# Conformal invariance and microscopic sensitivity in cosmic inflation

Aalst, T.A.F. van der

**Author**: Aalst, Ted Adrianus Franciscus van der
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# 5

# Worldsheet cosmology

In the previous chapter we have discussed the difficulties one faces when studying inflation in a separated but controlled environment in any supergravity theory. We have seen that there is a substantial worry that other parts of the theory will contribute to inflation in a non-negligible fashion. In this chapter we will capitalize on precisely this, employing the opportunity inflation provides to constrain unknown physics. To incorporate a complete system, we have to go back to the roots of string theory. Therefore, our approach starts from the worldsheet description of string theory, using conformal invariance to investigate the (coarse) constraints that inflation imposes on the theory. The chapter is based on [226].

## 5.1 Introduction

The last ten years many attempts have been made to understand inflation from a more fundamental level within string theory [1, 197, 227–230]. Cosmological observations strongly suggest an era of inflation in the early universe, and string theory, being a quantum theory of gravity with a unique UV-completion, should be able to describe this. In addition, inflation generically probes energy scales that are unobtainable in accelerator experiments, and there is a chance that string scale effects may be detectable in future cosmological observations [51, 231–236].

One of the essential characteristics of inflation is that it solves the flatness and horizon problem *within classical general relativity* [25–27]. Moreover, inflation is a very coarse phenomenon that only depends on the energy density and pressure in the universe without a need to specify any details of the matter content. In string theory the equations of motion of classical general relativity are the conditions of conformal invariance of the worldsheet string theory. As such, a string theoretic description of inflation should only depend on very generic scaling properties of the conformal field theory on the worldsheet.

Extending worldsheet descriptions of tachyon condensation scenarios [111, 237, 238], we will attempt to describe inflation with a worldsheet theory that is a combination of a spacetime and matter-part, which mix via spacetime dependent couplings $u^a(x)$ for operators $O_a$ of an abstract internal conformal field theory. From the viewpoint of the internal conformal field theory alone such a deformation induces an internal renormalization group flow. Total conformal invariance of the combined theory can only be kept if the background fields adjust themselves in such a way that the running induced by the scaling behavior of the operators $O_a$ of the internal conformal field theory is canceled. The renormalization group flow can therefore be seen to define the possible dependence of $u^a(x)$ on the spacetime coordinates $x^\mu$, or in other words the $\beta$ functions of the full theory determine the equations of motion for the background fields $u^a(x)$. These equations can be compared to slow-roll inflation to find conditions on the internal conformal field theory. We shall indeed find that, from the worldsheet perspective, the inflationary slow-roll parameters are completely characterized by the central charge and the scaling behavior of the couplings of the conformal field theory, in line with our expectation that inflation is a phenomenon that only depends on generic properties of the matter sector.

This is not to say that we have solved inflation in string theory. Describing strings in a time-dependent background is notoriously difficult. In a large part this is due to our lack of a background independent description of the theory. At low energies we can resort to a supergravity description, but inflation fits awkwardly in the low energy supergravity framework ($\eta$-problem, Lyth-bound, absence of de Sitter solutions [239]). As recently emphasized [240], one almost certainly needs stringy ingredients to describe accelerating backgrounds. The worldsheet approach is conceptually different from supergravity calculations, but it has its own drawbacks when trying to describe a string in a de Sitter-like background. At tree-level (in $g_s$), we are only able to describe small deviations from Minkowski spacetime rather than de Sitter spacetime, as is well known [241–245]. Inflationary solutions are a larger class of accelerating spacetimes than pure de Sitter, so one could optimistically hope for a better fit into string theory. Nevertheless, they are closely related to pure de Sitter and we may already anticipate problems to describe them for the same reason. Substituting the solutions to the $\beta$ functions into the formal expressions, we indeed find a similar divergence due to the fact that the dilaton cannot be stabilized in tree-level string theory and with a dynamical dilaton inflation does not occur. This is of course the Fischler-Susskind phenomenon [241, 242]. This, however, is not the main point. We wish to show that, inflation being a coarse phenomenon, it only depends on coarse details of the internal conformal field theory. That we do, formally, while at the same time we recover the known Fischler-Susskind result that any tree-level string theory model is ruled out as a theory for inflation.

This chapter is structured as follows: first we describe the worldsheet set-up suitable for inflation and derive the equations of motion. We review multi-field slow-roll inflation in section 5.3, so that in section 5.4 we can state our main result. We shortly discuss the possibility to generalize the results to higher loop order. We conclude discussing the relation between our results with results known from the literature [244, 245].

## 5.2 Background dynamics for a generic worldsheet theory

### 5.2.1 Conformal perturbation of a coupled gravity and matter system

We wish to describe a realistic model of inflation in string theory, i.e. there is a $3+1$-dimensional homogeneous and isotropic cosmological spacetime which experiences accelerated expansion. Similar to phenomenological model building, we are naturally led to consider a worldsheet conformal field theory consisting of two parts: a nonlinear $\sigma$ model accounting for four-dimensional gravity in combination with a matter/internal theory [111, 237]. The nonlinear sigma model is a curved bosonic string in four dimensions, $\mu, \nu \in \{0, 1, 2, 3\}$,

$$S_{\mathrm{NL}\sigma\mathrm{M}} = S_{g(x)} + S_{\Phi(x)}, \tag{5.1a}$$

$$S_{g(x)} = \frac{1}{2\pi\alpha'} \int \mathrm{d}^2 z\, g^{(S)}_{\mu\nu}(x)\partial x^\mu \bar{\partial} x^\nu, \tag{5.1b}$$

$$S_{\Phi(x)} = \frac{1}{4\pi} \int \mathrm{d}^2 z\, \sqrt{h}\, \Phi(x) R^{(2)}, \tag{5.1c}$$

with $g^{(S)}_{\mu\nu}$ the four-dimensional string frame metric and $h_{\alpha\beta}$ the Euclidean worldsheet metric. To keep the discussion simple we will set the Neveu-Schwarz form to zero, $B_{\mu\nu} = 0$, but we do consider the effect of the dilaton. The dilaton is a (light) scalar and is naturally a part of cosmological dynamics or any time-dependent scenario, e.g. tachyon condensation [111]. More importantly, the dilaton is closely related to the scale factor of the Einstein frame metric and as such could be driving part of the cosmological expansion.

