ and u, the operator O is seen to have scaling dimension $\Delta = 3 + \lambda_\pm$. In accordance with AdS/CFT reasoning, the λ_--solution vanishes at the boundary, i.e. at late times, and is therefore regarded as the vanishing mode. The $\lambda = \lambda_+$-solution defines a non-normalized mode, i.e. vanishing in the interior, which is sourced by the boundary field $\phi_0(\boldsymbol{x})$. Depending on the sign of λ, it describes a relevant ($\lambda < 0$) or irrelevant ($\lambda > 0$) perturbation from the field theory perspective. Via

$$\lambda = \frac{\partial \log \phi}{\partial \log a} = \frac{\partial \log u}{\partial \log \mu} = \frac{\beta}{u},$$

(6.3) is related to the β function $\beta = \frac{\partial u}{\partial \log \mu}$ of the operator O. We can define the non-marginal scaling dimension $\lambda(u)$ as

$$\lambda(u) = \frac{\partial \beta}{\partial u} + O(u), \tag{6.4}$$

in accordance with (3.17) [110]. In the limit $u \to 0$ that we consider, both definitions are equivalent and no ambiguity exists [263, 265].

In inflation, the pure de Sitter evolution is perturbed due to the varying inflaton scalar field $\phi(t, \boldsymbol{x})$. From the observation of the near scale invariant power spectrum, we know that the perturbation away from the de Sitter phase is only small, leading to a time dependent Hubble parameter $H(t)$ that is allowed to vary only slightly during the inflationary evolution. This is conveniently expressed by the requirement that the slow-roll parameters (2.5) are much smaller than unity. For $\epsilon = \eta = 0$, the evolution is that of a de Sitter spacetime.

From the field theory point of view, it will perturb away from the conformal fixed point S_0 to which the pure de Sitter phase corresponds, due to the non-marginal nature of O, i.e. $\lambda \neq 0$ to first order or, more precisely, its β function is non-vanishing, $\beta \neq 0$. It is therefore to be expected that λ and the β function of the operator O express the departure away from the pure de Sitter phase. Indeed the slow-roll parameters ϵ and η can be fully expressed in terms of the conformal field theory-data β and λ via

$$\beta = \frac{\partial u}{\partial \log \mu} = \frac{\partial \phi}{\partial \log a} = \frac{\dot{\phi}}{H} = -2\frac{H'}{H}, \tag{6.5}$$

where in the last step, the Friedmann equations are employed, which tell us that $\dot{\phi} = -2H'$. By taking multiple derivatives of this relation with respect to $\phi = u$, one readily obtains

$$\beta^2 = 2\epsilon, \qquad \epsilon = \frac{1}{2}\beta^2, \tag{6.6a}$$

$$\lambda = \epsilon - \eta, \qquad \eta = \frac{1}{2}\beta^2 - \lambda, \tag{6.6b}$$

$$\frac{\partial^2\beta}{\partial u^2} = \frac{1}{\sqrt{2\epsilon}}\left(2\xi^2 - 3\epsilon\eta + 2\epsilon^2\right), \qquad \xi^2 = \beta\left(\frac{1}{2}\frac{\partial^2\beta}{\partial u^2} + \frac{1}{8}\beta^3 - \frac{3}{4}\beta\lambda\right). \tag{6.6c}$$

We have included the second derivative $\frac{\partial^2\beta}{\partial u^2}$ and the third slow-roll parameter $\xi^2 = 2\frac{H'H'''}{H^2}$ as these will appear in later expressions.

The relation between scaling parameters of the field theory and the slow-roll parameters of the inflationary theory suggests that for the study of slow-roll inflation, for which $\epsilon, \eta, \xi \ll 1$, we can consider a nearly marginal deformation of the conformal field theory fixed point, $\beta, \lambda, \frac{\partial^2\beta}{\partial u^2} \approx 0$. With the relations (6.6), more substance has been given to the picture as presented in figure 6.1, in that the inflationary quasi-de Sitter phase can be approximated by a near conformal field theory.

Although it is tempting to also rely on techniques from conformal perturbation theory, the above relation between the β function (and its derivatives) and the slow-roll parameters does not necessarily imply the smallness of the coupling u. In fact, from the expression of $\beta(u)$ in conformal perturbation theory (3.18), we can immediately read off possible problems,

$$\beta = \lambda u + \ldots. \tag{6.7}$$

If $\beta^2 = O(\epsilon)$ and $\lambda = O(\epsilon)$, it means that u itself is of order $O(\epsilon^{-1/2})$. Hence, the slow-roll expansion seems to correspond to the *large* u-regime. Drawing a parallel with expansion in dimensionful parameters, we know that we should perhaps not attach too much value to this observation, but it does emphasize a subtle mismatch between the slow-roll expansion and conformal perturbation theory. Conformal perturbation theory requires a small deviation from marginality $\lambda \ll 1$ and a small coupling $u \ll 1$. The slow-roll expansion is an expansion for small $\beta \ll 1$ and its derivatives, but has no analogue for u. For this reason, we will try to keep the use of conformal perturbation theory and the $u \to 0$-limit to a minimum, although we will not succeed in doing this everywhere. In particular, as we have seen, the $u \to 0$-limit is necessary to relate the higher order slow-roll parameters with β [263, 265].

6.3 Holographic correlation functions

6.3.1 Wavefunction expansion

The previous section suggests that the physics of the inflationary epoch in our (four-dimensional) universe resembles the (three-dimensional) physics close to a conformal fixed point. We will investigate whether this suggested resemblance can be employed at the level of the correlation functions. As was summarized in chapter 2, the form of these correlation functions is well-known from gravity calculations [28, 37–43, 57]. To study these from the holographic viewpoint, we need a dictionary between the gravity correlation functions and the correlation functions of the boundary field theory. In the spirit of the AdS/CFT-correspondence, such a relation has been provided by [57].

Quantitatively, the holographic relation between de Sitter geometry and conformal field theory is given by an identification of the partition function of the field theory with the wavefunction of the de Sitter universe with appropriate boundary conditions,

$$\Psi_{dS} = Z_{CFT}. \tag{6.8}$$

The partition function (6.8) determines the correlation functions via

$$\langle O_1 \dots O_n \rangle = \left. \frac{\delta^n \Psi_{dS}[\phi]}{\delta\phi_1 \dots \delta\phi_n} \right|_{\phi=0}.$$

Therefore the wavefunction may be trivially expanded as

$$\Psi_{dS}[\phi] = \exp\Bigg(\frac{1}{2} \int \mathrm{d}^3\boldsymbol{k}_1 \mathrm{d}^3\boldsymbol{k}_2 \, \phi_{\boldsymbol{k}_1}\phi_{\boldsymbol{k}_2} \langle O_{\boldsymbol{k}_1} O_{\boldsymbol{k}_2} \rangle \\ + \frac{1}{6} \int \mathrm{d}^3\boldsymbol{k}_1 \mathrm{d}^3\boldsymbol{k}_2 \mathrm{d}^3\boldsymbol{k}_3 \, \phi_{\boldsymbol{k}_1}\phi_{\boldsymbol{k}_2}\phi_{\boldsymbol{k}_3} \langle O_{\boldsymbol{k}_1} O_{\boldsymbol{k}_2} O_{\boldsymbol{k}_3} \rangle + \dots \Bigg).$$

Using this expression, the dictionary follows immediately. The two-point function $\langle \phi_{\boldsymbol{k}_1}\phi_{\boldsymbol{k}_2} \rangle = \int \mathcal{D}\phi \, \phi_{\boldsymbol{k}_1}\phi_{\boldsymbol{k}_2} |\Psi_{dS}|^2$ can be rewritten as

$$\begin{aligned} \langle \phi_{\boldsymbol{k}_1}\phi_{\boldsymbol{k}_2} \rangle &= \int \mathcal{D}\phi \, \phi_{\boldsymbol{k}_1}\phi_{\boldsymbol{k}_2} e^{\int \mathrm{d}^3\boldsymbol{k} \mathrm{d}^3\boldsymbol{l} \, \phi_{\boldsymbol{k}}\phi_{\boldsymbol{l}} \mathrm{Re}\langle O_{\boldsymbol{k}} O_{\boldsymbol{l}} \rangle} \\ &= \frac{-1}{2\mathrm{Re}\langle O_{\boldsymbol{k}_1} O_{-\boldsymbol{k}_1} \rangle'} \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2) \int \mathcal{D}\tilde{\phi} \, \tilde{\phi}_{\boldsymbol{k}_1} \tilde{\phi}_{-\boldsymbol{k}_1} e^{-\frac{1}{2}\int \mathrm{d}^3\boldsymbol{k} \, \tilde{\phi}_{\boldsymbol{k}} \tilde{\phi}_{-\boldsymbol{k}}}, \end{aligned}$$

where we have employed the substitution of variables $\tilde{\phi}_{\boldsymbol{k}} = i\sqrt{2\mathrm{Re}\langle O_{\boldsymbol{k}} O_{-\boldsymbol{k}} \rangle'}$ and where we have assumed the path integral measure to be invariant under this substitution. A prime $'$ indicates that we consider the part of the correlation function multiplying the momentum conserving delta function. The path integral equals some number and hence the correlation functions are related via

$$\langle \phi_{\boldsymbol{k}} \phi_{-\boldsymbol{k}} \rangle' \propto \frac{-1}{\mathrm{Re}\langle O_{\boldsymbol{k}} O_{-\boldsymbol{k}} \rangle'}. \tag{6.9}$$

For the three-point function we can do a similar calculation,

$$\begin{aligned}
\langle \phi_{\boldsymbol{k}_1} \phi_{\boldsymbol{k}_2} \phi_{\boldsymbol{k}_3} \rangle &= \int \mathcal{D}\phi\, \phi_{\boldsymbol{k}_1} \phi_{\boldsymbol{k}_2} \phi_{\boldsymbol{k}_3} e^{\int \mathrm{d}^3\boldsymbol{k} \mathrm{d}^3\boldsymbol{l}\, \phi_{\boldsymbol{k}} \phi_{\boldsymbol{l}} \mathrm{Re}\langle O_{\boldsymbol{k}} O_{\boldsymbol{l}} \rangle + \frac{1}{3} \int \mathrm{d}^3\boldsymbol{k} \mathrm{d}^3\boldsymbol{l} \mathrm{d}^3\boldsymbol{m}\, \phi_{\boldsymbol{k}} \phi_{\boldsymbol{l}} \phi_{\boldsymbol{m}} \mathrm{Re}\langle O_{\boldsymbol{k}} O_{\boldsymbol{l}} O_{\boldsymbol{m}} \rangle} \\
&= \int \mathcal{D}\phi\, \phi_{\boldsymbol{k}_1} \phi_{\boldsymbol{k}_2} \phi_{\boldsymbol{k}_3} e^{\int \mathrm{d}^3\boldsymbol{k}\, \phi_{\boldsymbol{k}} \phi_{-\boldsymbol{k}} \mathrm{Re}\langle O_{\boldsymbol{k}} O_{-\boldsymbol{k}} \rangle'} \times \\
&\qquad \left(1 + \frac{1}{3} \int \mathrm{d}^3\boldsymbol{k} \mathrm{d}^3\boldsymbol{l} \mathrm{d}^3\boldsymbol{m}\, \phi_{\boldsymbol{k}} \phi_{\boldsymbol{l}} \phi_{\boldsymbol{m}} \mathrm{Re}\langle O_{\boldsymbol{k}} O_{\boldsymbol{l}} O_{\boldsymbol{m}} \rangle \right) \\
&= 2\mathrm{Re}\langle O_{\boldsymbol{k}_1} O_{\boldsymbol{k}_2} O_{\boldsymbol{k}_3} \rangle \int \mathcal{D}\phi\, \phi_{\boldsymbol{k}_1} \phi_{-\boldsymbol{k}_1} \phi_{\boldsymbol{k}_2} \phi_{-\boldsymbol{k}_2} \phi_{\boldsymbol{k}_3} \phi_{-\boldsymbol{k}_3} e^{\int \mathrm{d}^3\boldsymbol{k}\, \phi_{\boldsymbol{k}} \phi_{-\boldsymbol{k}} \mathrm{Re}\langle O_{\boldsymbol{k}} O_{-\boldsymbol{k}} \rangle'} \\
&= \frac{-\mathrm{Re}\langle O_{\boldsymbol{k}_1} O_{\boldsymbol{k}_2} O_{\boldsymbol{k}_3} \rangle}{4 \prod_{j=1}^3 \mathrm{Re}\langle O_{\boldsymbol{k}_j} O_{-\boldsymbol{k}_j} \rangle'} \int \mathcal{D}\tilde{\phi}\, \tilde{\phi}_{\boldsymbol{k}_1} \tilde{\phi}_{-\boldsymbol{k}_1} \tilde{\phi}_{\boldsymbol{k}_2} \tilde{\phi}_{-\boldsymbol{k}_2} \tilde{\phi}_{\boldsymbol{k}_3} \tilde{\phi}_{-\boldsymbol{k}_3} e^{-\frac{1}{2} \int \mathrm{d}^3\boldsymbol{k}\, \tilde{\phi}_{\boldsymbol{k}} \tilde{\phi}_{-\boldsymbol{k}}},
\end{aligned}$$

where we can approximate the exponent because $\langle OOO \rangle \ll \langle OO \rangle$. The zeroth order term in this approximation will integrate to 0 as it is an odd function. Hence

$$\langle \phi_{\boldsymbol{k}_1} \phi_{\boldsymbol{k}_2} \phi_{\boldsymbol{k}_3} \rangle \propto \frac{-\mathrm{Re}\langle O_{\boldsymbol{k}_1} O_{\boldsymbol{k}_2} O_{\boldsymbol{k}_3} \rangle}{\prod_{j=1}^3 \mathrm{Re}\langle O_{\boldsymbol{k}_j} O_{-\boldsymbol{k}_j} \rangle'}. \tag{6.10}$$

