A monodromy criterion for existence of Neron models and a result on semi-factoriality

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On the other hand, the base change of $\mathcal{X}/R$ by the étale map $R \to R' := \mathbb{Q}(i)[[t]]$ is semi-factorial, since its special fibre has split singularities and its graph is a tree. We see that, denoting by X_1 and X_2 the two components of the special fibre, the Weil divisors $s_{R'} - X_1$ and $s_{R'} - X_2$ are both Cartier, and both extend the Cartier divisor on $\mathcal{X}_{K'}$ given by $s_{K'}$.

13 Application to Néron lft-models of jacobians of nodal curves

13.1 Representability of the relative Picard functor

Let S be a scheme and $\mathcal{X} \to S$ a curve. We denote by $\mathrm{Pic}_{\mathcal{X}/S}$ the relative Picard functor, that is, the fppf-sheafification of the functor

$$\begin{array}{rcl} (\mathbf{Sch}/S)^{opp} & \to & \mathbf{Sets} \\ T & \mapsto & \{\text{invertible sheaves on } \mathcal{X}_T\}/\cong \end{array}$$

We start with a result on representability of the Picard functor:

Theorem 13.1 ([BLR90] 9.4/1)**.** *Let $f\colon \mathcal{X} \to S$ be a nodal curve. Then the relative Picard functor $\mathrm{Pic}_{\mathcal{X}/S}$ is representable by an algebraic space*[2]*, smooth over S.*

Lemma 13.2. *Let $f\colon \mathcal{X} \to S$ be a nodal curve admitting a section $s\colon S \to \mathcal{X}$. Then for any S-scheme T the natural map*

$$\mathrm{Pic}(\mathcal{X} \times_S T)/\mathrm{Pic}(T) \to \mathrm{Pic}_{\mathcal{X}/S}(T)$$

is an isomorphism.

Proof. See the discussion about rigidified line bundles on [BLR90] 8.1. □

13.2 Néron lft-models

Let S be a Dedekind scheme, that is, a noetherian normal scheme of dimension ≤ 1. Then S is a disjoint union of integral Dedekind schemes S_i. The *ring of rational functions* of S is the direct sum $K := \bigoplus_i k(\eta_i)$, where the points $\{\eta_i\}$ are the generic points of the S_i.

[2] Defined as in [BLR90] 8.3/4

Definition 13.3 ([BLR90], 10.1/1)**.** Let S be a Dedekind scheme, with ring of rational functions K. Let A be a K-scheme. A *Néron lft-model* over S for A is the datum of a smooth separated scheme $\mathcal{A} \to S$ and a K-isomorphism $\varphi\colon \mathcal{A} \times_S K \to A$ satisfying the following universal property: for any smooth map of schemes $T \to S$ and K-morphism $f\colon T_K \to A$, there exists a unique S-morphism $F\colon T \to \mathcal{A}$ with $F_K = f$.

A Néron lft-model differs from a Néron model in that the former is not required to be quasi-compact.

Proposition 13.4 ([BLR90], 10.1/2)**.** *Let S be a trait and G a smooth separated S-group scheme. The following are equivalent:*

i) G is a Néron lft-model of its generic fibre;

ii) for every essentially smooth local extension of traits $S' \to S$, with $K' = \operatorname{Frac}\Gamma(S, \mathcal{O}_S)$, the map $G(S') \to G(K')$ is surjective.

Lemma 13.5. *Let $\mathcal{X} \to S$ be a nodal curve over a trait. Let $\mathrm{cl}(e_K) \subset \mathrm{Pic}_{\mathcal{X}/S}$ be the schematic closure of the unit section $e_K\colon \operatorname{Spec} K \to \mathrm{Pic}_{\mathcal{X}_K/K}$. Then the fppf-quotient sheaf $\mathcal{N} = \mathrm{Pic}_{\mathcal{X}/S} / \mathrm{cl}(e_K)$ is representable by a smooth separated S-group scheme. Moreover, the quotient morphism $\mathrm{Pic}_{\mathcal{X}/S} \to \mathcal{N}$ is étale.*

