Conformal invariance and microscopic sensitivity in cosmic inflation

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Describing nature at its tiniest

Phenomenologically, inflation is a very successful theory that is compatible with all observations. However, its microscopic origin is not very clear and ideally one would like to have an embedding of inflation into a more fundamental framework. Maintaining the slow-roll condition, especially over 60 e-folds, turns out to be an extremely delicate exercise that is easily disturbed by the very quantum effects that provide the origin of density fluctuations. To understand inflation and the origin of our cosmos, we need to have an accurate description of the workings of nature at the smallest scales.

The quest to find out nature's workings at an ever more precise level is the tale of the history of physics, a progression that has happened in steps. Nature has been so kind to us that in order to understand a certain macroscopic phenomenon, we do not need to know the (full) details of the microscopic details within. An effective description of the phenomenon, in which the microscopic degrees of freedom decouple, is often sufficient to completely understand the relevant behavior. At a certain stage however, the details do become important and one should refine the fundamental theory. By successively focussing on the details of a given theory, we have come to understand more and more about nature. The current fundamental theory, which is believed to unify all known particles and interactions, is *string theory*.

In this chapter we will discuss the effective field theory description alluded to above, in particular the role played by *conformally invariant* theories. Furthermore we will discuss the relation between string theory and conformal invariance and we consider two additional aspects of string theory: supergravity and holography.[1]

[1] The material presented in this chapter can also be found in many terrific books and reviews. The information on renormalization and conformal field theory can be found in [72, 80–84]. String theory books and lecture notes include [85–87]. Supersymmetry and supergravity is discussed by [88, 89] and an

3.1 Conformal field theory

3.1.1 Nature at different energy scales

Renormalization and effective field theory

In technical terms, our stepwise progression into the details of nature's workings is understood through renormalization of the quantum field theory that is used to describe the world around us. Historically, renormalization was invented as a procedure of mathematical tricks, in order to extract finite answers from the divergent expressions [92], but Wilson's interpretation of renormalization in terms of coarse graining has played a key role in the conceptual understanding of renormalization and effective field theory [93–96]. The remarkable conclusion of renormalization is that the renormalized, *physical* coupling constants, i.e. the strengths of the interactions between particles, depend on the energy scale t at which a given process happens. In the Wilsonian context, this dependence is understood as the only remaining effect of the unknown underlying microphysics. The power of the renormalization group, however, is that the way the couplings run does not depend on the microscopic physics.

The scale dependence of the couplings has immediate consequences for the observability of different interactions. In a d-dimensional theory, a coupling constant u multiplying an operator of mass dimension Δ, has itself mass dimension $d - \Delta$, where the mass typically is of order of the cut-off scale Λ, used to regularize the theory. Hence, using the momentum-scale t of a given process to make a dimensionless quantity, the coupling scales as $u \sim \left(\frac{t}{\Lambda}\right)^{\Delta-d}$. This simple argument based on dimensional analysis shows that operators can be split into 3 categories: *relevant* operators with $\Delta < d$, whose coupling constants become increasingly important at low energy scales $t \ll \Lambda$, *marginal* operators with $\Delta = d$, whose coupling constants are scale independent, and *irrelevant* operators with $\Delta > d$, whose coupling constants are irrelevant at low energies, but all the more important at high energies. Therefore at low energies, an *effective field theory* in terms of only $\Delta \leq d$ operators is a sufficient description of nature, as long as one probes the theory at energies below the fundamental cut-off scale Λ [93–96]. This argument explains why nature is insensitive to microscopic details when it is only observed at a macroscopic level. All the details of the microscopic theory can be captured in terms of just a finite number of (relevant and marginal) coupling constants, which survive the small t-limit. It also implies that we are hard pressed to deduce anything about the tiniest scales in nature with our everyday, low energy experiments [97]. It is for this reason that we need to probe

introduction to holography is given in [90, 91].

high energy scales $t > \Lambda$, such as in the opportunity given by (indirect) observations from the inflationary epoch. By probing beyond the regime of validity of the effective field theory, we hope to find out what kind of irrelevant operators reside within the underlying, more fundamental theory.

Callan-Symanzik equation

The arguments using dimensional analysis give a first, qualitative indication of why a physical process depends on the energy scale t at which it occurs. The quantitative formalism to understand the precise energy dependence of a quantum field theory has been worked out by Callan and Symanzik [98–101], which results into the *Callan-Symanzik equation*, a differential equation that governs the energy dependence of n-point correlation functions. To derive it, we consider an n-point correlation function $G_0^{(n)}(p_j; u_0)$ of an operator O, given in terms of its bare coupling u_0 and depending on the momenta p_j of the operators. After regularization and renormalization, imposing renormalization group conditions at a certain scale μ, the correlation functions can also be expressed in terms of the renormalized coupling $u(\mu)$,

$$G^{(n)}(p_j; u(\mu), \mu) = Z^{-n/2} G_0^{(n)}(p_j; u_0),$$

where Z is the field rescaling factor $O \to Z^{-1/2} O$. The Callan-Symanzik equation results from the observation that the original, bare correlation function $G_0^{(n)}$ cannot depend on the choice of renormalization scale μ. This imposes a consistency condition on the renormalized n-point function $G^{(n)}$, which determines uniquely its dependence on the energy scale μ,

$$0 = Z^{-n/2} \mu \frac{\mathrm{d}}{\mathrm{d}\mu} G_0^{(n)}(p_j; u_0) = \left(\mu \frac{\partial}{\partial \mu} + \beta(u) \frac{\partial}{\partial u} + n\gamma(u) \right) G^{(n)}(p_j; u(\mu), \mu). \tag{3.1}$$

The *β function*, $\beta(u)$, and *anomalous dimension* $\gamma(u)$ of the operator O are defined through the use of the chain-rule,

$$\beta(u) = \mu \frac{\partial u}{\partial \mu}, \qquad \gamma(u) = \frac{1}{2} \frac{\mu}{Z} \frac{\partial Z}{\partial \mu}. \tag{3.2}$$

β function and anomalous dimension

The dependence of the renormalized n-point function $G^{(n)}$ on the renormalization scale μ, specified by β and γ through the Callan-Symanzik equation (3.1), automatically dictates the dependence of the theory on the physical scale t [102]. The mass dimension of a correlation function $G^{(n)}(x_j; u, \mu)$ of n operators $O(x_j)$ with scaling dimension Δ_0 is $n\Delta_0$ [72]. Extracting a momentum-conserving $\delta^{(d)}(p_1 + \ldots + p_n)$ function, the corresponding Fourier transformed correlation function has mass dimension $\Delta_n^{(p)} = n(\Delta_0 - d) + d$, which can be written in terms of the renormalization scale μ and some function of dimensionless ratios p_j/μ,

$$G^{(n)}(p_j; u, \mu) = \mu^{\Delta_n^{(p)}} f\left(\frac{p_j}{\mu}\right).$$

We can rescale all momenta with a common factor of t. It sets the energy at which the physical process is probed. Using the relation between t and μ,

$$t\frac{\partial}{\partial t} G^{(n)}(tp_j; u, \mu) = \left(-\mu\frac{\partial}{\partial \mu} + \Delta_n^{(p)}\right) G^{(n)}(tp_j; u, \mu),$$

the Callan-Symanzik equation is written completely in terms of the overall momentum dependence t,

$$\left(t\frac{\partial}{\partial t} - \beta(u)\frac{\partial}{\partial u} - (\Delta_n^{(p)} + n\gamma(u))\right) G^{(n)}(tp_j; u, \mu) = 0. \tag{3.3}$$

In this form, the Callan-Symanzik equation fixes the dependence of the correlation function with physical rescalings. A general solution to this equation is [72],

$$G^{(n)}(tp_j; u, \mu) = G^{(n)}(p_j; \tilde{u}(t; u), \mu) \exp\left(\int_{t'=1}^{t'=t} \mathrm{d}\log t' \left[\Delta_n^{(p)} + n\gamma(\tilde{u}(t'; u))\right]\right), \tag{3.4}$$

in terms of the function $\tilde{u}(t; u)$ defined through the differential equation (3.2),

$$t\frac{\partial}{\partial t}\tilde{u}(t; u) = \beta(\tilde{u}(t; u)), \qquad \tilde{u}(1; u) = u. \tag{3.5}$$

Usually, once a solution for $\tilde{u}(t; u)$ is found, it is denoted with $u(t)$ and simply referred to as the *running coupling* of the theory.