These expressions hold for any dual pair of fields and operators. In particular, the correlation functions of the curvature perturbation ζ are related to the correlation functions of the trace Θ of the stress-energy tensor via

$$\langle \zeta_{\boldsymbol{k}} \zeta_{-\boldsymbol{k}} \rangle' \propto \frac{-1}{\mathrm{Re}\langle \Theta_{\boldsymbol{k}} \Theta_{-\boldsymbol{k}} \rangle'}, \tag{6.11a}$$

$$\langle \zeta_{\boldsymbol{k}_1} \zeta_{\boldsymbol{k}_2} \zeta_{\boldsymbol{k}_3} \rangle \propto \frac{-\mathrm{Re}\langle \Theta_{\boldsymbol{k}_1} \Theta_{\boldsymbol{k}_2} \Theta_{\boldsymbol{k}_3} \rangle}{\prod_{j=1}^3 \mathrm{Re}\langle \Theta_{\boldsymbol{k}_j} \Theta_{-\boldsymbol{k}_j} \rangle'}. \tag{6.11b}$$

6.3.2 Ward identities

The trace of the stress-energy tensor is not a standard primary operator. Therefore, to compute (6.11) we can not simply resort to the standard expressions (3.12) for correlation functions of primary operators in a conformal field theory. However, in the gravity calculation we have seen that the curvature perturbations are defined in a gauge independent way and one has the freedom to choose a gauge in which the calculation is done. The gauge invariance of the gravity theory is translated to the fact that in a scale dependent field theory one can either change the dimensionful coupling u to the operator O or one can change the metric, relating ζ to Θ accordingly [57]. This gauge relation is reflected in the field theory, since both operators are related, to leading order, via [286, 287]

$$\Theta = -\beta O, \tag{6.12}$$

where the constant of proportionality β is the Weyl anomaly coefficient. For the purpose of this thesis it is taken to be equal to the standard renormalization β function for the coupling u to the operator O, cf. section 3.1.3. Making use of this relation, the holographic two- and three-point functions can be calculated from the n-point functions of the primary operator O.

The appearance of β is no coincidence, as was already explained in chapter 3. In a quantum field theory, scale transformations are associated with the regularization and renormalization of the theory. This can be described in terms of the Callan-Symanzik renormalization group equations [99–101], where the β functions in the Callan-Symanzik equation describe the dependence of the coupling constants on the renormalization scale. Equivalently —and historically, in the derivation of the Callan-Symanzik equation— the scale dependence is described in terms of the Ward identity of scale transformations.

In gravity, gauge invariance is really important, but at the end of the day the only meaningful physical quantity is ζ. This corresponds to the trace of the stress-energy tensor Θ of the boundary field theory. In general, gauge symmetries of a theory correspond to constraints. In the case of the gravity theory, these are the hamiltonian and momentum/reparameterization constraints of the ADM formalism [256]. In a field theory, the symmetries impose constraints on the correlation function through Ward identities. It is in this way that the gauge choices are implemented in the field theory.

In our particular case, we need to find the relations between the two- and three-point function of the trace of the stress-energy tensor Θ and the operator O. As the trace of the stress-energy tensor is the Noether current of Weyl transformations, we consider the Ward identities of (multiple) trace insertions. Initially the calculation follows directly from any textbook field theory calculation, particularly [83], but once multiple trace insertions have to be taken into account, more care is required. The details of the calculation can be found in appendix 6.A. The final result is given by

$$\langle \Theta_u(\boldsymbol{x}) X \rangle_u = -u(\Delta - 3)\langle O(\boldsymbol{x}) X \rangle_u + \sum_k \delta(\boldsymbol{x} - \boldsymbol{x}_k)\Delta_k \langle X \rangle_u, \tag{6.13a}$$

$$\begin{aligned}\langle\Theta_u(\boldsymbol{x})\Theta_u(\boldsymbol{y})X\rangle_u &= u^2(\Delta-3)^2\langle O(\boldsymbol{x})O(\boldsymbol{y})X\rangle_u - u(\Delta-3)^2\delta(\boldsymbol{x}-\boldsymbol{y})\langle O(\boldsymbol{x})X\rangle_u \\ &\quad - u(\Delta-3)\sum_k \delta(\boldsymbol{x}-\boldsymbol{x}_k)\Delta_k\langle O(\boldsymbol{y})X\rangle_u - \boldsymbol{x}\leftrightarrow\boldsymbol{y} \\ &\quad + \sum_{k,l}\delta(\boldsymbol{x}-\boldsymbol{x}_k)\Delta_k\delta(\boldsymbol{y}-\boldsymbol{x}_l)\Delta_l\langle X\rangle_u, \qquad (6.13b)\end{aligned}$$

$$\begin{aligned}\langle\Theta_u(\boldsymbol{x})\Theta_u(\boldsymbol{y})\Theta_u(\boldsymbol{z})X\rangle_u &= -u^3(\Delta-3)^3\langle O(\boldsymbol{x})O(\boldsymbol{y})O(\boldsymbol{z})X\rangle_u \qquad (6.13c)\\ &\quad + u^2(\Delta-3)^3\delta(\boldsymbol{y}-\boldsymbol{z})\langle O(\boldsymbol{x})O(\boldsymbol{y})X\rangle_u + \boldsymbol{x}\leftrightarrow\boldsymbol{y} + \boldsymbol{y}\leftrightarrow\boldsymbol{z} + \ldots,\end{aligned}$$

where the correlation function $\langle\ \rangle_u$ of the trace(s) of the stress-energy tensor Θ_u is evaluated in the perturbed conformal field theory (6.2), contracted with an arbitrary product of operators $X = O(\boldsymbol{x}_1)\ldots O(\boldsymbol{x}_n)$. Δ_k is the (full) scaling dimension of the k'th operator $O(\boldsymbol{x}_k)$ inside X, which in a single field scenario are all equal. The ... contain highly local contributions that are negligible for our purposes.

The Ward identities (6.13) are valid *throughout* the renormalization group flow, i.e. for each value of u. However, to write them in a more familiar form, we rely on conformal perturbation theory. Equation (6.13a) does not yet seem to contain the familiar $\beta(u)$-dependence, but it does contain the u-dependent conformal weight Δ_u, which carries similar information [286, 287]. Along the renormalization group flow, the scaling behavior of the operator will change and also the coupling will adjust accordingly. Near a conformal fixed point, $u \to 0$, or similarly, near our quasi-conformal fixed point dual to inflation $u \to u_* = 0$, one can make the relation between Δ_u and $\beta(u)$ more precise in a conformal perturbation expansion, cf. (3.18)

$$\beta(u) = u(\Delta_0 - 3) + 2\pi C u^2 + \ldots. \qquad (6.14)$$

Hence, as an expansion in the coupling u, one recognizes the first order contribution $u(\Delta_0 - 3)$ to $\beta(u)$ in (6.13a). The higher order contribution, as obtained via conformal perturbation theory methods, is proportional to the operator product coefficient C of the operators O [110, 111]. The combination Cu results from expanding the one-point correlation function $\langle O\rangle_u$ with respect to the unperturbed theory $\langle O\rangle_0$. It is of order $O(\epsilon)$ and appears with increasing power, Cu, $(Cu)^2$, etc., for higher orders in u. Hence, although we can not be certain of the validity of conformal perturbation theory itself, the expansion of perturbed correlation functions in terms of unperturbed correlation functions is very much similar to the slow-roll expansion.

Since we are only interested in the small u-behavior around the (quasi-)conformal fixed point, the lowest order contribution to β should be sufficient for our purposes to interpret the result. We insist on writing the expression in terms of β, as it nicely emphasizes the dependence on the renormalization group flow or equivalently, the slow-roll dependence. Hence,

$$\langle \Theta_u(\boldsymbol{x}) X \rangle_u = -\beta \langle O(\boldsymbol{x}) X \rangle_0 + \sum_k \delta(\boldsymbol{x} - \boldsymbol{x}_k) \Delta_k \langle X \rangle_u \tag{6.15}$$

is the more precise version of the familiar relation (6.12). These two equations are equal up to contact terms in real space. Similarly, we may write the other Ward identities (6.13b) and (6.13c) as

$$\langle \Theta_u(\boldsymbol{x}) \Theta_u(\boldsymbol{y}) \rangle_u = \beta^2 \langle O(\boldsymbol{x}) O(\boldsymbol{y}) \rangle_0 - \beta \lambda \delta(\boldsymbol{x} - \boldsymbol{y}) \langle O(\boldsymbol{x}) \rangle_0 \tag{6.16}$$

and

$$\begin{aligned} \langle \Theta_u(\boldsymbol{x}) \Theta_u(\boldsymbol{y}) \Theta_u(\boldsymbol{z}) \rangle_u = & -\beta^3 \langle O(\boldsymbol{x}) O(\boldsymbol{y}) O(\boldsymbol{z}) \rangle_0 \\ & + \beta^2 \lambda \left[\delta(\boldsymbol{y} - \boldsymbol{z}) \langle O(\boldsymbol{x}) O(\boldsymbol{y}) \rangle_0 + \boldsymbol{x} \leftrightarrow \boldsymbol{z} + \boldsymbol{y} \leftrightarrow \boldsymbol{z} \right] + \ldots \end{aligned} \tag{6.17}$$

respectively. The ellipsis contain lower n-point functions in the conformal fixed point, which do not contribute. These relations form the starting point of the calculation of the power spectrum and bispectrum through holographic means.

6.4 Slow-roll predictions from Ward identities

Using the holographic dictionary (6.11) from [57] between the two- and three-point functions of the scalar curvature perturbations ζ and the two- and three-point functions of the stress-energy tensor Θ_u of the (near) conformal field theory, we can interpret the Ward identities (6.16–6.17) as inflationary correlation functions, with their dependence on the slow-roll parameters captured by β and λ. In this section we will investigate the prediction for the two-point and three-point correlation functions on the basis of conformal symmetry of the field theory. Special care has to be taken to correctly interpret the renormalization group flow, which takes the expressions away from their conformal fixed point and can be seen as the transcription of the slow-roll dependence. We will first consider the, known [264, 265], holographic description of the two-point function. From this we can draw important lessons for the three-point function, in particular via the consistency condition that should be satisfied in the *squeezed limit* of the three-point function. We first consider the squeezed limit of the bispectrum and then turn our attention to its full expression.

6.4.1 Two-point function

Power spectrum in the conformal fixed point

The power spectrum $\langle\zeta\zeta\rangle$ of curvature perturbations can be found from the field theory via the stress-energy tensor two-point function $\langle\Theta_u\Theta_u\rangle_u$, which in its turn is fully determined by the two-point function of the dual operator O. As we have seen in chapter 3, in a conformal fixed point, the two-point function for an operator O with scaling dimension Δ is completely specified by conformal symmetry [83, 84],

$$\langle O(\boldsymbol{x})O(\boldsymbol{y})\rangle_0 = \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^{2\Delta}}.$$

In principle, this completely specifies the holographic power spectrum for the conformal fixed point. However, before we can compare with (2.14), we need to perform a Fourier transform. This is necessary as the constraints by conformal symmetry are naturally given in terms of the real space variables $\boldsymbol{x}_j$, whereas n-point functions in cosmology are naturally given in terms of the outgoing momenta $\boldsymbol{k}_j$. Any connection between conformal correlation functions and inflationary correlation functions is therefore necessarily obtained only after a Fourier transform. Although finding the Fourier transform for the two-point function is readily done, in general the Fourier transform leads to a technical obstruction for any quick use of the holographic correspondence [149, 150, 255]. As we will see, already for the three-point function this obstruction is difficult to overcome.