Proof. As $\mathrm{cl}(e_K)$ is flat over S, the fppf-quotient of sheaves $\mathcal{N} = \mathrm{Pic}_{\mathcal{X}/S} / \mathrm{cl}(e_K)$ is a group algebraic space, smooth over S because $\mathrm{Pic}_{\mathcal{X}/S}$ is; as $\mathrm{cl}(e_K)$ is closed in $\mathrm{Pic}_{\mathcal{X}/S}$, $\mathcal{N}$ is separated over S. In particular, $\mathcal{N}$ is a separated group algebraic space locally of finite type over S, so it is a group scheme by [Ana73], Chapter IV, Theorem 4.B. Finally, to show that $\mathrm{Pic}_{\mathcal{X}/S} \to \mathcal{N}$ is étale we prove that $\mathrm{cl}(e_K)$ is étale over S. As the property is étale local on S, we may assume that $\mathcal{X} \to S$ has special fibre with split singularities. The multidegree map $E(\mathcal{X}) \to \mathbb{Z}^V$ (lemma 12.2, ii)) is injective, hence the intersection of $\mathrm{cl}(e_K)$ with the identity component $\mathrm{Pic}^0_{\mathcal{X}/S} \subset \mathrm{Pic}_{\mathcal{X}/S}$ is trivial and it follows that $\mathrm{cl}(e_K)$ is étale over S. □

Given a nodal curve $\mathcal{X} \to S$ over a trait, we can associate to it the labelled graph (Γ, l) of the base change $\mathcal{X} \times_S S' \to S'$, where S' is the spectrum of the strict henselization of $\Gamma(S, \mathcal{O}_S)$ with respect to some algebraic closure of the residue field k. The graph (Γ, l) does not depend on the choice of an algebraic closure of k.

Theorem 13.6. *Let $\mathcal{X} \to S$ be a nodal curve over a trait with perfect fraction field K. The S-group scheme $\mathcal{N} = \mathrm{Pic}_{\mathcal{X}/S} / \mathrm{cl}(e_K)$ is a Néron lft-model for $\mathrm{Pic}_{\mathcal{X}_K/K}$ over S if and only if the labelled graph (Γ, l) of $\mathcal{X} \to S$ is circuit-coprime.*

Proof. Let $S^{sh} \to S$ be a strict henselization of S with respect to some algebraic closure of the residue field, and denote by K^{sh} its fraction field. If (Γ, l) is not circuit-coprime, the map

$$\mathrm{Pic}(\mathcal{X}_{S^{sh}}) \to \mathrm{Pic}(\mathcal{X}_{K^{sh}})$$

is not surjective, by theorem 12.3. Now, as the special fibre of $\mathcal{X}_{S^{sh}}/S^{sh}$ is generically smooth, $\mathcal{X}_{S^{sh}} \to S^{sh}$ admits a section; hence, we can apply lemma 13.2 and find that

$$\mathrm{Pic}_{\mathcal{X}/S}(S^{sh}) \to \mathrm{Pic}_{\mathcal{X}_K/K}(K^{sh})$$

is not surjective. As the quotient $\mathrm{Pic}_{\mathcal{X}/S} \to \mathcal{N}$ is an étale surjective morphism of S^{sh}-algebraic spaces (lemma 13.5), the map $\mathrm{Pic}_{\mathcal{X}/S}(S^{sh}) \to \mathcal{N}(S^{sh})$ is surjective. We deduce that $\mathcal{N}(S^{sh}) \to \mathrm{Pic}_{\mathcal{X}_K/K}(K^{sh})$ is not surjective. Then for some étale extension of discrete valuation rings $S' \to S$, $\mathcal{N}(S') \to \mathrm{Pic}_{\mathcal{X}_K/K}(K')$ is not surjective, hence $\mathcal{N}$ is not a Néron model of $\mathrm{Pic}_{\mathcal{X}_K/K}$.

Now assume that (Γ, l) is circuit coprime. Assume first that S is strictly henselian. By proposition 13.4 it is enough to prove that for all essentially smooth local extensions $R \to R'$ of discrete valuation rings, the map

$$\mathcal{N}(R') \to \mathrm{Pic}_{\mathcal{X}_K/K}(K')$$

is surjective. As $\mathcal{X} \to S$ admits a section, we may apply lemma 13.2 and just show that $\mathrm{Pic}(\mathcal{X}_{R'}) \to \mathrm{Pic}(\mathcal{X}_{K'})$ is surjective. The map $R \to R'$ has ramification index 1, i.e. it sends a uniformizer to a uniformizer. Therefore the labelled graph (Γ', l') associated to $\mathcal{X}_{R'}$ is again circuit-coprime, and in fact $(\Gamma', l') = (\Gamma, l)$. Now we conclude by theorem 12.3.

