A monodromy criterion for existence of Neron models and a result on semi-factoriality

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each of the points $P_i \in \mathcal{X}_K$ extends to $Q_i^{(N)} \in \mathcal{X}_N^{reg}(R_i)$. Therefore the Weil divisor D extends to a Weil divisor $\widetilde{D}$ on $\mathcal{X}_N$ that is supported on the union of the $Q_i^{(N)}$, hence on the regular locus of $\mathcal{X}_N$. This implies that $\widetilde{D}$ is a Cartier divisor, and the line bundle $\mathcal{O}_{\mathcal{X}_n}(\widetilde{D})$ restricts to $\mathcal{O}_{\mathcal{X}_K}(D) \cong L$ on $\mathcal{X}_K$. This completes the proof.

□

10 Descent of line bundles along blowing-ups

Lemma 10.1. *Let S be a trait and $\pi\colon \mathcal{Y} \to \mathcal{X}$ a proper morphism of flat S-schemes, which restricts to an isomorphism over the generic point of S. Assume that the special fibre $\mathcal{X}_k$ is reduced. Then $\pi_*\mathcal{O}_\mathcal{Y} \cong \mathcal{O}_\mathcal{X}$.*

Proof. Consider an affine open $W \subset \mathcal{X}$. The morphism $\mathcal{O}_\mathcal{X}(W) \to \pi_*\mathcal{O}_\mathcal{Y}(W)$ is integral ([Liu02], Prop.3.3.18). Denoting by t a uniformizer of $\Gamma(S, \mathcal{O}_S)$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_\mathcal{X}(W) & \longrightarrow & \pi_*\mathcal{O}_\mathcal{Y}(W) \\ \downarrow & & \downarrow \\ \mathcal{O}_\mathcal{X}(W)[t^{-1}] & \xrightarrow{\cong} & (\pi_*\mathcal{O}_\mathcal{Y}(W))[t^{-1}] \end{array}$$

The two vertical arrows are injective because $\mathcal{X}$ and $\mathcal{Y}$ are S-flat; the lower arrow is an isomorphism because π is generically an isomorphism and $(\pi_*\mathcal{O}_\mathcal{Y}(W))[t^{-1}] = \pi_*(\mathcal{O}_\mathcal{Y}(W)[t^{-1}])$. It follows that the upper arrow is injective. We claim that $\mathcal{O}_\mathcal{X}(W)$ is integrally closed in $\mathcal{O}_\mathcal{X}(W)[t^{-1}]$, so that the upper arrow is an isomorphism, which proves the lemma. Take then $g \in \mathcal{O}_\mathcal{X}(W)[t^{-1}]$ satisfying a monic polynomial equation $g^m + a_1 g^{m-1} + \ldots + a_m = 0$ with coefficients in $\mathcal{O}_\mathcal{X}(W)$ and write $g = f/t^n$ with $f \in \mathcal{O}_\mathcal{X}(W)$ and $n \geq 0$ minimal. We want to show that n is zero. We have

$$\frac{f^m}{t^{nm}} + a_1 \frac{f^{m-1}}{t^{n(m-1)}} + \ldots + a_m = 0.$$

Suppose by contradiction $n \geq 1$. Upon multiplying by t^{nm} the above relation, we find that $f^m \in t\mathcal{O}_\mathcal{X}(W)$. Because the special fibre of $\mathcal{X}$ is reduced, the ring $\mathcal{O}_\mathcal{X}(W)/t\mathcal{O}_X(W)$ is reduced, hence $f \in t\mathcal{O}_\mathcal{X}(W)$. This violates the hypothesis of minimality of n and we have a contradiction. Hence $n = 0$ and $g \in \mathcal{O}_\mathcal{X}(W)$, proving the claim.

