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ABSTRACT

The object of study of this paper is the asymptotic behaviour of sequences $\{M_n\}_{n\geq 1}$ of square matrices with real or complex entries. Two decomposition theorems are treated. These give conditions under which a sequence of non-singular square matrices whose terms are block-diagonal (diagonal, respectively) matrices plus some perturbation term can be transformed into a sequence $\{F_{n+1}^{-1} M_n F_n\}_{n\geq 1}$ whose terms are block-diagonal (diagonal) and where the sequence $\{F_n\}_{n\geq 1}$ converges to the identity. In the first section we introduce the concept of a matrix recurrence and some further notation. In §2 we present the first of the two decomposition theorems. As an application, we present, in §3, a generalization of the Theorem of Poincaré–Perron for linear recurrences, and in §4 we prove a decomposition theorem for matrix sequences that are the sum of a sequence of diagonal matrices and some (small) perturbation term. In the final section we use the second decomposition theorem to derive a result concerning the solutions of matrix recurrences in case the matrices converge fast to some limit matrix. All our results are quantitative as well.

1. PRELIMINARY CONCEPTS

Let $\mathcal{K}$ be the field of real or complex numbers. In this paper we study sequences $\{M_n\}_{n\geq N}$ ($N \in \mathbb{Z}$) of matrices in the set $\mathcal{K}^{k,k}$ of $k \times k$-matrices with entries in $\mathcal{K}$ that display a regular asymptotic behaviour. We call a sequence $\{M_n\}_{n\geq N}$ convergent to M if for all i,j the entries $(M_n)_{ij}$ converge to some number $M_{ij} \in \mathcal{K}$ (for a matrix $A \in \mathcal{K}^{k,l}$ we let A_{ij} denote the entry in the i-th row and the j-th column ($1 \leq i \leq k$, $1 \leq j \leq l$)). The limit matrix M will also be denoted by $\lim M_n$.

For $M \in \mathcal{K}^{k,l}$ we define the *norm* $\|M\|$ as the matrix norm induced by the Euclidian vector norm on $\mathcal{K}^l$:

$$\|M\| = \max_{x\neq 0} |Mx|/|x|.$$

In particular, we have that $\|MN\| \leq \|M\| \cdot \|N\|$ whenever the multiplication is well-defined.

A block-diagonal matrix is a matrix $M \in \mathcal{K}^{k,k}$ of the form

$$M = \begin{pmatrix} S_1 & & & 0 \\ & S_2 & & \\ & & \ddots & \\ 0 & & & S_h \end{pmatrix}$$

where $S_i \in \mathcal{K}^{k_i,k_i}$, $\sum_{i=1}^{h} k_i = k$. We shall denote such matrices by $M = \mathrm{diag}(S_1, S_2, \ldots, S_h)$. If some of the blocks are 1×1-matrices, we just write their value: $M = \mathrm{diag}(\alpha_1, S_2, \ldots, S_h)$ if $S_1 = (\alpha_1)$.

We recall the concept of a Jordan canonical form. For convenience, we denote by I_k (or by I, as well) the $k \times k$ identity matrix and by J_k the $k \times k$-matrix such that $(J_k)_{ij} = \delta_{i+1,j}$ $(1 \leq i, j \leq k)$. By $R(\phi)$ we denote the 2×2 rotation matrix:

$$R(\phi) = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}.$$

For each matrix $M \in \mathbb{R}^{k,k}$ there exists some matrix $U \in \mathbb{R}^{k,k}$ such that $U^{-1}MU = \mathrm{diag}(S_1, S_2, \ldots, S_h)$, where $S_i = \alpha_i I_{k_i} + J_{k_i}$ for some $\alpha_i \in \mathbb{R}$, or $S_i = \alpha_i \cdot \mathrm{diag}(R(\phi_i), \ldots, R(\phi_i)) + J_{k_i}^2$ for some $\alpha_i \in \mathbb{R}$, $\alpha_i > 0$, and $\phi_i \in \mathbb{R}$ $(1 \leq i \leq h)$. For $M \in \mathbb{C}^{k,k}$ a matrix $U \in \mathbb{C}^{k,k}$ can be found such that $U^{-1}MU = \mathrm{diag}(S_1, \ldots, S_h)$ with $S_i = \alpha_i I_{k_i} + J_{k_i}$ for $\alpha_i \in \mathbb{C}$. The matrices $U^{-1}MU$ are called the (real and complex, respectively) Jordan canonical form of M. We call $S_1, \ldots, S_h$ the *Jordan blocks*. The Jordan canonical form is unique up to permutation of the Jordan blocks.

We introduce some more notation: For $A \in \mathcal{K}^{k,k}$ and $z \in K$, $z \neq 0$, we define

$$z^A := \mathrm{e}^{A \log z} = \sum_{l=0}^{\infty} \frac{1}{l!}(A \log z)^l$$

for some branch of the logarithm. In this paper, we take $z \in \mathbb{R}$, $z > 0$ and $\log z \in \mathbb{R}$. Note that for A a diagonal matrix, z^A has a particularly simple form: it is a diagonal matrix with entries $(z^A)_{ij} = z^{A_{ij}}$ $(1 \leq i, j \leq k)$.

By $\mathcal{M}$ we denote the set of functions $f : \mathbb{N} \to \mathbb{R}_{>0}$ such that $\lim_{n\to\infty} f(n)$ exists in $\mathbb{R}$ or is infinity, and such that $f(n)/f(m)$ is bounded either from above or from below for $m, n \in \mathbb{N}$, $N \leq n \leq m$. The subset $\mathcal{M}^0 \subset \mathcal{M}$ consists of the functions $f \in \mathcal{M}$ for which $\lim_{n\to\infty}(f(n+1)/f(n)) = 1$, and $\mathcal{M}^1$ is the set of functions $f \in \mathcal{M}^0$ such that the functions $f(X) \cdot X^r$ lie in $\mathcal{M}$ for all $r \in \mathbb{R}$.

Finally, in asymptotic estimates we shall avail ourselves of the notations $\sum_{(n)}$ and $\sim$. Let $N \in \mathbb{Z}$ be fixed. If the series $\sum_{j=N}^{\infty} a_j$ $(a_j \in \mathcal{K})$ converges then $\sum_{(n)} a_j := \sum_{j=n}^{\infty} a_j$ $(n \geq N)$, and if it diverges, then $\sum_{(n)} a_j := \sum_{j=N}^{n-1} a_j$ $(n \geq N)$. If f, g are sequences of numbers, vectors or matrices, we write $f \sim g$ if $f(n) - g(n) = \mathrm{o}(|f(n)|)$ or, if $f(n)$ is a matrix, $\mathrm{o}(\|f(n)\|)$ as $n \to \infty$.

Let $\{M_n\}_{n\geq N}$ ($N \in \mathbb{Z}$) be a sequence of non-singular matrices in $\mathcal{K}^{k,k}$. Consider the recurrence relation

$$(1.1) \qquad M_n x_n = x_{n+1} \quad (x_n \in \mathcal{K}^{k,l},\ n \geq N,\ l = 1 \text{ or } k).$$

We call (1.1) the matrix recurrence induced by $\{M_n\}_{n\geq N}$, and $\{x_n\}_{n\geq N}$ a solution of (1.1). For $l = k$, we require that $\det x_n \neq 0$. Clearly, the set of solutions of (1.1) (with $l = 1$) is a k-dimensional linear subspace of the vector space of sequences $\{a_n\}_{n\geq N}$ ($a_n \in \mathcal{K}^k$) (with termwise addition and (scalar) multiplication). We identify two sequences if their terms are equal from a certain index on and we simply write $\{M_n\}$, $\{x_n\}$, etc, without specifying the starting index. If the starting index matters, we usually take 1 or N, without further specification.

2. A DECOMPOSITION THEOREM

In this section we treat the first decomposition theorem for matrix sequences (or matrix recurrences). If we have a matrix recurrence whose defining matrices can be written as the sum of a block-diagonal matrix with two blocks, one of which is (constantly) of smaller 'size' than the other one, plus some perturbation matrix, then, if the perturbation matrix is small enough, another matrix recurrence can be found whose defining matrices are block-diagonal, whereas its solutions $\{x_n\}$ correspond, in a 1-1 manner, to solutions $\{y_n\}$ of the original matrix recurrence such that $|x_n - y_n| = o(|x_n|)$. The use of the theorem lies in the fact that the second matrix recurrence is of simpler form than the first one, whereas the solutions of the former correspond to solutions of the latter which are asymptotically equal. (Note that the solutions of a matrix recurrence whose defining matrices are diagonal, or even upper (or lower) triangular can be calculated in an exact manner, i.e. an explicit expression for them can be given in terms of the coefficients of the matrices). The theorem precises what we mean by the size of the matrix blocks and gives conditions on the size of the perturbations. Moreover, upper bounds are given for the normalized differences $(|x_n - y_n|/|x_n|)$.

