A monodromy criterion for existence of Neron models and a result on semi-factoriality

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$\mathcal{X} \setminus \mathcal{Y}_0$ and in particular over the generic fibre. For $i \in \mathbb{Z}_{\geq 1}$ we let $Y_i := (\mathcal{X}_i)_k \cap (\mathcal{X}_i)^{nreg}$ with its reduced structure, and define $\mathcal{X}_{i+1} \to \mathcal{X}_i$ to be the blowing-up at Y_i. We obtain a (possibly infinite) chain of proper birational S-morphisms between nodal curves,

$$(\pi_n \colon \mathcal{X}_n \to \mathcal{X}_{n-1})_{n \in \mathbb{Z}_{\geq 1}}, \quad \mathcal{X}_0 := \mathcal{X} \tag{29}$$

which eventually stabilizes if and only if the generic fibre $\mathcal{X}_K$ is regular.

8.3 The case of split singularities

From the calculations of the lemma 8.1 we deduce how blowing-up alters the special fibre of a nodal curve whose special fibre has split singularities. Let $\mathcal{X} \to S$ be such a curve and let $p \in \mathcal{X}$ be a non-regular point of the special fibre. We have $k(p) = k$. Let $\pi \colon \widetilde{\mathcal{X}} \to \mathcal{X}$ be the blow-up at p, $Y = \operatorname{Spec} \widehat{\mathcal{O}}_p$, and $\widetilde{Y} = Y \times_{\mathcal{X}} \widetilde{\mathcal{X}}$. Then $\pi_Y \colon \widetilde{Y} \to Y$ is the blowing-up at the closed point q of Y. Explicit calculations show that the exceptional fibre $\pi_Y^{-1}(q) = \pi^{-1}(p)$ is a chain of projective lines meeting transversally at nodes defined over k.

We now distinguish all possible cases:

- If $\tau_p = \infty$, so that p is the specialization of a node ζ of $\mathcal{X}_K$, $\pi^{-1}(p)$ is given by two copies of $\mathbb{P}^1_k$ meeting at a k-rational node p' with $\tau_{p'} = \infty$;
- if $\tau_p = 2$, $\pi^{-1}(p)$ consists of one $\mathbb{P}^1_k$;
- finally, if $\tau_p > 2$, then $\pi^{-1}(p)$ consists again of two copies of $\mathbb{P}^1_k$, meeting at a k-rational node p' with $\tau_{p'} = \tau_p - 2$.

In all cases, the intersection points between $\pi^{-1}(p)$ and the closure of its complement in $\widetilde{\mathcal{X}}_k$ are regular in $\widetilde{\mathcal{X}}$, that is, they have thickness 1, and are k-rational. Moreover, $\widetilde{\mathcal{X}} \to S$ has special fibre with split singularities.

9 Extending line bundles to blowing-ups of a nodal curve

Our first aim is to prove that for any line bundle L on the generic fibre $\mathcal{X}_K$, there exists an $n \geq 0$ such that L extends to a line bundle on the surface $\mathcal{X}_n$ of the chain of nodal curves (29). In order to do this, we recall and slightly generalize the definition of Néron's measure for the defect of smoothness presented in [BLR90], Chapter 3.

Definition 9.1. Let R be a discrete valuation ring and $\mathcal{Z}$ an R-scheme of finite type. Let $R \to R'$ be a local flat morphism of discrete valuation rings. Let $a \in \mathcal{Z}(R')$ and denote by $\Omega^1_{\mathcal{Z}/R}$ the $\mathcal{O}_{\mathcal{Z}}$-module of R-differentials. The pullback $a^*\Omega^1_{\mathcal{Z}/R}$ is a finitely-generated R'-module, thus a direct sum of a free and a torsion sub-module. We define *Néron's measure for the defect of smoothness* of $\mathcal{Z}$ along a as

$$\delta(a) := \text{length of the torsion part of } a^*\Omega^1_{\mathcal{Z}/R}$$

Remark 9.2. In [BLR90] 3.3, the measure for the defect of smoothness is defined for points with values in the strict henselization R^{sh} of R (which amounts to considering only local étale morphisms $R \to R'$). We allow more general maps because we will need them in the proof of theorem 9.5.

The following two lemmas generalize two analogous results in [BLR90] 3.3, concerning Néron's measure for the defect of smoothness to the case of points $a \in \mathcal{Z}(R')$ with R' a (possibly ramified) local flat extension of R. In the following lemma, we denote by $\mathcal{Z}^{sm}$ the S-smooth locus of $\mathcal{Z}$.