The Fourier transform of the two-point function is

$$\begin{aligned}\langle OO\rangle_0 \xrightarrow{\text{F.T.}} &\int \mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{y}\,|\boldsymbol{x}-\boldsymbol{y}|^{-2\Delta}e^{i(\boldsymbol{k}\cdot\boldsymbol{x}+\boldsymbol{k}'\cdot\boldsymbol{y})} = \int \mathrm{d}^3\boldsymbol{u}\,e^{i\boldsymbol{u}\cdot(\boldsymbol{k}+\boldsymbol{k}')}\int \mathrm{d}^3\boldsymbol{v}\,v^{-2\Delta}e^{i\boldsymbol{v}\cdot(\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{k}_1)}\\ &= \delta(\boldsymbol{k}+\boldsymbol{k}')k^{2\Delta-3}\int \mathrm{d}\xi\,\xi^{-2\Delta+2}\int \mathrm{d}\theta\,\sin\theta e^{-i\xi\cos\theta}\\ &\propto \delta(\boldsymbol{k}+\boldsymbol{k}')k^{2\Delta-3},\end{aligned} \tag{6.18}$$

up to factors of 2 and 2π. Since $\langle O\rangle_0 = 0$, if we take $\beta(u)$ to be an overall constant, the stress-energy tensor correlation function in the conformal fixed point is

$$\langle\Theta_{\boldsymbol{k}}\Theta_{\boldsymbol{k}'}\rangle_0 \propto \delta(\boldsymbol{k}+\boldsymbol{k}')\beta^2k^{2\Delta_0-3}.$$

Using $\langle\zeta\zeta\rangle' \propto \frac{1}{\mathrm{Re}\langle\Theta\Theta\rangle'}$ and the expressions (6.6) for ϵ and η in the conformal fixed point,

$$\epsilon = \frac{1}{2}\beta^2 = 0, \qquad \eta = -(\Delta_0-3), \tag{6.19}$$

the Fourier transform of the two-point function of primary operators agrees with the standard result (2.14), where β^2 describes the singular behavior of the power spectrum as $\epsilon \to 0$ and where the spectral index is given by

$$n_s - 1 = -2(\Delta_0 - 3) = 2\eta. \tag{6.20}$$

Power spectrum in the renormalization group flow

As explained in section 6.2, away from the conformal fixed point, the perturbation by the operator O will lead to a renormalization group flow. Holographically this is understood as the deviation from the pure de Sitter phase to a (quasi-de Sitter) inflationary phase and was interpreted by [264] in the light of the known AdS/CFT holographic renormalization methods [139, 141, 142]. Conceptually, it is understood in the Wilsonian sense as a flow between theories, specified by the running of the coupling constants. Technically, the renormalization group flow is the result of the need to renormalize the operators O appearing in (6.16). At the level of the correlation functions, the differential Callan-Symanzik equation dictates the scale dependence of the correlation functions, which was introduced by the inclusion of non-marginal coupling constants. For the truly marginal stress-energy tensor Θ_u, the Callan-Symanzik equation determines its two-point function via

$$\left(\mu\frac{\partial}{\partial\mu} + \beta\frac{\partial}{\partial u}\right)\langle\Theta_u\Theta_u\rangle'_u = 0. \tag{6.21}$$

The μ-dependence can be traded for momentum dependence via dimensional analysis,

$$\langle\Theta_u\Theta_u\rangle'_u = k^3 F\left[\frac{k^2}{\mu^2}, u(\mu)\right],$$

telling us that

$$\left(\mu\frac{\partial}{\partial\mu} + k\frac{\partial}{\partial k}\right)\langle\Theta_u\Theta_u\rangle'_u = 3\langle\Theta_u\Theta_u\rangle'_u.$$

Following the literature [72], the Callan-Symanzik equation acting on the renormalized operators $\tilde{O} = Z(u)O$, can be solved by investigating the ansatz

$$\langle\Theta_u\Theta_u\rangle'_u = Z^2(u)\beta^2(u)k^3. \tag{6.22}$$

This seems to separate the k- and u-dependence completely, although the two are inherently related through the defining equation of the running coupling u,

$$\beta = k\frac{\partial u}{\partial k}.$$

Applying the Callan-Symanzik equation on this ansatz, the wavefunction renormalization Z is given by

$$\frac{\partial Z}{Z\partial u} + \frac{\partial \beta}{\beta \partial u} = 0,$$

which is solved by

$$Z(u) = Z_0 e^{-\int_0^u \mathrm{d}u' \frac{\lambda(u')}{\beta(u')}}. \tag{6.23}$$

Since $\beta = k\frac{\partial u}{\partial k}$, we can trade the integration variable for k, introducing μ as the only other scale in the problem,

$$Z(k) = Z_0 e^{\int_\mu^k \mathrm{d}\log(\frac{k'}{\mu})\,\lambda}. \tag{6.24}$$

In this form, it is clear that the wavefunction renormalization just introduces an anomalous dimension to the correlation function. For a constant $\lambda(u) = \Delta - 3$, the two-point function reads

$$\langle \Theta_u \Theta_u \rangle'_u = Z_0^2 \beta^2 k^{3+2(\Delta-3)}. \tag{6.25}$$

For $u \to 0$, this returns to the earlier found result with an exponent $2\Delta_0 - 3$. For completely arbitrary λ, the result is expressed through the integral in (6.24), which provides a possible method to go beyond the lowest order in slow-roll [265, 288].

As was mentioned in [265], and which deserves renewed emphasis, to connect the conformal correlation function with the inflationary power spectrum, one has to express the two-point function with respect to the *average Hubble flow*. The standard inflationary perturbation theory calculates correlation functions of the quantum fluctuations around the classical inflationary evolution. This evolution is driven by an almost —but not exactly— constant Hubble parameter $H(u)$. Of course, the fact that we have to consider a quasi-de Sitter phase rather than a pure de Sitter evolution is precisely expressed through the slow-roll approximation, something we have already incorporated in the holographic description by studying the renormalization group flow. Still, to correctly identify the fluctuations, we need to express the result with respect to the classical evolution. Since conformal perturbation theory only works around a fixed point of the renormalization group flow, one might wonder how we can express our results with respect to an arbitrary point on the flow, corresponding to the quasi-de Sitter phase. For the two-point function this can be remedied by isolating an explicit Hubble parameter dependence, expressed as an integrated effect of the slow-roll parameter (6.5),

$$H(u) = H_0 e^{-\frac{1}{2}\int_0^u \mathrm{d}u'\,\beta(u')} = H_0 e^{\frac{1}{2}\int_\mu^k \mathrm{d}\log(\frac{k'}{\mu})\beta^2}. \tag{6.26}$$

Using the expression in terms of k, we find

$$\langle \zeta_{\boldsymbol{k}} \zeta_{\boldsymbol{k}'} \rangle \propto \delta(\boldsymbol{k} + \boldsymbol{k}') k^{-3} \frac{H^2}{\beta^2} e^{-\int_\mu^k \mathrm{d}\log(\frac{k'}{\mu})(\beta^2+2\lambda)}. \tag{6.27}$$

The relation (6.6) between the slow-roll parameters and the (derivative of the) β function for constant $\beta(k)$ and $\lambda(k)$ then immediately gives

$$n_s - 1 = -\beta^2 - 2\lambda = 2\eta - 4\epsilon, \tag{6.28}$$

in agreement with (2.15).

6.4.2 Three-point function in the squeezed limit

In principle, using similar techniques, we should be able to analyze the holographic three-point function and give it a slow-roll interpretation. Before we will start this subtle endeavor, we will shape our understanding with a very useful consistency condition of the three-point function in the long wavelength limit.

The bispectrum describes the three-point correlation between three different Fourier modes of the curvature perturbation. If one of the three modes is very small, i.e. its wavelength is very long, it will leave the horizon earlier than the other two modes. This limit is called the *squeezed* limit, since the momentum conserving triangle of Fourier modes has a squeezed shape. Due to momentum conservation, the other two modes become equal in magnitude. Since one of the modes is frozen as a dynamical mode, effectively the three-point function reduces to a two-point function between the other two modes. The only effect of the long mode can be seen through the tilt $n_s - 1$ of the power spectrum, which describes the difference in horizon crossing between the modes.

This observation was first translated into a quantitative statement by [57] in the context of single field slow-roll inflation. It is known to hold for any inflationary scenario with a single clock [279], including our single field set-up. Taking the mode k_3 much smaller than the other two modes, $k_3 \ll k_1 \approx k_2$, the consistency condition in the squeezed limit is, to lowest order in k_3 [280],

$$\lim_{k_3 \to 0} \langle \zeta_{\boldsymbol{k}_1} \zeta_{\boldsymbol{k}_2} \zeta_{\boldsymbol{k}_3} \rangle = -\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3) \langle \zeta_{\boldsymbol{k}_1} \zeta_{-\boldsymbol{k}_1} \rangle' \langle \zeta_{\boldsymbol{k}_3} \zeta_{-\boldsymbol{k}_3} \rangle' \frac{\mathrm{d}\log k_1^3 \langle \zeta_{\boldsymbol{k}_1} \zeta_{-\boldsymbol{k}_1} \rangle'}{\mathrm{d}\log k_1}. \tag{6.29}$$

Corrections to this expression of order $1/k_3$ and beyond are also investigated [270, 271, 279, 280]. Given the general applicability of the relation, such corrections provide interesting criteria to observationally test (and possibly rule out) large classes of slow-roll inflationary models once non-Gaussianities become within reach of observations. At the same time the squeezed limit provides a robust prediction theoretically, which can be used as a first check on the consistency of any particular description for (single clock) inflation.

For our holographic formula (6.11b) of the three-point function, we can explicitly verify the consistency condition in the squeezed limit. First we need to find the analogous expression of (6.17) for the Fourier transformed correlation functions. To do so, note that the Fourier transform of the second term(s) on the right hand side of (6.17) follows directly from the fact that the two-point functions in a conformal field theory can only depend on the spatial separation of the arguments,

$$\begin{aligned}
\delta(\boldsymbol{x}_2-\boldsymbol{x}_3)\langle O(\boldsymbol{x}_1)O(\boldsymbol{x}_2)\rangle_0 &\xrightarrow{\text{F.T.}} \int \mathrm{d}^3\boldsymbol{x}_1\mathrm{d}^3\boldsymbol{x}_2\mathrm{d}^3\boldsymbol{x}_3\,\delta(\boldsymbol{x}_2-\boldsymbol{x}_3)\langle O(\boldsymbol{x}_1)O(\boldsymbol{x}_2)\rangle_0 e^{i\boldsymbol{k}_j\cdot\boldsymbol{x}_j}\\
&= 2^3\int \mathrm{d}^3\boldsymbol{u}\mathrm{d}^3\boldsymbol{v}\,\langle OO\rangle_0(v)e^{i\boldsymbol{u}\cdot(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)}e^{i\boldsymbol{v}\cdot(\boldsymbol{k}_2+\boldsymbol{k}_3-\boldsymbol{k}_1)}\\
&= 2^3\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\int \mathrm{d}^3\boldsymbol{v}\,\langle OO\rangle_0(v)e^{-2i\boldsymbol{v}\cdot\boldsymbol{k}_1}\\
&= \delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\langle O_{\boldsymbol{k}_1}O_{-\boldsymbol{k}_1}\rangle_0'.
\end{aligned} \tag{6.30}$$

As before in performing the Fourier transform, we have not paid particular attention to the conventional factors of 2π. When we would include these, only an overall contribution to (6.17) will be obtained, when taking the squeezed limit.