Theorem 2.1. *Let $\{M_n\}$ be a sequence of non-singular matrices in $\mathcal{K}^{k,k}$ of the form*

$$(2.1) \qquad M_n = \begin{pmatrix} R_n & Q_n \\ P_n & S_n \end{pmatrix}$$

where $R_n \in \mathcal{K}^{l,l}$ and $S_n \in \mathcal{K}^{k-l,k-l}$ are such that S_n is non-singular, and such that for some sequence $\{\delta_n\}$ with $\delta_n \in \mathbb{R}$, $\delta_n \geq 0$ ($n \in \mathbb{N}$) and $\sum_{n=1}^{\infty} \delta_n < \infty$,

$$(2.2) \qquad 0 \leq \|R_n\| \cdot \|S_n^{-1}\| < 1 + \delta_n \quad \textit{for all } n$$

$$(2.3) \qquad \sum_{n=1}^{\infty} (1 - \|R_n\| \cdot \|S_n^{-1}\|) \quad \textit{diverges,}$$

and moreover

$$(2.4) \qquad \lim_{n\to\infty} \frac{(\|P_n\| + \|Q_n\|)\, \|S_n^{-1}\|}{1 + \delta_n - \|R_n\| \cdot \|S_n^{-1}\|} = 0.$$

Then there exists a sequence $\{F_n\}$ of non-singular matrices $F_n \in \mathcal{K}^{k,k}$ such that

$$(2.5) \qquad F_{n+1}^{-1} M_n F_n = \operatorname{diag}(\tilde{R}_n, \tilde{S}_n)$$

with $\tilde{R}_n \in \mathcal{K}^{l,l}$, $\tilde{S}_n \in \mathcal{K}^{k-l,k-l}$,

$$(2.6) \qquad \|\tilde{R}_n - R_n\| + \|\tilde{S}_n - S_n\| = \mathrm{o}(\|P_n\|)$$

$$(2.7) \qquad \lim F_n = I$$

and, for each $\varepsilon > 0$,

$$(2.8) \qquad \begin{cases} \|F_n - I\| \ll_\varepsilon \sum\limits_{h=n}^{\infty} \|P_h\| \cdot \|\tilde{S}_h^{-1}\| \cdot \prod\limits_{m=n}^{h-1} \|\tilde{R}_m\| \cdot \|\tilde{S}_m^{-1}\| \\ \qquad + \sum\limits_{h=1}^{n-1} (\|Q_h\| + \varepsilon \|P_h\|) \cdot \|S_h^{-1}\| \cdot \prod\limits_{m=h+1}^{n-1} \|R_m\| \cdot \|S_m^{-1}\|. \end{cases}$$

Further, if the matrices R_n are non-singular, and

$$(2.9) \qquad \sum_{h=1}^{\infty} (\|P_h\| + \|Q_h\|) \cdot \|R_h^{-1}\| \cdot \prod_{m=1}^{h-1} \|S_m\| \cdot \|R_m^{-1}\|$$

converges, then the second term on the right-hand side of (2.8) *may be replaced by*

$$(2.10) \qquad \sum_{h=n}^{\infty} (\|Q_h\| + \varepsilon \|P_h\|) \cdot \|R_h^{-1}\| \cdot \prod_{m=n}^{h-1} \|S_m\| \cdot \|R_m^{-1}\|.$$

We prove the theorem in several steps.

Lemma 2.2. *Let $A, B \in \mathcal{K}^{k,k}$ be such that A is non-singular and $\|B\| < \|A^{-1}\|^{-1}$. Then $A + B$ is non-singular and*

$$\|(A+B)^{-1}\| \le \frac{1}{\|A^{-1}\|^{-1} - \|B\|}.$$

Proof. Let $x \in \mathcal{K}^k$, $x \neq 0$. Then

$$|(A+B)x| \ge \big||Ax| - |Bx|\big| \ge (\|A^{-1}\|^{-1} - \|B\|)\,|x| > 0,$$

hence $A + B$ is invertible. Furthermore,

$$\max_{y \neq 0} \frac{|(A+B)^{-1} y|}{|y|} = \max_{x \neq 0} \frac{|x|}{|(A+B)x|} \le \frac{1}{\|A^{-1}\|^{-1} - \|B\|}. \qquad \square$$

Lemma 2.3. (a) *Let $\{R_n\}, \{S_n\}, \{Q_n\}$ be as in Theorem 2.1 with $\{\delta_n\} = \{0\}$. Then, if $\{X_n\}$ ($X_n \in \mathcal{K}^{l,k-l}$) satisfies the recurrence*

$$(2.11) \qquad X_{n+1} S_n = R_n X_n + Q_n,$$

then, for $n \geq N$

$$
(2.12)\quad \begin{cases} \|X_n\| \leq \|X_N\| \cdot \prod_{h=N}^{n-1} \|R_h\| \cdot \|S_h^{-1}\| \\ \qquad + \sum_{l=N}^{n-1} \|Q_l\| \cdot \|S_l^{-1}\| \cdot \prod_{m=l+1}^{n-1} \|R_h\| \cdot \|S_h^{-1}\| \\ \quad \leq \|X_N\| \cdot \prod_{h=N}^{n-1} \|R_h\| \cdot \|S_h^{-1}\| + \max_{l \geq N} \dfrac{\|Q_l\| \cdot \|S_l^{-1}\|}{1 - \|R_l\| \cdot \|S_l^{-1}\|}, \end{cases}
$$

so that $\lim_{n\to\infty} X_n = 0$ *for every solution* $\{X_n\}$ *of* (2.11), ($X_n \in \mathcal{K}^{l,k-l}$).

(b) *Furthermore, the recurrence*

$$
(2.13)\quad Y_{n+1}R_n = S_n Y_n + Q_n
$$

has a solution $\{Y_n^{(0)}\}$, $Y_n^{(0)} \in \mathcal{K}^{k-l,l}$, *with*

$$
(2.14)\quad \|Y_n^{(0)}\| \leq \sum_{l=n}^{\infty} \|Q_l\| \cdot \|S_l^{-1}\| \cdot \prod_{h=n}^{l-1} \|R_h\| \cdot \|S_h^{-1}\|.
$$

Proof. (a) Solving (2.11) explicitly, we find, for $n \geq N$,

$$
(2.15)\quad \begin{cases} X_n = (R_{n-1} \cdot \cdots \cdot R_N)\, X_N (S_{n-1} \cdot \cdots \cdot S_N)^{-1} \\ \qquad + \sum_{l=N}^{n-1} (R_{n-1} \cdot \cdots \cdot R_{l+1})\, Q_l (S_{n-1} \cdot \cdots \cdot S_l)^{-1} \end{cases}
$$

from which (2.12) follows immediately, by $\|R_h\| \cdot \|S_h^{-1}\| < 1$ for all $h \geq N$.

(b) Since the sum $T_n := \sum_{l=N}^{\infty} (S_l \cdot \cdots \cdot S_N)^{-1} Q_l (R_{l-1} \cdot \cdots \cdot R_N)$ converges, by

$$
(2.16)\quad \begin{cases} \left\| \sum_{l=p}^{q} (S_l \cdot \cdots \cdot S_N)^{-1} Q_l (R_{l-1} \cdot \cdots \cdot R_N) \right\| \\ \quad \leq \max_{l \geq p} \dfrac{\|Q_l\| \cdot \|S_l^{-1}\|}{1 - \|R_l\| \cdot \|S_l^{-1}\|} \cdot \prod_{h=N}^{p-1} \|R_h\| \cdot \|S_h^{-1}\| \end{cases}
$$

we may choose $Y_N = -T_N$. Then

$$
Y_n = -T_n = -\sum_{l=n}^{\infty} (S_l \cdot \cdots \cdot S_n)^{-1} Q_l (R_{l-1} \cdot \cdots \cdot R_n) \quad \text{for } n \geq N,
$$

which yields (2.14) (take $Y_n^{(0)} = Y_n$). In particular, by (2.16),

$$
\|Y_N^{(0)}\| \leq \max_{l \geq n} \frac{\|Q_l\| \cdot \|S_l^{-1}\|}{1 - \|R_l\| \cdot \|S_l^{-1}\|}
$$

so $\lim_{n\to\infty} Y_n^{(0)} = 0$. □

Lemma 2.4. (a) *Let* $\{R_n\}, \{S_n\}, \{P_n\}, \{Q_n\}$ *be as in Theorem* 2.1 *with* $\{\delta_n\} = \{0\}$. *Then the recurrence*

$$
(2.17)\quad X_{n+1} = (R_n X_n + Q_n)(S_n + P_n X_n)^{-1}
$$

has a solution $\{X_n^{(0)}\}$, $X_n^{(0)} \in \mathcal{K}^{l,k-l}$, *such that* $\lim_{n\to\infty} X_n^{(0)} = 0$ *and, for any* $\varepsilon > 0$,

$$(2.18)\qquad \|X_n^{(0)}\| \le \sum_{l=N}^{n-1} (\|Q_l\| + \varepsilon\,\|P_l\|)\cdot\|S_l^{-1}\|\cdot\prod_{h=l+1}^{n-1}\|R_h\|\cdot\|S_h^{-1}\|$$

for $N \ge N(\varepsilon)$.

(b) *Moreover, if* R_n *is non-singular too for* $n \in \mathbb{N}$ *and the sum*

$$\sum_{l=N}^{\infty} (\|P_l\| + \|Q_l\|)\cdot\|R_l^{-1}\|\cdot\prod_{h=N}^{l-1}\|R_h^{-1}\|\cdot\|S_h\|$$

converges for some $N \in \mathbb{N}$, *then* (2.17) *has a solution* $\{X_n^{(1)}\}$, $X_n^{(1)} \in \mathcal{K}^{l,k-l}$ *with*

$$(2.19)\qquad \|X_n^{(1)}\| \le \sum_{l=n}^{\infty} (\|Q_l\| + \varepsilon\,\|P_l\|)\cdot\|R_l^{-1}\|\cdot\prod_{h=n}^{l-1}\|R_h^{-1}\|\cdot\|S_h\|$$

for $n \ge N'(\varepsilon)$. *In particular*, $\lim_{n\to\infty} X_n^{(1)} = 0$.