□

Proposition 10.2. *Let S be a trait, $\mathcal{X}/S$ a nodal curve, and $\mathcal{Y} \to \mathcal{X}$ the blowing-up of $\mathcal{X}$ at a closed point $p \in \mathcal{X}$. Let $\mathcal{L}$ be a line bundle on $\mathcal{Y}$ such that its restriction to every irreducible component of the exceptional locus of π has degree zero. Then $\pi_*\mathcal{L}$ is a line bundle on $\mathcal{X}$.*

Proof. We first consider the case where S is the spectrum of a strictly henselian discrete valuation ring. In this case, the special fibre of $\mathcal{X} \to S$ has split singularities, hence, as seen in section 8.3, the exceptional fibre E of $\mathcal{Y} \to \mathcal{X}$ consists either of a projective line, or of two projective lines meeting at a k-rational node.

The sheaf $\pi_*\mathcal{L}$ is a coherent $\mathcal{O}_{\mathcal{X}}$-module. Since the curve $\mathcal{X}$ is reduced, to show that $\pi_*\mathcal{L}$ is a line bundle it is enough to check that $\dim_{k(x)} \pi_*\mathcal{L} \otimes_{\mathcal{O}_{\mathcal{X}}} k(x) = 1$ for all $x \in \mathcal{X}$. This clearly holds for $x \in \mathcal{X}$ different from p. We remain with the case $x = p$. Denote by $\mathcal{O}_p$ the local ring of $\mathcal{X}$ at p. Let

$$\mathcal{Z} := \mathcal{Y} \times_{\mathcal{X}} \operatorname{Spec} \mathcal{O}_p$$

so that $\mathcal{Z}$ is the blow-up of $\operatorname{Spec} \mathcal{O}_p$ at its closed point. We write $\mathcal{I}$ for the ideal sheaf $\mathfrak{m}_p\mathcal{O}_{\mathcal{Z}} \subset \mathcal{O}_{\mathcal{Z}}$. For every $n \geq 1$ define

$$\mathcal{Z}_n := \mathcal{Y} \times_{\mathcal{X}} \operatorname{Spec} \mathcal{O}_p/\mathfrak{m}_p^n$$

so we have $\mathcal{O}_{\mathcal{Z}_n} = \mathcal{O}_{\mathcal{Z}}/\mathcal{I}^n$. In particular, $\mathcal{Z}_1$ is the exceptional fibre of the blowing-up $\mathcal{Z} \to \operatorname{Spec} \mathcal{O}_p$, which coincides with the exceptional fibre E of $\pi\colon \mathcal{Y} \to \mathcal{X}$.

The formal function theorem tells us that there is a natural isomorphism

$$\Phi\colon \lim_n (\pi_*\mathcal{L}) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_p/\mathfrak{m}_p^n \to \lim_n H^0(\mathcal{Z}_n, \mathcal{L}_{|\mathcal{Z}_n}).$$

We claim that $\mathcal{L}_{|\mathcal{Z}_n}$ is trivial for all $n \geq 1$. We start with the case $n = 1$: the dual graph of the curve $\mathcal{Z}_1$ is a tree, hence $\operatorname{Pic}(\mathcal{Z}_1)$ is the product of the Picard groups of the components of $\mathcal{Z}_1$ (this can be checked via the Mayer-Vietoris sequence for $\mathcal{O}^\times$, for example). In other words, a line bundle on $\mathcal{Z}_1$ is determined by its restrictions to the components of $\mathcal{Z}_1$. As $\operatorname{Pic}(\mathbb{P}^1_k) = \mathbb{Z}$ via the degree map, we have $\mathcal{L}_{|\mathcal{Z}_1} = \mathcal{O}_{\mathcal{Z}_1}$. Now let $n \geq 1$ and assume that $\mathcal{L}_{|\mathcal{Z}_n}$ is trivial. There is an exact sequence of sheaves of groups on $\mathcal{Z}$