Next we consider the Fourier transform $\langle O_{\boldsymbol{k}_1}O_{\boldsymbol{k}_2}O_{\boldsymbol{k}_3}\rangle_0$ of the first term of (6.17) in the limit $k_3 \to 0$,

$$\begin{aligned}
\lim_{k_3\to 0}\langle O_{\boldsymbol{k}_1}O_{\boldsymbol{k}_2}O_{\boldsymbol{k}_3}\rangle_0' &= \int \mathrm{d}^3\boldsymbol{x}\langle O_{\boldsymbol{k}_1}O_{-\boldsymbol{k}_1}O(\boldsymbol{x})\rangle_0' = -\frac{\partial}{\partial u}\langle O_{\boldsymbol{k}_1}O_{-\boldsymbol{k}_1}\rangle_u' + O(u)\\
&= \frac{1}{\beta}\left(-k_1\frac{\partial}{\partial k_1}+3+2\lambda\right)\langle O_{\boldsymbol{k}_1}O_{-\boldsymbol{k}_1}\rangle_u' + O(u)\\
&= \frac{-1}{\beta}\left(k_1\frac{\partial}{\partial k_1}-3\right)\langle O_{\boldsymbol{k}_1}O_{-\boldsymbol{k}_1}\rangle_0' + \frac{2}{\beta}\lambda\langle O_{\boldsymbol{k}_1}O_{-\boldsymbol{k}_1}\rangle_0' + O(u),
\end{aligned} \tag{6.31}$$

where in the second line, the Callan-Symanzik equation (3.3) is applied to the two-point function,

$$\left(k\frac{\partial}{\partial k}-\beta\frac{\partial}{\partial u}-2\lambda-3\right)\langle O_{\boldsymbol{k}}O_{-\boldsymbol{k}}\rangle_u' = 0.$$

Taking into account the prefactors $-\beta^3$ and β^2 in the Ward identity of three stress-energy tensors (6.17) and using $\lim_{k_3\to 0}\langle O_{\boldsymbol{k}_3} O_{-\boldsymbol{k}_3}\rangle'_0 = 0$, the Ward identity yields

$$\lim_{k_3\to 0}\langle \Theta_u(\boldsymbol{k}_1)\Theta_u(\boldsymbol{k}_2)\Theta_u(\boldsymbol{k}_3)\rangle'_u = \beta^2\left(k_1\frac{\partial}{\partial k_1} - 3\right)\langle O_{\boldsymbol{k}_1} O_{-\boldsymbol{k}_1}\rangle'_0 - 2\beta^2\lambda\langle O_{\boldsymbol{k}_1} O_{-\boldsymbol{k}_1}\rangle'_0 + \beta^2\lambda\left(2\langle O_{\boldsymbol{k}_1} O_{-\boldsymbol{k}_1}\rangle'_0 + 0\right) \tag{6.32}$$

in the squeezed limit. The local term of (6.17) exactly cancels against the $\frac{2}{\beta}\lambda$-contribution of the squeezed limit of $\langle OOO\rangle'_0$. Therefore, the squeezed limit of (6.11b) gives

$$\begin{aligned}\lim_{k_3\to 0}\langle \zeta_{\boldsymbol{k}_1}\zeta_{\boldsymbol{k}_2}\zeta_{\boldsymbol{k}_3}\rangle' &\propto \frac{\left(k_1\frac{\partial}{\partial k_1} - 3\right)\langle O_{\boldsymbol{k}_1} O_{-\boldsymbol{k}_1}\rangle'}{\beta^4\langle O_{\boldsymbol{k}_1} O_{-\boldsymbol{k}_1}\rangle'\langle O_{\boldsymbol{k}_2} O_{-\boldsymbol{k}_2}\rangle'\langle O_{\boldsymbol{k}_3} O_{-\boldsymbol{k}_3}\rangle'} \\ &= \langle \zeta_{\boldsymbol{k}_1}\zeta_{-\boldsymbol{k}_1}\rangle'^2\langle \zeta_{\boldsymbol{k}_3}\zeta_{-\boldsymbol{k}_3}\rangle'\left[-\frac{1}{\langle \zeta_{\boldsymbol{k}_1}\zeta_{-\boldsymbol{k}_1}\rangle'}\left(k_1\frac{\partial}{\partial k_1}\log\langle \zeta_{\boldsymbol{k}_1}\zeta_{-\boldsymbol{k}_1}\rangle' + 3\right)\right] \\ &= -\langle \zeta_{\boldsymbol{k}_1}\zeta_{-\boldsymbol{k}_1}\rangle'\langle \zeta_{\boldsymbol{k}_3}\zeta_{-\boldsymbol{k}_3}\rangle'\frac{\partial\log\left(k_1^3\langle \zeta_{\boldsymbol{k}_1}\zeta_{-\boldsymbol{k}_1}\rangle'\right)}{\partial\log k_1}.\end{aligned} \tag{6.33}$$

Hence, the squeezed limit consistency condition to lowest order in k_3, is immediate in the holographic description. The crucial step in the derivation is the second equality in (6.31), in which the three-point function with a zero Fourier mode is recognized as the first order contribution to the two-point function in a perturbed conformal field theory. In the squeezed limit the three-point function appears as a small perturbation of the two-point function around the conformal fixed point, leading to the tilt of the power spectrum. The other steps follow from a rewriting of this dependence, which can be seen as a slight rescaling, i.e. the infinitesimal coordinate transformation induced by the insertion of a stress-energy tensor. This interpretation is consistent with the original motivation behind the consistency condition, which observes that, once frozen, the only effect of the long wavelength mode to the bispectrum is to cause a local rescaling of the spatial distance scales, cf. (2.17) [57, 279].

A separate, independent derivation of the consistency condition (6.29) using similar ingredients has been given in [271], in which the (broken) conformal symmetry is described using a Ward identity. This Ward identity is equivalent to the Callan-Symanzik equation in our formalism, whereas our Ward identity relating Θ and O has no equivalent in the description of [271], which work directly with the gauge-invariant curvature perturbation ζ. Since Ward identities naturally relate an $n+1$-point correlation function with the variation of an n-point function, the observation in [271, 277] is that the consistency condition essentially *is* a Ward identity, applied to a particular conserved current. The current under consideration corresponds to a combination of a shift and dilational transformation, perturbing the system much in the same way as the Callan-Symanzik equation in our formalism. It would be very interesting to further investigate the connection between [271] and our work.

The consistency condition provides a powerful technique in the investigation of the structure of the correlation functions of curvature perturbations generated during inflation. Several approaches are considered in the literature to use the relation [271, 272] or possible generalizations [270] in order to restrict the n-point functions. In our approach, we use it as a consistency check and as an important guide to the full holographic bispectrum. In particular, the consistency condition explicitly shows that the local contributions in the bispectrum should combine in such a way that there is an overall contribution proportional to $n_s - 1 = 2\eta - 4\epsilon$. As we will see, in the full expression of the holographic bispectrum, the dominating slow-roll contribution is not at all obvious. With the consistency condition at our disposal, we have a strong indication where the important contributions should reside.

6.4.3 Three-point function

Bispectrum in a quasi-conformal fixed point

To understand inflationary non-Gaussianities, we now employ an analysis of the conformal three-point functions, similar to section 6.4.1. The Ward identity (6.17) consists of two contributions. The main contribution appears to come from the operator three-point function $\langle OOO \rangle_0$, but also a contact term proportional to the two-point function $\langle OO \rangle_0$ appears. Both of these contributions are again constrained by conformal symmetry, in particular [83, 84],

$$\langle O(\boldsymbol{x}_1)O(\boldsymbol{x}_2)O(\boldsymbol{x}_3)\rangle_0 = \frac{C}{(x_{12}x_{13}x_{23})^{\Delta}} + \text{contact terms}, \tag{6.34}$$

where $x_{jl} = |\boldsymbol{x}_j - \boldsymbol{x}_l|$ and C is the coefficient from the operator product expansion. The local contribution is generally not included in the literature, as it only contributes at coincident points. We have included it for completeness and wish to note that its contribution may well be relevant in the final expression.

Before we can compare any of the conformal structure with the inflationary bispectrum, we will need to Fourier transform these expressions. The contact terms are analyzed straightforwardly from (6.18) and (6.30),

$$\begin{aligned}\delta(\boldsymbol{x}_2 - \boldsymbol{x}_3)\langle O(\boldsymbol{x}_1)O(\boldsymbol{x}_2)\rangle_0 &\xrightarrow{\text{F.T.}} \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)\langle O_{\boldsymbol{k}_1}O_{-\boldsymbol{k}_1}\rangle_0' \\ &\propto \delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)k_1^{2\Delta-3}.\end{aligned}$$

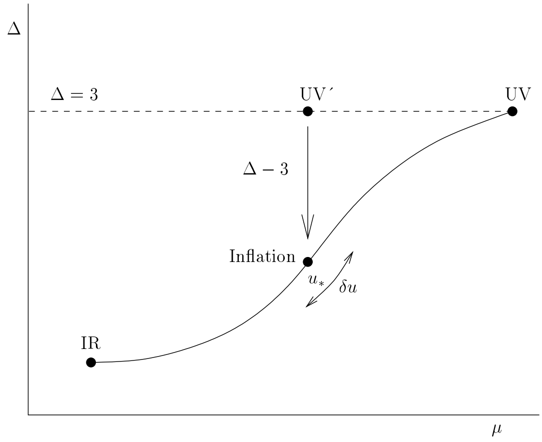

Figure 6.2: *The inflationary phase at an intermediate point in the renormalization group flow may be approximated by a conformal fixed point. The dashed line indicates a marginal renormalization group flow from one UV theory to another, for an operator with exactly marginal dimension* $\Delta = 3$. *The validity of the slow-roll approximation suggests that expressions in the quasi-fixed point can be approximated by a* $\Delta - 3$*-expansion. Around the quasi-fixed point, with coupling* u_*, *the effect of inflation can be found through a further dependence on the renormalization group flow.*

For nearly marginal operators, this is a local contribution k_1^3. Adding the symmetrized terms, yields a contribution

$$Q(k_1, k_2, k_3) = k_1^3 + k_2^3 + k_3^3. \tag{6.35}$$

Fourier transforming (6.34) for arbitrary Δ is technically more involved [255] and requires some ingenuity in the analysis, cf. appendix 6.B

To understand the Fourier transform, we will make full use of the conceptual relation between slow-roll expansion and renormalization group flow, cf. section 6.2. As emphasized earlier, the dual to the inflationary phase appears as a point on (or short section of) the renormalization group flow at an intermediate stage. Because of the slow-roll expansion, at a given instance the inflationary expansion is that of a de Sitter evolution. Therefore, we can approximate the intermediate dual point on the renormalization group flow, by a nearby conformal fixed point, cf. figure 6.2. The difference between the conformal fixed point and the quasi-conformal fixed point is that the operator O does not describe a marginal renormalization group flow, i.e. $\Delta \neq 3$. Since the slow-roll expansion indicates that the operator is nearly marginal, we can approximate (6.34) as a Taylor series expansion with respect to $\lambda = \Delta - 3$,

$$\begin{aligned}\langle O(\boldsymbol{x}_1)O(\boldsymbol{x}_2)O(\boldsymbol{x}_3)\rangle_0 &\xrightarrow{\text{F.T.}} C\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)R_\Delta(k_1,k_2,k_3)\\ &= C\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\left[R_3(k_1,k_2,k_3)+\lambda R_3'(k_1,k_2,k_3)+\ldots\right].\end{aligned}$$

The evaluation of R_Δ to first order is a technical exercise, which we detail in appendix 6.B. Using the expression for three traces (6.17) and the relation (6.6) between slow-roll parameters and β and λ, the holographic prediction (6.11b) for the bispectrum in the quasi-conformal fixed point is

$$\langle \zeta_{\boldsymbol{k}_1}\zeta_{\boldsymbol{k}_2}\zeta_{\boldsymbol{k}_3}\rangle' \propto \frac{1}{\epsilon^2}\frac{1}{k_1^3k_2^3k_3^3}\left[\beta C\left(R_3+(\epsilon-\eta)R_3'\right)+(\epsilon-\eta)Q\right]. \tag{6.36}$$

Next we will interpret this holographic prediction in the light of the gravitational calculation (2.22).

Local and non-local contributions to the bispectrum

The terms involving R_3 and Q have a clear interpretation. The local contribution Q seems to match precisely with the local contribution from the bispectrum (2.21),

$$\lambda Q(k_1,k_2,k_3) = (\epsilon-\eta)(k_1^3+k_2^3+k_3^3). \tag{6.37}$$

The momentum dependence as well as the parametric dependence on the slow-roll parameters agree, except for a relative factor of 2 between the ϵ- and η-terms.

The contribution $\beta C R_3$ has a clear interpretation as well. This zeroth order contribution, at $\Delta = 3$, has already been considered in [257], as a contribution from a direct three-point interaction $V'''\delta\phi^3$ [76–78, 257], cf. 6.B.3,

$$R_3(k_1,k_2,k_3) = (-1+\gamma+\log[-k_t\tau_*])\sum_{j=1}^{3}k_j^3 + k_1k_2k_3 - \sum_{j\neq l}k_jk_l^2. \tag{6.38}$$

For a massless spectator field it is the leading contribution, but as was argued in [57, 76], it appears at second order in the slow-roll expansion for the curvature perturbations. The prefactor βC can indeed be seen to be related to the third order slow-roll parameter, in agreement with (2.22). In a similar fashion to what we argued that $\lambda = \frac{\partial\beta}{\partial u}$ in the limit $u \to 0$, it is clear from the expression (6.14) of β in the conformal perturbation theory limit that the second derivative of the β function is equal to the operator product coefficient,

$$\frac{\partial^2\beta}{\partial u^2}(u) = 4\pi C + O(u).$$

Hence, from (6.6) we find

$$\beta C R_3 = \frac{1}{4\pi}\beta \frac{\partial^2 \beta}{\partial u^2} R_3 = \frac{1}{2\pi}(\xi^2 - \frac{3}{2}\epsilon\eta + \epsilon^2) R_3, \tag{6.39}$$

which matches the result in (2.22).