For a free field theory, with $\beta = \gamma = 0$, the solution (3.4) indeed reproduces the correct scaling $G^{(n)}(tp_j; u, \mu) \sim t^{\Delta_n^{(p)}}$. When β and γ are nonzero, the momentum dependence changes. In a given theory, the β function can be calculated by computing the counterterms in a renormalization procedure. The differential equation (3.5) then determines the running of the coupling constant $u = u(t)$ as a function of the energy scale t at which the specific process is considered. As such, the renormalization group equations can be seen as a *flow* on the space of coupling constants of the theory: depending on the sign of $\beta(u)$, coupling constants are increasingly dominant or less and less important along the renormalization group flow. As we will see shortly, close to a free field theory, $u \approx 0$, the β functions determine a flow that indeed approximates the anticipated $u \sim t^{\Delta-d}$ behavior. However, once $u \neq 0$, the β function might change and as a result the precise scaling behavior of $u(t)$ will also change. Whether operators are *truly* relevant, marginal or irrelevant therefore depends on whether $\beta < 0$, $\beta = 0$ or $\beta > 0$.

A nonzero anomalous dimension γ changes the scaling behavior of the correlation function and induces a change in the scaling dimension of the operators $\mathcal{O}$ inside the correlation function. This is best seen around a non-trivial fixed point of the theory, where $\beta(u_*) = 0$ and $u(t)$ is constant $u(t) = u_* \neq 0$. Then the solution (3.4) yields

$$G^{(n)}(tp_j; u_*, \mu) = t^{\Delta_n^{(p)} + n\gamma(u_*)} G^{(n)}(p_j; u_*, \mu).$$

Again, the n-point function scales with a power law of t, but now the exponent is different than the usual $\Delta_n^{(p)}$. Tracing back to the scaling dimension of the operator $\mathcal{O}(x_j)$, it appears as if its scaling dimension Δ_0 is changed to

$$\Delta = \Delta_0 + \gamma, \tag{3.6}$$

which explains why γ is called the anomalous dimension.

3.1.2 Field theory without a scale

Conformal transformations

It is clear that theories with vanishing β functions play a special role in the study of renormalization group flow on the space of theories. Such a scale invariant theory acts as a fixed point for the renormalization group flow: the coupling constants are scale invariant and remain scale invariant. As such, they form an ideal starting point to study the renormalization group flow perturbatively. Before explaining the perturbative approach, let us consider the *conformal field theories* themselves [83, 84].

Conformal field theories are invariant under conformal transformations, i.e. transformations $x \mapsto x'(x)$ such that the metric $h_{\alpha\beta}$ changes with an overall spacetime dependent factor,

$$h'_{\alpha\beta}(x') = \Lambda(x) h_{\alpha\beta}(x).$$

Together with the standard Lorentz transformations, they form a group, the conformal group, whose transformations in dimensions $d > 2$ are translations, dilations, rotations[2] and special conformal transformations,

$$x'^\alpha = x^\alpha + a^\alpha, \qquad x'^\alpha = \lambda x^\alpha, \tag{3.7a}$$

$$x'^\alpha = L^\alpha{}_\beta x^\beta, \qquad x'^\alpha = \frac{x^\alpha - b^\alpha x^2}{1 - 2b^\alpha x_\alpha + b^2 x^2}, \tag{3.7b}$$

respectively. The first three transformations together form the Poincaré group extended with dilations, i.e. the symmetry group of scale invariant theories. Hence, conformal invariance, or *local* scale invariance, implies scale invariance, which is why they form a good starting point to study renormalization group fixed points.

Rotations and special conformal transformations shall not play a large role in this thesis and we shall focus on translations and dilations. On a field O, these two transformations act as $O'(x) = (1 - iG_a\omega_a(x))O(x)$, where ω_a is the infinitesimal parameter of the transformation and the generators G_a are given by

$$G_{T,\alpha} = -i\partial_\alpha, \qquad G_D = -i\left(x^\alpha\partial_\alpha + \Delta\right),$$

respectively. Δ is the scaling dimension of the field O, $O'(\lambda x) = \lambda^{-\Delta}O(x)$. The conserved current associated with translational symmetry is the stress-energy tensor $j_{T,\alpha\beta} = T_{\alpha\beta}$. Canonically it can be expressed through a standard Noether procedure, i.e. under translations $x'^\alpha = x^\alpha + a^\alpha$ the action changes infinitesimally

$$\delta S = \int \mathrm{d}^d x \sqrt{h}\, T^c_{\alpha\beta}\nabla^\alpha a^\beta.$$

The disadvantage of this definition is that $T^c_{\alpha\beta}$ will not necessarily be symmetric. Therefore, a new, improved stress-energy tensor can be defined [83, 98] which plays the same role as $T^c_{\alpha\beta}$ and which is symmetric. Another way to define the stress-energy tensor is by considering a dynamical metric $h_{\alpha\beta}$ for the theory. Under the diffeomorphism $x'^\alpha = x^\alpha + a^\alpha(x)$ the metric changes as a tensor, $\delta h^{\alpha\beta} = \nabla^\alpha a^\beta + \nabla^\beta a^\alpha$. Hence in an invariant theory, the metric itself must transform opposite to this,

$$\delta S = -\frac{1}{2}\int \mathrm{d}^d x \sqrt{h}\, T_{\alpha\beta}\delta h^{\alpha\beta}. \tag{3.8}$$

A manifestly symmetric stress-energy tensor can therefore also be obtained via

$$T_{\alpha\beta} = -\frac{2}{\sqrt{h}}\frac{\delta S}{\delta h^{\alpha\beta}}, \tag{3.9}$$

[2] Although we will partly be interested in conformal field theories that are of Lorentzian signature, we can always Wick-rotate to a Euclidian signature. For this reason, the inner products in this section are always taken to be Euclidean. Moreover, when studying conformal field theory, the metric is fixed, which for many purposes may assumed to be flat.

where the normalization depends on convention. In particular the stress-energy tensor in string theory often contains factors of π in its definition [85, 86].

Under an infinitesimal conformal transformation $x'^{\alpha} = x^{\alpha} + \epsilon^{\alpha}(x)$, which can be shown to satisfy the conformal Killing equation,

$$\nabla_{\alpha}\epsilon_{\beta} + \nabla_{\beta}\epsilon_{\alpha} = \frac{2}{d}\nabla_{\gamma}\epsilon^{\gamma}h_{\alpha\beta}, \tag{3.10}$$

the action is invariant,

$$0 = \delta S = -\frac{1}{d}\int \mathrm{d}^d x\,\sqrt{h}\,T^{\alpha}{}_{\alpha}\nabla_{\beta}\epsilon^{\beta}, \tag{3.11}$$

if the stress-energy tensor is traceless,

$$\Theta = T^{\alpha}{}_{\alpha} = 0.$$

Hence, a theory with a traceless stress-energy tensor is conformally invariant. One might be tempted to think that the reverse is also true, but since $\epsilon(x)$ has to satisfy (3.10), it is not an arbitrary function. Nevertheless, for most conformal field theories, the stress-energy tensor can indeed be made traceless by a procedure similar to the one used to make it symmetric [83]. In those cases, the stress-energy tensor is related to the dilational current,

$$j_D^{\alpha} = T^{\alpha}{}_{\beta}x^{\beta},$$

and tracelessness follows from (translational and) scale invariance.

