A monodromy criterion for existence of Neron models and a result on semi-factoriality

Orecchia, G.

this point on, the rest of the proof coincides with the proof of proposition 11.16; we only mention that, at the point when $x(e)$ is defined, one can assign to it any value if $l(e) = 0$.

□

12 Semi-factoriality of nodal curves

Let S be the spectrum of a discrete valuation ring R having perfect fraction field K, residue field k and uniformizer t. Let $f\colon \mathcal{X} \to S$ be a nodal curve whose special fibre has split singularities, and $\Gamma = (V, E)$ be the dual graph of the special fibre $\mathcal{X}_k$. For any $v \in V$, we denote by X_v the corresponding irreducible component of the special fibre $\mathcal{X}_k$.

Definition 12.1. The *labelled graph* of $\mathcal{X} \to S$ is the $\mathbb{N}_\infty$-labelled graph (Γ, l) whose labelling l assigns to each edge of Γ the thickness (see section 7.1) of the corresponding singular point of $\mathcal{X}_k$.

Our aim is to relate the property of being circuit-coprime for the graph (Γ, l) to the semi-factoriality of $f\colon \mathcal{X} \to S$. To this end, we are going to provide a dictionary between the geometry of $\mathcal{X}/S$ and the combinatorial objects introduced in section 11.

Denote by $\mathrm{Div}_k(\mathcal{X})$ the group of Weil divisors on $\mathcal{X}$ supported on the special fibre $\mathcal{X}_k$. It is the free abelian group generated by the irreducible components of $\mathcal{X}_k$. Hence we obtain a natural isomorphism $\mathrm{Div}_k(\mathcal{X}) \to \mathbb{Z}^V$.

Let $\mathcal{C}(\mathcal{X})$ be the group of Cartier divisors on $\mathcal{X}$ whose restriction to the generic fibre $\mathcal{X}_K$ is trivial. We claim that the natural map $\mathcal{C}(\mathcal{X}) \to \mathrm{Div}_k(\mathcal{X})$ is injective. This follows from ([GD67], 21.6.9 (i)) under the assumption that $\mathcal{X}$ is normal, which is not satisfied if $\mathcal{X}/S$ has singular generic fibre. However, the proof only requires that for all $x \in \mathcal{X}_k$, $\mathrm{depth}(\mathcal{O}_{\mathcal{X},x}) = 1$ implies $\dim \mathcal{O}_{\mathcal{X},x} = 1$. This is immediately checked: let $x \in \mathcal{X}_k$ with $\dim \mathcal{O}_{\mathcal{X},x} \neq 1$; then x is a closed point of $\mathcal{X}_k$. By S-flatness of $\mathcal{X}$, the uniformizer t is not a zero divisor in $\mathcal{O}_{\mathcal{X},x}$; as $\mathcal{X}_k$ is reduced, $\mathcal{O}_{\mathcal{X},x}/t\mathcal{O}_{\mathcal{X},x}$ is reduced. Every reduced noetherian ring of dimension 1 is Cohen-Macaulay, hence $\mathrm{depth}(\mathcal{O}_{\mathcal{X},x}/t\mathcal{O}_{\mathcal{X},x}) = 1$, and we deduce by [Sta16]TAG 0AUI that $\mathrm{depth}(\mathcal{O}_{\mathcal{X},x}) = 2$, establishing the claim. Hence $\mathcal{C}(\mathcal{X})$ is in a natural way a subgroup of $\mathrm{Div}_k(\mathcal{X})$.

Finally, denote by $E(\mathcal{X})$ the kernel of the restriction map $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_K)$, so that $E(\mathcal{X})$ is the group of isomorphism classes of line bundles on $\mathcal{X}$ that are generically trivial. We have an exact sequence of groups

$$0 \to \mathbb{Z} \to \mathcal{C}(\mathcal{X}) \to E(\mathcal{X}) \to 0$$

where the first map sends 1 to $\operatorname{div}(t)$ and the second map sends D to $\mathcal{O}_{\mathcal{X}}(D)$. Indeed, every principal Cartier divisor supported on the special fibre belongs to $\mathbb{Z}\operatorname{div}(t)$. For this we can reduce to showing that every regular function on $\mathcal{X}$ that is generically invertible is of the form $t^n u$ for some $n \in \mathbb{Z}_{\geq 0}$ and $u \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})^\times$. By [Sta16]TAG 0AY8 we have $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_S$, from which the claim easily follows.