The internal theory will be some two-dimensional conformal field theory $S_0$ with central charge $c$ and (primary and descendant) operators $O_a$ with scaling dimensions $\Delta_a$. We purposely leave the theory unspecified. The goal of this study is to deduce what type of internal conformal field theory, i.e. which constraints on the central charge and operator dimensions and couplings, could give rise to a realistic model for inflation. Since FLRW cosmological dynamics only cares about coarse characteristics of the matter, viz. pressure and energy, we expect that only coarse information about the internal conformal field theory should be needed to deduce cosmological dynamics. Because time-dependent backgrounds must break supersymmetry, we can incorporate all the fermionic partners to $x^\mu$ and the worldsheet diff $\times$ Weyl- and supersymmetry ghosts into the internal conformal field theory.[1] The internal conformal field theory will exhibit characteristic scaling behavior under a deformation by nonzero couplings $u^a$ to the primary operators,

$$S = S_0 + S_\Phi + S_u, \tag{5.2a}$$

$$S_\Phi = \frac{1}{4\pi}\int \mathrm{d}^2z\,\sqrt{h}\,\Phi R^{(2)}, \tag{5.2b}$$

$$S_u = \int \mathrm{d}^2z\, u^a O_a. \tag{5.2c}$$

This behavior is intrinsic to the internal theory and fully captured by the $\beta$ functions $\overline{\beta}^a(u)$ of the couplings $u^a$, whose lowest order (classical) contribution is given by $(\Delta_a - 2)u^a$. We have again included the (constant part of the) dilaton $\Phi$ here as a (non $x$-dependent) coupling to the worldsheet curvature $R^{(2)}$ in order to easily incorporate the Weyl anomaly contribution of the internal theory. At a renormalization group fixed point of this perturbed conformal field theory, $\overline{\beta}^\Phi(u)$ will just be proportional to the central charge of the internal conformal field theory, cf. (3.23),

$$\overline{\beta}^\Phi(u) = \frac{c}{6} + O(u).$$

Due to the conformal perturbations of the internal theory, higher order effects in $u$ will result in a "running" of $\overline{\beta}^\Phi$ [110, 246].

To obtain spacetime dynamics driven by the matter sector, we couple the internal theory plus dilaton to the Polyakov nonlinear $\sigma$ model into a full worldsheet theory

[1] One could keep supersymmetry manifest in principle but it is technically far more involved: with the worldsheet supersymmetric string one needs to track the GSO projection carefully whereas the superspace Green-Schwarz string does not lend itself easily to non-supersymmetric backgrounds. Essentially all these technicalities reside in the internal sector and it is not clear what one would gain by tracking them closely.

with a cross-coupling $u^a(x)O_a$ between the two sectors,[2]

$$S_{\rm tot} = S_{g(x)} + S_0 + S_{\Phi(x)} + S_{u(x)}, \tag{5.3a}$$

$$S_{u(x)} = \int {\rm d}^2 z\, u^a(x) O_a. \tag{5.3b}$$

The couplings $u^a(x)$ to the internal conformal field theory operators $O_a$ depend on the spacetime coordinates $x^\mu$. Since a consistent string theory is described by a conformal worldsheet theory, the *full* operators $u^a(x)O_a$ are assumed to be exactly marginal deformations of the theory. That is, the total theory must remain conformally invariant and the spacetime equations of motion are given by the requirement that the $\beta$ functions of the full theory vanish [103, 105, 117].

The $\beta$ functions of the coupling functionals $g_{\mu\nu}(x)$, $u^a(x)$ and $\Phi(x)$ are readily computed using worldsheet techniques and conformal perturbation theory [111]. We give a brief summary in appendix 5.A. Here we simply state the result,

$$0 = \frac{1}{\alpha'}\beta^g_{\mu\nu} = R_{\mu\nu} - M_{ab}(u)\nabla_\mu u^a \nabla_\nu u^b + 2\nabla_\mu\nabla_\nu\Phi, \tag{5.4a}$$

$$0 = \frac{1}{\alpha'}\beta^a = \frac{1}{\alpha'}\overline{\beta}^a(u) - \frac{1}{2}{\rm D}\nabla u^a + \nabla^\rho\Phi\nabla_\rho u^a, \tag{5.4b}$$

$$0 = \frac{1}{\alpha'}\beta^\Phi = U(u) - \frac{1}{2}\nabla^2\Phi + (\nabla\Phi)^2\,, \tag{5.4c}$$

where $M_{ab}(u)$ is the *positive definite* Zamolodchikov metric on the space of coupling constants [110, 111],

$$M_{ab}(u) = 4\pi^2 \langle O_a(\epsilon) O_b(0)\rangle_u.$$

We denote its connection by $K^a_{bc}$ and we have defined a covariant derivative [193]

[2] As in [111] we do not include cross couplings in the dilatonic sector, $\int {\rm d}^2 z\, \Phi^a(x) O_a R^{(2)}$, nor do we consider a further dependence of the spacetime metric on the internal degrees of freedom through a "warped geometry" cross coupling $\int {\rm d}^2 z\, g^a_{\mu\nu}(x)\partial x^\mu \overline{\partial} x^\nu O_a$. Conformal perturbation theory is only valid when all operators are marginal or nearly marginal, in which case the corresponding couplings describe nearly massless string excitations and the deviation away from the conformal product structure is small. When we assume the operators $O_a$ to be nearly marginal, i.e. $|\Delta_a - 2| \ll 1$, the couplings $\Phi^a$ and $g^a_{\mu\nu}$ are highly irrelevant and describe very massive string excitations. As such they will fall outside the range of validity. In order to describe these more general cross couplings, a different set of operators $\mathcal{A}_\alpha$ with dimension nearly zero would have to be introduced to combine with the $\Phi^\alpha(x)R^{(2)}$- and $g^\alpha_{\mu\nu}(x)\partial x^\mu \overline{\partial} x^\nu$-operators. Considering the ubiquity of warped solutions in string inflation, it would be interesting to extend the computation below to such solutions. One should bear in mind however that almost all warped solutions other than a non-trivial dilaton involve contributions from different worldsheet topologies [102], cf. the discussion in section 5.4.2.

$D\nabla u^a$ and scalar function $U(u)$ respectively by[3]

$$D\nabla u^a = \nabla^\rho \nabla_\rho u^a + K^a_{bc} \nabla^\rho u^b \nabla_\rho u^c, \tag{5.5a}$$

$$U(u) = \frac{c_x}{6\alpha'} + \frac{1}{\alpha'} \overline{\beta}^\Phi(u). \tag{5.5b}$$

The scalar function $U(u)$ accounts for the different quantum Weyl anomalous effects. There are contributions from the central charges of the two components of the theory, $c_x = 4$ and $c \equiv 6\overline{\beta}^\Phi(0)$, and in addition there are higher order effects in $u$, which are collected in the non-constant parts of $\overline{\beta}^\Phi(u)$.