Correlation functions in a conformal field theory

The symmetries in a conformal field theory impose powerful constraints on the functional dependence of n-point correlation functions, particularly the two- and three-point functions. For example, translational and rotational invariance of the theory tell us that the dependence on the arguments x_a can only appear via $|x_a - x_b|$. Including dilational invariance and special conformal transformations, the two- and three-point functions of operators O_a with scaling dimension Δ_a are fixed to have the form [83]

$$\langle O_1(x_1)O_2(x_2)\rangle = \frac{N\delta_{\Delta_1\Delta_2}}{x_{12}^{2\Delta_1}}, \tag{3.12a}$$

$$\langle O_1(x_1)O_2(x_2)O_3(x_3)\rangle = \frac{C^1_{23}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3}\,x_{23}^{-\Delta_1+\Delta_2+\Delta_3}\,x_{13}^{\Delta_1-\Delta_2+\Delta_3}}, \tag{3.12b}$$

where $x_{ab} = |x_a - x_b|$. The overall coefficients N and C^1_{23} are not fixed by any symmetry constraints. N determines the overall normalization of the field O_a. C^1_{23} is called the *OPE coefficient* because it is the coefficient in the operator product expansion, an expansion similar to a Taylor expansion that relates the product of two operators $O_a(x)$ and $O_b(y)$ to the other operators in the theory in the limit $x \to y$,

$$O_b(x)O_c(y) = \sum_a C^a_{bc}|x-y|^{\Delta_a-\Delta_b-\Delta_c} O_a\left(\frac{x+y}{2}\right).$$

Higher order n-point functions are less constrained than the two- and three-point correlation functions. For example, the four-point function can have an arbitrary functional dependence on *cross ratios*,

$$\langle O(x_1)O(x_2)O(x_3)O(x_4)\rangle = F\left[\frac{x_{12}x_{34}}{x_{13}x_{24}}, \frac{x_{12}x_{34}}{x_{23}x_{14}}\right]\prod_{a<b}^{4} x_{ab}^{\sum_c \Delta_c/3-\Delta_a-\Delta_b}.$$

Conformal invariance in two dimensions

Conformal symmetry is particularly powerful in two dimensions. This is clear from the condition (3.10) on the infinitesimal parameter $\epsilon^\alpha(x)$, which on a flat metric, $h_{\alpha\beta} = \delta_{\alpha\beta}$, reduces to the Cauchy-Riemann equations

$$\partial_1\epsilon_1 = \partial_2\epsilon_2, \qquad \partial_1\epsilon_2 = -\partial_2\epsilon_1.$$

This suggests we should actually express the transformation parameter $\epsilon(z) = \epsilon^1 + i\epsilon^2$ and $\bar{\epsilon}(\bar{z}) = \epsilon^1 - i\epsilon^2$ in terms of the complex coordinates $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$. The functions $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ are otherwise unconstrained, giving an infinite set of symmetry generators $z' = z+\epsilon(z)$, $\bar{z}' = \bar{z}+\bar{\epsilon}(\bar{z})$, rather than the finite set given by (the infinitesimal version of) (3.7).

By momentarily promoting x^1 and x^2 to elements in $\mathbb{C}$, the transformation between x^α and $z, \bar{z}$ is a coordinate transformation of independent coordinates. The symmetry algebras for $\epsilon(z)$ and $\bar{\epsilon}(\bar{z})$ are then independent copies of the same algebra. Only at the end of a calculation is the reality condition $\bar{z} = z^*$ imposed.

Mathematically the restrictive power of two-dimensional conformal symmetry is equivalent to the conditions imposed on (anti)-holomorphic functions in complex analysis. Since complex analysis is such a rich and well-developed branch of mathematics, many of the techniques can be applied successfully to two-dimensional conformal field theory [83, 84]. Although part of this thesis deals with conformal invariance in two dimensions, its remarkable structure is not heavily or actively built upon. For this reason, we do not elaborate much further on the special two-dimensional case.

3.1.3 Conformal perturbation theory

Weyl anomaly coefficients

In the previous section we have mostly been interested in the classical behavior of conformal field theories. When considering quantum field theories with conformal invariance, the situation becomes more involved. To make sense out of a quantum field theory, its expressions need to be regularized and renormalized, thereby automatically introducing a scale into the classically scale invariant theory [83]. For this reason, it might be that a classically conformally invariant theory loses its conformality as a quantum field theory. The departure from conformal invariance, can be expressed in terms of a violation of the hallmark of a conformal field theory, i.e. the trace of the stress-energy tensor is no longer vanishing,

$$\Theta = -\beta(u)\frac{\delta\mathcal{L}}{\delta u}. \tag{3.13}$$

The notation of the coefficients, $\beta(u)$, is no accident, as they are closely related to the renormalization group β functions (3.2) [103–106]. Conceptually this is easy to understand. Only a quantum field theory with a vanishing β function will remain a fixed point for the renormalization group flow, without the introduction of any new scale into the theory. All relevant and irrelevant operators induce a renormalization of their couplings and therefore a scale dependence.

Technically the relation may be seen by the effect of an explicit scale transformation $x'^{\alpha} = e^{\omega}x^{\alpha}$ on the coupling $\delta u = \omega\beta(u)$, where $\beta(u)$ now really is the renormalization group β function [72]. As a result the action changes as

$$\delta S = \int \mathrm{d}^d x\, \delta\mathcal{L} = \int \mathrm{d}^d x\, \omega\beta(u)\frac{\delta\mathcal{L}}{\delta u}.$$

Compared to the definition (3.8) of the stress-energy tensor in terms of a scale transformation of the metric $h'_{\alpha\beta}(x') = e^{-2\omega}h_{\alpha\beta}(x)$,

$$\delta S = -\frac{1}{2}\int \mathrm{d}^d x\, T_{\alpha\beta}\delta h^{\alpha\beta} = -\int \mathrm{d}^d x\, \Theta\omega, \tag{3.14}$$

the renormalization group β functions appear as coefficients in

$$\int \mathrm{d}^d x\, \Theta = -\int \mathrm{d}^d x \beta(u)\frac{\delta\mathcal{L}}{\delta u}.$$

Hence, the renormalization group β functions and the *Weyl anomaly coefficients* β appearing in (3.13) are related in the same way as global and local scale invariance are related. The requirements $\int \Theta = 0$ and $\Theta = 0$ for global and local scale invariant theories respectively are directly transferred to the β functions. Although this relation can and is used [107–109] to simplify the computation of Weyl anomaly coefficients, we will not emphasize the distinction.

In this section we have specifically restricted ourselves to flat metrics only. As we will see when we turn our attention to string theory, conformal symmetry on a curved space introduces another source for Weyl anomaly, due to the curvature scale that is introduced.