Lemma 12.2. *Hypotheses as in the beginning of this section.*

i) The natural isomorphism $\operatorname{Div}_k(\mathcal{X}) \to \mathbb{Z}^V$ identifies $\mathcal{C}(\mathcal{X}) \subset \operatorname{Div}_k(\mathcal{X})$ with the subgroup $\mathcal{C} \subset \mathbb{Z}^V$ of Cartier vertex labellings (definition 11.6).

Let

$$0 \to \mathbb{Z} \to \mathcal{C} \xrightarrow{\delta} \mathbb{Z}^V$$

be the exact sequence of lemma 11.8, where δ is the multi-degree operator (definition 11.7).

ii) The isomorphism $\mathcal{C}(\mathcal{X}) \to \mathcal{C}$ induces an exact sequence

$$0 \to \mathbb{Z} \to \mathcal{C}(\mathcal{X}) \xrightarrow{\delta_{\mathcal{X}}} \mathbb{Z}^V.$$

The first arrow is the map $1 \mapsto \operatorname{div}(t)$; the map $\delta_{\mathcal{X}}$ factors via the map $E(\mathcal{X}) \to \mathbb{Z}^V$, which sends a line bundle $\mathcal{L}$ to the vertex labelling

$$v \mapsto \deg \mathcal{L}_{|X_v}.$$

Let

$$\ldots \to \mathcal{X}_n \to \ldots \to \mathcal{X}_1 \to \mathcal{X}_0 = \mathcal{X}$$

be the chain of blowing-ups (29). *Denote by π_n the composition $\mathcal{X}_n \to \mathcal{X}$.*

iii) For every $n \geq 0$ the labelled graph of $\mathcal{X}_n \to S$ is the n-th blow-up graph (Γ_n, l_n) of (Γ, l) (definition 11.17). The new vertices of (Γ_n, l_n) correspond to the irreducible components of the exceptional fibre of $\mathcal{X}_n \to \mathcal{X}$.

iv) Let $\mathcal{C}_n$ be the group of Cartier vertex labellings on $\mathcal{X}_n$. The map $\mathcal{C}(\mathcal{X}) \to \mathcal{C}(\mathcal{X}_n)$ induced by $\iota \colon \mathcal{C} \to \mathcal{C}_n$ (section 11.3) descends to the pullback map $\pi_n^* \colon E(\mathcal{X}) \to E(\mathcal{X}_n)$.

Proof.

i) Let $D = \sum_v n_v X_v \in \operatorname{Div}_k(\mathcal{X})$. We want to show the equivalence of the two conditions:

a) for every node $p \in \mathcal{X}_k$ lying on distinct components X_w, X_z of $\mathcal{X}_k$, the thickness τ_p divides $n_w - n_z$ (with the convention that ∞ divides only 0);

b) D is Cartier.

As every Weil divisor D is Cartier on the generic fibre and on the regular locus of $\mathcal{X}$, we may fix a node $p \in \mathcal{X}_k$ and reduce to work on the complete local ring $\widehat{\mathcal{O}}_{\mathcal{X},p}$. We identify $\widehat{\mathcal{O}}_{\mathcal{X},p}$ with $A = \widehat{R}[[x,y]]/xy - t^{\tau_p}$. Let X_w and X_z be the components of $\mathcal{X}_k$ through p, and let Y_w, Y_z be their preimages in $\operatorname{Spec} A$, which are given by the ideals (x,t) and (y,t) of A respectively.