$$0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to (\mathcal{O}_{\mathcal{Z}}/\mathcal{I}^{n+1})^\times \to (\mathcal{O}_{\mathcal{Z}}/\mathcal{I}^n)^\times \to 1$$

with the first map sending α to $1 + \alpha$. The ideal sheaf $\mathcal{I}$ is canonically isomorphic to the invertible sheaf $\mathcal{O}_{\mathcal{Z}}(1)$. Hence $\mathcal{I}^n/\mathcal{I}^{n+1} = \mathcal{O}_{\mathcal{Z}_1}(n)$. Taking the long exact sequence of cohomology we obtain

$$H^1(\mathcal{Z}_1, \mathcal{O}_{\mathcal{Z}_1}(n)) \to H^1(\mathcal{Z}_{n+1}, \mathcal{O}_{\mathcal{Z}_{n+1}}^\times) \to H^1(\mathcal{Z}_n, \mathcal{O}_{\mathcal{Z}_n}^\times) \to 0.$$

We find that the term $H^1(\mathcal{Z}_1, \mathcal{O}_{\mathcal{Z}_1}(n))$ vanishes using Mayer-Vietoris exact sequence and the fact that $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(n)) = 0$. It follows that the restriction map $\mathrm{Pic}(\mathcal{Z}_{n+1}) \to \mathrm{Pic}(\mathcal{Z}_n)$ is an isomorphism. Since the sheaf $\mathcal{L}_{|\mathcal{Z}_{n+1}}$ restricts to the trivial sheaf on $\mathcal{Z}_n$, it is itself trivial, establishing the claim.

We obtain

$$\lim_n H^0(\mathcal{Z}_n, \mathcal{L}_{|\mathcal{Z}_n}) \cong \lim_n H^0(\mathcal{Z}_n, \mathcal{O}_{\mathcal{Z}_n}) \cong \lim_n (\pi_*\mathcal{O}_\mathcal{Y}) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_p/\mathfrak{m}_p^n \cong \widehat{\mathcal{O}}_p$$

the second isomorphism coming again from the formal function theorem applied to $\mathcal{O}_\mathcal{Y}$ and the third coming from lemma 10.1. Finally, we obtain by composition with Φ an isomorphism

$$\lim_n (\pi_*\mathcal{L}) \otimes_\mathcal{O} \mathcal{O}_p/\mathfrak{m}_p^n \to \widehat{\mathcal{O}}_p$$

which induces an isomorphism $\pi_*\mathcal{L} \otimes_\mathcal{O} \mathcal{O}_p/\mathfrak{m}_p \to \mathcal{O}_p/\mathfrak{m}_p = k(p)$, as desired.

Now we drop the assumption of strict henselianity on the base, so let S be the spectrum of a discrete valuation ring. Let S' be the étale local ring of S with respect to some separable closure of the residue field of S. The cartesian diagram

$$\begin{array}{ccc} \mathcal{Y}_{S'} & \xrightarrow{f} & \mathcal{Y} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \mathcal{X}_{S'} & \xrightarrow{g} & \mathcal{X} \end{array}$$

has faithfully flat horizontal arrows, and $\mathcal{Y}_{S'} \to \mathcal{X}_{S'}$ is the blowing-up at $g^{-1}(p)$. Let $\mathcal{L}$ be a line bundle on $\mathcal{Y}$ as in the hypotheses. The restrictions of $f^*\mathcal{L}$ to the irreducible components of the exceptional fibre of π' have degree zero, hence $\pi'_* f^*\mathcal{L}$ is a line bundle. Moreover the canonical map

$$g^*\pi_*\mathcal{L} \to \pi'_* f^*\mathcal{L}$$

is an isomorphism, because g is flat. Hence $g^*\pi_*\mathcal{L}$ is a line bundle, and so is $\pi_*\mathcal{L}$ by faithful flatness of g. □

11 Graph theory

In this section we develop some graph-theoretic results that, together with the results of sections 9 and 10, will be needed to prove theorem 12.3.