Computation of the β functions

A conformal field theory, with vanishing β functions and traceless stress-energy tensor, is a fixed point for the renormalization group flow. To study the flow perturbatively, we consider a perturbation of the conformal field theory S_0 by operators O_a with coupling u^a and dimension $\Delta_{0,a}$,

$$S_u = S_0 + \int \mathrm{d}^d x\, u^a O_a(x). \tag{3.15}$$

The trace of the stress-energy tensor is

$$\Theta = -\beta^a(u) O_a. \tag{3.16}$$

The coefficients $\beta^a(u)$ can be calculated perturbatively by considering the Callan-Symanzik equation (3.1). The anomalous dimension γ appearing in the Callan-Symanzik equation actually becomes a matrix of anomalous dimensions γ^a_b for the multi-operator case under consideration. It can be related to the β functions [110] via[3]

$$\gamma^a_b(u) = \frac{\partial \beta^a}{\partial u^b} - (\Delta_{0,b} - d)\delta^a_b. \tag{3.17}$$

Writing $\beta^a(u) = A^a + B^a_b u^b + \ldots$ and remembering that u^a has mass dimension $\mu^{d-\Delta_{0,a}}$, applying the Callan-Symanzik equation to the partition function of the perturbed theory,

$$Z = \langle e^{-\int \mathrm{d}^d x\, u^a O_a} \rangle_0 = \langle 1 - \int \mathrm{d}^d x\, u^a O_a + \ldots \rangle_0,$$

enables us to compute $\beta(u)$ recursively as a perturbation series in u [86, 110–112]. To compute the higher order coefficients of $\beta(u)$, an operator product expansion is necessary, whose singularities have to be regularized. The regularization scheme

[3] We note that different conventions compared to [110, 111] are used.

dependence is carefully explained in [111]. In the limit where the operators are nearly marginal, $|\Delta_{0,a} - d| \ll 1$, the result in the Zamolodchikov scheme [110, 111] is

$$\beta^a(u) = (\Delta_{0,a} - d)u^a + 2\pi C^a_{bc} u^b u^c + \dots \tag{3.18}$$

to second order in u. In the first term, there is no summation over the a-index.

As we see, conformal perturbation theory enables us to study the renormalization group flow around a conformal fixed point. The flow is determined by the β functions, which can be expressed in terms of the scaling dimension $\Delta_{0,a}$ of the operator $\mathcal{O}_a$ in the unperturbed conformal field theory, as long as the deviation from the fixed point is small, $u^a \ll 1$ and the operators under consideration are nearly marginal $|\Delta_a - d| \ll 1$.

3.2 String theory

3.2.1 Worldsheet physics

Strings in a flat background

The previous section mostly dealt with conformal field theories with a fixed flat metric. In essence, string theory is the study of two-dimensional conformal field theory with a dynamical, and hence curved, metric. The motivation to study such a theory follows from a direct generalization of the first quantization description of a point particle. Similar to a point particle, the classical trajectory of a string is determined by minimizing its worldvolume, which is called a worldsheet for a one-dimensional extended object. The string is described by embedding the worldsheet, with coordinates $\sigma^\alpha = (\sigma^0, \sigma^1)$ into the d-dimensional target spacetime $\sigma \mapsto x^\mu(\sigma)$. In a flat target spacetime, the worldsheet area is minimized by the minimization of the Polyakov action

$$S[x,h] = -\frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \, \sqrt{h}\, h^{\alpha\beta} \eta_{\mu\nu} \partial_\alpha x^\mu \partial_\beta x^\nu. \tag{3.19}$$

α' is a coupling constant for the two-dimensional field theory, which determines the string's tension. Both the fields x^μ and the worldsheet metric $h_{\alpha\beta}$ are dynamical objects. As a two-dimensional field theory, the Polyakov action describes d scalar fields x^μ coupled to two-dimensional gravity. Varying the action with respect to the metric $h_{\alpha\beta}$ tells us that the two-dimensional stress-energy tensor $T_{\alpha\beta}$ should vanish. The equations of motion from a variation with respect to the fields x^μ yield a free wave equation, determining the string's propagation in target spacetime. The latter variation also specifies boundary conditions, allowing both open and closed string solutions [85, 86].

The Polyakov action is invariant under several symmetries. It is invariant under d-dimensional Poincaré transformations,

$$x'^{\mu}(\sigma) = \Lambda^{\mu}{}_{\nu} x^{\nu}(\sigma) + a^{\mu}, \qquad h'_{\alpha\beta}(\sigma) = h_{\alpha\beta}(\sigma),$$

and under two-dimensional reparameterizations $\sigma \mapsto \sigma'(\sigma)$,

$$x'^{\mu}(\sigma') = x^{\mu}(\sigma), \qquad h'_{\alpha\beta}(\sigma') = \frac{\partial\sigma^{\gamma}}{\partial\sigma'^{\alpha}} \frac{\partial\sigma^{\delta}}{\partial\sigma'^{\beta}} h_{\gamma\delta}(\sigma).$$

Most importantly, and very specific to the two-dimensional nature of the worldsheet, it is invariant under *Weyl transformations*,

$$x'^{\mu}(\sigma) = x^{\mu}(\sigma), \qquad h'_{\alpha\beta}(\sigma) = e^{2\omega(\sigma)} h_{\alpha\beta}(\sigma),$$

which are, again, local scale transformations of the theory. From (3.14) it is clear that the stress-energy tensor is traceless if and only if a theory is invariant under Weyl transformations, explaining the terminology for the coefficients in (3.13). In two dimensions, the Weyl symmetry is special, as it ensures that all three metric modes can be gauged away. This also shows why there are no gravitational dynamics in two dimensions.

The Weyl invariance of the worldsheet action is reminiscent of conformal invariance. The relation can best be seen by noting that the symmetries of the Polyakov action are gauge symmetries. Gauge symmetries describe a redundancy in the theory, introduced for mathematical convenience but at the same time introducing more degrees of freedom than just the physical ones. In the Polyakov action the redundancy can be removed by fixing a gauge, $h_{\alpha\beta} = \eta_{\alpha\beta}$. After gauge fixing, the (Wick)-rotated action,

$$S[x] = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma\, \delta^{\alpha\beta} \partial_{\alpha} x^{\mu} \partial_{\beta} x^{\nu} \eta_{\mu\nu}, \tag{3.20}$$

is conformally invariant. Conformal transformations describe a residual gauge symmetry, a particular combination of diffeomorphisms that can be undone by a Weyl transformation [87].

Weyl invariance and background dynamics

Before and after gauge fixing, string theory can equivalently be described by a Weyl invariant worldsheet action with dynamical metric or by a two-dimensional conformal field theory with a fixed metric respectively. As these are gauge symmetries, it is important that invariance is maintained both at the classical as well as at the quantum level. An anomalous gauge symmetry would introduce a dependence on the gauge choice, promoting unphysical degrees of freedom. For a conformal field theory the risk of a quantum anomaly is not improbable. We have already seen that, even if a theory is classically locally scale invariant, renormalization effects can easily introduce anomalous contributions to the stress-energy tensor at the quantum level. Therefore, imposing conformal invariance on the worldsheet even at the quantum level, introduces severe constraints on the possible worldsheet theory.

One of the quantum excitations of the solutions x^μ of the Polyakov action (3.19) is the spacetime graviton. This introduces a quantum deviation from the flat Minkowski metric $\eta_{\mu\nu}$ through which the strings propagate. Building a full coherent target spacetime metric $g_{\mu\nu}$ from such gravitons, the string's trajectory is determined by the spacetime curvature determined from the two-dimensional worldsheet action. At the same time, conformal invariance dictates what kind of conformal theory, including its quantum excitations such as the graviton, is allowed. The subtle interplay between the string's propagation in the background metric and the background dynamics built up from string excitations is a non-trivial consistency check on the two-dimensional worldsheet. Weyl (or, equivalently, conformal) invariance is at the heart of this consistency of string theory. The relation between background dynamics and Weyl invariance is one of the best understood and most studied features of string theory, going back to the advent of the theory in the early 1980s [85–87, 113–117]. We will now explain how Weyl invariance determines the dynamics of the background fields and as a result how general relativity follows from string theory.