Assume a) is true; we are going to deduce that D is Cartier at p. We may assume that the two components X_w and X_z are distinct, otherwise D is given by $\operatorname{div}(t^{n_w})$ locally at p and is automatically Cartier at p. As $\operatorname{div}(x) = \tau_p Y_w$, we have that $(n_w - n_z)Y_w = \operatorname{div}(x^{\frac{n_w - n_z}{\tau_p}})$ is Cartier. Therefore $D - \operatorname{div}(t^{n_z}) = \sum_v (n_v - n_z)X_v$ is Cartier at p, and also D is.

Assume now b) and that p lies on distinct components X_w, X_z of $\mathcal{X}_k$. We may assume that the restriction of D to $\operatorname{Spec} A$, $n_w Y_w + n_z Y_z$, is the divisor of some regular function $f \in A = \widehat{R}[[x,y]]/xy - t^{\tau_p}$. We first consider the case $\tau_p = \infty$. As f is a unit in $A[t^{-1}]$, there exists $g \in A$ and $n \geq 0$ such that $fg = t^n$. Now, let f_x be the image of f in A/xA. As the latter is a unique factorization domain, $f_x = t^{m_1}u_1$ for some unit $u_1 \in (A/xA)^\times$ and $m_1 \leq n$. Moreover, we have $m_1 = n_w$. Similarly, we write $f_y = t^{m_2}u_2 \in A/yA$, with $m_2 = n_z$. As the images of f_x and f_y in $A/(x,y)A = R$ coincide, we find that $m_1 = m_2$, that is, $n_w = n_z$, as desired. Now we remain with the case $\tau_p \neq \infty$. Replacing f by ft^{-n_z}, we get $\operatorname{div}(f) = (n_w - n_z)Y_w$. We want to show that τ_p divides $m := n_w - n_z$. Let $d = \gcd(m, \tau_p)$. As $\operatorname{div}(x) = \tau_p Y_w$, we may replace f by a product of powers of f and x and assume that $m = d$. Write $\tau_p = m\alpha$, for some $\alpha \in \mathbb{Z}$. We have $\operatorname{div}(f^\alpha/x) = 0$, hence, as $\operatorname{Spec} A$ is normal, f^α/x is a unit in A. Now, reducing modulo t, one can easily see that α has to be 1, so $m = \tau_p$ as desired.

ii) The composition $\mathbb{Z} \to \mathcal{C} \to \mathcal{C}(\mathcal{X})$ sends 1 to $\sum_v X_v = \mathcal{X}_k = \operatorname{div}(t)$. The map $\delta_{\mathcal{X}}$ factors via the cokernel of $\mathbb{Z} \to \mathcal{C}(\mathcal{X})$, which is indeed $E(\mathcal{X})$. For the characterization of the map $\delta_{\mathcal{X}}$, recall first that $\delta\colon \mathcal{C} \to \mathbb{Z}^V$ sends a Cartier vertex labelling φ to the vertex labelling

$$v \mapsto \sum_{\substack{\text{edges } e \\ \text{incident to } v}} \frac{\varphi(w) - \varphi(v)}{l(e)}$$

where w denotes the other endpoint of e. The composition $\delta_{\mathcal{X}}\colon \mathcal{C}(\mathcal{X}) \to$

$\mathcal{C} \to \mathbb{Z}^V$ sends a Cartier divisor $D = \sum_v n_v X_v$ to

$$v \mapsto \sum_{\substack{\text{nodes } p \\ \text{lying on } X_v}} \frac{n_w - n_v}{\tau_p}$$