Proof. (a) Without loss of generality we take $\varepsilon < 1$. First of all we show that small solutions remain bounded. Put, for $n \in \mathbb{N}$, $r_n = \|R_n\|\cdot\|S_n^{-1}\|$, $p_n = \|P_n\|\cdot\|S_n^{-1}\|$, and $q_n = \|Q_n\|\cdot\|S_n^{-1}\|$. Let $N(\varepsilon)$ be such that $q_n + \varepsilon p_n < \sqrt{\varepsilon}(1 - r_n)$ for $n \ge N(\varepsilon)$. By (2.17) and Lemma 2.2, we have

$$\|X_{n+1}\| \le \frac{r_n\cdot\|X_n\| + q_n}{1 - p_n\cdot\|X_n\|}$$

provided that $\|X_n\| < 1/p_n$. Hence, if for some $N' \ge N(\varepsilon)$ we have $\|X_{N'}\| \le \sqrt{\varepsilon}$, then $\|X_n\| \le \sqrt{\varepsilon}$ for all $n \ge N'$. We can write (2.17) as

$$(2.20)\qquad X_{n+1}\,S_n = R_n\,X_n + \tilde{Q}_n$$

with $\tilde{Q}_n = Q_n - X_{n+1}\,P_n\,X_n$, for a fixed solution $\{X_n\}$. If $N \ge N(\varepsilon)$ and we choose $\{X_n^{(0)}\}$ such that $X_N^{(0)} = 0$, then we have $\|X_n^{(0)}\| \le \sqrt{\varepsilon}$ for all $n \ge N$. Application of Lemma 2.3(a) to (2.20) (with $\tilde{Q}_n$ instead of Q_n) yields the desired result.

(b) Put $\sigma_n = \|R_n^{-1}\|\cdot\|S_n\|$, $\pi_n = \|R_n^{-1}\|\cdot\|P_n\|$, and $\rho_n = \|R_n^{-1}\|\cdot\|Q_n\|$ $(n \ge N)$. Since $\sum_{l=N}^{\infty} (\pi_l + \rho_l)\,\sigma_{l-1}\cdot\dots\cdot\sigma_N$ converges, we may put

$$D_n = \sum_{l=n}^{\infty} (\varepsilon\pi_l + \rho_l)\,\sigma_{l-1}\cdot\dots\cdot\sigma_n \quad (n \ge N).$$

Then $D_n = \sigma_n\,D_{n+1} + \rho_n + \varepsilon\pi_n$ for $n \ge N$ and $\lim_{n\to\infty} D_n = 0$ since $\sigma_n\,r_n \ge 1$ for all n. Let $N'(\varepsilon) \ge N$ be so large that $D_n\,D_{n+1} \le \varepsilon$ for $N \ge N'(\varepsilon)$. Then $\rho_n' := \rho_n + \varepsilon\pi_n - \pi_n\,D_n\,D_{n+1} \ge \rho_n$ and

$$D_{n+1} = \frac{D_n - \rho_n'}{\rho_n + \pi_n\,D_n} \quad (n \ge N'(\varepsilon)).$$

For $n \ge N'(\varepsilon)$ we define compact sets $U_n = \{X \in \mathcal{K}^{l,k-l} : \|X\| \le D_n\} \subset \mathcal{K}^{l,k-l}$

(with the topology induced by $\| \ \|$), and for each $h \geq N'(\varepsilon)$ we choose solutions $\{X_n^{(h)}\}$ of (2.17) with $X_h^{(h)} \in U_h$. Then $X_n^{(h)} \in U_n$ for $N'(\varepsilon) \leq n \leq h$. For we have

$$\|X_n^{(h)}\| \leq \frac{\|X_{n+1}^{(h)}\| \, \sigma_n + \rho_n}{1 - \pi_n \, \|X_{n+1}^{(h)}\|}$$

provided that $\pi_n \, \|X_{n+1}^{(h)}\| < 1$. But if $X_{n+1}^{(h)} \in U_{n+1}$, then $\pi_n \, \|X_{n+1}^{(h)}\| \leq \pi_n \, D_{n+1} < \pi_n \, \varepsilon / D_n < 1$. Thus, if $X_{n+1}^{(h)} \in U_{n+1}$ for $n > N'(\varepsilon)$, we have

$$\|X_n^{(h)}\| \leq \frac{\sigma_n \, D_{n+1} + \rho_n}{1 - \pi_n \, D_{n+1}} \leq D_n$$

so that $X_n^{(h)} \in U_n$. Since the U_n are compact sets, the sequence $\{X_{N'(\varepsilon)}^{(h)}\}$ has at least one limit point in $U_{N'(\varepsilon)}$, say $\tilde{X}_{N'(\varepsilon)}$. Let $\{\tilde{X}_n\}_{n > N'(\varepsilon)}$ be defined by (2.17). By continuity of (2.17), $\tilde{X}_n \in U_n$, hence $\|\tilde{X}_n\| \leq D_n$ for $n \geq N'(\varepsilon)$. □

Proof of Theorem 2.1. We first suppose that $\{\delta_n\} = \{0\}$. By Lemma 2.4 the recurrence (2.17) has a solution $\{X_n^{(0)}\}$ such that $X_n^{(0)} \in \mathcal{K}^{l, k-l}$, $\lim X_n^{(0)} = 0$ and such that (2.18) holds. Moreover, if R_n is non-singular, and the sum (2.9) converges for some $\varepsilon > 0$, then, again by Lemma 2.4, (2.17) has a solution $\{X_n^{(0)}\}$ for which (2.19) holds. Put

$$B_n = \begin{pmatrix} I_l & X_n^{(0)} \\ 0 & I_{k-l} \end{pmatrix} \quad (n \geq N).$$

Then $\|B_n - I\| = \|X_n^{(0)}\|$ and

$$B_{n+1}^{-1} \, M_n \, B_n = \begin{pmatrix} R_n - X_{n+1}^{(0)} \, P_n & 0 \\ P_n & P_n \, X_n^{(0)} + S_n \end{pmatrix}.$$

Since B_n, M_n are invertible for $n \geq N$, so is $B_{n+1}^{-1} \, M_n \, B_n$. Further, since by Lemma 2.2 we have $1 - \|R_n - X_{n+1}^{(0)} \, P_n\| \cdot \|(S_n + P_n \, X_n^{(0)})^{-1}\| \geq \frac{1}{2}(1 - \|R_n\| \cdot \|S_n^{-1}\|)$ for n large enough, say $n \geq N$, we may apply Lemma 2.3(b) to the recurrence

$$X_{n+1}(R_n - X_{n+1}^{(0)} \, P_n) = (S_n + P_n \, X_n^{(0)}) \, X_n + P_n$$

and find that there is a solution $\{X_n^{(1)}\}_{n \geq N}$, $X_n^{(1)} \in \mathcal{K}^{k-l, l}$, with

$$\|X_n^{(1)}\| \leq \sum_{h=n}^{\infty} \|P_h\| \cdot \|(S_h + P_h \, X_h^{(0)})^{-1}\|$$
$$\times \left\{ \prod_{m=n}^{h-1} \|R_m - X_{m+1}^{(0)} \, P_m\| \cdot \|(S_m + P_m \, X_m^{(0)})^{-1}\| \right\}$$

for $n \geq N$, and $\lim X_n^{(1)} = 0$. Define

$$F_n = B_n \cdot \begin{pmatrix} I_l & 0 \\ X_n^{(1)} & I_{k-l} \end{pmatrix} \quad (n \geq N).$$

Then $F_n \in \mathcal{K}^{k,k}$, $F_{n+1}^{-1} \, M_n \, F_n = \operatorname{diag}(R_n - X_{n+1}^{(0)} \, P_n, S_n + P_n \, X_n^{(0)})$ $(n \geq N)$. It is now easy to check properties (2.5)–(2.8) and (2.10).

Now consider the general case. Put $b_n = \prod_{h=1}^{n-1}(1+\delta_h)$. Further, put $U_n = \operatorname{diag}(b_n I_l, I_{k-l})$ $(n \in \mathbb{N})$. Then $\lim_{n\to\infty} b_n$ exists and is not zero. Moreover

$$U_{n+1}^{-1} M_n U_n = \begin{pmatrix} R_n/(1+\delta_n) & b_{n+1}^{-1} Q_n \\ b_n P_n & S_n \end{pmatrix} \quad (n \in \mathbb{N})$$

so we may apply Theorem 2.1 with $\{\delta_n\} = \{0\}$ to $\{U_{n+1}^{-1} M_n U_n\}$, thus obtaining a sequence $\{\tilde{F}_n\}$, $\tilde{F}_n \in \mathcal{K}^{k,k}$ such that

$$\tilde{F}_{n+1}^{-1} U_{n+1}^{-1} M_n U_n \tilde{F}_n = \operatorname{diag}(\tilde{R}_n/(1+\delta_n), \tilde{S}_n)$$

with $\|\tilde{R}_n - R_n\| + \|\tilde{S}_n - S_n\| = \mathrm{o}(\|P_n\|)$ and such that (2.7)–(2.10) hold for $\{\tilde{F}_{n+1}^{-1} U_{n+1}^{-1} M_n U_n \tilde{F}_n\}$ and $\{\tilde{F}_n\}$. Finally, put $F_n = U_n \tilde{F}_n U_n^{-1}$ $(n \in \mathbb{N})$. Since U_n converges to some non-singular matrix U, we find that (2.5) to (2.10) hold for $\{M_n\}$ and $\{F_n\}$. □

Remark 2.1. If, in Theorem 2.1, $\{\delta_n\} = \{0\}$, it follows from the proof that, in (2.8), $\ll_\varepsilon$ can be replaced by $\leq$, provided that n is large enough.