Lemma 9.3. *Let R be a discrete valuation ring and $\mathcal{Z}$ an R-scheme of finite type. Let $a \in \mathcal{Z}(R')$ for some local flat extension $R \to R'$ of discrete valuation rings. Assume that the restriction to the generic fibre $a_{K'} \colon \operatorname{Spec} K' \to \mathcal{Z}_{K'}$ factors through the smooth locus $\mathcal{Z}^{sm}_{K'}$ of $\mathcal{Z}_{K'}$. Then*

$$\delta(a) = 0 \Leftrightarrow a \in \mathcal{Z}^{sm}(R')$$

Proof. See [BLR90] 3.3/1, for a proof in the case of smooth generic fibre and $R \to R'$ a local étale map of discrete valuation rings. The same proof works for non-smooth generic fibre, as long as a_K factors through $\mathcal{Z}^{sm}$. Now notice that $a^*\Omega_{\mathcal{Z}/R} \cong (a')^*\Omega_{\mathcal{Z}_{R'}/R'}$, where $a' \colon \operatorname{Spec} R' \to \mathcal{Z}_{R'}$ is the section induced by a. We conclude by the fact that the smooth locus of $\mathcal{Z}/R$ is preserved under the faithfully flat base change $\operatorname{Spec} R' \to \operatorname{Spec} R$. □

Proposition 9.4. *Let R be a discrete valuation ring, $\mathcal{Z}/R$ a nodal curve, $f \colon R \to R'$ a finite locally free extension of discrete valuation rings with ramification index $r \in \mathbb{Z}_{\geq 1}$. Suppose $a \in \mathcal{Z}(R')$ is such that the restriction to the generic fibre $a_{K'}$ factors through the smooth locus of $\mathcal{Z}_K$, and that the restriction to the special fibre a_k is contained in the non-regular locus $\mathcal{Z}^{nreg}$. Let $\pi \colon \widetilde{\mathcal{Z}} \to \mathcal{Z}$ be the blowing-up at the closed point $p = a \cap \mathcal{Z}_k$ with its reduced structure and denote by $\widetilde{a} \in \widetilde{\mathcal{Z}}(R')$ the unique lifting of a to $\widetilde{\mathcal{Z}}$. Then, either $\widetilde{a}$ is contained in the regular locus of $\widetilde{\mathcal{Z}}$, or*

$$\delta(\widetilde{a}) \leq \max(\delta(a) - r, 0).$$

Proof. For $R' = R$, proposition 9.4 is a particular case of [BLR90] 3.6/3. The strategy of the proof is to reduce to this case.

Denote by t a uniformizer for R, and by u a uniformizer for R', with $u^r = t$ in R'. Since $\mathcal{Z}(R') = \mathcal{Z}_{R'}(R')$ the section a can be interpreted as a section $b \in \mathcal{Z}_{R'}(R')$. Because $\Omega^1_{\mathcal{Z}_{R'}/R'} \cong \Omega^1_{\mathcal{Z}/R} \otimes_R R'$, we have $\delta(a) = \delta(b)$. The flat map $f\colon R \to R'$ induces a cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{Z}}_{R'} & \longrightarrow & \widetilde{\mathcal{Z}} \\ \downarrow{\scriptstyle \pi_{R'}} & & \downarrow{\scriptstyle \pi} \\ \mathcal{Z}_{R'} & \xrightarrow{g} & \mathcal{Z} \end{array}$$

where $\pi_{R'}\colon \widetilde{\mathcal{Z}}_{R'} \to \mathcal{Z}_{R'}$ is the blowing-up of the preimage $g^{-1}(p)$ of p via $g\colon \mathcal{Z}_{R'} \to \mathcal{Z}$. Then the lifting $\widetilde{a} \in \mathcal{Z}(R')$ factors via the unique lifting of b to $\widetilde{b} \in \widetilde{\mathcal{Z}}_{R'}(R')$. All we need to prove is that $\delta(\widetilde{b}) \le \max\{\delta(b) - r, 0\}$. We may work locally around p, and assume $\mathcal{Z} = \operatorname{Spec} A$ for some R-algebra A, and write $\mathcal{Z}_{R'} = \operatorname{Spec} B$ with $B = A \otimes_R R'$. By restricting $\mathcal{Z}$, we may also assume that p is the only non-smooth point of $\mathcal{Z}$. We let $(t, x_1, \ldots, x_n) \subset A$ be the maximal ideal corresponding to p. The ideal of the closed subscheme $g^{-1}(p) \subset \mathcal{Z}_{R'} = \operatorname{Spec} B$ is then $I = (u^r, x_1, \ldots, x_n) \subset B$, so in particular $g^{-1}(p)$ is a non-reduced point for $r > 1$.