The actual computation of the $\beta$ functions combines two methods with distinct perturbative expansions: conformal perturbation theory where $u^a$ and $\Delta_a - 2$ are small and $\overline{\beta}^a(u) = (\Delta_a - 2)u^a + \ldots$ is known exactly, and separately the background field method where $u^a$ can be large but $\overline{\beta}^a(u)$ and $\nabla u^a$ are required to be small. By allowing for arbitrary $\overline{\beta}^a(u)$ and $\overline{\beta}^\Phi(u)$ these methods can be combined in a mixed $\alpha'$-expansion: it can be made "exact" to all orders in $u^a$, but only to second order in $\nabla u^a$ by capturing all $u$-dependence in the arbitrary unknown functions $M_{ab}(u)$, $\overline{\beta}^a(u)$ and $\overline{\beta}^\Phi(u)$. Note that $\beta^g_{\mu\nu}(u)$ only depends on $\nabla u^a$ as the two sectors of the total theory decouple when $u^a$ is $x$-independent. Limiting ourselves to two derivatives is not an impediment, since inflation should be captured by a two derivative description, especially slow-roll inflation.

### 5.2.2 String dynamics from an action

The condition for Weyl invariance $\beta^g_{\mu\nu} = \beta^a = \beta^\Phi = 0$ determines the equations of motion for the background fields $\Phi(x)$, $g_{\mu\nu}(x)$ and $u^a(x)$. A crucial ingredient for the consistency of this interpretation is the coupling between the dilaton field $\Phi(x)$ and the other matter fields $u^a(x)$. The potential terms, $\overline{\beta}^a(u)$ and $\overline{\beta}^\Phi(u)$ in (5.4), are not independent but related via

$$M_{ab}(u)\overline{\beta}^b(u) = \partial_a \overline{\beta}^\Phi(u), \tag{5.6}$$

*to all orders in* $u^a$. This result may be derived from the fact that the conformal anomaly $\beta^\Phi$ is a $c$-number rather than an operator by the Wess-Zumino consistency

[3]For later convenience we have rescaled the metric by a factor of $4\pi^2$ compared to more conventional definitions. Furthermore, as can be read in the appendix, the $M_{ab}$ and $U$ used in the main text differ from the corresponding objects in conformal perturbation theory by $u^a$-corrections that are beyond the order of perturbation of interest to us.

condition [105, 108, 111, 117, 120] and cf. section 3.2.1. In particular $\beta^\Phi$ is $x$-independent and hence $\nabla_\mu\beta^\Phi$ vanishes. Since $\beta^g_{\mu\nu} = \beta^a = 0$, we can verify

$$0 = \nabla_\nu\beta^\Phi = \nabla_\nu\left(\beta^\Phi - \frac{1}{4}\beta^{g\mu}{}_\mu\right) = \left(\partial_a\overline{\beta}^\Phi - M_{ab}\overline{\beta}^b\right)\nabla_\nu u^a.$$

The last step follows from the explicit formulae for the $\beta$ functions (5.4). Recall that the $\beta$ functions are derived up to second order in $\nabla u^a$ but are *exact* in powers of zeroth derivatives of $u$ due to the incorporation of all zeroth derivatives of $u$ in the potential functions $\overline{\beta}^a(u)$ and $\overline{\beta}^\Phi(u)$. Whereas our result is only an effective description for the connection between the spacetime and matter sector, the matter sector itself is described exactly.

As a result of the relation (5.6) between $\overline{\beta}^\Phi(u)$ and $\overline{\beta}^a(u)$ the equations of motion can be integrated to an action

$$S_{\text{SF}} = \frac{1}{2\kappa_0^2}\int \mathrm{d}^4x\,\sqrt{g}e^{-2\Phi}\left[R + 4(\nabla\Phi)^2 - M_{ab}\nabla_\mu u^a\nabla^\mu u^b - 4U(u)\right]. \tag{5.7}$$

Transforming to the Einstein frame $\tilde{g}^{(E)}_{\mu\nu} = e^{\Phi_0-\Phi}g^{(S)}_{\mu\nu} = e^{-\tilde{\Phi}}g^{(S)}_{\mu\nu}$, we obtain an action that can be directly compared to standard cosmological models,

$$S_{\text{EF}} = \frac{1}{2\kappa^2}\int \mathrm{d}^4x\,\sqrt{\tilde{g}}\left[\tilde{R} - 2\tilde{\nabla}_\mu\tilde{\Phi}\tilde{\nabla}^\mu\tilde{\Phi} - M_{ab}\tilde{\nabla}_\mu u^a\tilde{\nabla}^\mu u^b - 4e^{2\tilde{\Phi}}U(u)\right]. \tag{5.8}$$

Again, $\kappa = \kappa_0 e^{\Phi_0} = \sqrt{8\pi G_N}$ is the gravitational coupling. The action (5.8) is simply that of a multi-scalar field model coupled to gravity,

$$S_{\text{inflation}} = \frac{1}{\kappa^2}\int \mathrm{d}^4x\,\sqrt{g}\left[\frac{1}{2}R - \frac{1}{2}G_{ij}\partial^\mu\phi^i\partial_\mu\phi^j - V(\phi)\right], \tag{5.9}$$

with the potential

$$V(\phi) = 2e^{-2\Phi_0}e^{2\Phi}U(u), \tag{5.10}$$

where we have defined a multi-scalar field $\phi^i = (\Phi, u^a)^t$ and a metric on the space of fields $G_{ij} = \left(\begin{smallmatrix}2 & 0\\ 0 & M_{ab}\end{smallmatrix}\right)$. Since we will be working in the Einstein frame from here on, we have dropped the tilde on the spacetime metric $g_{\mu\nu}(x)$. The question we wish to investigate is whether the potential (5.10) is flat enough to provide realistic slow-roll inflation. Since $V(\phi)$ is proportional to the $\beta$ function $\overline{\beta}^\Phi(u)$ of the internal sector and the central charge $c_{\text{tot}}$ of the total theory, demanding slow-roll inflation is equivalent to a set of phenomenological constraints on the internal conformal field theory. Before we turn to this question, we quickly review slow-roll inflation in multi-field models.

## 5.3 Multi-field slow-roll inflation

The rapid acceleration of the universe that characterizes inflation arises when the system is potential energy dominated. Current observations favor an adiabatic slow-roll inflationary model of early universe cosmology, whose phenomenology can be described by gravity coupled to a single scalar field. The single field inflationary case was formalized in [34] and shortly explained in section 2.2. Fundamentally there is no reason to have only one scalar field. Indeed in string theory or supergravity one generically has multiple scalar fields, although its characteristic signature, isocurvature fluctuations, is at most 10% of the primordial power spectrum and is at this time not a better fit to the data [10]. The connection to the power spectrum for multi-field slow-roll inflation [26, 27, 247, 248] was formalized in [193, 212, 213]. We shall follow [193].