Remark 2.2. If $\|R_n^{-1}\| = \|R_n\|^{-1}$ and $\|S_n^{-1}\| = \|S_n\|^{-1}$ for all n, we can take (2.8) and (2.9) together, writing

$$(2.21)\quad \begin{cases} \|F_n - I\| \ll_\varepsilon \sum\limits_{h=n}^{\infty} \|P_h\| \cdot \|\tilde{S}_h^{-1}\| \cdot \prod\limits_{m=n}^{h-1} \|\tilde{R}_m\| \cdot \|\tilde{S}_m^{-1}\| + \prod\limits_{m=1}^{n-1} \|R_m\| \cdot \|S_m^{-1}\| \\ \qquad \times \left\{ \sum\limits_{(n)} (\|Q_h\| + \varepsilon\, \|P_h\|) \cdot \|S_h^{-1}\| \cdot \prod\limits_{m=1}^{h} \|R_m^{-1}\| \cdot \|S_m\| \right\}. \end{cases}$$

3. APPLICATION: SEPARATION OF EIGENVALUES WITH DISTINCT MODULI

In this section we study converging sequences $\{M_n\}$ of square non-singular matrices. We show that a converging sequence $\{U_n\}$ can be found with $\lim U_n$ non-singular and such that $U_{n+1}^{-1} M_n U_n$ is a block-diagonal matrix with each of its blocks converging to some matrix whose eigenvalues have equal moduli. Moreover, the rate of convergence of $\{U_{n+1}^{-1} M_n U_n\}$ is about the same as the rate of convergence of $\{M_n\}$.

For a matrix $M \in \mathcal{K}^{k,k}$, we denote by $\rho(M)$ its special radius.

Theorem 3.1. *Let $M \in \mathcal{K}^{k,k}$ be a matrix of the form $M = \operatorname{diag}(R_1, R_2, \ldots, R_L)$ where $R_j \in \mathcal{K}^{k_j,k_j}$ such that al eigenvalues of R_j have smaller moduli than those of R_i $(1 \leq j < i \leq L)$. Let $f \in \mathcal{M}^0$ and $\{M_n\}$ a sequence of matrices in $\mathcal{K}^{k,k}$ such that*

$$M_n = \begin{pmatrix} M_n^{(1,1)} & \cdots & M_n^{(1,L)} \\ \vdots & \ddots & \vdots \\ M_n^{(L,1)} & \cdots & M_n^{(L,L)} \end{pmatrix}, \quad M_n^{(i,j)} \in \mathcal{K}^{k_i,k_j},$$

for $n \in \mathbb{N}$ and $1 \leq i, j \leq L$, and

$$(3.1)\quad \|M_n - M\| = \mathrm{o}(f(n)) \quad (n \to \infty).$$

Then there exists a sequence $\{F_n\}$ *of non-singular matrices in* $\mathcal{K}^{k,k}$ *such that*

$$F_{n+1}^{-1} M_n F_n = \operatorname{diag}(R_{1n}, R_{2n}, \dots, R_{Ln}) \tag{3.2}$$

with $\lim R_{jn} = R_j$ $(j = 1, \dots, L)$,

$$\|R_{jn} - M_n^{(j,j)}\| = \mathrm{o}(\|M_n - M\|) \quad (n \to \infty) \tag{3.3}$$

$$\lim F_n = I \tag{3.4}$$

and

$$\|F_n - I\| = \mathrm{o}(f(n)) \quad (n \to \infty). \tag{3.5}$$

Moreover, if $\sum_{n=1}^{\infty} (1/f(n))\, \|M_n - M\|$ *converges, then* $\{F_n\}$ *can be found such that, in addition,*

$$\sum_{n=1}^{\infty} \frac{1}{f(n)} \|F_n - I\| \quad \textit{converges.} \tag{3.6}$$

Lemma 3.2. *Let* $M \in K^{k,k}$. *There exists for each number* $\varepsilon > 0$ *some invertible matrix* $A(\varepsilon) \in \mathcal{K}^{k,k}$ *such that* $\|A(\varepsilon)^{-1} M A(\varepsilon)\| \le \rho(M) + \varepsilon$.

Proof. Note that if $M = \operatorname{diag}(M_1, \dots, M_L)$, then $\|M\| = \max(\|M_1\|, \dots, \|M_L\|)$. Since a matrix $U \in \mathcal{K}^{k,k}$ exists such that $U^{-1}MU$ is in Jordan canonical form, it suffices to construct $A(\varepsilon)$ for the Jordan blocks of M. First consider a Jordan block of the form $\alpha I_l + J_l$. Put

$$E_l := \operatorname{diag}(0, 1, 2, \dots, l-1) \in \mathcal{K}^{l,l}. \tag{3.7}$$

For $z \in \mathcal{K},\ z \ne 0$,

$$(z^{E_l} \cdot J_l \cdot z^{-E_l})_{ij} = z^{i-1} (J_l)_{ij} z^{1-j} = z^{i-j} \delta_{i+1,j}.$$

Hence

$$z^{E_l} \cdot J_l \cdot z^{-E_l} = z^{-1} J_l. \tag{3.8}$$

Thus, $\|\varepsilon^{-E_l}(\alpha I_l + J_l)\, \varepsilon^{E_l}\| = \|\alpha I_l + \varepsilon J_l\| \le |\alpha| + \varepsilon$. Now consider a Jordan block of the form $\beta \cdot \operatorname{diag}(R(\varphi), \dots, R(\varphi)) + J_l^2$. (Here we assume $\mathcal{K} = \mathbb{R}$). Consider the (right-)Kronecker product $E_{l/2} \otimes I_2$. Since $E_{l/2} \otimes I_2 = \operatorname{diag}(0 \cdot I_2, 1 \cdot I_2, \dots, ((l/2) - 1)\, I_2)$, it commutes with $\operatorname{diag}(R(\varphi), \dots, R(\varphi))$. Furthermore, by $J_l^2 = J_{l/2} \otimes I_2$, we have

$$z^{E_{l/2} \otimes I_2} \cdot J_l^2 \cdot z^{-E_{l/2} \otimes I_2} = z^{-1} J_l^2. \tag{3.9}$$

Hence,

$$\begin{aligned} &\|\varepsilon^{-E_l \otimes I_2} (\beta \cdot \operatorname{diag}(R(\varphi), \dots, R(\varphi)) + J_l^2)\, \varepsilon^{E_l \otimes I_2}\| \\ &\quad \le |\beta| \cdot \|R(\varphi)\| + \varepsilon = |\beta| + \varepsilon, \end{aligned}$$

since $R(\varphi)$ is an orthogonal matrix. □

Proof of Theorem 3.1. We proceed by induction on L. First take $L = 2$. Put $R := R_1$, $S := R_2$. Let $\beta := \rho(R)$, $\gamma := \rho(S^{-1})^{-1}$. By assumption, $\gamma > \beta$ and we may choose some number $\varepsilon \in \mathbb{R}$, $0 < \varepsilon < (\gamma - \beta)/6$. By Lemma 3.2 there exist matrices $U \in \mathcal{K}^{k_1,k_1}$ and $V \in \mathcal{K}^{k_2,k_2}$ such that

$$\|U^{-1}RU\| \leq \beta + \varepsilon, \qquad \|V^{-1}S^{-1}V\| \leq \frac{1}{\gamma - \varepsilon}.$$

Put $W = \operatorname{diag}(U, V)$ and choose $N \in \mathbb{N}$ so large that

$$\|M_n - M\| < \frac{\varepsilon}{\|W\| \cdot \|W^{-1}\|} \quad \text{for } n \geq N.$$