We want to decompose the blowing-up $\pi_{R'}\colon \widetilde{\mathcal{Z}}_{R'} \to \mathcal{Z}_{R'}$ into a chain of r blowing-ups and then apply to each of these the known case described in the beginning. We construct the chain as follows: we first blow up the ideal $I_1 = (u, x_1, \ldots, x_n) \subset B$ and obtain a blowing-up map $\mathcal{Z}_1 \to \mathcal{Z}_{R'}$. The scheme $\mathcal{Z}_1$ is a closed subscheme of $\mathbb{P}^n_B$, whose defining homogeneous ideal is the kernel of the map of graded B-algebras

$$B[u^{(1)}, x_1^{(1)}, \ldots, x_n^{(1)}] \to \oplus_{d \ge 0} I_1^d$$

given by sending $u^{(1)}$ to u and $x_i^{(1)}$ to x_i for all $i = 1, \ldots, n$. The locus $D^+(u^{(1)}) \subset \mathcal{Z}_1$ where $u^{(1)}$ does not vanish is affine, and we denote it by $\mathcal{Y}_1$. We blow up its closed subscheme given by the ideal $(u, x_1^{(1)}/u^{(1)}, x_2^{(1)}/u^{(1)}, \ldots, x_n^{(1)}/u^{(1)})$, and obtain a map

$$\mathcal{Z}_2 \to \mathcal{Y}_1.$$

Next we consider the affine $\mathcal{Y}_2 := D^+(u^{(2)}) \subset \mathcal{Z}_2$ and reiterating the procedure r times, we end up with a chain of morphisms

$$\mathcal{Y}_r \to \mathcal{Y}_{r-1} \to \ldots \to \mathcal{Y}_1 \to \mathcal{Z}_{R'}$$

of affine schemes.

Every blow-up $\mathcal{Y}_{i+1} \to \mathcal{Y}_i$ is the blow-up at a closed point, with reduced structure. Moreover, by the description in section 8.3, we can see that every $\mathcal{Y}_i$ has only one non-regular point p_i in the special fibre; working étale locally one sees that $\mathcal{Y}_{i+1} \to \mathcal{Y}_i$ is exactly the blowing-up at p_i.

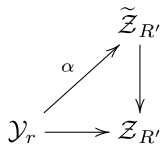
Let's now relate this chain of maps to the blowing-up $\widetilde{\mathcal{Z}}_{R'} \to \mathcal{Z}_{R'}$ given by the ideal $(u^r, x_1, \ldots, x_n)$. Combining the relations

$$\frac{x_i^{(j-1)}}{u^{(j-1)}} u^{(j)} = u x_i^{(j)}$$

for all $j = 1, \ldots, r$ (where we also set $x_i^{(0)} := x_i$ and $u^{(0)} := u$), we obtain in $\mathcal{Y}_r$ the equality

$$x_i = \frac{x_i^{(r)}}{u^{(r)}} u^r$$

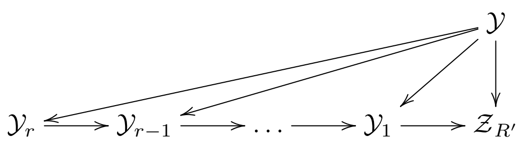
for all $i = 1, \ldots, n$. Hence the ideal sheaf $(u^r, x_1, \ldots, x_n)$ on $\mathcal{Z}_{R'}$ has preimage in $\mathcal{Y}_r$ which is free of rank 1, generated by u^r. By the universal property of blowing-up we obtain a unique map $\alpha\colon \mathcal{Y}_r \to \widetilde{\mathcal{Z}}_{R'}$ such that the diagram



commutes. Next, we focus on the blow-up map $\widetilde{\mathcal{Z}}_{R'} \to \mathcal{Z}_R$. The scheme $\widetilde{\mathcal{Z}}_{R'}$ is a closed subscheme of $\mathbb{P}^n_B$, whose defining homogeneous ideal is the kernel of the map of graded B-algebras