Minimally coupled multi-field inflation is described by the action (5.9), where $V(\phi)$ is the scalar potential and $G_{ij}$ is the *positive definite* metric on the space of scalar fields. For a flat, homogeneous and isotropic FLRW universe, the independent equations of motion for the generic multi-field action (5.9) are[4]

$$H^2 = \frac{1}{3}\left(\frac{1}{2}G_{ij}\dot{\phi}^i\dot{\phi}^j + V\right), \tag{5.11a}$$

$$0 = \mathrm{D}\dot{\phi}^i + 3H\dot{\phi}^i + g^{ij}\partial_j V, \tag{5.11b}$$

where $\Gamma^i_{jk}$ are the connection coefficients for the metric $G_{ij}$ and where we define

$$\mathrm{D}\dot{\phi}^i = \ddot{\phi}^i + \Gamma^i_{jk}\dot{\phi}^j\dot{\phi}^k,$$

similar to (5.5a). The field equations (5.11) completely determine the dynamics of the model, but are difficult to solve exactly. Therefore, we again consider the slow-roll approximation using the slow-roll parameters [193],

$$\epsilon = -\frac{\dot{H}}{H^2}, \qquad \eta^i = \frac{\mathrm{D}\dot{\phi}^i}{H|\dot{\phi}|}. \tag{5.12}$$

The vector $\boldsymbol{\eta}$ can be decomposed in components parallel $\eta^{\parallel}$ and perpendicular $\eta^{\perp}$ to the field velocity $\dot{\boldsymbol{\phi}}$. Define

$$\boldsymbol{e}_1^i = \frac{\dot{\phi}^i}{|\dot{\phi}|}, \qquad \boldsymbol{e}_2^i = \frac{\mathrm{D}\dot{\phi}^i - \frac{\mathrm{D}\dot{\phi}\cdot\dot{\phi}}{|\dot{\phi}|^2}\dot{\phi}^i}{\left|\mathrm{D}\dot{\phi} - \frac{\mathrm{D}\dot{\phi}\cdot\dot{\phi}}{|\dot{\phi}|^2}\dot{\phi}\right|},$$

[4] There is another equation of motion, $\dot{H} = -|\dot{\phi}|^2/2$, from the spatial part of the variation with respect to the metric, but this also follows from (5.11).

then

$$\eta^{\parallel} = \boldsymbol{e}_1 \cdot \boldsymbol{\eta} = \frac{\mathrm{D}\dot{\phi} \cdot \dot{\phi}}{H|\dot{\phi}|^2}, \qquad \eta^{\perp} = \boldsymbol{e}_2 \cdot \boldsymbol{\eta} = \frac{\left|\mathrm{D}\dot{\phi} - \frac{\mathrm{D}\dot{\phi}\cdot\dot{\phi}}{|\dot{\phi}|^2}\dot{\phi}\right|}{H|\dot{\phi}|}, \tag{5.13}$$

and

$$\eta^i = \eta^{\parallel}\boldsymbol{e}_1^i + \eta^{\perp}\boldsymbol{e}_2^i.$$

Recall that the parameter $\epsilon$ is a direct measure for inflation [34],

$$\ddot{a} > 0 \qquad \Leftrightarrow \qquad \epsilon < 1.$$

$\epsilon$ and $\eta$ together quantify the relative energy contributions of kinetic and potential energy. One can reexpress (5.11) in terms of the slow-roll parameters,

$$H^2 = \frac{V}{3}(1 - \frac{1}{3}\epsilon)^{-1}, \tag{5.14a}$$

$$\dot{\phi}^i + \frac{1}{\sqrt{3V}} g^{ij}\partial_j V = -\frac{1}{3}\sqrt{\frac{2}{3}}\frac{\sqrt{\epsilon V}}{1-\frac{1}{3}\epsilon}\left(\eta^i + \frac{\epsilon\frac{\dot{\phi}^i}{|\dot{\phi}|}}{1+\sqrt{1-\frac{1}{3}\epsilon}}\right). \tag{5.14b}$$

As it is given here, equation (5.14) is exact. It shows precisely which approximation is made by assuming that "potential energy strictly dominates over kinetic energy", which is often the explanation behind slow-roll inflation. Using (5.14) one could obtain results at any order in slow-roll [34, 193]. Limiting ourselves to first order in the approximation, in which $\epsilon$, $\sqrt{\epsilon}\eta^{\parallel}$, $\sqrt{\epsilon}\eta^{\perp} \ll 1$, equation (5.14) reduces to

$$H^2 = \frac{1}{3}V,$$
$$\dot{\phi}^i = -\frac{1}{\sqrt{3V}} g^{ij}\partial_j V.$$

The second equation tells us that slow-roll approximation implies *gradient flow*. Using these equations we see that in the slow-roll approximation

$$\dot{H} = \frac{1}{6\sqrt{\frac{V}{3}}}\partial_i V\dot{\phi}^i = -\frac{\sqrt{3}}{6\sqrt{V}}\frac{1}{\sqrt{3V}} g^{ij}\partial_i V\partial_j V = -\frac{1}{6V}|\nabla V|^2, \tag{5.15a}$$

$$\mathrm{D}\dot{\phi}^i = \partial_t\left(-\frac{1}{\sqrt{3V}} g^{ij}\partial_j V\right) + \Gamma^i_{jk}\frac{1}{3V} g^{jl} g^{km}\partial_l V\partial_m V = \frac{1}{6}\nabla^i\frac{|\nabla V|^2}{V}, \tag{5.15b}$$

and hence in the slow-roll regime,

$$\epsilon = -\frac{\dot{H}}{H^2} = \frac{1}{2}\frac{|\nabla V|^2}{V^2}, \tag{5.16a}$$

$$\eta^i = \frac{\mathrm{D}\dot{\phi}^i}{H|\dot{\phi}|} = \frac{1}{2\,|\nabla V|}\nabla^i \frac{|\nabla V|^2}{V}, \tag{5.16b}$$

$$\eta^{\|} = \frac{\mathrm{D}\dot{\phi}\cdot\dot{\phi}}{H|\dot{\phi}|^2} = \frac{-1}{2|\nabla V|^2}\nabla V\cdot\nabla\frac{|\nabla V|^2}{V} = \epsilon - \frac{\nabla^i V\nabla^j V\nabla_i\nabla_j V}{V|\nabla V|^2}, \tag{5.16c}$$

$$\begin{aligned}\eta^{\perp} &= \frac{\left|\mathrm{D}\dot{\phi} - \frac{\mathrm{D}\dot{\phi}\cdot\dot{\phi}}{|\dot{\phi}|^2}\dot{\phi}\right|}{H|\dot{\phi}|} = \frac{1}{2|\nabla V|}\sqrt{\left|\nabla\frac{|\nabla V|^2}{V}\right|^2 - \frac{\left(\nabla V\cdot\nabla\frac{|\nabla V|^2}{V}\right)^2}{|\nabla V|^2}} \\ &= \sqrt{\frac{1}{4|\nabla V|^2}\left|\nabla\frac{|\nabla V|^2}{V}\right|^2 - (\eta^{\|})^2}.\end{aligned} \tag{5.16d}$$