Observe that $W^{-1} M_n W =: \tilde{M}_n$ can be written as

$$\tilde{M}_n = \begin{pmatrix} R_n & Q_n \\ P_n & S_n \end{pmatrix}$$

with $\|R_n - U^{-1}RU\| < \varepsilon$, $\|S_n - V^{-1}SV\| < \varepsilon$, $\|P_n\| < \varepsilon$, $\|Q_n\| < \varepsilon$ for $n \geq N$. Thus, by Lemma 2.2, $\|R_n\| < \beta + 2\varepsilon$, and $\|S_n^{-1}\| < 1/(\gamma - 2\varepsilon)$ for $n \geq N$. Hence,

$$\|R_n\| \cdot \|S_n^{-1}\| < 1 - \delta \quad \text{for some } \delta > 0 \ (n \geq N),$$

$$\sum_{n=N}^{\infty} (1 - \|R_n\| \cdot \|S_n^{-1}\|) = \infty$$

and

$$\lim_{n \to \infty} (\|P_n\| + \|Q_n\|) \cdot \|S_n^{-1}\| = 0.$$

Applying Theorem 2.1 to $\{\tilde{M}_n\}$ we obtain a sequence $\{\tilde{F}_n\}_{n \geq N}$, $\tilde{F}_n \in \mathcal{K}^{k,k}$, such that

$$\lim \tilde{F}_n = I$$

$$\tilde{F}_{n+1}^{-1} \tilde{M}_n \tilde{F}_n = \operatorname{diag}(\hat{R}_n, \hat{S}_n)$$

and such that (2.6), (2.8) hold for $\{\tilde{M}_n\}$ and $\{\tilde{F}_n\}$, hence $\|\hat{R}_n - R_n\| + \|\hat{S}_n - S_n\| = o(\|P_n\|)$ for $n \to \infty$. Put $F_n = W\tilde{F}_n W^{-1}$ $(n \geq N)$. Then

$$\lim F_n = I, \qquad F_{n+1}^{-1} M_n F_n = \operatorname{diag}(\tilde{R}_n, \tilde{S}_n)$$

with

$$\|\tilde{R}_n - R_{1n}\| + \|\tilde{S}_n - R_{2n}\| = o(\|M_n - M\|) \quad (n \to \infty).$$

Put

$$p_n = \|P_n\| \cdot \|S_n^{-1}\|, \quad q_n = \|Q_n\| \cdot \|S_n^{-1}\|, \quad r_n = \|R_n\| \cdot \|S_n^{-1}\|$$

for all $n \geq N$. Then

$$r_n < 1 - \delta, \quad p_n = O(\|M_n - M\|), \quad q_n = O(\|M_n - M\|).$$

Furthermore,

$$\sum_{h=n}^{\infty} p_h \prod_{m=n}^{h-1} (r_m + \varepsilon p_m) \ll \sum_{h=n}^{\infty} \|M_h - M\| \cdot \zeta^{h-n}$$

and

$$\sum_{h=1}^{n-1} q_h \prod_{m=h+1}^{n-1} (r_m + \varepsilon p_m) \ll \sum_{h=1}^{n-1} \|M_h - M\| \cdot \zeta^{n-h}$$

for some number $\zeta < 1$. So, by (2.8), and by $\|\tilde{F}_n - I\| \sim \|F_n - I\|$, we have

$$\|F_n - I\| \ll \sum_{h=n}^{\infty} \|M_h - M\| \cdot \zeta^{h-n} + \sum_{h=1}^{n-1} \|M_h - M\| \cdot \zeta^{n-h}.$$

We show that

$$\sum_{h=n}^{\infty} f(h) \cdot \zeta^{h-n} + \sum_{h=1}^{n-1} f(h) \cdot \zeta^{n-h} = \mathrm{O}(f(n)) \quad (n \to \infty).$$

Set $A = \max_{n \in \mathbb{N}} f(n)$ and let N' be so large that $|(f(m+1)/f(m)) - 1| < (1-\zeta)/2$ for $m \geq N'$. Choose $n > N'$. Since $\zeta(3-\zeta)/2 < 1$, $2\zeta/(1+\zeta) < 1$ and $\zeta^n/f(n) \to 0$ $(n \to \infty)$, we obtain

$$\begin{aligned}
&\frac{1}{f(n)} \sum_{h=n}^{\infty} f(h) \cdot \zeta^{h-n} + \frac{1}{f(n)} \sum_{h=N'}^{n-1} f(h) \cdot \zeta^{n-h} + \frac{1}{f(n)} \sum_{h=1}^{N'-1} f(h) \cdot \zeta^{n-h} \\
&\leq \sum_{h=n}^{\infty} \left(\frac{(3-\zeta)\zeta}{2}\right)^{h-n} + \sum_{h=N'}^{n-1} \left(\frac{2\zeta}{1+\zeta}\right)^{n-h} + \frac{A}{f(n)} \cdot \zeta^n \sum_{h=1}^{N'-1} \zeta^{-h} = \mathrm{O}(1)
\end{aligned}$$

$(n \to \infty)$. By (3.1) this implies (3.5). Suppose that $\sum_{n=1}^{\infty} (1/f(n)) \|M_n - M\|$ converges. Then

$$\begin{aligned}
&\sum_{n=N'}^{\infty} \frac{1}{f(n)} \left\{ \sum_{h=n}^{\infty} \|M_h - M\| \cdot \zeta^{h-n} + \sum_{h=N'}^{n-1} \|M_h - M\| \cdot \zeta^{n-h} \right\} \\
&\leq \sum_{h=N'}^{\infty} \frac{1}{f(h)} \cdot \|M_h - M\| \cdot \sum_{n=N'}^{h} \left(\frac{\zeta(3-\zeta)}{2}\right)^{h-n} \\
&\quad + \sum_{h=N'}^{\infty} \frac{1}{f(h)} \cdot \|M_h - M\| \cdot \sum_{n=h+1}^{\infty} \left(\frac{2\zeta}{1+\zeta}\right)^{n-h} \\
&\ll \sum_{h=N'}^{\infty} \frac{1}{f(h)} \cdot \|M_h - M\| < \infty.
\end{aligned}$$

So, $\sum_{n=N}^{\infty} (1/f(n)) \|F_n - I\|$ converges and we have proved Theorem 3.1 in case $L = 2$.

Now suppose $L = L_0 > 2$. We assume that the theorem holds for $L < L_0$. We denote the matrix that is obtained from M_n by omitting the first k_1 rows and columns by M_n' $(n \in \mathbb{N})$. Thus

$$M_n = \begin{pmatrix} R_n' & Q_n \\ P_n & M_n' \end{pmatrix}$$

with $\lim R_n' = R_1$. By the theorem for $L = 2$ we can find a sequence of non-singular matrices $\{F_n'\}$ such that

$$(F_{n+1}')^{-1} M_n F_n' = \operatorname{diag}(R_{1n}, M_{1n}')$$

with $\lim M'_{1n} = \lim M'_n$ and $\lim R_{1n} = \lim R_n$ and such that (3.3)–(3.6) hold for $\{M_n\}$ and $\{F'_n\}$. Applying the theorem for $L = L_0 - 1$ yields a sequence $\{F''_n\}$ such that $(F''_{n+1})^{-1} M'_{1n} F''_n = \mathrm{diag}(R_{2n}, \ldots, R_{Ln})$ and such that (3.3)–(3.6) hold for $\{M'_{1n}\}$ and $\{F''_n\}$. (Note that since $\|(F'_{n+1})^{-1} M_n F'_n - M\| \sim \|M_n - M\|$ by (3.3), the order of convergence of $\{M'_{1n}\}$ is not essentially larger than the order of convergence of $\{M_n\}$). Put

$$F_n = F'_n \cdot \mathrm{diag}(I_{k_1}, F''_n) \quad (n \in \mathbb{N}).$$

It is not difficult to check that $\{F_n\}$ satisfies the requirements. □

Note that for any matrix $M \in \mathcal{K}^{k,k}$ it is always possible to find a matrix $U \in \mathcal{K}^{k,k}$ such that $U^{-1} M U$ has the form prescribed in Theorem 3.1. Moreover, we can find U and $R_1, \ldots, R_L$ such that all eigenvalues of R_j have the same modulus $(1 \leq j \leq L)$.

We apply Theorem 3.1 to matrix recurrences:

Corollary 3.3. *Let $\{M_n\}$ be a sequence of non-singular matrices in $\mathcal{K}^{k,k}$ with limit matrix M. Suppose that M has some eigenvalue $\alpha \in \mathcal{K}$ with multiplicity* 1 *and such that $|\beta| \neq |\alpha|$ for all other complex eigenvalues β of M. Then the matrix recurrence* (1.1) *induced by $\{M_n\}$ has a solution $\{x_n^{(0)}\}$, $x_n^{(0)} \in \mathcal{K}^k$, such that $(x_n^{(0)}/|x_n^{(0)}|) - f_n = \mathrm{o}(1)$ with f_n an eigenvector of M with eigenvalue α. Moreover, if $\sum_{n=1}^{\infty} \|M_n - M\| < \infty$ and $\alpha \neq 0$, then $\{x_n^{(0)}/\alpha^n\}$ converges.*

Proof. We can find a matrix $U \in \mathcal{K}^{k,k}$ such that $U^{-1} M U = \mathrm{diag}(\alpha, R)$ for some $R \in \mathcal{K}^{k-1,k-1}$. By Theorem 3.1, there exists some sequence $\{F_n\}$, $F_n \in \mathcal{K}^{k,k}$, such that

$$F_{n+1}^{-1} U^{-1} M_n U F_n = \mathrm{diag}(\alpha_n, R_n)$$

with $\lim_{n\to\infty} \alpha_n = \alpha$, $\lim R_n = R$, $\lim F_n = I$. Further, if $\sum_{n=1}^{\infty} \|M_n - M\| < \infty$, then $\sum_{n=1}^{\infty} |\alpha_n - \alpha| < \infty$ as well. So, the matrix recurrence induced by $\{F_{n+1}^{-1} U^{-1} M_n U F_n\}$ has some solution $\{y_n^{(0)}\}$ with $y_n^{(0)}/|y_n^{(0)}| = \lambda_n e_1$, where $\lambda_n \in \mathcal{K}$, $|\lambda_n| = 1$ and $e_1 = (1, 0, \ldots, 0)^t$ is the first unit vector in $\mathcal{K}^k$ $(n \in \mathbb{N})$. Then $\{x_n^{(0)}\} := \{U F_n y_n^{(0)}\}$ is a solution of (1.1) and $(x_n^{(0)}/|x_n^{(0)}|) - (\lambda_n U e_1/|U e_1|) = \mathrm{o}(1)$ and $U e_1$ is clearly an eigenvector of M with eigenvalue α. If $\alpha \neq 0$ and $\sum_{n=1}^{\infty} |\alpha_n - \alpha| < \infty$, then

$$\frac{y_n^{(0)}}{\alpha^{n-1}} = \left(\prod_{l=1}^{n-1} \frac{\alpha_l}{\alpha} \right) \cdot e_1 \quad (n \geq 1)$$

and the product on the right-hand side converges. Then $\{x_n^{(0)} \alpha^{-n}\}$ converges as well. □