with τ_p being the thickness of the node p, X_w the second component passing through p. We want to check that $\delta_{\mathcal{X}}(D)$ is the vertex labelling $v \mapsto \deg \mathcal{O}(D)_{|X_v}$. Fix a vertex z; multiplication by t^{n_z} gives an isomorphism $\mathcal{O}(D) \cong \mathcal{O}(D')$ where $D' = \sum_v (n_v - n_z) X_v$. We reduce to computing the contribution to $\deg \mathcal{O}(D')_{|X_z}$ coming from $(n_v - n_z)X_v$, where $v \in V$ is some vertex different from z. The contribution is zero if $\mathcal{X}_v$ and $\mathcal{X}_z$ do not intersect; otherwise, let $p \in X_v \cap X_z$, with thickness τ_p. Notice that $\tau_p | n_v - n_z$. Locally at p, the divisor $(n_v - n_z)X_v$ is given by the fractional ideal $I = (x^{(n_v - n_z)/\tau_p}, t^{n_v - n_z}) = (x^{(n_v - n_z)/\tau_p})$ of $\widehat{\mathcal{O}}_{\mathcal{X},p} \cong \widehat{R}[[x,y]]/xy - t^{\tau_p}$. Restricting to the branch $y = 0, t = 0$, we obtain the fractional ideal $I \otimes \widehat{\mathcal{O}}_{\mathcal{X},p}/y = (x^{(n_v - n_z)/\tau_p})$ of $k[[x]]$, hence a contribution of $(n_v - n_z)/\tau_p$ to the degree of $\mathcal{O}(D')_{|X_z}$. Summing over all the nodes in $X_v \cap X_z$, we recover the map $\delta_{\mathcal{X}}$.

iii) This can be read directly in the description of the effect of blowing-up on the special fibre provided in section 8.3.

iv) The commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{C}(\mathcal{X}) & \longrightarrow & E(\mathcal{X}) & \longrightarrow & 0 \\
 & & \downarrow \text{id} & & \downarrow \iota & & \downarrow \bar{\iota} & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{C}(\mathcal{X}_n) & \longrightarrow & E(\mathcal{X}_n) & \longrightarrow & 0
\end{array}$$

yields a map $\bar{\iota} \colon E(\mathcal{X}) \to E(\mathcal{X}_n)$. Such map fits into the commutative diagram

$$\begin{array}{ccc}
E(\mathcal{X}) & \xrightarrow{\delta_{\mathcal{X}}} & \mathbb{Z}^V \\
\downarrow \bar{\iota} & & \downarrow \epsilon \\
E(\mathcal{X}_n) & \xrightarrow{\delta_{\mathcal{X}_n}} & \mathbb{Z}^{V_n}
\end{array}$$

where $\epsilon \colon \mathbb{Z}^V \to \mathbb{Z}^{V_n}$ is the extension by zero map, and the two horizontal maps are induced by the exact sequences as in ii) for $\mathcal{X}$ and $\mathcal{X}_n$. They associate to a line bundle its multi-degree on the special fibre, and are injective. The pullback map $\pi_n^* \colon E(\mathcal{X}) \to E(\mathcal{X}_n)$ makes the diagram above commutative as well; it follows that it coincides with $\bar{\iota}$.

□

Theorem 12.3. *Let $\mathcal{X} \to S$ be a nodal curve over a trait with perfect fraction field K, and assume that the special fibre $\mathcal{X}_k$ has split singularities.*

i) If the labelled graph (Γ, l) is circuit-coprime then $\mathcal{X} \to S$ is semi-factorial.

ii) Suppose that $\Gamma(S, \mathcal{O}_S)$ is strictly-henselian. If $\mathcal{X}$ is semi-factorial over S, then the labelled graph (Γ, l) is circuit-coprime.

Proof. We start with part i). Suppose Γ is circuit-coprime. Let L be a line bundle on $\mathcal{X}_K$. By theorem 9.5, there exists an integer $n \geq 0$ such that L extends to a line bundle $\widetilde{\mathcal{L}}$ on $\mathcal{X}_n$. Let (Γ_n, l_n) be the labelled graph of $\mathcal{X}_n$, which is the n-th blow-up graph of Γ. Denote by $\alpha \in \mathbb{Z}^{V_n}$ the vertex-labelling assigning to each vertex v the degree of the restriction of $\widetilde{\mathcal{L}}$ to the component of $(\mathcal{X}_n)_k$ corresponding to v. By proposition 11.21, the map $H \to H_n$ is an isomorphism; hence there exists a Cartier vertex labelling φ on (Γ_n, l_n) such that $\delta(\varphi) + \alpha$ is in the image of the map $\mathbb{Z}^V \to \mathbb{Z}^{V_n}$. Equivalently (by lemma 12.2) there exists a Cartier divisor $D \in \mathcal{C}(\mathcal{X}_n)$, such that $\delta_{\mathcal{X}_n}(D) + \alpha$ is in the image of $\mathbb{Z}^V \to \mathbb{Z}^{V_n}$, i.e., $\delta_{\mathcal{X}}(D) + \alpha$ has value zero on all new vertices of Γ_n. This means precisely that $\mathcal{O}_{\mathcal{X}_n}(D) \otimes \widetilde{\mathcal{L}}$ has degree zero on every component of the exceptional locus of $\pi_n \colon \mathcal{X}_n \to \mathcal{X}$. By proposition 10.2, $\mathcal{L} := (\pi_n)_*(\widetilde{\mathcal{L}} \otimes \mathcal{O}(D))$ is a line bundle on $\mathcal{X}$, which restricts to L on the generic fibre.