In the holographic prediction (6.36), the final term $\lambda R_3'$ should then account for the remaining terms in (2.22). In particular, we are looking for contributions to the holographic bispectrum, which are both linear in the slow-roll parameters as well as have an interesting momentum behavior which mixes different momenta and has a $\frac{1}{k_t}$-dependence. As is clear from 6.B.4, the contribution from R_3' does contain more involved momentum dependence,

$$R_3' \subset \sum_j k_j^3 (a + b\gamma) \log[-k_t \tau_*] + \frac{1}{k_t^3}\left(\sum_j k_j^6 + k_1^5 k_2 \log[-k_t \tau_*] + k_1^2 k_3^4 \mathrm{Li}_2 \left[\frac{k_t}{k_t - 2k_1} \right] \right),$$

with a and b numerical constants. However, this contribution is multiplied by $\beta C \lambda$, which seems to be higher order in slow-roll, cf. [79].

From the squeezed limit we know that the latter observation is misleading. In the consistency condition we have found more terms that are linear in the slow-roll parameters than just the local term Q. Although the squeezed limit is often interpreted as a small momentum limit, $k_3 \to 0$, in principle it is a relative statement, $k_3 \ll k_1, k_2$. Hence, because of momentum conservation, the squeezed limit could also be seen as a high frequency limit $k_1 \to \infty$. In this form, the squeezed limit tells us that the high frequency dominant part of the holographic bispectrum does contain additional linear dependence on the slow-roll parameters, despite the explicit second order dependence of βC.

The question is how one could extract the "hidden" linear parametric dependence. Since the bispectrum reveals its hidden parametric dependence in the high energy regime, an obvious suggestion is to consider the counterterms of the regulated expressions. One objection might be that a counterterm is not able of producing a non-local contribution of the form $\frac{1}{k_t}$. Counterterms generically have at most polynomial dependence on the momenta, which vanish in the small frequency limit. A $\frac{1}{k_t}$-behavior looks awfully divergent in this limit. However, in the bispectrum (2.21), the numerator of the $\frac{1}{k_t}$-terms ensures that there is no divergent behavior for low frequency modes. We conclude that counterterms can produce $\frac{1}{k_t}$-terms and should therefore be studied in more detail. Although R_3' comes with explicit divergent terms, cf. (6.68), these are only homogeneous of degree 1 in the momenta and therefore do not resemble any of the terms in (2.22).

Therefore, an alternative analysis is called for. In fact, the derivation of the squeezed limit (6.31) and in particular the explicit division by $\beta \propto (\Delta - 3) = \lambda$ necessary for the cancelation against the two-point correlation function contribution of (6.17), indicates that non-analytic behavior in λ plays an essential role. This underlines the need for a Laurent series rather than a Taylor series. By analyzing the Laurent series of $\langle OOO \rangle_0$ in λ, one should be able to uncover the dominant contributions, which is the topic of future work.

The renormalization group flow

As with the power spectrum, to truly compare the inflationary bispectrum with the holographic prediction, we will need to deviate away from the quasi-de Sitter phase using the renormalization group flow. Compared to the two-point case, the sought-for change in functional dependence is different. Whereas the slow-roll result for the two-point function has slow-roll dependence in the exponent of the momentum k, the slow-roll dependence of the three-point function is usually found in the overall amplitude of the bispectrum, cf. (2.21).

To calculate $\langle \zeta_{\boldsymbol{k}_1} \zeta_{\boldsymbol{k}_2} \zeta_{\boldsymbol{k}_3} \rangle$, we will have to find the renormalized expressions for $\langle \Theta_{\boldsymbol{k}_j} \Theta_{-\boldsymbol{k}_j} \rangle'_u$ and $\langle \Theta_{\boldsymbol{k}_1} \Theta_{\boldsymbol{k}_2} \Theta_{\boldsymbol{k}_3} \rangle'_u$. As we have seen in the previous section, each of the $\langle \Theta_{\boldsymbol{k}_j} \Theta_{-\boldsymbol{k}_j} \rangle'_u$ in the denominator of (6.11b), is given by

$$\langle \Theta_{\boldsymbol{k}_j} \Theta_{-\boldsymbol{k}_j} \rangle'_u = Z^2 \beta^2 k^3,$$

where $Z(u)$ is given by (6.23). The stress-energy tensor three-point function can be found by a comparable calculation. Using dimensional analysis,

$$\left(\mu \frac{\partial}{\partial \mu} + \sum_j k_j \frac{\partial}{\partial k_j} \right) \langle \Theta_u(\boldsymbol{k}_1) \Theta_u(\boldsymbol{k}_2) \Theta_u(\boldsymbol{k}_3) \rangle'_u = 3 \langle \Theta_u(\boldsymbol{k}_1) \Theta_u(\boldsymbol{k}_2) \Theta_u(\boldsymbol{k}_3) \rangle'_u,$$

we can write the Callan-Symanzik equation for the three-point function of the exactly marginal stress-energy tensor as

$$\left(k_j \frac{\partial}{\partial k_j} - 3 - \beta \frac{\partial}{\partial u} \right) \langle \Theta_u(\boldsymbol{k}_1) \Theta_u(\boldsymbol{k}_2) \Theta_u(\boldsymbol{k}_3) \rangle'_u = 0. \tag{6.40}$$

This equation determines $\tilde{Z}(u)$ in the ansatz

$$\langle \Theta_u(\boldsymbol{k}_1) \Theta_u(\boldsymbol{k}_2) \Theta_u(\boldsymbol{k}_3) \rangle'_u = -\tilde{Z}^3 \beta^3 C R(k_1, k_2, k_3) + \tilde{Z}^2 \beta^2 \lambda Q(k_1, k_2, k_3),$$

where both R and Q are homogeneous in k_1, k_2, k_3 of degree 3. This latter fact follows from the approximation we consider, in which the quasi-fixed point describing inflation is expanded with respect to a marginal conformal dimension, which ensures the degree of homogeneity of the three-point function to be 3.

As a first attempt of finding a solution for $\tilde{Z}$, we consider the case in which $\lambda(u)$ is constant. Then, $\tilde{Z}$ is again given by (6.23). Collecting results,

$$\begin{aligned}\langle\zeta_{\boldsymbol{k}_1}\zeta_{\boldsymbol{k}_2}\zeta_{\boldsymbol{k}_3}\rangle' &= Z^{-6}\beta^{-6}(k_1k_2k_3)^{-3}\left(-Z^3\beta^3 CR(k_1,k_2,k_3)+Z^2\beta^2\lambda Q(k_1,k_2,k_3)\right)\\ &= H^4\beta^{-4}(k_1k_2k_3)^{-3}(HZ)^{-4}\left(-Z\beta CR(k_1,k_2,k_3)+\lambda Q(k_1,k_2,k_3)\right). \quad (6.41)\end{aligned}$$

Again we have chosen to explicitly isolate the required H-dependence of the bispectrum. The overall factor $(HZ)^{-4}$ contributes additional factors of β and λ via

$$(HZ)^{-4} = e^{2\int_0^u \mathrm{d}u'\left(\beta(u')+2\frac{\lambda(u')}{\beta(u')}\right)} = 1+2\int_0^u \mathrm{d}u'\left(\beta(u')+2\frac{\lambda(u')}{\beta(u')}\right)+\ldots. \quad (6.42)$$

Performing a similar expansion for $Z = e^{-\int\lambda/\beta}$, the holographic bispectrum is given by

$$\begin{aligned}B_\zeta(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) =&\frac{H^4}{\beta^4(k_1k_2k_3)^3}\left(1+2\int_0^u \mathrm{d}u'\left(\beta(u')+2\frac{\lambda(u')}{\beta(u')}\right)\right)\times\\ &\left[\lambda Q(k_1,k_2,k_3)-\beta C\left(1-\int_0^u \mathrm{d}u'\frac{\lambda(u')}{\beta(u')}\right)R(k_1,k_2,k_3)\right]. \quad (6.43)\end{aligned}$$

This expression makes clear that a renormalization of the bispectrum will only contribute to higher order in slow-roll. Since we explicitly consider the renormalization as a variation of the slow-roll parameters, this is little surprising, but it also means that, for the time being, we do not have to consider the effect of renormalization on the bispectrum.

6.5 Conclusions

In this chapter we have considered the late-time de Sitter symmetry constraints on the two- and three-point correlation functions of curvature perturbations generated during single field slow-roll inflation. The fact that the inflationary evolution is a near-de Sitter phase is captured by considering the renormalization of the Ward identities relating the stress-energy tensor and the operators holographically dual to the inflaton field.

In the case of the power spectrum, the symmetry constraints are sufficient to retrieve the near scale invariance that is characteristic for slow-roll inflation. The fact that the power spectrum can be retrieved from the renormalization group flow, suggests a type of universality in the two-point function. If a dS/CFT-correspondence would be found, this universality contrasts with the finetuning problem for inflation, whose dynamics does seem to depend sensitively on the slow-roll parameters.

In the case of the bispectrum, our study is, as yet, inconclusive as to whether symmetries are sufficient to specify the three-point function completely. Several ingredients of the bispectrum can be seen to appear directly from our holographic study, but other ingredients, most notably the appearance of a $\frac{1}{k_t}$-term at linear order in slow-roll, remain hidden in the approach. Both ingredients do appear in one or more of the terms that contribute to the bispectrum, but the necessary combination does not appear. From the correctness of the consistency condition in the squeezed limit, we obtain tantalizing hints that counterterms and/or a Laurent expansion in λ should contain the required momentum-dependence, at the right order in slow-roll. However, a first study into the subtle regularization procedure, has not proven to be successful.

The methods we have used resemble techniques from the AdS/CFT-correspondence. If a holographic understanding of the primordial bispectrum could be found, then our analysis should be a first step into a further understanding of a possible dS/CFT-correspondence. At this stage, however, the dS/CFT-correspondence is only considered at the level of the symmetries, which is a far more general statement than the intricate details of a holographic correspondence. Moreover, one could worry in which regime we consider the field theory. Since we are set out to study a phenomenon in classical general relativity, a direct application of holography would suggest the dual theory to be strongly coupled. However, for us, this question is irrelevant, since we never consider the coupling constant of the putative dual theory and base our results solely on the restrictive power of the symmetries.

A subtle issue in our methods is the use of conformal perturbation theory. Conformal perturbation theory requires that the coupling that drives the theory away from the conformal fixed point, is small $u \ll 1$. This requirement does not seem to follow immediately from the slow-roll expansion. For this reason, it is unclear whether or when we are entitled to rely upon conformal perturbation techniques. Possibly, some of the unresolved puzzles are caused by the absence of a full understanding of the applicability of conformal perturbation theory in this context.

In conclusion, we have presented a detailed but not yet finalized understanding of how constraints from the asymptotic conformal symmetry of de Sitter space may restrict the two- and three-point functions of primordial density fluctuations generated during inflation. The three-point correlation function seems to subtly depend on regularization and renormalization, which is partly beyond the scope of this study. Clearly, it would be interesting to fully develop the necessary techniques to study the three-point function. The bulk of the ingredients essential for this analysis has been laid out in this thesis.