## 5.4 Inflation from the worldsheet

### 5.4.1 Slow-roll parameters for tree-level worldsheet string theory

We are now in a position to address our question: how do we describe slow-roll inflation in terms of worldsheet dynamics? That is, we need to verify that the potential $V(\Phi, u) = 2\left(\frac{\kappa_0}{\kappa}\right)^2 e^{2\Phi}U(u)$ is capable of driving a slowly rolling inflaton field. We shall assume the spacetime part of the worldsheet theory to describe an accelerating (i.e. inflationary) flat, homogeneous and isotropic FLRW universe, $g^{(E)} = \mathrm{diag}(-1, a^2(t), a^2(t), a^2(t))$, which is driven by a homogeneous dilaton $\Phi(t, \boldsymbol{x}) = \Phi(t)$ and homogeneous internal fields $u(t, \boldsymbol{x}) = u(t)$. The demand that the slow-roll parameters are small then provides restrictions on $V(\Phi, u)$ and hence, as conjectured, on the coarse characteristics of the internal conformal field theory, $c, \overline{\beta}^{\Phi}(u)$ and $\overline{\beta}^{a}(u)$. Direct calculation of (5.16) for $V(\phi) = 2\left(\frac{\kappa_0}{\kappa}\right)^2 e^{2\Phi}U(u)$ reveals

$$\epsilon = 1 + \frac{1}{2}\gamma^2, \tag{5.17a}$$

$$\eta^{\|} = -\epsilon - \frac{D}{2+\gamma^2}, \tag{5.17b}$$

$$\eta^{\perp} = \sqrt{\frac{1}{4}(2+\gamma^2)^2 + D + \frac{\frac{\gamma^b\gamma^c\nabla_a\overline{\beta}_b\nabla^a\overline{\beta}_c}{\alpha'^2 U^2} - 2\gamma^2 D - 2\gamma^6 + \gamma^4}{2+\gamma^2} - (\eta^{\|})^2}, \tag{5.17c}$$

where we have defined the combinations,

$$\gamma_a(u) = \frac{M_{ab}\overline{\beta}^b}{\alpha' U} = \partial_a \log U = \partial_a \log\left[\frac{c_x}{6\alpha'} + \frac{1}{\alpha'}\overline{\beta}^\Phi\right], \tag{5.18a}$$

$$D(u) = \frac{\gamma^a\gamma^b\nabla_a\nabla_b U}{U} - \gamma^4. \tag{5.18b}$$

From (5.17a) we immediately see that $V(\Phi, u) = 2\left(\frac{\kappa_0}{\kappa}\right)^2 e^{2\Phi}U(u)$ is incapable of driving inflation: $\epsilon$ is always larger than unity. Regardless of the specific form of $\gamma_a = \partial_a \log\left[\frac{c_x}{6\alpha'} + \frac{1}{\alpha'}\overline{\beta}^\Phi\right]$, the positive definiteness of the Zamolodchikov metric $M_{ab}$ ensures that $\gamma^2 \geq 0$.

Tracing back we see that the coefficient 1 in $\epsilon$, characteristic of an exponential potential, is due to the dynamics of the dilaton. One could wonder whether taking $\Phi$ constant, i.e. excluding it from the cosmological dynamics, would modify the model into one which does allow for inflation. Because the field space metric $G_{ij}$ is block diagonal, equation (5.11) implies that for a constant $\Phi$, $\Phi$ must be stabilized at $\partial_\Phi V = 4\left(\frac{\kappa_0}{\kappa}\right)^2 e^{2\Phi}U = 0$. However, excluding $\Phi = -\infty$, the relation (5.6) precludes a constant dilaton, as $U$ is not allowed to vanish. In our set-up, fields $u^a(x)$ that undergo a time evolution in four-dimensional spacetime are described by a renormalization group flow of the couplings, i.e. $\overline{\beta}^a \neq 0$. Equation (5.6) then implies that $U$ cannot vanish, which forces the dilaton to be non-constant by the requirement (5.4c) of a vanishing $\beta^\Phi$. Turning the argument around, suppose one magically stabilizes the dilaton at tree-level. Then $\epsilon = U^{-2}\overline{\beta}^a\overline{\beta}_a$ but $U \sim \partial_\Phi V$ which must vanish by the assumption that the dilaton is stabilized.

Within tree-level worldsheet string theory, the dilaton is therefore always part of the cosmological dynamics and its tree-level exponential potential rules out an inflationary universe.

### 5.4.2 Inflation from the Ramond sector, string loop corrections and inflation from open strings

Clearly to describe inflation in string theory we must have a more complicated potential for the dilaton. One guess could be to supersymmetrize the worldsheet and include RR fields, i.e. the background fields corresponding to string states with fermionic boundary conditions. Technically this is a far from trivial task, as it is not yet known how to compute $\beta$ functions for RR vertex operators. However at the end of the day, even including fermionic dynamics, the resulting worldsheet theory must be of the form (5.3). On the worldsheet, the dilaton/vertex operator interactions are such that they always lead to an effective action $S = \int e^{-2\Phi}\mathcal{L}$ in the string frame [86]. Thus one always deduces equation (5.7) and the remainder of the analysis is the same.

Let us be more specific in light of the known examples of string-inspired supergravity inflation built on RR- and NS-flux compactifications [199, 249, 250]. In all global compactifications one needs O-planes to ensure tadpole cancelation. O-planes correspond to non-oriented worldsheets, which occur at higher order in the string loop expansion and are therefore not considered here. Secondly, a persistent issue in all these constructions is the stabilization of the volume modulus of the compact space. In essence this is the same absence of a potential as we exhibit for the dilaton. In current models the stabilization is thought to happen through non-perturbative $D$-brane effects [198, 251]. $D$-branes, i.e. open strings, are similarly higher order in the loop expansion.