Let's turn to part ii). Suppose that Γ is not circuit-coprime. Then there exists $n \geq 0$ such that the map $H \to H_n$ is not surjective. Let α be a basis element of $\mathbb{Z}^{V_n}$ such that the image of α in $H_n = \mathbb{Z}^{V_n}/\delta_n(\mathcal{C}_n)$ is not in the image of $H \to H_n$. Then α takes value 1 on some vertex v of Γ_n and value zero on all other vertices. The vertex v corresponds to an exceptional component $C \cong \mathbb{P}^1_k$ of $\pi_n \colon \mathcal{X}_n \to \mathcal{X}$. Let p be a k-rational point of $(\mathcal{X}_n)_k^{sm}$ lying on C, which exists as k is separably closed. Since the base is henselian, p can be extended to a section $s \colon S \to \mathcal{X}_n$. The image $D \subset \mathcal{X}_n$ of s defines a Cartier divisor. Let $L := \mathcal{O}(D)_{|K}$ be its restriction to the generic fibre. Assume by contradiction that L can be extended to a line bundle $\mathcal{L}$ on $\mathcal{X}$. Then $\mathcal{F} := \mathcal{O}(D) \otimes \pi_n^* \mathcal{L}^{-1}$ is generically trivial. Let D' be a Cartier divisor supported on the special fibre of $\mathcal{X}_n$ such that $\mathcal{O}(D') \cong \mathcal{F}$. Then D' corresponds to a Cartier-vertex labelling φ of Γ_n, and $\alpha - \delta_n(\varphi)$ is the vertex-labelling associated to the multidegree of $\pi_n^* \mathcal{L}$. As $\pi_n^* \mathcal{L}$ has degree zero on every component of the exceptional fibre of $\pi_n \colon \mathcal{X}_n \to \mathcal{X}$, $\alpha - \delta_n(\varphi)$ has value zero on every new vertex of Γ_n. In particular, $\alpha \delta_n(\varphi)$ is in the image of $H \to H_n$, and so is α, yielding a contradiction.

□

Remark 12.4. The assumption that $\Gamma(S, \mathcal{O}_S)$ is strictly-henselian can be replaced by the weaker assumption: for each irreducible component Y of $\mathcal{X}_k$, there exists a line bundle $\mathcal{L}_Y$ on $\mathcal{X}$ whose restriction to $\mathcal{X}_k$ has degree 1 on Y and degree 0 on all other components.

Corollary 12.5. *Hypotheses as in theorem 12.3. Let* $\pi\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ *be the blowing-up of* $\mathcal{X}$ *at the finite union of closed points* $\mathcal{X}^{nreg} \cap \mathcal{X}_k$*. The restriction map*

$$\mathrm{Pic}(\widetilde{\mathcal{X}}) \to \mathrm{Pic}(\mathcal{X}_K)$$

is surjective.

Proof. Let (Γ, l) be the labelled graph of $\mathcal{X} \to S$. The labelled graph $(\widetilde{\Gamma}, \widetilde{l})$ of $\widetilde{\mathcal{X}} \to S$ is the first-blow-up graph of Γ (definition 11.17). Every edge of $\widetilde{\Gamma}$ with a label different from 1 is adjacent to exactly two edges, both with label 1. Hence $\widetilde{\Gamma}$ is circuit-coprime, and we conclude by theorem 12.3.