6.A Ward identities of multiple trace insertions

6.A.1 Ward identities for a perturbed action

The Ward identity of any symmetry generator can be derived from considering infinitesimal transformations of the correlation function,

$$\langle X\rangle_u = \frac{1}{Z}\int \mathcal{D}O\, X e^{-S_u[O]}, \tag{6.44}$$

of a product of operators $X = O(\boldsymbol{x}_1)\ldots O(\boldsymbol{x}_n)$. We specifically evaluate the expectation value with respect to a *perturbed* conformal field theory,

$$S_u[O] = S_0[O] + \int \mathrm{d}^3\boldsymbol{x}\, uO(\boldsymbol{x}) + S_{\text{c.t.}}(\mu). \tag{6.45}$$

The last term is necessary to regulate any divergences, which we have introduced by turning on a scale in the form of a non-marginal operator O. The precise form of $S_{\text{c.t.}}$ is not clear at this stage, but its presence can later be used to regulate any divergences in the correlators. Under the transformation $O \to O' = O(\boldsymbol{x}) - i\omega_a G_a(\boldsymbol{x})O(\boldsymbol{x})$, the action transforms as

$$\begin{aligned}S_u[O'] = S_u[O] &- \int \mathrm{d}^3\boldsymbol{x}\,\omega_a(\boldsymbol{x})\partial_\mu j^\mu_a(\boldsymbol{x}) - i\int \mathrm{d}^3\boldsymbol{x}\,\omega_a(\boldsymbol{x})G_a(\boldsymbol{x})uO(\boldsymbol{x})\\ &- \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\,\omega_a(\boldsymbol{x})G_a(\boldsymbol{x})\omega_b(\boldsymbol{x})G_b(\boldsymbol{x})uO(\boldsymbol{x}) + \ldots,\end{aligned}$$

to second order in ω_a, where j^μ_a is the Noether current of the transformation in the conformal field theory $S_0[O]$ at the fixed point. The last term is a contact term, which we have included because it is of second order in ω_a. It stems from the transformation $O \to O'$ in the perturbation-part of the action (6.45). In principle, the unperturbed conformal action $S_0[O]$ also obtains a contribution at second order as a result of the transformation to second order. However, this contribution is difficult to retrieve from first principles, as the Noether current of the transformation is only defined infinitesimally. It is therefore left implicit in the ..., while its effect on the Ward identity will later be inferred by different means. Hence, for the moment we find

$$e^{-S_u[O']} = e^{-S_u[O]}\Big(1 - \int \mathrm{d}^3\boldsymbol{x}\, \omega_a(\boldsymbol{x})\delta L_a(\boldsymbol{x}) + \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\mathrm{d}\boldsymbol{y}\, \omega_a(\boldsymbol{x})\omega_b(\boldsymbol{y})\delta L_a(\boldsymbol{x})\delta L_b(\boldsymbol{y})\Big)$$
$$\times\Big(1 + \frac{1}{2}\int \mathrm{d}^3\boldsymbol{x}\, \omega_a(\boldsymbol{x})G_a(\boldsymbol{x})\omega_b(\boldsymbol{x})G_b(\boldsymbol{x})uO(\boldsymbol{x})\Big),$$

up to second order, where $\delta L_a(\boldsymbol{x})$ is shorthand notation for

$$\delta L_a(\boldsymbol{x}) = -\partial_\mu j_a^\mu(\boldsymbol{x}) - iuG_a(\boldsymbol{x})O(\boldsymbol{x}).$$

Infinitesimally transforming X gives

$$X' = e^{-i\omega_a(\boldsymbol{x})G_a(\boldsymbol{x})}X = X - i\sum_k \omega_{a,k}G_{a,k}X - \frac{1}{2}\sum_{k,l}\omega_{a,k}G_{a,k}\omega_{b,l}G_{b,l}X,$$

where $\omega_{a,k} = \omega_a(\boldsymbol{x}_k)$ and $G_{a,k} = G_a(\boldsymbol{x}_k)$ acts on the k'th $O(\boldsymbol{x}_k)$ inside X.

Assuming the measure is invariant, $\mathcal{D}O' = \mathcal{D}O$, comparison of the transformed expression for $\langle X\rangle_u$ and (6.44) gives

$$0 = \int \mathrm{d}^3\boldsymbol{x}\, \omega_a(\boldsymbol{x})\langle \delta L_a(\boldsymbol{x})X\rangle_u + i\sum_k \omega_{a,k}G_{a,k}\langle X\rangle_u, \tag{6.46}$$

to first order in ω. Similarly, to second order it gives

$$\begin{aligned}
0 &= \int \mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{y}\, \omega_a(\boldsymbol{x})\omega_b(\boldsymbol{y})\langle \delta L_a(\boldsymbol{x})\delta L_b(\boldsymbol{y})X\rangle_u + 2i\sum_k \omega_{a,k}G_{a,k}\int \mathrm{d}^3\boldsymbol{x}\, \omega_b(\boldsymbol{x})\langle \delta L_b(\boldsymbol{x})X\rangle_u \\
&\quad - \sum_{k,l}\omega_{a,k}G_{a,k}\omega_{b,l}G_{b,l}\langle X\rangle_u + \int \mathrm{d}^3\boldsymbol{x}\, \omega_a(\boldsymbol{x})G_a(\boldsymbol{x})\omega_b(\boldsymbol{x})G_b(\boldsymbol{x})u\langle O(\boldsymbol{x})X\rangle_u \\
&= \int \mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{y}\, \omega_a(\boldsymbol{x})\omega_b(\boldsymbol{y})\langle \delta L_a(\boldsymbol{x})\delta L_b(\boldsymbol{y})X\rangle_u + \sum_{k,l}\omega_{a,k}G_{a,k}\omega_{b,l}G_{b,l}\langle X\rangle_u \\
&\quad + \int \mathrm{d}^3\boldsymbol{x}\, \omega_a(\boldsymbol{x})G_a(\boldsymbol{x})\omega_b(\boldsymbol{x})G_b(\boldsymbol{x})u\langle O(\boldsymbol{x})X\rangle_u.
\end{aligned} \tag{6.47}$$

6.A.2 Alternative derivation

As emphasized, these expressions have been derived with respect to the perturbed theory. Since Ward identities are usually derived with respect to an invariant theory, our approach may raise questions on its correctness. We therefore present a different calculation, which is independent from the previous one. We use the fact that

$$\langle X\rangle_u = \langle X e^{-\int \mathrm{d}^3\boldsymbol{x}\, uO(\boldsymbol{x})}\rangle_0$$

and apply it to the standard $u = 0$ Ward identities,

$$\langle \partial_\mu j^\mu_a(\boldsymbol{x}) X\rangle_0 = i \sum_k \delta(\boldsymbol{x} - \boldsymbol{x}_k) G_{a,k} \langle X\rangle_0, \tag{6.48a}$$

$$\langle \partial_\mu \partial_\nu j^\mu_a(\boldsymbol{x}) j^\nu_b(\boldsymbol{y}) X\rangle_0 = -\sum_{k,l} \delta(\boldsymbol{x} - \boldsymbol{x}_k)\delta(\boldsymbol{y} - \boldsymbol{x}_l) G_{a,k} G_{b,l} \langle X\rangle_0. \tag{6.48b}$$

Applying (6.48a) repeatedly to the Taylor expansion of the exponential yields

$$\begin{aligned}
\langle \partial_\mu j^\mu_a(\boldsymbol{x}) X\rangle_u &= \sum_n \frac{(-u)^n}{n!} \int \mathrm{d}^3\boldsymbol{x}_1 \dots \mathrm{d}^3\boldsymbol{x}_n \, \langle \partial_\mu j^\mu_a(\boldsymbol{x}) O(\boldsymbol{x}_1) \dots O(\boldsymbol{x}_n) X\rangle_0 \\
&= \sum_n \frac{(-u)^n}{n!} \int \mathrm{d}^3\boldsymbol{x}_1 \dots \mathrm{d}^3\boldsymbol{x}_n \times \\
&\qquad i \left(\sum_{k=1}^n + \sum_{k=X} \right) \delta(\boldsymbol{x} - \boldsymbol{x}_k) G_{a,k} \langle O(\boldsymbol{x}_1) \dots O(\boldsymbol{x}_n) X\rangle_0 \\
&= -iuG_a(\boldsymbol{x}) \langle O(\boldsymbol{x}) X\rangle_u + i \sum_k \delta(\boldsymbol{x} - \boldsymbol{x}_k) G_{a,k} \langle X\rangle_u,
\end{aligned} \tag{6.49}$$

where the summation $\sum_{k=1}^n$ runs over all first n operators O coming from the exponent and where $\sum_{k=X}$ runs over all remaining operators O inside X. Similarly, for the double Ward identity,

$$\begin{aligned}
\langle \partial_\mu \partial_\nu j^\mu_a(\boldsymbol{x}) j^\nu_b(\boldsymbol{y}) X\rangle_u &= \sum_n \frac{(-u)^n}{n!} \int \mathrm{d}^3\boldsymbol{x}_1 \dots \mathrm{d}^3\boldsymbol{x}_n \times \\
&\Bigg(- \left[\sum_{k \neq l}^n + \sum_{k=1}^n \sum_{l=X} + \sum_{k=X} \sum_{l=1}^n + \sum_{k,l=X} \right] \delta(\boldsymbol{x} - \boldsymbol{x}_k)\delta(\boldsymbol{y} - \boldsymbol{x}_l) G_{a,k} G_{b,l} \\
&\qquad - \delta(\boldsymbol{x} - \boldsymbol{y}) \sum_{k=1}^n \delta(\boldsymbol{x} - \boldsymbol{x}_k) G_{a,k} G_{b,k} \Bigg) \langle O(\boldsymbol{x}_1) \dots O(\boldsymbol{x}_n) X\rangle_0 \\
&= -u^2 G_a(\boldsymbol{x}) G_b(\boldsymbol{y}) \langle O(\boldsymbol{x}) O(\boldsymbol{y}) X\rangle_u + u\delta(\boldsymbol{x} - \boldsymbol{y}) G_a(\boldsymbol{x}) G_b(\boldsymbol{x}) \langle O(\boldsymbol{x}) X\rangle_u \\
&\quad + uG_a(\boldsymbol{x}) \sum_k \delta(\boldsymbol{y} - \boldsymbol{x}_k) G_{b,k} \langle O(\boldsymbol{x}) X\rangle_u + \boldsymbol{x} \leftrightarrow \boldsymbol{y} \\
&\quad - \sum_{k,l} \delta(\boldsymbol{x} - \boldsymbol{x}_k)\delta(\boldsymbol{y} - \boldsymbol{x}_l) G_{a,k} G_{b,l} \langle X\rangle_u,
\end{aligned} \tag{6.50}$$

where the summation $\sum_k$ on the right hand side of the last equation only runs over the operators inside X. We verify that

$$\langle \delta L_a(\boldsymbol{x}) X\rangle_u = -i \sum_k \delta(\boldsymbol{x}-\boldsymbol{x}_k) G_{a,k} \langle X\rangle_u, \tag{6.51a}$$

$$\begin{aligned}\langle \delta L_a(\boldsymbol{x}) \delta L_b(\boldsymbol{y}) X\rangle_u &= \langle \partial_\mu \partial_\nu j_a^\mu(\boldsymbol{x}) j_b^\nu(\boldsymbol{y}) X\rangle_u - u^2 G_a(\boldsymbol{x}) G_b(\boldsymbol{y}) \langle O(\boldsymbol{x}) O(\boldsymbol{y}) X\rangle_u \\ &\quad + iu G_a(\boldsymbol{x}) \langle \partial_\nu j_b^\nu(\boldsymbol{y}) O(\boldsymbol{x}) X\rangle_u + \boldsymbol{x} \leftrightarrow \boldsymbol{y} \\ &= -\sum_{k,l} \delta(\boldsymbol{x}-\boldsymbol{x}_k)\delta(\boldsymbol{y}-\boldsymbol{x}_l) G_{a,k} G_{b,l} \langle X\rangle_u \\ &\quad - u\delta(\boldsymbol{x}-\boldsymbol{y}) G_a(\boldsymbol{x}) G_b(\boldsymbol{x}) \langle O(\boldsymbol{x}) X\rangle_u,\end{aligned} \tag{6.51b}$$

in agreement with (6.46–6.47).