Strings in a curved background

Strings moving in a curved background target spacetime metric can be described by the Euclidean action

$$S[x,h] = \frac{1}{4\pi\alpha'} \int \mathrm{d}^2\sigma \sqrt{h} \left[h^{\alpha\beta} g_{\mu\nu}(x) \partial_\alpha x^\mu \partial_\beta x^\nu + \alpha' \Phi(x) R^{(2)} \right], \tag{3.21}$$

where $g_{\mu\nu}(x)$ is the target spacetime metric and $\Phi(x)$ is the dilaton field. The action (3.21) is a straightforward generalization of (3.19), promoting the flat spacetime metric $\eta_{\mu\nu}$ to the (dynamical) metric $g_{\mu\nu}(x)$ of the curved background spacetime. The dilaton contribution gives rise to a weight factor in the path integral sum over all geometries. When the dilaton Φ is constant, it multiplies the topologically invariant Euler number of the worldsheet, that counts the genus of the two-dimensional Riemann surface. The vacuum expectation value of the dilaton is therefore directly related to the *string coupling constant* $g_s = e^{\Phi_0}$ which determines the likelihood of strings joining or splitting. Usually one also includes a contribution from the anti-symmetric Kalb-Ramond field $B_{\mu\nu}(x)$, which will be assumed to vanish throughout this thesis.

Equation (3.21) can be obtained from an exponentiation of the massless quantum excitations of the string. In a flat Minkowski target spacetime metric, the massless excitations of the string decompose into three irreducible representations of the Poincaré algebra: a traceless symmetric representation giving rise to the graviton, an anti-symmetric representation leading to $B_{\mu\nu}$ and a trace, i.e. singlet, representation for the dilaton Φ. Each of the excitations is described by a corresponding vertex operator, inserting the required excitation in the far past, via the state-operator correspondence. The vertex operators may be combined into the fields $g_{\mu\nu}(x)$ and $\Phi(x)$, which give rise to (3.21) after exponentiation of the vertex operators [85, 86, 118].

As expected after our plea for Weyl invariance of the worldsheet theory, the first term in the action (3.21) is (classically) Weyl invariant. The generalization from (3.19) by promoting the Minkowski metric $\eta_{\mu\nu}$ to a general metric $g_{\mu\nu}(x)$ has no effect on the Weyl invariance of the theory. As a two-dimensional theory, it is just a change in the functional of couplings in front of the kinetic terms for the scalar fields. The way in which this is done is known as a *nonlinear σ model*. The second term of (3.21) is all the more surprising, as it already breaks Weyl invariance at the classical level for a non-constant dilaton profile $\Phi(x)$. However, it is necessary to include the dilaton in order to take into account the full multiplet of massless quantum string excitations. For this reason, the term appearing in the action was introduced by [119] and was shown to behave consistently with the other massless degrees of freedom. As we will see shortly, the tree level Weyl variation of the dilaton can be combined with the one-loop Weyl anomalies arising from the other terms [120]. The additional factor of α' helps ordering the different contributions to the breaking of Weyl invariance.

To preserve Weyl invariance at the quantum level, we again impose a vanishing trace of the stress-energy tensor Θ of the two-dimensional worldsheet tensor. Expanding Θ in terms of the operators appearing in (3.21),

$$\Theta = -\frac{1}{2\alpha'}\beta^g_{\mu\nu}h^{\alpha\beta}\partial_\alpha x^\mu \partial_\beta x^\nu - \frac{1}{2}\beta^\Phi R^{(2)},$$

the stress-energy tensor is traceless if the β functions,

$$\beta^g_{\mu\nu} = \alpha' R_{\mu\nu} + 2\alpha'\nabla_\mu\nabla_\nu\Phi + O(\alpha'^2), \tag{3.22a}$$

$$\beta^\Phi = \frac{d-26}{6} - \frac{\alpha'}{2}\nabla^2\Phi + \alpha'\nabla_\mu\Phi\nabla^\mu\Phi + O(\alpha'^2), \tag{3.22b}$$

vanish. The β functions are given up to first order in α'. To compute them, one has to combine several contributions, which we consider individually.

The first term in β^{Φ} is a pure quantum anomaly, first calculated in the context of string theory by [116]. In general, a conformal field theory with a curved metric $h_{\alpha\beta}$ has an anomaly proportional to the two-dimensional Ricci scalar

$$\Theta = -\frac{c}{12} R^{(2)}. \tag{3.23}$$

The constant of proportionality is called the *central charge*, as it appears as a central charge in the quantum algebra of the generators of conformal transformations. This anomaly tells us that, in the quantum theory, the trace of the stress-energy tensor no longer vanishes and conformal invariance is broken, i.e. the theory has become scale dependent. Since in string theory the conformal symmetry follows from the local Weyl *gauge* symmetry, the metric is dynamical and the only way to ensure an anomaly-free quantum theory is to consider conformal field theories which have central charge $c = 0$. The value of $c = d - 26$ in β^{Φ} can be understood by the study of the path integral of the string worldsheet [116]. Due to the gauge redundancy in the path integral measure, one has to be careful to not overcount the number of (inequivalent) physical configurations. This can be done by the use of a Faddeev-Popov determinant, which can be written in terms of a ghost action. The ghost action for Weyl transformations is a conformal field theory with central charge $c = -26$. This is why any worldsheet action, with the ghost action left implicit, has to have central charge $c = 26$. The curved Polyakov action (3.21) achieves this geometrically by considering d scalar fields, which explains the much emphasized critical dimension for string theory. However, the curved Polyakov action only serves as a motivational starting point for string theory. In principle, any conformal field theory with central charge $c = 26$ would describe some solution to string theory, emphasizing it is not the dimension but the central charge that is critical.

The other terms in (3.22) are conceptually more straightforward to understand, but technically still quite involved to compute [85, 86, 115, 117]. The metric profile $g_{\mu\nu}(x)$ and dilaton profile $\Phi(x)$ act as coupling functionals to the operators in (3.21). Renormalizing these coupling constants will lead to β functions much in the same way as we explained previously. The Ricci tensor $R_{\mu\nu}$ of the target spacetime and the second term of β^{Φ} arise due to these renormalization effects. The remaining terms are due to classical breaking of Weyl invariance by the dilaton term, which appear at the same order as the renormalization effects from the other terms as predicted.

It can be shown explicitly that the β functions (3.22) are proportional to the equations of motion for the background fields, $g_{\mu\nu}(x)$ and $\Phi(x)$, that one would compute in string perturbation theory [105, 121]. The stringy degrees of freedom, i.e. excitations with a mass proportional to α', do not play a role, as the worldsheet perturbation is derived in the limit of small α', or in other words for strings with a very large tension. In this limit, stringy excitations cost a lot of energy, which justifies the name "low energy equations of motion" for (3.22). In fact, (3.22) can be integrated to a *low energy effective action*,

$$S[g,\Phi] = \frac{1}{2\kappa_0^2}\int \mathrm{d}^d x\,\sqrt{g}\,e^{-2\Phi}\left[-\frac{2(d-26)}{3\alpha'} + R + 4\,(\nabla\Phi)^2 + O(\alpha')\right]. \tag{3.24}$$

By a field redefinition $\Phi^{\text{new}} = \Phi^{\text{old}} - \Phi_0$ and $g^{\text{new}}_{\mu\nu} = e^{-4\Phi^{\text{new}}/(d-2)} g^{\text{old}}_{\mu\nu}$, the action can be written in a more familiar form,

$$S[g,\Phi] = \frac{1}{2\kappa^2}\int \mathrm{d}^d x\,\sqrt{g}\left[-\frac{2(d-26)}{3\alpha'}e^{4\Phi/(d-2)} + R - \frac{4}{d-2}\,(\nabla\Phi)^2 + O(\alpha')\right], \tag{3.25}$$

where $\kappa = \kappa_0 e^{\Phi_0} = \sqrt{8\pi G_N}$ is the gravitational coupling constant. Equation (3.25) describes a scalar field Φ coupled to Einstein gravity in d dimensions, showing explicitly that string theory is a theory of spacetime quantum gravity.