$$B[v, y_1, \ldots, y_n] \to \oplus_{d \geq 0} I^d$$

given by sending v to u^r and y_i to x_i for all $i = 1, \ldots, n$. So we have relations $v x_i = u^r y_i$ for all $i = 1, \ldots, n$. Then the map $\alpha^*\colon \mathcal{O}_{\widetilde{\mathcal{X}}_{R'}} \to \mathcal{O}_{\mathcal{Y}_r}$ sends y_i to $x_i^{(r)}$ and v to $u^{(r)}$. We restrict our attention to the open affine $\mathcal{Y} \subset \widetilde{\mathcal{Z}}_{R'}$ where v does not vanish. Since v is mapped by α^* to $u^{(r)}$, which does not vanish on $\mathcal{Y}_r$, the map α factors as a map $\alpha'\colon \mathcal{Y}_r \to \mathcal{Y}$ followed by the inclusion $\mathcal{Y} \subset \widetilde{\mathcal{Z}}_{R'}$. Now we produce an inverse to α'. One checks that the ideal sheaf $(u, x_1, \ldots, x_n)$ of $\mathcal{Z}_{R'}$ becomes free in $\mathcal{Y}$ (generated by u), hence we obtain a unique map $\mathcal{Y} \to \mathcal{Y}_1$ compatible with the maps to $\mathcal{Z}_{R'}$. Then the argument can be reiterated to produce a commutative diagram



In particular we obtain a map $\beta\colon \mathcal{Y} \to \mathcal{Y}_r$. It is an easy check that the maps α' and β produced between $\mathcal{Y}$ and $\mathcal{Y}_r$ are inverse one to another, hence they give an isomorphism $\mathcal{Y}_r \to \mathcal{Y}$.

If we let b_i be the unique lift to $\mathcal{Y}_i$ of $b_0 := b\colon R' \to \mathcal{Z}_{R'}$, wither b_i is in the regular locus of $\mathcal{Y}_i$, or $\mathcal{Y}_{i+1} \to \mathcal{Y}_i$ is the blowing-up at $b_i \cap (Y_i)_k$, in which case we obtain, by [BLR90] 3.3/5, that $\delta(b_{i+1}) \leq \max\{\delta(b_i) - 1, 0\}$. Now, if for some $1 \leq i \leq r$ the section b_i is contained in the regular locus of $\mathcal{Y}_i$, then also $\widetilde{b}$ is contained in the regular locus of $\mathcal{Y}$. Otherwise, $\delta(\widetilde{b}) \leq \max\{\delta(b) - r, 0\}$ as desired.

□

We now have the tools to prove our main result on extending line bundles to blowing-ups in the chain of morphisms (29).

Theorem 9.5. *Let S be a trait, with perfect fraction field K, $\mathcal{X}/S$ a nodal curve. Let L be a line bundle on $\mathcal{X}_K$. Let $(\pi_i\colon \mathcal{X}_i \to \mathcal{X}_{i-1})_i$ be the chain* (29) *of blow-ups. Then there exists $N \geq 0$ for which L extends to a line bundle $\mathcal{L}$ on $\mathcal{X}_N$.*

Proof. Let L be an invertible sheaf on $\mathcal{X}_K$, and D be a Cartier divisor with $\mathcal{O}_{\mathcal{X}_K}(D) \cong L$. We may take D to be supported on the smooth locus of $\mathcal{X}_K$ ([Sha13], Theorem 1.3.1) and see it as a Weil divisor. We may also assume that D is effective, since any Weil divisor is the difference of two effective Weil divisors.

The closed subscheme D_{red} given by the support of D with its reduced structure is a disjoint union of finitely many closed points of the smooth locus of $\mathcal{X}_K$. We write