□

Corollary 12.6. *Hypotheses as in theorem 12.3. Suppose that the special fibre* $\mathcal{X}_k$ *is of compact-type (i.e. its dual graph* Γ *is a tree). Then the restriction map*

$$\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_K)$$

is surjective.

Proof. The dual graph Γ of the special fibre has no circuits, hence the labelled graph (Γ, l) is circuit-coprime. □

In general, semi-factoriality of nodal curves over traits does not descend along étale base change, and we cannot drop the assumption in theorem 12.3 that the special fibre of the curve has split singularities. Here is an example.

Example 12.7. Let $R = \mathbb{Q}[[t]]$, $K = \mathrm{Frac}\, R$, $S = \mathrm{Spec}\, R$, and

$$\mathcal{X} = \mathrm{Proj}\, \frac{R[x,y,z]}{x^2 + y^2 - t^2 z^2}.$$

The curve $\mathcal{X} \to S$ has smooth generic fibre $\mathcal{X}_K/K$, and a node $P = (t = 0, x = 0, y = 0, z = 1)$ on the special fibre. The section $s\colon S \to \mathcal{X}$ given by $x = t, y = 0, z = 1$ goes through the node P . The Cartier divisor on $\mathcal{X}_K$ given by the image of $s_K\colon \mathrm{Spec}\, K \to \mathcal{X}_K$ does not extend to a Cartier divisor on $\mathcal{X}$. Indeed, if by contradiction it extended to a Cartier divisor D on $\mathcal{X}$, the difference $D - s$ as Weil divisors would be a Weil divisor supported on the special fibre; hence a Weil divisor linearly equivalent to zero, since the special fibre is irreducible. Then s would be Cartier, which it is not, and we have the contradiction.

On the other hand, the base change of $\mathcal{X}/R$ by the étale map $R \to R' := \mathbb{Q}(i)[[t]]$ is semi-factorial, since its special fibre has split singularities and its graph is a tree. We see that, denoting by X_1 and X_2 the two components of the special fibre, the Weil divisors $s_{R'} - X_1$ and $s_{R'} - X_2$ are both Cartier, and both extend the Cartier divisor on $\mathcal{X}_{K'}$ given by $s_{K'}$.

13 Application to Néron lft-models of jacobians of nodal curves

13.1 Representability of the relative Picard functor

Let S be a scheme and $\mathcal{X} \to S$ a curve. We denote by $\mathrm{Pic}_{\mathcal{X}/S}$ the relative Picard functor, that is, the fppf-sheafification of the functor

$$\begin{array}{rcl} (\mathbf{Sch}/S)^{opp} & \to & \mathbf{Sets} \\ T & \mapsto & \{\text{invertible sheaves on } \mathcal{X}_T\}/\cong \end{array}$$

We start with a result on representability of the Picard functor:

Theorem 13.1 ([BLR90] 9.4/1)**.** *Let $f\colon \mathcal{X} \to S$ be a nodal curve. Then the relative Picard functor $\mathrm{Pic}_{\mathcal{X}/S}$ is representable by an algebraic space*[2]*, smooth over S.*

Lemma 13.2. *Let $f\colon \mathcal{X} \to S$ be a nodal curve admitting a section $s\colon S \to \mathcal{X}$. Then for any S-scheme T the natural map*

$$\mathrm{Pic}(\mathcal{X} \times_S T)/\mathrm{Pic}(T) \to \mathrm{Pic}_{\mathcal{X}/S}(T)$$

is an isomorphism.

Proof. See the discussion about rigidified line bundles on [BLR90] 8.1. □

13.2 Néron lft-models

Let S be a Dedekind scheme, that is, a noetherian normal scheme of dimension ≤ 1. Then S is a disjoint union of integral Dedekind schemes S_i. The *ring of rational functions* of S is the direct sum $K := \bigoplus_i k(\eta_i)$, where the points $\{\eta_i\}$ are the generic points of the S_i.

[2] Defined as in [BLR90] 8.3/4