Now let $\mathcal{X} \to S$ be any nodal curve with circuit-coprime labelled graph. Let $p\colon S' \to S$ be a strict henselization of S. Consider the smooth separated S-group scheme $\mathcal{N} = \mathrm{Pic}_{\mathcal{X}/S} / \mathrm{cl}(e_K)$. As taking the schematic closure commutes with flat base change, $p^*\mathcal{N}$ is canonically isomorphic to $\mathrm{Pic}_{\mathcal{X}'/S'} / \mathrm{cl}(e_{K'})$, hence is a Néron lft-model for $\mathrm{Pic}_{\mathcal{X}_{K'}/K'}$ over S'. We show that $\mathcal{N}$ is a Néron lft-model of its generic fibre. Let $T \to S$ be a smooth S-scheme, $f\colon T_K \to \mathrm{Pic}_{\mathcal{X}_K/K}$ a K-morphism. The base change $p^*f\colon T_{K'} \to \mathrm{Pic}_{\mathcal{X}_{K'}/K'}$ extends uniquely to an S'-morphism $g\colon p^*T \to \mathcal{N}'$. Let $S'' := S' \times_S S'$, $p_1, p_2\colon S'' \to S'$ the two projections, and $q\colon S'' \to S$ the composition. The two maps $p_1^*g, p_2^*g\colon q^*T \to q^*\mathcal{N}$ both coincide with q^*f when restricted to q^*T_K. As $q^*T \to S$ is flat, q^*T_K is schematically dense in q^*T. Since moreover $q^*\mathcal{N}$ is separated, we have that $p_1^*g = p_2^*g$. Hence g descends to a morphism $T \to \mathcal{N}$ extending f. Again, the extension is unique because $\mathcal{N} \to S$ is separated and T_K is schematically dense in T.

□

Corollary 13.7. *Let $\mathcal{X} \to S$ be a nodal curve over a trait. Let $\pi\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ be the blowing-up of $\mathcal{X}$ at the finite union of closed points $\mathcal{X}^{nreg} \cap \mathcal{X}_k$. Then $\mathcal{N} = \mathrm{Pic}_{\widetilde{\mathcal{X}}/S} / \mathrm{cl}(e_K)$ is a Néron lft-model for $\mathrm{Pic}_{\mathcal{X}_K/K}$ over S.*

Proof. It is enough to check that the labelled graph $(\widetilde{\Gamma}, \widetilde{l})$ of $\widetilde{\mathcal{X}} \to S$ is circuit-coprime, by the previous Theorem. As labelled graphs are preserved under étale extensions of the base trait, we may assume that $\mathcal{X} \to S$ has special fibre with split singularities. Then the same argument as in the proof of corollary 12.5 shows that $(\widetilde{\Gamma}, \widetilde{l})$ is circuit-coprime. □

Corollary 13.8. *Let $\mathcal{X} \to S$ be a nodal curve over a trait with perfect fraction field K. Let $\overline{k}$ be a separable closure of the residue field of S and suppose that the graph of $\mathcal{X}_{\overline{k}}$ is a tree. Then $\mathcal{N} = \mathrm{Pic}_{\mathcal{X}/S} / \mathrm{cl}(e_K)$ is a Néron lft-model for $\mathrm{Pic}_{\mathcal{X}_K/K}$ over S.*

We have shown how to construct Néron lft-models for the group scheme $\mathrm{Pic}_{\mathcal{X}_K/K}$, without ever imposing bounds on the degree of line bundles; the following lemma allows us to retrieve lft-Néron models for subgroup schemes of $\mathrm{Pic}_{\mathcal{X}_K/K}$, and applies in particular to subgroup schemes that are open and closed, such as the connected component of the identity $\mathrm{Pic}^{[0]}_{\mathcal{X}_K/K}$.

Lemma 13.9. *Let $\mathcal{X}/S$ be a nodal curve over a trait, and $H \subset \mathrm{Pic}_{\mathcal{X}_K/K}$ a K-smooth closed subgroup scheme of $\mathrm{Pic}_{\mathcal{X}_K/K}$. Let $\mathcal{N} \to S$ be the Néron model of $\mathrm{Pic}_{\mathcal{X}_K/K}$. Then H admits a Néron lft-model $\mathcal{H}$ over S, which is obtained as a group smoothening of the schematic closure of H inside $\mathcal{N}$.*

Proof. This is a special case of [BLR90], 10.1/4.

□

We remark that, if the generic fibre $\mathcal{X}_K/K$ is not smooth, $\mathrm{Pic}^{[0]}_{\mathcal{X}_K/K}$ is an extension of an abelian variety by a torus; if the torus contains a copy of $\mathbb{G}_{m,K}$, the Néron lft-model of $\mathrm{Pic}^{[0]}_{\mathcal{X}_K/K}$ is not quasi-compact.

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