Remark. If $M \in \mathcal{K}^{k,k}$ has k distinct eigenvalues $\alpha_1, \ldots, \alpha_k$ with $|\alpha_1| < |\alpha_2| < \cdots < |\alpha_k|$, we have, by Corollary 3.3, that (1.1) has k solutions $\{x_n^{(1)}\}, \ldots, \{x_n^{(k)}\}$ such that $\lambda_n^{(i)} x_n^{(i)}/|x_n^{(i)}|$ converges to an eigenvector of M with eigenvalue α_i, for certain numbers $\lambda_n^{(i)} \in \mathcal{K}$, $|\lambda_n^{(i)}| = 1$ $(i = 1, \ldots, k)$. Clearly,

$\{x_n^{(1)}\}, \ldots, \{x_n^{(k)}\}$ constitute a basis of solutions of (1.1). This is essentially the Poincaré–Perron theorem for matrix recurrences (see e.g. [3], [4] (p. 300–313), [5], [6].)

4. SEQUENCES OF ALMOST-DIAGONAL MATRICES

It appears useful to have a separate lemma for the case that the matrices M_n are almost diagonal, i.e. they can be written as the sum of some diagonal matrix and a perturbation matrix. We impose a few regularity conditions on the diagonal parts of M_n, which are often fulfilled in practice.

Lemma 4.1. *Let $b_1, \ldots, b_k$ be $\mathcal{K}$-valued functions on integers such that $|b_j(n)/b_i(n)| - 1$ is either non-negative or non-positive for all $n \in \mathbb{N}$ up to addition of a term $d_{i,j}(n)$ where $\sum_{n=1}^{\infty} |d_{i,j}(n)|$ converges $(1 \le i,\ j \le k)$. Let $\{D_n\}$ be such that $D_n \in \mathcal{K}^{k,k}$ and $\sum_{n=1}^{\infty} \|D_n\|/|b_i(n)|$ converges for all i. Put $B_n = \operatorname{diag}(b_1(n), \ldots, b_k(n))$ $(n \in \mathbb{N})$. There exists a sequence $\{F_n\}$ of matrices in $\mathcal{K}^{k,k}$ such that*

$$(4.1)\qquad F_{n+1}^{-1}(B_n + D_n)\, F_n = B_n \quad (n \in \mathbb{N})$$

and

$$(4.2)\qquad \|F_n - I\| \ll \max_{1 \le i,j \le k} \prod_{\mu=1}^{n-1} \left| \frac{b_i(\mu)}{b_j(\mu)} \right| \cdot \sum_{(n)} \frac{\|D_h\|}{|b_i(h)|} \prod_{m=1}^{h-1} \left| \frac{b_j(m)}{b_i(m)} \right|.$$

In particular, $\lim F_n = I$.

For the proof of Lemma 4.1 we need an auxiliary result:

Lemma 4.2. *Let $\{\gamma_n\}$, $\{\delta_n\}$ be sequences of complex numbers such that both $\sum_{n=1}^{\infty} |\gamma_n|$ and $\sum_{n=1}^{\infty} |\delta_n|$ converge. Then for every $\alpha \in \mathbb{C}$ the recurrence*

$$(4.3)\qquad y_{n+1} = (1 + \gamma_n)\, y_n + \delta_n \quad (n \in \mathbb{N})$$

has a solution $\{y_n(\alpha)\}$ such that $\lim_{n \to \infty} y_n = \alpha$. *Moreover,* $|y_n(0)| \ll \sum_{h=n}^{\infty} |\delta_n|$.

Proof. Solving the recurrence (4.3) explicitly (compare Lemma 2.3), we obtain

$$y_n = \prod_{m=1}^{n-1} (1 + \gamma_m) \cdot \left\{ y_1 + \sum_{h=1}^{n-1} \delta_h \cdot \prod_{l=1}^{h} (1 + \gamma_l)^{-1} \right\}.$$

Since $\prod_{m=1}^{\infty} (1 + \gamma_m)$ converges to, say, $\beta \in \mathbb{C}^*$ and since the sum $\sum_{h=1}^{\infty} \delta_h \cdot \prod_{l=1}^{h} (1 + \gamma_l)^{-1}$ converges too, we may put

$$y_1(\alpha) = \frac{\alpha}{\beta} - \sum_{h=1}^{\infty} \delta_h \cdot \prod_{l=1}^{h} (1 + \gamma_l)^{-1}$$

and let $\{y_n(\alpha)\}_{n \ge 1}$ be a solution of (4.3). Then $\lim_{n \to \infty} y_n(\alpha) = \alpha$ and

$$|y_n(0)| \le \sum_{h=n}^{\infty} |\delta_h| \cdot \prod_{l=n}^{h} |1 + \gamma_l|^{-1} \ll \sum_{h=n}^{\infty} |\delta_h|. \qquad \square$$

Proof of Lemma 4.1. Since the lemma remains valid if we take $\{P^{-1}(B_n+D_n)P\}$ instead of $\{B_n + D_n\}$ for any permutation matrix P, we may suppose that $|b_i(n)| \leq |b_{i+1}(n)| + d_i(n)$ for all n and $i \in \{1, \ldots, k-1\}$, with $d_i(n)$ non-negative real numbers such that $\sum_{n=1}^{\infty} d_i(n)$ converges for all i. We first look for a sequence of unipotent upper triangle matrices (i.e. with diagonal elements 1) $\{F_n^{(1)}\}$ such that

$$((F_{n+1}^{(1)})^{-1}(B_n + D_n)\, F_n^{(1)})_{ij} = 0 \quad \text{for } i < j$$

where A_{ij} denotes the entry in the i-th row and j-th column of the matrix A. We set $f_{ij}(n) = (F_n^{(1)})_{ij}$, $b_{ij}(n) = (B_n + D_n)_{ij}$ and $\tilde{b}_{ij}(n) = ((F_{n+1}^{(1)})^{-1} \cdot (B_n + D_n) \cdot F_n^{(1)})_{ij}$. Then

$$\text{(4.4a)} \quad \sum_{h<j} b_{ih}(n)\, f_{hj}(n) + b_{ij}(n) = \sum_{m \geq j} f_{im}(n+1)\, \tilde{b}_{mj}(n) \quad \text{if } i < j$$

$$\text{(4.4b)} \quad \sum_{h<j} b_{ih}(n)\, f_{hj}(n) + b_{ij}(n) = \sum_{m>i} f_{im}(n+1)\, \tilde{b}_{mj}(n) + \tilde{b}_{ij}(n) \quad \text{if } i \geq j.$$

Before choosing $|f_{ij}(n)|$ as small as possible, we show that it can be chosen small. Set $d(n) = \sum_{i,j} |(D_n)_{ij}| = \|D_n\|_1$. Then $\sum_{n=1}^{\infty} (d(n)/|b_{ii}(n)|) < \infty$ for all i. Set $f(n) = \max_{i \neq j} |f_{ij}(n)|$ and let N be so large that $(d(n)/b_{jj}(n)) < 2^{-k}$ for $n \geq N$ and $j = 1, \ldots, k$. Let $f(n) \leq 1$ for some $n \geq N$. We show that then $f(n+1) \leq 1$ too. Suppose it has been shown that $|f_{ij}(n+1)| \leq 1$ for $j = J+1, \ldots, k$ and that $|b_{ij}(n) - \tilde{b}_{ij}(n)| \leq 2^{k-i} \cdot d(n)$ for $i = J+1, \ldots, k$. Applying (4.4b) for $i = J$ and $j \leq J \leq k$ we obtain that

$$|b_{Jj}(n) - \tilde{b}_{Jj}(n)| \leq d(n) + \sum_{m>J} |b_{mj}(n) - \tilde{b}_{mj}(n)| \leq d(n) \cdot 2^{k-J}.$$