6.A.3 Trace Ward identities

Returning to the integral expressions (6.46–6.47), we consider the special choice for $\omega_a = \omega(\boldsymbol{x})(1_D, -x^\nu 1_T)$ to derive the trace insertion formulae, where 1_D and 1_T means we consider the dilational and translational transformations. The combined effect of this familiar combination [98, 271, 289] yields

$$\begin{aligned}\omega_a(\boldsymbol{x}) G_a(\boldsymbol{x}) &= \omega(\boldsymbol{x}) \left(-i(x^\mu \partial_\mu + \Delta) - x^\nu (-i\partial_\nu)\right) = -i\omega(\boldsymbol{x})\Delta, \\ \omega_a(x) \delta L_a(\boldsymbol{x}) &= \omega(\boldsymbol{x}) \left(-\Theta_0(\boldsymbol{x}) - u\Delta O(\boldsymbol{x})\right) = \omega(\boldsymbol{x}) \left(-\Theta_u(\boldsymbol{x}) - u(\Delta-3) O(\boldsymbol{x})\right).\end{aligned}$$

In the last line we rewrite the answer in terms of the stress-energy tensor of the *perturbed* theory,

$$\Theta_u = \left.\frac{-2}{\sqrt{h}} h^{\alpha\beta} \frac{\delta S}{\delta h^{\alpha\beta}}\right|_{h_{\alpha\beta}=\delta_{\alpha\beta}} = \Theta_0 + \frac{-2}{\sqrt{h}} h^{\alpha\beta} \left.\frac{-1}{2} \sqrt{h} h_{\alpha\beta}\right|_{h_{\alpha\beta}=\delta_{\alpha\beta}} uO = \Theta_0 + 3uO. \tag{6.52}$$

The Ward identities can then be written as

$$\langle \Theta_u(\boldsymbol{x}) X\rangle_u = -u(\Delta-3)\langle O(\boldsymbol{x}) X\rangle_u + \sum_k \delta(\boldsymbol{x}-\boldsymbol{x}_k) \Delta_k \langle X\rangle_u \tag{6.53}$$

and

$$\begin{aligned}\langle\Theta_u(\boldsymbol{x})\Theta_u(\boldsymbol{y})X\rangle_u &= \sum_{k,l}\delta(\boldsymbol{x}-\boldsymbol{x}_k)\Delta_k\delta(\boldsymbol{y}-\boldsymbol{x}_l)\Delta_l\langle X\rangle_u + u\delta(\boldsymbol{x}-\boldsymbol{y})\Delta^2\langle O(\boldsymbol{x})X\rangle_u \\ &\quad - u(\Delta-3)\langle\Theta_u(\boldsymbol{x})O(\boldsymbol{y})X\rangle_u - \boldsymbol{x}\leftrightarrow\boldsymbol{y} - u^2(\Delta-3)^2\langle O(\boldsymbol{x})O(\boldsymbol{y})X\rangle_u \\ &= \sum_{k,l}\delta(\boldsymbol{x}-\boldsymbol{x}_k)\Delta_k\delta(\boldsymbol{y}-\boldsymbol{x}_l)\Delta_l\langle X\rangle_u \\ &\quad + u\delta(\boldsymbol{x}-\boldsymbol{y})\left(\Delta^2-2(\Delta-3)\Delta\right)\langle O(\boldsymbol{x})X\rangle_u \\ &\quad - u(\Delta-3)\sum_k\delta(\boldsymbol{x}-\boldsymbol{x}_k)\Delta_k\langle O(\boldsymbol{y})X\rangle_u - \boldsymbol{x}\leftrightarrow\boldsymbol{y} \\ &\quad + u^2(\Delta-3)^2\langle O(\boldsymbol{x})O(\boldsymbol{y})X\rangle_u, \end{aligned} \tag{6.54}$$

where Δ_k is the (full) scaling dimension of the k'th operator $O(\boldsymbol{x}_k)$ inside X.

At this stage, we have to reflect on the correctness of the expressions by performing a consistency check on the two-point function (6.54). When we perturb the conformal field theory with a purely marginal operator, $\Delta = 3$, the renormalization group flow remains in a (different) conformal field theory. The trace Θ_u of this perturbed theory is still vanishing. Hence, the two-point correlation function of the perturbed stress-energy tensor should vanish with respect to the perturbed theory. However, substituting $\Delta = 3$ into our expression,

$$\langle\Theta_u(\boldsymbol{x})\Theta_u(\boldsymbol{y})\rangle_u = u^2(\Delta-3)^2\langle O(\boldsymbol{x})O(\boldsymbol{y})\rangle_u - u\delta(\boldsymbol{x}-\boldsymbol{y})\left((\Delta-3)^2-3^2\right)\langle O(\boldsymbol{x})\rangle_u,$$

does not yield zero. Clearly in our derivation we must have missed a term of the form $-3^2u\delta(\boldsymbol{x}-\boldsymbol{y})\langle O(\boldsymbol{x})\rangle_u$. This term is a contact term and should stem from the neglected second order transformation of $S_0[O]$, which will contain a contribution from the variation of the Noether current. In [281–283] such a contribution is explicitly included for the consistency of the expressions. In our case we can infer the final result based on conceptual reasoning. We thus employ the expression

$$\begin{aligned}\langle\Theta_u(\boldsymbol{x})\Theta_u(\boldsymbol{y})X\rangle_u &= u^2(\Delta-3)^2\langle O(\boldsymbol{x})O(\boldsymbol{y})X\rangle_u - u(\Delta-3)^2\delta(\boldsymbol{x}-\boldsymbol{y})\langle O(\boldsymbol{x})X\rangle_u \\ &\quad - u(\Delta-3)\sum_k\delta(\boldsymbol{x}-\boldsymbol{x}_k)\Delta_k\langle O(\boldsymbol{y})X\rangle_u - \boldsymbol{x}\leftrightarrow\boldsymbol{y} \\ &\quad + \sum_{k,l}\delta(\boldsymbol{x}-\boldsymbol{x}_k)\Delta_k\delta(\boldsymbol{y}-\boldsymbol{x}_l)\Delta_l\langle X\rangle_u, \end{aligned} \tag{6.55}$$

for the double Ward identity.

We similarly derive the correlation function of three traces. Using either of the two methods described above to third order in ω, the Ward identity of three current insertions gives

$$0 = \int \mathrm{d}^3\boldsymbol{x}\mathrm{d}^3\boldsymbol{y}\mathrm{d}^3\boldsymbol{z}\, \omega_a(\boldsymbol{x})\omega_b(\boldsymbol{y})\omega_c(\boldsymbol{z})\langle \delta L_a(\boldsymbol{x})\delta L_b(\boldsymbol{y})\delta L_c(\boldsymbol{z}) X\rangle_u \\ - i \sum_{k,l,m} \omega_{a,k} G_{a,k} \omega_{b,l} G_{b,l} \omega_{c,m} G_{c,m} \langle X\rangle_u + \ldots, \tag{6.56}$$

where the ... contain terms of order u. These terms are double contact terms and, as is clear from the main text, are not relevant for our purposes. Choosing again $\omega_a = \omega(\boldsymbol{x})(1_D, -x^\nu 1_T)$ and using the single and double trace-inserted Ward identities (6.53), (6.55), the three-point function of the stress-energy tensor reads

$$\begin{aligned}\langle \Theta_u(\boldsymbol{x})\Theta_u(\boldsymbol{y})\Theta_u(\boldsymbol{z})X\rangle_u &= -u^3(\Delta-3)^3\langle O(\boldsymbol{x})O(\boldsymbol{y})O(\boldsymbol{z})X\rangle_u \\ &\quad - u^2(\Delta-3)^2\langle \Theta_u(\boldsymbol{x})O(\boldsymbol{y})O(\boldsymbol{z})X\rangle_u - \boldsymbol{x}\leftrightarrow\boldsymbol{y} - \boldsymbol{x}\leftrightarrow\boldsymbol{z} \\ &\quad - u(\Delta-3)\langle \Theta_u(\boldsymbol{x})\Theta_u(\boldsymbol{y})O(\boldsymbol{z})X\rangle_u - \boldsymbol{z}\leftrightarrow\boldsymbol{x} - \boldsymbol{z}\leftrightarrow\boldsymbol{y} + \ldots \\ &= -u^3(\Delta-3)^3\langle O(\boldsymbol{x})O(\boldsymbol{y})O(\boldsymbol{z})X\rangle_u \\ &\quad + u^2(\Delta-3)^3\delta(\boldsymbol{y}-\boldsymbol{z})\langle O(\boldsymbol{x})O(\boldsymbol{y})X\rangle_u + \boldsymbol{x}\leftrightarrow\boldsymbol{y} + \boldsymbol{y}\leftrightarrow\boldsymbol{z} + \ldots,\end{aligned} \tag{6.57}$$

where the ... contain highly local contributions.

6.B The Fourier transform of the three-point function

6.B.1 Feynman parameters

In a conformal field theory, the three-point correlation function of an operator O with conformal dimension Δ is determined by the symmetries to be of the form

$$\langle O(\boldsymbol{x}_1)O(\boldsymbol{x}_2)O(\boldsymbol{x}_3)\rangle_0 = \frac{C}{(x_{12}x_{13}x_{23})^\Delta},$$

in position-space. To find the momentum dependence, one has to perform a Fourier transform,

$$F_\Delta(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = C\int \mathrm{d}^3\boldsymbol{x}_1\mathrm{d}^3\boldsymbol{x}_2\mathrm{d}^3\boldsymbol{x}_3\, e^{i\sum_j \boldsymbol{k}_j\cdot\boldsymbol{x}_j}(x_{12}x_{13}x_{23})^{-\Delta}.$$

In practice, for arbitrary Δ, this quickly becomes difficult. In the appendix of [255] it is explained how the result can be written as an integral over Feynman parameters. For completeness, we shortly review this approach here.

Inserting the Fourier transform,

$$|\boldsymbol{x}|^{-\Delta} = B(\Delta)(2\pi)^{-3}\int \mathrm{d}^3\boldsymbol{p}\,|\boldsymbol{p}|^{\Delta-3}e^{-i\boldsymbol{p}\cdot\boldsymbol{x}}, \qquad \text{where } B(\Delta) = 2^{3-\Delta}\pi^{3/2}\frac{\Gamma\left(\frac{3-\Delta}{2}\right)}{\Gamma\left(\frac{\Delta}{2}\right)},$$

into F_Δ, the $\boldsymbol{x}_j$ integrals can be performed explicitly and also two of the momentum integrals can be done to give

$$F_\Delta(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = CB^3(\Delta)\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)\int \mathrm{d}^3\boldsymbol{p}\,(p^2|\boldsymbol{p}-\boldsymbol{k}_1|^2|\boldsymbol{p}+\boldsymbol{k}_2|^2)^{\lambda/2}, \tag{6.58}$$

where $\lambda = \Delta - 3$. As is usual, the difficult dot-product dependence in the integrand can be rewritten with the use of Feynman parameters. Using

$$\begin{aligned}A^{-\alpha}B^{-\beta}C^{-\gamma} = \frac{\Gamma(\alpha+\beta+\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}&\int_0^1 \mathrm{d}u\int_0^1 \mathrm{d}v\,\times \\ &u^{\alpha-1}(1-u)^{\beta-1}v^{\alpha+\beta-1}(1-v)^{\gamma-1}\left[uvA+(1-u)vB+(1-v)C\right]^{-\alpha-\beta-\gamma},\end{aligned}$$

the integral in (6.58) equals

$$\frac{\Gamma(-3\lambda/2)}{\Gamma^3(-\lambda/2)}\int_0^1\int_0^1 \mathrm{d}u\mathrm{d}v\;\left[(1-u)(1-v)\right]^{-(1+\lambda/2)}\,v^{-(1+\lambda)}G(u,v,\boldsymbol{k}_1,\boldsymbol{k}_2),$$

where

$$\begin{aligned}G(u,v,\boldsymbol{k}_1,\boldsymbol{k}_2) &= \int \mathrm{d}^3\boldsymbol{p}\left[uvp^2+v(1-u)|\boldsymbol{p}-\boldsymbol{k}_1|^2+(1-v)|\boldsymbol{p}+\boldsymbol{k}_2|^2\right]^{3\lambda/2} \\ &= \int \mathrm{d}^3\boldsymbol{p}\,(p^2+a^2)^{3\lambda/2} = \pi^{3/2}\frac{\Gamma(-\frac{3}{2}(1+\lambda))}{\Gamma(-3\lambda/2)}(a^2)^{\frac{3}{2}(1+\lambda)}.\end{aligned}$$

The second identity follows from a shift of the momentum $\boldsymbol{p}$. We have written a as a shorthand notation for

$$a^2 = (1-u)v(1-(1-u)v)k_1^2 + v(1-v)k_2^2 + 2v(1-u)(1-v)\boldsymbol{k}_1\cdot\boldsymbol{k}_2,$$

which can be expressed fully in terms of the sizes of the three momenta, k_1, k_2 and k_3 due to momentum conservation. Collecting results and changing variables $u\to 1-u$, $v\to 1-v$, the Fourier transform of the three-point function is given by

$$F_\Delta(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3) = C\frac{2^{3-3\Delta}(2\pi)^6}{\Gamma^2(\frac{\Delta}{2})}\delta(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3)k_1^{3\Delta-6}S_\Delta(X,Y), \tag{6.59}$$

where the shape function $S_\Delta(X, Y)$ of the ratios $X = \frac{k_2^2}{k_1^2}$, $Y = \frac{k_3^2}{k_1^2}$ is given by [255]

$$S_\Delta(X, Y) = \frac{\Gamma\left(3 - \frac{3\Delta}{2}\right)}{\Gamma\left(\frac{\Delta}{2}\right)} \int_0^1 \mathrm{d}u \int_0^1 \mathrm{d}v \, \frac{[u(1-u)v]^{\frac{1}{2}-\frac{\Delta}{2}}(1-v)^{\frac{\Delta}{2}-1}}{[u(1-u)(1-v) + (1-u)vX + uvY]^{3-\frac{3\Delta}{2}}}$$
$$= \frac{2}{\sqrt{\pi}}\Gamma(3 - \frac{3\Delta}{2})\Gamma(\frac{3}{2} - \frac{\Delta}{2})\times$$
$$\int_0^1 \mathrm{d}u \, \frac{[u(1-u)]^{\frac{1}{2}-\frac{\Delta}{2}}}{[(1-u)X + uY]^{3-\frac{3\Delta}{2}}} {}_2F_1\left(3 - \frac{3\Delta}{2}, \frac{\Delta}{2}; \frac{3}{2}; Z(X, Y, u)\right). \quad (6.60)$$