Thus one is naturally led to consider string loop corrections or non-perturbative effects, i.e. open strings. From the worldsheet point of view these two additions roughly boil down to the same thing. Both are obtained by including more general worldsheet topologies than just the spherical worldsheet of tree-level string theory. The corrections from including closed string loops could convert $\epsilon$ into a more sensible expression. We can expect this based on the well-known dilaton tadpoles of Fischler-Susskind [241, 242]. Our results are an extension of the Fischler-Susskind result that to obtain a worldsheet description of strings in a de Sitter space, there must be a one loop (in $g_s$) contribution to the dilaton to have vanishing $\beta$ functions, i.e. to satisfy the equations of motion. Slow-roll inflation is in essence an adiabatic continuation of de Sitter space to a slowly varying vacuum energy.

It is interesting to see what happens if we suppose that the higher loop contributions allow us to consistently stabilize the dilaton at weak coupling independent of the value of $u^a$. Then one finds the slow-roll parameters

$$\epsilon = \frac{\overline{\beta}^a\overline{\beta}_a}{2(\overline{\beta}^\Phi + \frac{c_x}{6})^2}, \tag{5.19a}$$

$$\eta^{\|} = \epsilon - \frac{\overline{\beta}^a\overline{\beta}^b\nabla_a\overline{\beta}_b}{(\overline{\beta}^\Phi + \frac{c_x}{6})\overline{\beta}^c\overline{\beta}_c}, \tag{5.19b}$$

$$\eta^{\perp} = \sqrt{\frac{1}{4\overline{\beta}^c\overline{\beta}_c}\left|\nabla_a\frac{\overline{\beta}^b\overline{\beta}_b}{(\overline{\beta}^\Phi + \frac{c_x}{6})}\right|^2 - (\eta^{\|})^2}. \tag{5.19c}$$

The dilaton stabilization needs to be such that $\alpha' U = \overline{\beta}^\Phi + \frac{c_x}{6}$ is no longer proportional to $\partial_\Phi V$ and hence the above expressions make sense. Of course dilaton stabilization at weak coupling has its own problems [244, 245].

The inclusion of open strings, in addition to the closed strings considered here, may yield more promising results for describing worldsheet theories on inflationary backgrounds. In the supergravity literature the usefulness/necessity of open string corrections has already been recognized [1, 197, 198, 227–230, 251].[5] Open strings have been extensively investigated from a low energy effective field theory point of view, e.g. DBI inflation, and all known viable supergravity inflationary models have an open string component.

## 5.5 Conclusions

Inflation does not care about anything but very coarse features of the matter sector, only its pressure and energy. This suggests that in string theory inflation is determined by coarse features of the internal conformal field theory on the worldsheet. Qualitatively this is what we find. At the same time our result shows that it is not possible to have an inflationary cosmology described by a tree-level string worldsheet. The exponential potential for the dilaton ensures that $\epsilon$ is strictly larger than unity, completely independent of the internal conformal field theory. At first sight this conclusion may be puzzling, as inflation is a classical phenomenon and one therefore may expect tree-level string theory to be sufficient for a consistent description. Nevertheless the result simply recovers that de Sitter backgrounds arise only at one-loop level in worldsheet string theory through the Fischler-Susskind mechanism [241–243]. For inflation to occur, the dilaton must be stabilized through such higher loop effects. If this stabilization happens at weak coupling, then inflation is possible with slow-roll parameters that only depend on the $\beta$ functions of the internal conformal field theory.

In a way Fischler-Susskind and the result here are special cases of Dine-Seiberg runaway [244, 245]: within string theory one cannot probe a nearby vacuum from the original vacuum because in string perturbation theory, as currently understood, all higher order corrections are larger than the first order — string theory is either free or strongly coupled. Whereas the result in [244, 245] is obtained by general reasoning, Fischler-Susskind specifically attempt to describe a de Sitter cosmology from a Minkowski worldsheet, and we attempt to obtain inflation. We can be even more explicit: in our tree-level analysis the time-dependent process of inflation requires on the one hand a non-constant dilaton to satisfy the equations of motion, while on the

[5] In supergravity language open strings add $D$-terms in addition to $F$-term potentials. The closed string worldsheet only captures a dilaton type $F$-term inflation.

other hand only a constant dilaton makes sense observationally. In the tree-level limit we therefore have found a clear inconsistency of the approach. A strong coupling analysis is necessary to realize inflation within string theory. The reader should be aware that we have *not* ruled out a non-constant dilaton scenario at all, we simply have found out that a zeroth order weak coupling approach is insufficient to describe inflation. In the strong coupling regime the dilaton may turn out to be non-constant after all.

It is interesting to note that our result confirms a conjecture in [245], that a cosmological solution in which the world is slowly sliding to its free Minkowski vacuum cannot be studied from this final state. From the reasoning in [244, 245] this appears to be a perfectly fine solution, if unlikely. Our result confirms their expectation that such a slow-roll inflationary scenario is not possible within tree-level worldsheet string theory.

To conclude: we have provided a proof of principle that the coarse characteristics of the internal conformal field theory determine whether and how inflation occurs, by expressing the slow-roll parameters in terms of the $\beta$ functions of the internal conformal field theory. As de Sitter-like solutions only arise at one-loop in a Minkowski string worldsheet, a necessary requirement for real and realistic worldsheet models of string inflation is to include higher order string loop corrections to the analysis. This remains subject to further investigation.

## 5.A Calculating the $\beta$ functions

In this appendix we will review the calculation of the $\beta$ functions (5.4) of the total theory (5.3). For more details concerning this calculation we refer to [111].

### 5.A.1 Conformal perturbation theory

For a general conformal field theory that is perturbed by adding operators to the action,

$$S = S_0 + \int \mathrm{d}^2 z\, u^I O_I,$$

the $\beta$ functions $\beta^I$ for the couplings $u^I$ can be defined as the coefficients of the trace of the stress-energy tensor

$$\Theta = -\pi \beta^I O_I, \tag{5.20}$$

where the factor of $\pi$ is convenient within a string theory context. In the Zamolodchikov renormalization group scheme these can be computed in an expansion in $u^I$

with $\Delta_I - 2$ small [86, 111, 252],

$$\beta^I = (\Delta_I - 2)u^I + 2\pi C^I_{JK} u^J u^K + O(u^3), \tag{5.21}$$

where $C^I_{JK}$ are the OPE coefficients defined via

$$O_J(y)O_K(z) = \sum_I C^I_{JK} |y-z|^{\Delta_I - \Delta_J - \Delta_K} O_I\left(\frac{y+z}{2}\right).$$