Applying (4.4a) for $j = J$ and $i < J$ we find

$$|f_{iJ}(n+1)\, \tilde{b}_{JJ}(n)| \leq d(n) + \sum_{m>J} |b_{mJ}(n) - \tilde{b}_{mJ}(n)| \leq d(n) \cdot 2^{k-J}$$

so that

$$|f_{iJ}(n+1)| \leq \frac{d(n) \cdot 2^{k-J}}{|\tilde{b}_{JJ}(n)|} \leq 1$$

for $i = 1, \ldots, J-1$. So we find that, if $|f(n)| \leq 1$ for some $n \geq N$ then $|f(\nu)| \leq 1$ and $|b_{ij}(\nu) - \tilde{b}_{ij}(\nu)| \leq d(\nu) \cdot 2^k$ for all $\nu \geq n$. We may now apply Lemma 2.4 to (4.4a) with $S_n = \tilde{b}_{jj}(n)$, $R_n = b_{ii}(n)$, $Q_n = \sum_{h<j, h \neq i} b_{ih}(n)\, f_{hj}(n) - \sum_{m>j} f_{im}(n+1)\, \tilde{b}_{mj}(n)$, and $P_n = 0$, whence there exists $X_n = f_{ij}(n)$ with (compare Remark 2.2)

$$\text{(4.5)} \quad |f_{ij}(n)| \ll \prod_{m=N}^{n-1} \left| \frac{b_{ii}(m)}{b_{jj}(m)} \right| \cdot \sum_{(n)} \frac{d(h)}{|b_{jj}(h)|} \cdot \prod_{m=N}^{h} \left| \frac{b_{jj}(m)}{b_{ii}(m)} \right|.$$

We now look for a sequence of unipotent lower triangular matrices $\{F_n^{(2)}\}$ such that $(F_{n+1}^{(2)})^{-1} \cdot (\tilde{b}_{ij}(n)) \cdot F_n^{(2)}$ is a diagonal matrix for all n. If we set $g_{ij}(n) = (F_n^{(2)})_{ij}$ we get, using that $g_{ii}(n) = 1$,

$$(4.6) \qquad \sum_{h=j}^{i} \tilde{b}_{ih}(n)\, g_{hj}(n) = g_{ij}(n+1)\, \tilde{b}_{jj}(n) \quad \text{for } i \geq j.$$

If $\prod_{m=N}^{\infty} |b_{jj}(n)/b_{ii}(n)| = 0$ and $i > j$, we may apply Lemma 2.3(b) to (4.6) with $S_n = \tilde{b}_{ii}(n)$, $R_n = \tilde{b}_{jj}(n)$, $Q_n = \sum_{h=j+1}^{i} \tilde{b}_{ih}(n)\, g_{hj}(n)$. Assuming that we know that $|g_{hj}(n)| \leq 1$ for $n \geq N'$ and $h > i$, we find that (4.6) has a solution $\{Y_n^{(0)}\} = \{g_{ij}(n)\}$ which tends to zero, so in particular $|g_{ij}(n)| \leq 1$ for $n \geq N'' \geq N'$ and

$$(4.7) \qquad |g_{ij}(n)| \ll \prod_{m=N}^{n-1} \left|\frac{b_{ii}(m)}{b_{jj}(m)}\right| \cdot \sum_{(n)} \frac{d(h)}{|b_{jj}(h)|} \cdot \prod_{m=N}^{h} \left|\frac{b_{jj}(m)}{b_{ii}(m)}\right|.$$

On the other hand, if $\sum_{n=N}^{\infty} (|b_{jj}(n)/b_{ii}(n)| - 1)$ converges, then so does $\sum_{n=N}^{\infty} (|\tilde{b}_{jj}(n)/\tilde{b}_{ii}(n)| - 1)$. Defining R_n, S_n, Q_n as above and assuming that $|g_{hj}(n)| \leq 1$ for $h > i$ and n large enough, we have that $\sum_{n=N}^{\infty} |Q_n/R_n|$ and $\sum_{n=N}^{\infty} |S_n/R_n - 1|$ converge. Hence, by Lemma 4.2, the recurrence (4.6) has a solution $\{g_{ij}(n)\}$ which tends to zero as $n \to \infty$ and

$$|g_{ij}(n)| \ll \sum_{(n)} \frac{d(h)}{|\tilde{b}_{jj}(h)|}$$

from which (4.7) follows. Moreover, we see that the assumption that $|g_{hj}(n)| \leq 1$ for $h > i$ and n large enough is indeed consistent. Finally, since $|b_{ij}(n) - \tilde{b}_{ij}(n)| \leq d(n) \cdot 2^k$ for $i, j \in \{1, \ldots, k\}$ we may find sequences of numbers $\{h_i(n)\}$ such that $\tilde{b}_{ii}(n)\, h_i(n) = h_i(n+1)\, b_{ii}(n)$ $(i = 1, \ldots, k,\ n \in \mathbb{N})$ and $\lim_{n\to\infty} h_i(n) = 1$. In this case, again by Lemma 4.2,

$$(4.8) \qquad |h_i(n)| \ll \sum_{(n)} \frac{d(h)}{|b_{ii}(h)|}.$$

Putting

$$F_n = F_n^{(1)} \cdot F_n^{(2)} \cdot \operatorname{diag}(h_1(n), \ldots, h_k(n))$$

we find

$$F_{n+1}^{-1}(B_n + D_n)\, F_n = B_n$$

and, combining (4.5), (4.7) and (4.8) we obtain

$$\|F_n - I\| \ll \max_{1 \leq i,j \leq k} \prod_{\mu=N}^{n-1} \left|\frac{b_{ii}(\mu)}{b_{jj}(\mu)}\right| \cdot \sum_{(n)} \frac{\|D_h\|_1}{|b_{jj}(h)|} \prod_{m=N}^{h} \left|\frac{b_{jj}(m)}{b_{ii}(m)}\right|$$

from which (4.2) follows since $\|\ \|$ and $\|\ \|_1$ are equivalent norms. □

5. FAST CONVERGING SEQUENCES

Again we consider converging sequences $\{M_n\}$, $M_n \in \mathcal{K}^{k,k}$. Let L be the maximum of the multiplicities of the zeros of the minimal polynomial of $\lim M_n =: M$. We shall say that $\{M_n\}$ *converges fast* if the sum $\sum_{n=1}^{\infty} n^{L-1} \cdot \|M_n - M\|$

converges. We show that in this case the solutions of the matrix recurrence (1.1) have a similar behaviour as the solutions of the matrix recurrence

$$(5.1) \qquad Mx_n = x_{n+1} \quad (n \in \mathbb{N})$$

provided that M is non-singular. More precisely, the following result holds:

Theorem 5.1. *Suppose that the sequence $\{M_n\}$, $M_n \in \mathcal{K}^{k,k}$, converges fast to some non-singular matrix M. Let $f \in \mathcal{M}^1$ such that $\sum_{n=1}^{\infty} (1/f(n)) \cdot n^{L-1} \cdot \|M_n - M\|$ converges. Then there is a bijection between solutions $\{x_n^{(0)}\}$ of (1.1) and solutions $\{x_n^{(1)}\}$ of (5.1) such that*

$$(5.2) \qquad |x_n^{(0)} - x_n^{(1)}| = |x_n^{(0)}| \cdot \mathrm{o}(f(n)).$$

(Note that (5.2) is in fact a symmetric relation, since it implies that

$$(5.3) \qquad |x_n^{(0)} - x_n^{(1)}| = |x_n^{(1)}| \cdot \mathrm{o}(f(n))$$

holds as well).

It is useful to make a few observations before proceeding to the proof of the theorem.

Remark 5.1. Note that we may assume $\mathcal{K} = \mathbb{C}$. For if $M, M_n, x_n^{(0)} \in \mathbb{R}$ for all n and (5.2), (5.3) hold for some solution $\{x_n^{(1)}\}$ of (5.1), then it also holds for $\{\mathrm{Re}\, x_n^{(1)}\}$, by

$$|\mathrm{Re}\, x_n^{(1)} - x_n^{(0)}| \le |x_n^{(1)} - x_n^{(0)}| = |x_n^{(0)}| \cdot \mathrm{o}(f(n)).$$

Remark 5.2. (5.1) has a basis of solutions $\{M^n e_1\}, \dots, \{M^n e_k\}$ (where e_i is the i-th unit vector in $\mathbb{C}^n$). If $M = \alpha I_k + J_k$, $\alpha \neq 0$, then $M^n = \alpha^n \cdot \sum_{j=1}^{k} \binom{n}{j-1} \alpha^{1-j} \cdot J_k^{j-1}$, so that

$$|M^n e_j| \sim \frac{|\alpha|^{n-j+1} \cdot n^{j-1}}{(j-1)!} \quad (j = 1, \dots, k).$$

Hence, for any arbitrary non-trivial solution $\{x_n\}$, we have $x_n = \lambda_1 M^n e_1 + \dots + \lambda_k M^n e_k$, so that

$$(5.4) \qquad |x_n| \sim |\lambda_j| \cdot |M^n e_j|$$

where j is chosen such that $\lambda_j \neq 0$ and $\lambda_i = 0$ for $j < i \le k$. If $M = \mathrm{diag}(S_1, \dots, S_h)$ with $S_1, , \dots, S_h$ Jordan blocks, $S_i \in \mathbb{C}^{k_i, k_i}$ and $\{x_n\}$ is a solution of (5.1), then there exist $u_i \in \mathbb{C}^{k_i}$ $(1 \le i \le h)$ such that $x_n = (S_1^n u_1, \dots, S_h^n u_h)$ $(n \in \mathbb{N})$, hence $|x_n|^2 = |S_1^n u_1|^2 + \dots + |S_h^n u_h|^2$ $(n \in \mathbb{N})$. Then, by (5.4), we have

$$(5.5) \qquad |x_n| \sim c \cdot |M^n e_j|$$

for some $c \in \mathbb{R}_{\ge 0}$, $j \in \{1, \dots, k\}$.