The hypergeometric function ${}_2F_1$ depends on u and the shape of the momentum conserving triangle via

$$Z(X, Y, u) = 1 - \frac{u(1-u)}{(1-u)X + uY}.$$

6.B.2 Bulk-boundary identity

The integral (6.60) over the hypergeometric function can not be evaluated for arbitrary values of Δ. Moreover it has divergences when Δ has integer values. Regulating the divergences is not easy, since the Feynman parameters u and v do not have a clear-cut physical meaning. This is unfortunate, since from the gravity calculation we have reasons to believe that the three-point function actually has a clean momentum dependence, which for (6.60) remains hidden in the Feynman integral. For this reason, we pursue a different approach to find the Fourier transforms. Although our technique is borrowed from AdS/CFT and is reminiscent of the actual gravity in-in calculation [57, 254], the calculation is to be understood as a pure mathematical identity, whose AdS/CFT-origin is not of particular relevance. The identity we will use is, cf. (3.36) [149, 150],

$$\frac{a(\Delta)}{(x_{12}x_{23}x_{13})^\Delta} = \quad (6.61)$$
$$\int_{z_0}^\infty \frac{\mathrm{d}z}{z^4} \mathrm{d}^3\boldsymbol{x}_1 \mathrm{d}^3\boldsymbol{x}_2 \mathrm{d}^3\boldsymbol{x}_3 \mathrm{d}^3\boldsymbol{w} \, K_{\mathrm{bb}}(\Delta; z, |\boldsymbol{x}_1 - \boldsymbol{w}|) K_{\mathrm{bb}}(\Delta; z, |\boldsymbol{x}_2 - \boldsymbol{w}|) K_{\mathrm{bb}}(\Delta; z, |\boldsymbol{x}_3 - \boldsymbol{w}|),$$

where $a(\Delta)$ depends on Δ,

$$a(\Delta) = \frac{\Gamma\left(\frac{3}{2}(\Delta - 1)\right)\Gamma\left(\frac{\Delta}{2}\right)^3}{2\pi^3\Gamma\left(\Delta - \frac{3}{2}\right)^3},$$

and where $K_{\rm bb}(\Delta; z, \boldsymbol{z}, \boldsymbol{x})$ is the bulk-boundary propagator in anti-de Sitter spacetime,

$$K_{\rm bb}(\Delta; z, \boldsymbol{z}, \boldsymbol{x}) = \frac{\Gamma(\Delta)}{\pi^{\frac{3}{2}}\Gamma(\Delta - \frac{d}{2})}\left(\frac{z}{z^2 + (\boldsymbol{z} - \boldsymbol{x})^2}\right)^{\Delta}.$$

Equation (6.61) has to be understood as a regulated expression, picking out the regular part as $z \to 0$.

We can Fourier transform the left hand side of (6.61) by Fourier transforming each of the bulk-boundary propagators [148]

$$K_{\rm bb}(\Delta; z, z_0, k) = \left(\frac{z}{z_0}\right)^{\frac{3}{2}} \frac{K_\nu(kz)}{K_\nu(kz_0)},$$

where $K_\nu(x)$ is the modified Bessel function of the second kind and $\nu = \Delta - \frac{3}{2}$. The special point z_0 is used to normalize the bulk-boundary propagator. In Fourier-space the three-point function $\langle O_\Delta(k_1)O_\Delta(k_2)O_\Delta(k_3)\rangle_0 = C\delta(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_3)R_\Delta(k_1, k_2, k_3)$ is then equal to

$$R_\Delta(k_1, k_2, k_3) = \frac{1}{a(\Delta)}\int_{z_0}^{\infty} \frac{{\rm d}z}{z^4} K_{\rm bb}(\Delta; z, z_0, k_1)K_{\rm bb}(\Delta; z, z_0, k_2)K_{\rm bb}(\Delta; z, z_0, k_3). \quad (6.62)$$

6.B.3 The marginal case

The expression (6.62) is particularly simple for marginal operators O with $\Delta = 3$ or equivalently $\nu = \frac{3}{2}$. In that case, the Taylor series of the Bessel function terminates, leaving a simple expression for the bulk-boundary propagator

$$K_{\rm bb}(3; z, z_0, k) = e^{-k(z-z_0)}\frac{1 + kz}{1 + kz_0}.$$

The integral (6.62) can be done explicitly and is of the form

$$R_3(k_1, k_2, k_3) = \frac{I_{-3}}{z_0^3} + \frac{I_{-1}}{z_0} + I_0(k_1, k_2, k_3)z_0^0 + O(z_0). \quad (6.63)$$

The first two terms are singular in the limit $z_0 \to 0$. At the same time, these terms are odd in z_0, indicating that they are imaginary contributions if we would do the Wick-rotation from the anti-de Sitter z_0 to conformal time τ_* [281]. Therefore, the leading contribution comes from the regular coefficient multiplying z_0^0. This term gives the well-known result [77, 78, 257]

$$I_0(k_1, k_2, k_3) = -\frac{1}{3a(3)}\left((-1 + \gamma + \log[-k_t\tau_*])\sum_{j=1}^{3} k_j^3 + k_1k_2k_3 - \sum_{j\neq l} k_jk_l^2\right), \quad (6.64)$$

for spectator fields in a de Sitter background, cf. (2.22).

6.B.4 The nearly marginal case

For general Δ, the expression (6.62) is less easy to evaluate. When Δ is nearly marginal, $\Delta \approx 3$, we may approximate the result via a Taylor series,

$$R_\Delta(k_1, k_2, k_3) = R_3(k_1, k_2, k_3) + (\Delta - 3)\, \partial_\Delta R_\Delta(k_1, k_2, k_3)|_{\Delta=3} + O(\lambda^2). \tag{6.65}$$

This approach is very similar to the one employed in the appendix of [79], in which higher order corrections in the slow-roll expansion are calculated. The zeroth order term $R_3(k_1, k_2, k_3)$ equals the result of the previous section, which now gets corrections of order $\Delta - 3$ proportional to the derivative of R_Δ at $\Delta = 3$.

Using one of the explicit formulae for the Bessel function [290], $K_\nu = \frac{\sqrt{\pi}}{\Gamma(\nu+\frac{1}{2})}\tilde{K}_\nu$, where

$$\tilde{K}_\nu(Z) = \left(\frac{Z}{2}\right)^\nu \int_1^\infty \mathrm{d}x\, e^{-Zx}(x^2-1)^{\nu-\frac{1}{2}},$$

the first order contribution $\partial_\Delta R_\Delta|_{\Delta=3}$ can be written as

$$\begin{aligned}\partial_\Delta R_\Delta|_{\Delta=3} = & -R_3(k_1,k_2,k_3)\left(\frac{\partial_\Delta a(\Delta)}{a(\Delta)} + \frac{\partial_\nu \tilde{K}_\nu(k_1 z_0)}{\tilde{K}_\nu(k_1 z_0)} + \frac{\partial_\nu \tilde{K}_\nu(k_2 z_0)}{\tilde{K}_\nu(k_2 z_0)} + \frac{\partial_\nu \tilde{K}_\nu(k_3 z_0)}{\tilde{K}_\nu(k_3 z_0)}\right)\Bigg|_{\Delta=3} \\ & + L(k_1;k_2,k_3) + L(k_2;k_1,k_3) + L(k_3;k_1,k_2), \qquad \text{where} \end{aligned} \tag{6.66a}$$

$$L(k_1;k_2,k_3) = \frac{1}{a(3)} \int_{z_0}^\infty \frac{\mathrm{d}z}{z^4} \partial_\nu \tilde{K}_\nu(k_1 z) \frac{\tilde{K}_\nu(k_2 z)\tilde{K}_\nu(k_3 z)}{\tilde{K}_\nu(k_1 z_0)\tilde{K}_\nu(k_2 z_0)\tilde{K}_\nu(k_3 z_0)}\Bigg|_{\nu=\frac{3}{2}}. \tag{6.66b}$$

The advantage of using $\tilde{K}_\nu$ is that its derivative can be explicitly evaluated,

$$\begin{aligned}\partial_\nu \tilde{K}_\nu(Z)\big|_{\nu=\frac{3}{2}} &= \left(\frac{Z}{2}\right)^{3/2} \int_1^\infty \mathrm{d}x\, e^{-Zx}(x^2-1)\log\left[\frac{Z}{2}(x^2-1)\right] \\ &= 2(2Z)^{-3/2}e^{-Z}\left(3 + Z - \gamma(1+Z) + (-1+Z)e^{2Z}\mathrm{Ei}(-2Z)\right), \end{aligned} \tag{6.67}$$

where $\mathrm{Ei}(Z) = -\int_{-Z}^\infty \mathrm{d}t\, \frac{e^{-t}}{t}$ is the exponential integral function. Hence, the contribution from the first line of (6.66a) can be directly found from an expansion in z_0. Again, since all singular terms are odd, the leading order contribution comes from the z_0^0-term. The result resembles (6.64), but now also includes terms proportional to $\log[-k_j\tau_*]$.

To find $L(k_1;k_2,k_3)$, the derivative of the Bessel function (6.67) needs to be integrated over $z \in (z_0, \infty)$. The total contribution consists of two parts, one part $L^{(1)}$ coming from the integration of $e^{2k_1 z}\mathrm{Ei}(-2k_1 z)$ and the other part $L^{(2)}$ from the integration over the other terms. The latter contribution can be readily done and it gives an answer of the form

$$L^{(2)}(k_1;k_2,k_3) = \frac{L^{(2)}_{-3}}{z_0^3} + \frac{L^{(2)}_{-2}}{z_0^2} + \frac{L^{(2)}_{-1}}{z_0} + L^{(2)}_0 z_0^0 + O(z_0). \tag{6.68}$$

In this case, the leading order contribution comes from the even, divergent, contribution $L^{(2)}_{-2}$, which is given by

$$L^{(2)}_{-2}(k_1;k_2,k_3) = -k_1\pi^{5/2}. \tag{6.69}$$

Interestingly, this contribution is not homogeneous of degree 3 in the momenta. The contribution $L^{(2)}_0$ is

$$\begin{aligned} L^{(2)}_0(k_1;k_2,k_3) = \frac{\pi^{5/2}}{3}\Big(&(k_2^3+k_3^3)((\gamma-1)(\gamma-3)+\gamma\log[-k_t\tau_*]) \\ &+ k_1^3(\gamma(\gamma-1)-3+\gamma\log[-k_t\tau_*]) \\ &- \gamma(k_1^2k_2+k_1^2k_3) + (k_1k_2^2+k_1k_3^2)(3-4\gamma-3\log[-k_t\tau_*]) \\ &+ \gamma k_1k_2k_3 + (k_2^2k_3+k_2k_3^2)(\gamma-3)(\gamma-2)\Big). \end{aligned} \tag{6.70}$$

It is again a term that looks very much like (6.64), except for its coefficients.

In order to calculate the integral $L^{(1)}$ over the exponential integral function, we consider the integral of each term of its Taylor series expansion separately,

$$e^Z\mathrm{Ei}(-Z) = \sum_{n=0}^{\infty}\left[\frac{1}{n!}(\gamma+\log[Z])\right]Z^n + \sum_{n=1}^{\infty}\left[\sum_{j=1}^{n}\left(\frac{(-1)^j}{jj!(n-j)!}\right)\right]Z^n.$$

The integral over each of the summands can be straightforwardly computed. Furthermore the restrictions $L^{(1)}_{-2}$ and $L^{(1)}_0$ of $L^{(1)}$ to the z_0^{-2}- and z_0^0-contributions can be resummed into closed expressions. Both expressions are too long to include in this appendix. Most important is the generic momentum dependence, which can already be appreciated by inspection. The contribution $L^{(1)}_{-2}(k_1;k_2,k_3)$ is a polynomial of degree 1 in the momenta, with logarithmic dependence on each of the k_j's. The contribution $L^{(1)}_0(k_1;k_2,k_3)$ is a sixth degree polynomial multiplied by $\frac{1}{k_t^3}$ with polylogarithmic dependence on k_t, but also on $k_1-k_2-k_3$. These expressions show the type of mixing that can occur between the different momenta. After a decent rewriting they may unveil the typical $1/k_t$-momentum dependence of the slow-roll result, although at the moment it seems that there is a much more exotic dependence on the momenta.