In the coupled system $CFT_x \otimes CFT_O$ that is deformed by the term $S_{u(x)} = \int u^a(x) O_a$ as described in the main text, the operators in (5.20) are the three (types of) operators,

$$O_g^{\mu\nu} = \frac{1}{2\pi\alpha'} \partial x^\mu \bar{\partial} x^\nu, \qquad O_a, \qquad O_\Phi = \frac{1}{8\pi} R^{(2)},$$

which couple to the coupling functionals $g_{\mu\nu}(x)$, $u^a(x)$ and $\Phi(x)$ respectively. By a Fourier transform these coupling functionals may be seen as an infinite set of coupling constants $g_{\mu\nu}(p)$, $u^a(k)$ and $\Phi(q)$ that couple to the dressed operators $O_p^{\mu\nu} = \frac{1}{2\pi\alpha'} \partial x^\mu \bar{\partial} x^\nu e^{ip\cdot x} \mathbf{1}$, $O_{(k,a)} = O_a e^{ik\cdot x}$ and $O_q^\Phi = \frac{1}{8\pi} R^{(2)} e^{iq\cdot x}$ with dimensions

$$\Delta_p^g = 2 + \frac{\alpha'}{2} p^2, \qquad \Delta_{(k,a)} = \Delta_a + \frac{\alpha'}{2} k^2, \qquad \Delta_q^\Phi = 2 + \frac{\alpha'}{2} q^2. \tag{5.22}$$

We are not constraining the graviton momentum or dilaton momentum to be lightlike. $p^2 = 0$ and $q^2 = 0$ would be the on-shell condition for a *free* graviton and *free* dilaton, whereas we wish to consider the coupled gravity-matter system. The OPE coefficients can be readily computed to be

$$C^{(p,\mathbf{1})}_{(k_1,a)(k_2,b)} = -\frac{\alpha'}{8\pi}(k_1 - k_2)_\mu (k_1 - k_2)_\nu \delta(p - k_1 - k_2) M_{ab}, \tag{5.23a}$$

$$C^{(k_1,a)}_{(k_2,b)(k_3,c)} = \delta(k_1 - k_2 - k_3) C^a_{bc}, \tag{5.23b}$$

where $C^a_{bc}$ are the OPE coefficients of the internal conformal field theory and we have denoted the Zamolodchikov metric by $M_{ab} = 4\pi^2 C^{\mathbf{1}}_{ab}$. Applying (5.22) and (5.23) to (5.21) and Fourier-transforming back to position-space, yields

$$\frac{1}{\alpha'}\beta^g_{\mu\nu} = -\frac{1}{2}\partial^\rho \partial_\rho g_{\mu\nu} + \frac{1}{2} M_{ab}\left(u^a \partial_\mu \partial_\nu u^b - \partial_\mu u^a \partial_\nu u^b\right), \tag{5.24a}$$

$$\begin{aligned} \frac{1}{\alpha'}\beta^a &= \frac{1}{\alpha'}\left((\Delta_a - 2)u^a + 2\pi C^a_{bc} u^b u^c\right) - \frac{1}{2}\partial^\rho \partial_\rho u^a \\ &= \frac{1}{\alpha'}\bar{\beta}^a(u) - \frac{1}{2}\partial^\rho \partial_\rho u^a, \end{aligned} \tag{5.24b}$$

$$\frac{1}{\alpha'}\beta^\Phi = -\frac{1}{2}\partial^\rho \partial_\rho \Phi, \tag{5.24c}$$

where we use (5.21) in reverse to express $\beta^a$ in terms of $\bar{\beta}^a$.

### 5.A.2 Weyl anomaly and classical dilatonic contribution

In addition to the operator effects from (5.24c), the $\beta$ function for $\Phi$ receives a further contribution from the well-known Weyl anomaly, a worldsheet contribution proportional to the worldsheet curvature. Its contribution is determined by the central charge of the spacetime nonlinear $\sigma$ model as well as by that from the (perturbed) internal theory as explained in the main text,

$$\Theta_{1-\text{loop}} = -\frac{c_x}{48}R^{(2)} - \frac{1}{8}\overline{\beta}^{\Phi} R^{(2)} = -\pi\left(\frac{c_x}{6} + \overline{\beta}^{\Phi}(u)\right)\frac{1}{8\pi}R^{(2)}. \tag{5.25}$$

Comparing this expression with the definition of the $\beta$ functions as coefficients in the stress-energy tensor (5.20), we find a contribution $\beta^{\Phi}_{1-\text{loop}} = \frac{c_x}{6} + \overline{\beta}^{\Phi}(u) = \alpha' U(u)$ to the $\beta$ function of the dilaton.

The final contribution to all of the $\beta$ functions comes from the dilaton term (5.1c) in the worldsheet action, which breaks Weyl invariance already at the classical level. Due to an additional overall $\alpha'$-factor compared to the other terms in the worldsheet, this classical contribution to the $\beta$ functions is of the same order as loop effects from the classically Weyl invariant terms. On a curved worldsheet the easiest way to determine deviation from Weyl invariance is by calculating the trace of the stress-energy tensor via

$$\Theta = \frac{-\pi}{\sqrt{h}}\frac{\delta S}{\delta h^{\alpha\beta}}h^{\alpha\beta}.$$

This definition for $\Theta$ in terms of a variation of the worldsheet metric differs by a factor from more common definitions, which is necessary to relate the result properly with our earlier definition (5.20). One can check that this leads to the right result by looking at the metric and dilaton field, whose contributions are well-known [86, 117]. Making use of the equations of motion for $x^{\mu}$,

$$\partial\overline{\partial}x^{\rho} = -\Gamma^{\rho}_{\mu\nu}\partial x^{\mu}\overline{\partial}x^{\nu} + \pi\alpha'\partial^{\rho}u^{a}O_{a} + \frac{\alpha'}{8}\partial^{\rho}\Phi R^{(2)},$$

the classical violation of Weyl invariance by the dilaton term (5.1c) is

$$\begin{aligned}\Theta_{\text{classical}} = \left.\frac{-\pi}{\sqrt{h}}\frac{\delta S_{\Phi(x)}}{\delta h^{\alpha\beta}}h^{\alpha\beta}\right|_{h_{z\bar{z}}=1/2} &= -\partial\overline{\partial}\Phi(x) = -\left(\partial_{\mu}\partial_{\nu}\Phi\partial x^{\mu}\overline{\partial}x^{\nu} + \partial_{\rho}\Phi\partial\overline{\partial}x^{\rho}\right)\\ &= -\pi\left(2\alpha'\nabla_{\mu}\nabla_{\nu}\Phi O_{g}^{\mu\nu} + \alpha'\nabla^{\rho}\Phi\nabla_{\rho}u^{a}O_{a} + \alpha'(\nabla\Phi)^{2}O_{\Phi}\right).\end{aligned} \tag{5.26}$$