$$D_{red} = \bigcup_{i=1}^{s} P_i$$

where $P_i \in \mathcal{X}_K^{sm}(K_i)$ for finite (separable) extensions $K \hookrightarrow K_i$, $i = 1, \ldots, s$. For each $i = 1, \ldots, s$, we let R_i be the localization at some prime of the integral closure of R in K_i, so that each R_i is a discrete valuation ring with fraction field K_i, and $R \to R_i$ is finite locally free. The curve $\mathcal{X}/R$ being proper, each P_i extends to $Q_i \in \mathcal{X}(R_i)$. Write $\mathcal{X}^{nsm}$ for the non-smooth locus of $\mathcal{X}/R$ and $\mathcal{X}^{nreg}$ for the non-regular locus of $\mathcal{X}$. Notice that $\delta(Q_i) > 0$ if and only if $Q_i \cap \mathcal{X}_k \in \mathcal{X}^{nsm}$, by lemma 9.3. Assume that the point $Q_i \cap \mathcal{X}_k$ lies in $\mathcal{X}^{nreg} \subset \mathcal{X}^{nsm}$. In this case, it is one of the closed points that are the center of the blowing-up $\mathcal{X}_1 \to \mathcal{X}$. By proposition 9.4, the unique lifting Q_i' of Q_i to $\mathcal{X}_1$ either is contained in the regular locus of $\mathcal{X}_1$, or it satisfies $\delta(Q_i') \leq \max(0, \delta(Q_i) - r_i)$, where $r_i \geq 1$ is the ramification index of $R \to R_i$. Applying repeatedly proposition 9.4, we see that there is $N > 0$ such that each of the points $P_i \in \mathcal{X}_K$ extends to $Q_i^{(N)} \in \mathcal{X}_N^{reg}(R_i)$. Therefore the Weil divisor D extends to a Weil divisor $\widetilde{D}$ on $\mathcal{X}_N$ that is supported on the union of the $Q_i^{(N)}$, hence on the regular locus of $\mathcal{X}_N$. This implies that $\widetilde{D}$ is a Cartier divisor, and the line bundle $\mathcal{O}_{\mathcal{X}_n}(\widetilde{D})$ restricts to $\mathcal{O}_{\mathcal{X}_K}(D) \cong L$ on $\mathcal{X}_K$. This completes the proof.

□

10 Descent of line bundles along blowing-ups

Lemma 10.1. *Let S be a trait and $\pi\colon \mathcal{Y} \to \mathcal{X}$ a proper morphism of flat S-schemes, which restricts to an isomorphism over the generic point of S. Assume that the special fibre $\mathcal{X}_k$ is reduced. Then $\pi_*\mathcal{O}_{\mathcal{Y}} \cong \mathcal{O}_{\mathcal{X}}$.*

Proof. Consider an affine open $W \subset \mathcal{X}$. The morphism $\mathcal{O}_{\mathcal{X}}(W) \to \pi_*\mathcal{O}_{\mathcal{Y}}(W)$ is integral ([Liu02], Prop.3.3.18). Denoting by t a uniformizer of $\Gamma(S, \mathcal{O}_S)$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X}}(W) & \longrightarrow & \pi_*\mathcal{O}_{\mathcal{Y}}(W) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{X}}(W)[t^{-1}] & \xrightarrow{\cong} & (\pi_*\mathcal{O}_{\mathcal{Y}}(W))[t^{-1}] \end{array}$$

The two vertical arrows are injective because $\mathcal{X}$ and $\mathcal{Y}$ are S-flat; the lower arrow is an isomorphism because π is generically an isomorphism and $(\pi_*\mathcal{O}_{\mathcal{Y}}(W))[t^{-1}] = \pi_*(\mathcal{O}_{\mathcal{Y}}(W)[t^{-1}])$. It follows that the upper arrow is injective. We claim that $\mathcal{O}_{\mathcal{X}}(W)$ is integrally closed in $\mathcal{O}_{\mathcal{X}}(W)[t^{-1}]$, so that the upper arrow is an isomorphism, which proves the lemma. Take then $g \in \mathcal{O}_{\mathcal{X}}(W)[t^{-1}]$ satisfying a monic polynomial equation $g^m + a_1 g^{m-1} + \ldots + a_m = 0$ with coefficients in $\mathcal{O}_{\mathcal{X}}(W)$ and write $g = f/t^n$ with $f \in \mathcal{O}_{\mathcal{X}}(W)$ and $n \geq 0$ minimal. We want to show that n is zero. We have

$$\frac{f^m}{t^{nm}} + a_1 \frac{f^{m-1}}{t^{n(m-1)}} + \ldots + a_m = 0.$$

Suppose by contradiction $n \geq 1$. Upon multiplying by t^{nm} the above relation, we find that $f^m \in t\mathcal{O}_{\mathcal{X}}(W)$. Because the special fibre of $\mathcal{X}$ is reduced, the ring $\mathcal{O}_{\mathcal{X}}(W)/t\mathcal{O}_X(W)$ is reduced, hence $f \in t\mathcal{O}_{\mathcal{X}}(W)$. This violates the hypothesis of minimality of n and we have a contradiction. Hence $n = 0$ and $g \in \mathcal{O}_{\mathcal{X}}(W)$, proving the claim.

□