Lemma 5.2. *Let $f \in \mathcal{M}$, $f \neq 0$ and let $\{a_n\}$ be a sequence of non-negative real numbers such that $\sum_{k=1}^{\infty} a_k$ converges. Then*

$$(5.6) \qquad \sum_{(n)} a_n f(h) = \mathrm{o}(f(n)).$$

Proof. First suppose that $\lim_{n\to\infty} f(n) = 0$. Then $\sum_{h=1}^{\infty} a_h f(h)$ converges and $f(h)/f(n) < A$ if $h \geq n$. Hence $\sum_{(n)} a_h f(h) < A \cdot f(n) \sum_{(n)} a_h = \mathrm{o}(f(n))$, so that (5.6) holds. Further, if $\lim_{n\to\infty} f(n)$ exists and is not zero, the assertion is trivial. Finally, suppose that $\lim_{n\to\infty} f(n) = \infty$. If $\sum_{h=1}^{\infty} a_h f(h)$ converges, then $\sum_{(n)} a_h f(h) = \mathrm{o}(1)$, so (5.6) holds a fortiori. Suppose that $\sum_{h=1}^{\infty} a_h f(h)$ diverges. Let A be such that $f(h)/f(n) < A$ for $h \leq n$. We choose some number $\varepsilon > 0$ and let $N \in \mathbb{N}$ be so large that $\sum_{(N)} a_h < \varepsilon/A$. Then, for $n \geq N$,

$$\begin{aligned}\sum_{h=1}^{n-1} a_h f(h) &= \sum_{h=1}^{N-1} a_h f(h) + \sum_{h=N}^{n-1} a_h f(h) \\ &\leq A \cdot f(N) \sum_{(1)} a_h + \varepsilon \cdot f(n) < 2\varepsilon \cdot f(n)\end{aligned}$$

for n large enough. □

Lemma 5.3. *Let $M_n = (1 + (1/n))^B + D_n$ $(n \in \mathbb{N})$ for some diagonal matrix $B \in \mathbb{R}^{k,k}$ and matrices $D_n \in \mathcal{K}^{k,k}$. Suppose there exists some $f \in \mathcal{M}^1$ such that $\sum_{h=1}^{\infty} (1/f(h)) \|D_h\| < \infty$. Then there exists a sequence $\{F_n\}$ of non-singular matrices in $\mathcal{K}^{k,k}$ such that*

$$(5.7) \qquad \|F_n - I\| = \mathrm{o}(f(n))$$

and

$$(5.8) \qquad M_n F_n n^B = F_{n+1}(n+1)^B.$$

Proof. Put $B_n = (1 + (1/n))^B$. Applying Lemma 4.1 to $\{M_n\} = \{B_n + D_n\}$ yields the result. (5.8) follows directly from (4.1). We show that (4.2) implies (5.7). The numbers $b_i(n) := (B_n)_{ii}$ are of the form $b_i(n) = (1 + (1/n))^{\beta_i}$ for $\beta_i \in \mathbb{R}$ $(1 \leq i \leq k)$. Put $g_{ij}(n) = \prod_{l=1}^{n-1} (b_i(l)/(b_j(l))$ $(1 \leq i, j \leq k)$. Since $g_{ij}(n) f(n) \in \mathcal{M}$ and $\sum_{h=1}^{\infty} (1/f(h)) \|D_h\| < \infty$, Lemma 5.2 yields that

$$\sum_{(n)} \|D_h\| \cdot g_{ij}(h) = \mathrm{o}(g_{ij}(n) \cdot f(n)).$$

Hence,

$$g_{ji}(n) \cdot \sum_{(n)} \|D_h\| \cdot g_{ij}(h) = \mathrm{o}(f(n))$$

for all $1 \leq i, j \leq k$. □

Proof of Theorem 5.1. (a) We first assume that all eigenvalues of M have the same modulus. Since the assertions of the theorem remain valid if we multiply all

M_n and M by some constant $c \neq 0$, c independent of n, we may as well assume that all eigenvalues of M have modulus one. We may further assume that M is in Jordan canonical form, thus $M = \operatorname{diag}(S_1, \dots, S_h)$ with $S_i \in \mathbb{C}^{k_i, k_i}$ the Jordan blocks of M $(1 \leq i \leq h)$. Put $E = \operatorname{diag}(E_{k_1}, \dots, E_{k_h})$, with E_j $(j \in \mathbb{N})$ as in (3.7). We define the sequence $\{G_n\}$ by $G_n = M^n \cdot n^{-E}$ $(n \in \mathbb{N})$. By (3.8), for each $\lambda \in \mathbb{N}$, $\alpha \in \mathbb{C} \setminus \{0\}$, we have

$$(\alpha I_\lambda + J_\lambda)^n \cdot n^{-E_\lambda} = n^{-E_\lambda}(\alpha I_\lambda + J_\lambda/n)^n \sim n^{-E_\lambda} \alpha^n e^{J_\lambda/\alpha}$$

and

$$((\alpha I_\lambda + J_\lambda)^n \cdot n^{-E_\lambda})_{1j} = (n^{-E_\lambda})_{11} \cdot |(I_\lambda + J_\lambda/n\alpha)^n_{1j}| \sim |(e^{J_\lambda/\alpha})_{1j}| \neq 0.$$

Hence, since G_n is a block-diagonal matrix with blocks of the form $S_i^n \cdot n^{-E_{k_i}}$ $(n \in \mathbb{N})$, we have that $\{G_n\}$ converges to some matrix G such that $Ge_j \neq 0$ for $1 \leq j \leq k$. Further, $G_n^{-1} = n^E \cdot M^{-n}$ and $n^{E_\lambda}(\alpha I_\lambda + J_\lambda)^n = (\alpha I_\lambda + J_\lambda/n)^n \cdot n^{E_\lambda}$, so that

$$\|G_n^{-1}\| = \mathrm{O}(\|n^E\|) = \mathrm{O}(n^{L-1}).$$

Now, $G_{n+1}^{-1} MG_n = (1 + (1/n))^E$, with E some diagonal matrix, and $\|G_{n+1}^{-1}(M_n - M)\, G_n\| \ll n^{L-1} \|M_n - M\|$, so

$$G_{n+1}^{-1} M_n G_n = \left(1 + \frac{1}{n}\right)^E + D_n$$

where $\sum_{n=1}^{\infty} (1/f(n)) \|D_n\|$ converges. By Lemma 5.3 there exists some sequence $\{F_n\}$, $F_n \in \mathbb{C}^{k,k}$ $(n \in \mathbb{N})$ such that

$$\|F_n - I\| = \mathrm{o}(f(n))$$

and

$$G_{n+1}^{-1} M_n G_n F_n n^E = F_{n+1}(n+1)^E.$$

Put $X_n^{(0)} = G_n F_n n^E$ $(n \in \mathbb{N})$. Then $\{X_n^{(0)}\}$ is a k-dimensional solution of the matrix recurrence determined by $\{M_n\}$. On the other hand, we have, for $\{X_n^{(1)}\} := \{G_n n^E\}$:

$$MX_n^{(1)} = X_{n+1}^{(1)} \quad (n \in \mathbb{N}).$$

Thus,

$$X_n^{(0)} - X_n^{(1)} = G_n(F_n - I)\, n^E \quad (n \in \mathbb{N})$$

so that for $1 \leq j \leq k$

$$\frac{|(X_n^{(0)} - X_n^{(1)})\, e_j|}{|X_n^{(1)} e_j|} \ll \frac{\|F_n - I\| \cdot |n^E e_j|}{|Ge_j| \cdot |n^E e_j|} \ll \|F_n - I\| = \mathrm{o}(f(n))$$

since $Ge_j \neq 0$ for all $1 \leq j \leq k$.

For any arbitrary non-trivial solution $\{x_n^{(0)}\}$ of (1.1), we have $x_n^{(0)} = X_n^{(0)} u = X_n^{(0)}(\lambda_1 e_1 + \cdots + \lambda_k e_k)$ for some tuple $(\lambda_1, \ldots, \lambda_k) \neq (0, \ldots, 0)$. Then, by Remark 5.2, and taking into account the special form of M, we have

$$\frac{|(X_n^{(0)} - X_n^{(1)})\, u|}{|X_n^{(1)}\, u|} \ll \frac{|(X_n^{(0)} - X_n^{(1)})\, e_j|}{|X_n^{(1)} e_j|} = \mathrm{o}(f(n))$$

for some $j \in \{1, \ldots, k\}$.

(b) In the general case, by Theorem 3.1 we can find some matrix $U \in \mathbb{C}^{k,k}$ and some sequence $\{F_n'\}$ of matrices in $\mathbb{C}^{k,k}$ with $\sum_{n=1}^{\infty} (1/f(n)) \, \|F_n' - I\| < \infty$, such that

$$(F_{n+1}')^{-1}\, U^{-1} M_n U F_n' = \mathrm{diag}(M_n^{(1)}, \ldots, M_n^{(h)}) \quad (n \in \mathbb{N}) \tag{5.9}$$

where $M_n^{(1)}, \ldots, M_n^{(h)}$ are square matrices such that for each of them all of its eigenvalues have the same modulus and such that $\rho(M_n^{(1)}) < \cdots < \rho(M_n^{(h)})$ for n large enough. Then the theorem follows easily from (5.9) and the special case (a). □

The results obtained in this paper for sequences of matrices can be applied to linear recurrences as well. A more detailed account, in particular for second-order recurrences, is given in [1], [2], where also more references concerning this subject can be found.

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