Again comparing with (5.26), we find contributions

$$\beta^g_{\text{classical}} = 2\alpha' \nabla_\mu \nabla_\nu \Phi, \tag{5.27a}$$

$$\beta^a_{\text{classical}} = \alpha' \partial^\rho \Phi \partial_\rho u^a, \tag{5.27b}$$

$$\beta^\Phi_{\text{classical}} = \alpha' \partial^\rho \Phi \partial_\rho \Phi. \tag{5.27c}$$

Therefore the full $\beta$ functions read

$$\frac{1}{\alpha'}\beta^g_{\mu\nu} = -\frac{1}{2}\partial^\rho \partial_\rho g_{\mu\nu} + \frac{1}{2}M_{ab}\left(u^a \partial_\mu \partial_\nu u^b - \partial_\mu u^a \partial_\nu u^b\right) + 2\nabla_\mu \nabla_\nu \Phi, \tag{5.28a}$$

$$\frac{1}{\alpha'}\beta^a = \frac{1}{\alpha'}\overline{\beta}^a(u) - \frac{1}{2}\partial^\rho \partial_\rho u^a + \partial^\rho \Phi \partial_\rho u^a, \tag{5.28b}$$

$$\frac{1}{\alpha'}\beta^\Phi = U(u) - \frac{1}{2}\partial^\rho \partial_\rho \Phi + \partial^\rho \Phi \partial_\rho \Phi. \tag{5.28c}$$

### 5.A.3 Covariantization

The $\beta$ functions (5.28) are (partially) non-covariant. For example, $\beta^a$ is not covariant on the space of couplings $u^a(x)$. The right expression for $\beta^a$ should be

$$\frac{1}{\alpha'}\beta^a = \frac{1}{\alpha'}\overline{\beta}^a(u) - \frac{1}{2}\partial^2 u^a - \frac{1}{2}K^a_{bc}(u)\partial^\rho u^b \partial_\rho u^c + \partial^\rho \Phi \partial_\rho u^a, \tag{5.29}$$

where $K^a_{bc}$ is the connection coefficient associated to the Zamolodchikov metric $M_{ab}$ [252]. In a general renormalization scheme it arises from contact terms in the OPE. It has not appeared explicitly in the Zamolodchikov scheme because in that scheme $K^a_{bc}$ is already of first order in $u$ [111], as a result of which $K^a_{bc}(u)\partial^\rho u^b(x)\partial_\rho u^c(x)$ is beyond leading order in the calculation of the $\beta$ functions. In the Zamolodchikov scheme (5.29) is correct to leading order and by general covariance it holds in any renormalization scheme.

Furthermore, the terms obtained using conformal perturbation methods are not spacetime covariant at first. This is inherent to the conformal perturbation method, which uses correlation functions defined with respect to *flat* spacetime. Conformal perturbation is an expansion in $\delta g_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ which is only sensitive to the transverse traceless part of the graviton. The longitudinal and trace part of the graviton are not encoded in (nearly) marginal operators and thus fall outside conformal perturbation theory. If one corrects for this by evaluating the Weyl transformation of all terms of the coherent state of gravitons $g_{\mu\nu}(x)$, the expressions will become covariant. Covariantization is necessary because the true $\beta$ functions are gravitationally only consistent when all orders and all polarizations in $\delta g_{\mu\nu}$ are taken into account.

Using background field methods one can obtain these spacetime covariant expressions [86, 111].

We propose a different method to see how the covariant expressions (5.4) may follow from the $\beta$ functions derived using conformal perturbation theory methods (5.28), by relating them at the level of their action functionals.[6] We will do this only up to second order in $u^a$ in the integrand, i.e. to first order in the equations of motion, $\beta^a = 0$, for the fields $u^a$. The necessity of this approximation can directly be inferred from the appearance of the non-tensorial object $u^a$ in the integrand.

Objects from the appendix are denoted with a tilde, while quantities without a tilde refer to the fields and couplings in the main text. If we restrict ourselves to transverse traceless variations in the metric, the covariant action[7]

$$S = \int \sqrt{\tilde{g}} e^{-2\tilde{\Phi}} \left[ \tilde{R} + 4(\tilde{\nabla}\Phi)^2 + \frac{1}{2}\tilde{M}_{ab}\left(u^a \tilde{\mathrm{D}}\tilde{\nabla}u^b - \tilde{\nabla}u^a\tilde{\nabla}u^b\right) - 4\tilde{U}\right], \tag{5.30}$$

generates the equations of motion given by the vanishing of (5.28), to leading order in $u^a$, provided

$$\frac{1}{\alpha'}\tilde{M}_{ab}\widetilde{\overline{\beta}^b} = \partial_a \tilde{U} + \frac{1}{2}\tilde{M}_{ab}u^b\tilde{U}.$$

The latter expression should be equivalent to the consistency condition (5.6), although it is probably rather involved to derive this for the non-covariant (5.28).

Being a covariantly consistent expression, we expect the action (5.30) to provide the true (spacetime and field space) covariant expressions for the $\beta$ functions as we would have found by background field methods [86, 111]. The double derivative of $u^a$ is non-standard. However, we can now consider the field redefinition

$$\tilde{\Phi} = \Phi + \frac{1}{8}\tilde{M}_{ab}u^a u^b, \qquad \tilde{g}_{\mu\nu} = e^{\frac{1}{4}\tilde{M}_{ab}u^a u^b} g_{\mu\nu}.$$

Together with the identifications

$$M_{ab} = \tilde{M}_{ab}, \qquad U = e^{\frac{1}{4}\tilde{M}_{ab}u^a u^b}\tilde{U},$$

the action (5.30) transforms to the conventional covariant action

$$S = \int \sqrt{g} e^{-2\Phi}\left[R + 4(\nabla\Phi)^2 - M_{ab}\nabla u^a \nabla u^b - 4U\right], \tag{5.31}$$

[6] In [111] this is done by way of a diffeomorphism that is not entirely clear to the authors.

[7] Note that this restriction means that the contraction of the variation of the connection in $M_{ab}u^a\tilde{\mathrm{D}}\tilde{\nabla}u^b$ does not contribute to the equations of motion. It is orthogonal to the transverse traceless fluctuations

$$\tilde{g}^{\mu\nu}\delta\tilde{\Gamma}^{\rho}_{\mu\nu} = -\frac{1}{2}\left(2\nabla_\mu \delta g^{\rho\mu} - g_{\mu\nu}\nabla^\rho \delta g^{\mu\nu}\right) = 0.$$

up to second order in $u^a$.