The relation between the two-dimensional worldsheet action and how it describes general relativity in the d-dimensional target spacetime is an intricate result. The fact that we can express the equations of motion given by (3.22) in terms of a target spacetime action guarantees that the equations are mutually consistent [120]. Crucial for the inner consistency is the interdependence among the β functions. The dilaton β function β^Φ acts as the central charge of the full nonlinear σ model. Although it looks like an x-dependent quantity, it is really a c-number due to the vanishing of $\beta^g_{\mu\nu}$ (and $\beta^B_{\mu\nu}$ if we would not have set $B_{\mu\nu}$ to zero to begin with) [117, 120]. It is this central charge, or rather the combination $\beta^\Phi - g^{\mu\nu}\beta^g_{\mu\nu}$, that effectively acts as the integrand for the low energy effective action (3.24). For the first order equations (3.22) we can verify these statements explicitly, but it can be proven to hold on general grounds for all order α'-corrections [107, 108, 122]. The possibility to interpret the conditions set by worldsheet Weyl invariance as a spacetime low energy effective action is one of the most remarkable results from string theory.

3.2.2 Supergravity

A super symmetry in our universe

In the previous section we considered the bosonic string, i.e. a string whose worldsheet theory is defined in terms of bosonic scalar fields x^μ only. It is a very interesting theory to study the relation between the worldsheet and spacetime theories, but it is unsure to what extent this version of string theory can describe our universe. Apart from the massless quantum excitations we have just considered, the bosonic string also contains a tachyonic mode, indicating that the theory suffers from an instability [85]. To remove the tachyon from the spectrum, the worldsheet is extended to a *superstring* theory, in which worldsheet bosons and fermions are related by a symmetry called *supersymmetry*. Similar to the previous discussion, the superstring worldsheet theory defines a low energy effective field theory for the background fields on the target spacetime. Because the value of the central charge c of a worldsheet theory with superconformal symmetry is $c = \frac{3d}{2} - 15$, the spacetime theory is a ten-dimensional theory.

In order to relate to our four-dimensional spacetime, the spacetime has to be compactified on an internal six-dimensional manifold [123]. The internal manifold is a compact manifold, which is too small for us to detect at low energies, giving rise to an effective four-dimensional action for the spacetime theory after compactification. In this thesis we will study superstring theory only through its four-dimensional low energy effective action, except for a short excursion in chapter 5 where we discuss how to possibly generalize the result of that chapter to open strings, the D-branes that they end on and the background RR fields sourced by the D-branes. At the level of the low energy effective action, supersymmetry remains a fundamental aspect for the theory, since the spacetime bosons and fermions are also invariant under the supersymmetry transformations [86].

The low energy effective action of superstring theory is an example of a *supergravity* theory, but the framework of supergravity is more general than just the supergravity theories arising from superstring theory. In the 1970s supersymmetry was discovered as a way to regulate UV-divergences in phenomenological particle physics models [124–127]. As gauge symmetries were particularly popular at the time for the successful way in which they describe particle physics, it was only a natural step to consider a theory which is invariant under local supersymmetry [128]. The surprising result is that such a theory necessarily incorporates gravity [88], hence the name "supergravity". Initial hope that supergravity theory might be a "theory of everything", unifying particle physics theories with general relativity, soon proved incorrect, because supergravity is not renormalizable. Therefore, in the beginning of the 1980s superstring theory started to replace supergravity as the new candidate theory for quantum gravity [85]. Nevertheless, through the relation between worldsheet superstring theory and its supergravity low energy effective theory, supergravity models have never really left the stage, providing an interesting playground at the effective field theory level for the study of quantum gravity.

Super dynamics

As with any symmetry, supersymmetrically invariant theories are constrained. The bosons and fermions of the theory have to reside in supermultiplets, which are irreducible representations of the supersymmetry algebra. To ensure that the algebra is closed off-shell as well, each supermultiplet also contains an auxiliary field. Conventionally for the chiral supermultiplets, i.e. the simplest four-dimensional supermultiplet that contains a scalar, the auxiliary field is denoted by F^I, where I is an index running over the number of chiral supermultiplets. Similarly, gauge vector supermultiplets have an auxiliary field denoted by D^A, where A runs over the number of vector supermultiplets. These auxiliary fields do not have a kinetic term in the action and therefore contain no propagating (physical) degrees of freedom. It turns out that potentials in supersymmetric field theories are precisely generated by integrating out these non-dynamical fields [88, 89]. Supersymmetry and supergravity potentials therefore naturally fall into two categories. The scalar potentials built from F are called F-terms, those built from the D-fields are called D-terms. In this thesis we will be concerned with the scalars ξ^I of the chiral multiplets. We will assume they are neutral under the gauge group, allowing us to concentrate on the F-terms.

In global supersymmetry the action for the complex scalars ξ^I in the chiral supermultiplets can be written as

$$S = -\int \mathrm{d}^4x\,\sqrt{g}\left[g^{\mu\nu}K_{I\bar{J}}(\xi,\bar{\xi})\nabla_\mu\xi^I\nabla_\nu\bar{\xi}^{\bar{J}} + V(\xi,\bar{\xi})\right]. \tag{3.26}$$

Supersymmetry has restricted the kinetic term to be a nonlinear σ model describing a Kähler manifold. The *Kähler potential* $K(\xi,\bar{\xi})$ is a real function which completely specifies the metric $G_{I\bar{J}}(\xi,\bar{\xi})$ of the target manifold,

$$G_{I\bar{J}} = \partial_I\partial_{\bar{J}}K \equiv K_{I\bar{J}}, \qquad G_{IJ} = G_{\bar{I}\bar{J}} = 0.$$

The F-term potential V is determined by the holomorphic *superpotential* $W(\xi)$ [127] via

$$V = K^{I\bar{J}}W_I\overline{W}_{\bar{J}},$$

where we denote derivatives with respect to the fields ξ^I and $\bar{\xi}^{\bar{J}}$ with a subscript, e.g. $W_I = \frac{\partial}{\partial\xi^I}W$. In supersymmetric theories, supersymmetry is broken precisely if the vacuum expectation value for F^I is non-vanishing, which via the equations of motion for F in the original action

$$F^I = K^{I\bar{J}}\overline{W}_{\bar{J}},$$

implies that supersymmetry is broken if and only if $W_I = \partial_I W = 0$ [88, 127].

In supergravity, an important change happens to the potential. The action for the complex scalars ξ^I in the chiral supermultiplets can now be written as

$$S = \int \mathrm{d}^4x\,\sqrt{g}\left[\frac{\mathrm{M}_{\mathrm{pl}}^2}{2}R - g^{\mu\nu}K_{I\bar{J}}(\xi,\bar{\xi})\nabla_\mu\xi^I\nabla_\nu\bar{\xi}^{\bar{J}} - V(\xi,\bar{\xi})\right]. \tag{3.27}$$

Again the nonlinear σ model target manifold is restricted to be a Kähler manifold with Kähler potential $K(\xi,\bar{\xi})$, but the F-term potential is now given by

$$V = e^{K/\mathrm{M}_{\mathrm{pl}}^2}\left(K^{I\bar{J}}\mathrm{D}_I W\overline{\mathrm{D}_J W} - \frac{3}{\mathrm{M}_{\mathrm{pl}}^2}W\overline{W}\right), \tag{3.28}$$

where $\mathrm{D}_I W$ denotes the Kähler covariant derivative

$$\mathrm{D}_I W = \partial_I W - \frac{\partial_I K}{\mathrm{M}_{\mathrm{pl}}^2}W.$$

The supergravity action is invariant under Kähler transformations

$$K(\xi,\bar{\xi}) \to K(\xi,\bar{\xi}) + f(\xi) + \bar{f}(\bar{\xi}), \qquad W(\xi) \to e^{-f(\xi)/\mathrm{M}_{\mathrm{pl}}^2}W(\xi),$$

by an arbitrary holomorphic function $f(\xi)$, which suggests to rewrite the theory in terms of one real, Kähler invariant function $G(\xi,\bar{\xi})$ that is related to the Kähler potential and superpotential via

$$G(\xi,\bar{\xi}) = K(\xi,\bar{\xi}) + \mathrm{M}_{\mathrm{pl}}^2\log\left(\frac{W(\xi)}{\mathrm{M}_{\mathrm{pl}}^3}\right) + \mathrm{M}_{\mathrm{pl}}^2\log\left(\frac{\overline{W}(\bar{\xi})}{\mathrm{M}_{\mathrm{pl}}^3}\right). \tag{3.29}$$

This definition is only valid for $W \neq 0$. A vanishing superpotential is a fixed point under Kähler transformations and deserves special treatment. Throughout this thesis we will therefore assume that $W \neq 0$. In terms of the *Kähler function* $G(\xi,\bar{\xi})$, the F-term potential reads

$$V = e^{G/\mathrm{M}_{\mathrm{pl}}^2}\left(G^{I\bar{J}}G_I G_{\bar{J}} - 3\mathrm{M}_{\mathrm{pl}}^2\right)\mathrm{M}_{\mathrm{pl}}^2. \tag{3.30}$$

Since

$$F^I = e^{G/2\mathrm{M}_{\mathrm{pl}}^2}G^{I\bar{J}}G_{\bar{J}}$$

in supergravity theories, supersymmetry is broken if and only if $G_I = \frac{\mathrm{D}_I W}{W} = 0$ [88].

The action (3.27) provides an interesting starting point for the purpose of inflationary model building. Ideally, one would like to study inflation directly from the superstring theory point of view, but since superstring theory is not yet fully known, our investigations are restricted to its low energy effective limit. The supergravity theories that appear as the low energy effective action of superstring theory are a subset of all supergravity theories. However, in the literature this distinction is not always made, because we first need to focus on the characteristic effects and possible issues in supergravity in general. For example, from (3.30) we can already infer one of these generalities, that a quasi-de Sitter phase with $V > 0$ requires supersymmetry to be broken during inflation. Although string inspired work can be found throughout the supergravity literature [1, 2], the construction of a model completely rooted in a consistent superstring theory set-up is still to be found. Until such a model exist, the rich but yet restricted character of supergravity make it an interesting framework for the study of inflation in quantum gravity.

3.2.3 Holography

Gauge/gravity duality

A final ingredient we shall need for the studies following, is holography and the AdS/CFT-correspondence. The discovery, about fifteen years ago, that string theory realizes the holographic principle, is a major development in theoretical physics. The holographic principle is a (crazy) hypothesis that the physics of a d-dimensional gauge theory can also be described by a $d+1$-dimensional theory with gravity and vice versa [129, 130]. The motivation for such a hypothesis derives from black hole physics, in which all the information of the black hole can be encoded by way of its event horizon.

Inspired by the work of others in this direction [131–135], a conjectured realization of two dual theories was constructed by [136]. In this realization we consider a system of D-branes[4] in a flat background geometry. This configuration has two distinct limits, each with its own description. One description considers the supergravity approximation around the branes, which is that of an anti-de Sitter AdS_5-geometry. The other description decouples the interacting brane-bulk system, leaving only the

[4]The known examples of the holographic duality are all advanced constructs in superstring theory. As a result, they contain elements not explained in this text elsewhere. The particular system in [136] is a set of N parallel $D3$-branes in a ten-dimensional flat background, which one can view as the supergravity limit of a type IIB superstring in an $AdS_5 \times S^5$-background on the one hand or as a decoupled brane-bulk system on the other hand, with the gauge theory on the brane being a four-dimensional $\mathcal{N} = 4$ superconformal $SU(N)$ Yang-Mills theory.

gauge theory on the brane, a specific four-dimensional conformal field theory. Since both descriptions originate from the same system, they present a dual description of the same physics [90, 91, 136]. We say that the theory in the bulk is dual to the conformal field theory living on its boundary, because the brane resides at the boundary of the anti-de Sitter space. The holographic duality in this construction is called the *AdS/CFT-correspondence* as it is a duality between anti-de Sitter geometry and conformal field theory.

Since a d-dimensional field theory has one dimension less than a d+1-dimensional gravity theory, the natural question arises how holography manages to encode the additional dimension of the bulk theory into the boundary theory. The example of [136] provides a clear indication of how this happens. The (Euclidean) AdS_{d+1}-metric is given by

$$\mathrm{d}s^2 = \mathrm{d}y^2 + e^{-2y/R}\mathrm{d}\boldsymbol{x}^2,$$

or

$$\mathrm{d}s^2 = \frac{R^2}{z^2}\left(\mathrm{d}\boldsymbol{x}^2 + \mathrm{d}z^2\right),$$

in Poincaré coordinates, where R is the anti-de Sitter-radius and where the boundary is located at $z = 0$. It is invariant under a scale transformation $\boldsymbol{x} \to \lambda\boldsymbol{x}$, $z \to \lambda z$. The d coordinates $\boldsymbol{x}$ are naturally identified with the coordinates of the conformal field theory, setting $z = 0$. The interpretation of the additional coordinate z becomes clear when we consider a scale transformation $\boldsymbol{x} \to \lambda\boldsymbol{x}$ in the field theory as well. The theory is scale invariant when such a scale transformation is accompanied by a rescaling of the energy scale $\mu \to \lambda^{-1}\mu$ [137]. Hence, the additional coordinate of the gravity theory corresponds to the energy scale in the gauge theory,

$$z \sim \frac{1}{\mu},$$

and the direction towards the interior of the bulk corresponds to a renormalization group flow from high energies to low energies. This immediately suggests that the AdS/CFT correspondence could be generalized to a bulk theory that is asymptotically anti-de Sitter with a dual gauge theory that approaches a conformal fixed point in the ultraviolet [137, 138]. Renormalization of the ultraviolet divergences of the gauge theory is completely understood in terms of regularizing and renormalizing the large distance, i.e. near-boundary, behavior of the bulk theory [138–142].

An important aspect of AdS/CFT is that the two limits where either the field theory or the gravitational description arises, correspond to opposite limits of the intrinsic CFT coupling constant [136, 143]. This means that the strongly coupled physics of one theory is equivalently described by the weakly coupled dual theory. On the one hand, the strong/weak-aspect of the duality makes it very difficult to verify a conjectured holographic correspondence, since a perturbative approach can only work for one of the two dual theories at a time. Making use of protecting symmetries of the theory, it is possible to match some of the properties of each of the two systems, indicating that the conjecture might hold. On the other hand, once a correspondence between theories has been (reasonably) established, the strong/weak duality provides a truly powerful approach to understand strongly coupled physics, by considering the weakly coupled dual theory.

The holographic correspondence is conjectured to hold for more general gauge and gravity theories than the AdS/CFT-correspondence of [136]. Finding other examples is difficult, but possible [90, 137]. As said, the hallmark strong/weak-duality of dual theories gives ample motivation to search for holographic examples, for the unique orthogonal approach the duality provides to the study of strongly coupled systems. Particularly relevant for cosmology would be if a correspondence between de Sitter space and some gauge theory is found. In principle dS/CFT should be closely related to AdS/CFT, as both gravity theories have a great resemblance [144–146]. This is immediate at the level of their symmetries, which in both cases is $\mathrm{O}(1,d)$ for a d-dimensional spacetime. In practice it proves difficult to actually find an explicit realization of the dS/CFT-correspondence. Nevertheless, the possibility of having a holographic description of (quasi)-de Sitter geometry provides the motivation behind chapter 6 of this thesis. In particular, in that chapter we will see to what extent conformal invariance dictates the correlation functions of the gravity theory.

Correlation functions

The real power of the AdS/CFT-correspondence is the precise quantitative dictionary, described in [147, 148], between the two perspectives. In these descriptions, a Euclideanized version of the gravity theory is considered. A field $\phi(z,\boldsymbol{x})$ in the bulk of the $d+1$-dimensional gravity theory has an asymptotic value $\phi_0(\boldsymbol{x})$ on the d-dimensional boundary, which acts as a coupling constant for an operator $O(\boldsymbol{x})$ of the boundary field theory. The duality is then summarized by the statement that the partition functions Z_{CFT} and Z_{AdS} are equal,

$$Z_{AdS}[\phi(\phi_0)] = Z_{CFT}[\phi_0] = \left\langle e^{-\int \mathrm{d}^d\boldsymbol{x}\,\phi_0 O}\right\rangle_{CFT}. \tag{3.31}$$

The partition function $Z_{AdS}[\phi(\phi_0)] = \int_{\phi_0} \mathcal{D}\phi e^{-S_{AdS}(\phi)}$ is evaluated in the semiclassical limit, i.e. a classical solution for the field $\phi(z,\boldsymbol{x})$ is found subject to the boundary condition $\phi_0(\boldsymbol{x})$ around which the action is perturbed. n-point correlation functions of the operators in the conformal field theory can then be calculated in the usual way through functional differentiation with respect to the boundary conditions,

$$\langle O(\boldsymbol{x}_1)\dots O(\boldsymbol{x}_n)\rangle = \frac{\delta}{\delta\phi_0(\boldsymbol{x}_1)}\cdots\frac{\delta}{\delta\phi_0(\boldsymbol{x}_n)} Z_{AdS}\left[\phi(\phi_0)\right]\Big|_{\phi_0=0}, \tag{3.32}$$

which act as sources to the operators.

To get finite answers in the matching of the asymptotic values for the bulk fields ϕ with the boundary couplings ϕ_0, the boundary fields are renormalized, which leads to a relation between the scaling dimension of the operator O to which ϕ_0 couples and the mass m of the bulk field [137, 147, 148],

$$\Delta = \frac{d}{2} + \sqrt{\frac{d^2}{4} + m^2R^2}, \tag{3.33}$$

where R is again the anti-de Sitter curvature radius. A massless field m corresponds to a marginal operator $\Delta = d$. With the identification given above, the (physical degrees of freedom of the) metric field $g_{\mu\nu}(z, \boldsymbol{x})$ corresponds to the stress-energy tensor operator $T_{\alpha\beta}(\boldsymbol{x})$ of the conformal field theory. The stress-energy tensor is a marginal operator that is always part of the conformal field theory, which is why the bulk theory always has to include gravity [137].

As an illustrative example of how the correspondence works, we consider an interacting massive scalar field ϕ in $d+1$-dimensional AdS, with action

$$S_{AdS} = \frac{1}{2}\int \mathrm{d}^d\boldsymbol{x}\mathrm{d}z\,\sqrt{g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2 + \frac{\lambda}{3}\phi^3\right]. \tag{3.34}$$

We have to solve the classical equation of motion subject to the boundary condition $\phi_0(\boldsymbol{x})$. This can be achieved conveniently by first finding the Green's function for the equation of motion of the quadratic part of the action [148, 149],

$$K_\Delta(z, \boldsymbol{x}, \boldsymbol{x}') = \frac{\Gamma(\Delta)}{\pi^{\frac{d}{2}}\Gamma(\Delta - \frac{d}{2})}\left(\frac{z}{z^2 + (\boldsymbol{x}-\boldsymbol{x}')^2}\right)^\Delta,$$

where Δ is related to the mass m via (3.33). The function $K_\Delta(z, \boldsymbol{x}, \boldsymbol{x}')$ is called the bulk-to-boundary propagator, which has to be normalized such that it is regular in the interior and provides the required singular behavior for $z \to 0$. The classical (homogeneous) solution is then automatically determined by the boundary value $\phi_0(\boldsymbol{x})$ via

$$\phi(z, \boldsymbol{x}) = \int \mathrm{d}^d\boldsymbol{x}'\, K_\Delta(z, \boldsymbol{x}, \boldsymbol{x}')\phi_0(\boldsymbol{x}').$$

To find the three-point function of the dual operator O (3.32), we can substitute this expression into (3.34) and find

$$\begin{aligned}\langle O(\boldsymbol{x}_1)O(\boldsymbol{x}_2)O(\boldsymbol{x}_3)\rangle &= \frac{\delta}{\delta\phi_0(\boldsymbol{x}_1)}\frac{\delta}{\delta\phi_0(\boldsymbol{x}_2)}\frac{\delta}{\delta\phi_0(\boldsymbol{x}_3)}Z_{AdS}\left[\phi(\phi_0)\right]\Big|_{\phi_0=0} \qquad (3.35)\\ &= -\lambda\int\frac{\mathrm{d}^d\boldsymbol{x}\mathrm{d}z}{z^{d+1}}K_\Delta(z,\boldsymbol{x},\boldsymbol{x}_1)K_\Delta(z,\boldsymbol{x},\boldsymbol{x}_2)K_\Delta(z,\boldsymbol{x},\boldsymbol{x}_3).\end{aligned}$$

Since this is a three-point correlation function in a conformal field theory, it should be of the form (3.12b). One can explicitly verify that this is so and determine the coefficient from the explicit form of the bulk-to-boundary propagator [149, 150],

$$\langle O(\boldsymbol{x}_1)O(\boldsymbol{x}_2)O(\boldsymbol{x}_3)\rangle = \frac{\lambda a(\Delta)}{(x_{12}x_{23}x_{13})^\Delta}, \qquad (3.36a)$$

$$a(\Delta) = -\frac{\Gamma\left(\frac{1}{2}(3\Delta-d)\right)\Gamma\left(\frac{\Delta}{2}\right)^3}{2\pi^d\Gamma\left(\Delta-\frac{d}{2}\right)^3}. \qquad (3.36b)$$

The matching of (3.35) with (3.12b) is a necessary requirement for the correspondence to hold. It is an explicit check that the anti-de Sitter space is constrained by the same symmetries as the conformal field theory. In general, for a duality to hold, the theories need to be invariant under the same symmetries. This is of course not a sufficient condition. Nevertheless, it is interesting to see what one can already derive based solely upon symmetry arguments. We will take the latter approach in our study of a hypothesized dS/CFT-correspondence in chapter 6.