A monodromy criterion for existence of Neron models and a result on semi-factoriality

Orecchia, G.

Author: Orecchia, G.
Title: A monodromy criterion for existence of Neron models and a result on semi-factoriality
Issue Date: 2018-02-27

Remark 7.4. If the hypothesis that the special fibre has split singularities is dropped, the same result holds after replacing R and $\mathcal{O}_{\mathcal{X},x}$ by their strict henselizations.

Lemma 7.5. *Let X be a nodal curve over a field k, $x \in X$ a split ordinary double point such that at least two irreducible components of X pass through x. Then x belongs to exactly two irreducible components Z_1, Z_2 which are smooth at x and meet transversally.*

In view of lemma 7.5, if X/k is a nodal curve with split singularities, the *dual graph* G of X can be defined. The vertices of G correspond to the irreducible components of X, while every edge e between vertices v, w corresponds to a an ordinary double point contained in the components corresponding to v and w.

7.2 Semi-factoriality

Definition 7.6 ([Pép13] 1.1.)**.** Let $\mathcal{X} \to S$ be a scheme over a trait. We say that $\mathcal{X}$ is *semi-factorial* over S if the restriction map

$$\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_K)$$

is surjective.

8 Blowing-up nodal curves

Let $f\colon \mathcal{X} \to S$ be a nodal curve over a trait. In this section we study the effects of blowing-up non-regular points of $\mathcal{X}$ lying on the special fibre of $\mathcal{X} \to S$.

8.1 Blowing-up a closed non-regular point

Lemma 8.1. *Let $\mathcal{X} \to S$ be a nodal curve over a trait. Let x be a non-regular point lying on the special fibre of $\mathcal{X} \to S$. The blowing-up $\pi\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ centered at (the reduced closed subscheme given by) x gives by composition a nodal curve $\widetilde{\mathcal{X}} \to S$.*

Proof. The map $\pi\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ is proper, hence so is the composition $\widetilde{\mathcal{X}} \to S$. Let $\overline{x}$ be a geometric point of $\mathcal{X}$ lying over x. We write $A := \widehat{R}^{sh}$ for the completion at its maximal ideal of the strict henselization of R induced by $\overline{x}$. Similarly we let $B := \widehat{\mathcal{O}}^{ét}_{\mathcal{X},\overline{x}}$ be the completion of the étale local ring of $\mathcal{X}$ at

$\overline{x}$. We have $B \cong A[[u,v]]/uv - c$ for some $c \in A$; we will assume that $c = 0$, as the reader can refer to [Liu02], Example 8.3.53 for the case $c \neq 0$.

The blowing-up $\mathcal{Z} \to \operatorname{Spec} B$ at the maximal ideal $\mathfrak{m} = (t, u, v) \subset B$ fits in a cartesian diagram

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \widetilde{\mathcal{X}} \\ \big\downarrow & & \big\downarrow{\scriptstyle \pi} \\ \operatorname{Spec} B & \longrightarrow & \mathcal{X} \end{array}$$

with flat horizontal maps and is given by

$$\mathcal{Z} = \operatorname{Proj} \frac{B[S,U,V]}{I}$$

where I is the homogenous ideal

$$I = (uS - tU, vS - tV, uV, vU, UV).$$

The scheme $\mathcal{Z}$ is covered by three affine patches, given respectively by the loci where S, U, V are invertible. Namely we have:

$$D^+(S) \cong \operatorname{Spec} \frac{A[U,V]}{UV},\ D^+(U) \cong \operatorname{Spec} \frac{A[[u]][S]}{t - uS},\ D^+(V) \cong \operatorname{Spec} \frac{A[[v]][S]}{t - vS}.$$

To see that $\widetilde{\mathcal{X}}$ is S-flat, we check that the image of the uniformizer $t \in R$ is torsion-free in $\mathcal{O}_{\widetilde{\mathcal{X}}}$, which is immediate upon inspection of the coordinate rings of $D^+(S), D^+(U), D^+(V)$. Also, for all field valued points $y\colon \operatorname{Spec} L \to \operatorname{Spec} A$ lying over the closed point of $\operatorname{Spec} A$, the completed local rings at the singular points of $\mathcal{Z}_y$ are of the form $L[[x,y]]/xy$, as desired.

□

8.2 An infinite chain of blowing-ups

Write now $\mathcal{X}^{nreg}$ for the non-regular locus of $\mathcal{X}$. By the very definition of nodal curve, the locus $\mathcal{X}^{nreg}$ is a closed subset of $\mathcal{X}$, and in particular its intersection with the special fibre $\mathcal{X}_k \cap \mathcal{X}^{nreg}$ is a finite union of closed points. We inductively construct a chain of proper birational maps of nodal curves as follows.

Construction 8.2. Let Y_0 be the closed subscheme given by $\mathcal{X}_k \cap \mathcal{X}^{nreg}$ with its reduced structure. Blowing-up $\mathcal{Y}_0$ in $\mathcal{X}$ we obtain a proper birational morphism $\pi_1\colon \mathcal{X}_1 \to \mathcal{X}$, which restricts to an isomorphism on the dense open

$\mathcal{X} \setminus \mathcal{Y}_0$ and in particular over the generic fibre. For $i \in \mathbb{Z}_{\geq 1}$ we let $Y_i := (\mathcal{X}_i)_k \cap (\mathcal{X}_i)^{nreg}$ with its reduced structure, and define $\mathcal{X}_{i+1} \to \mathcal{X}_i$ to be the blowing-up at Y_i. We obtain a (possibly infinite) chain of proper birational S-morphisms between nodal curves,

$$(\pi_n \colon \mathcal{X}_n \to \mathcal{X}_{n-1})_{n \in \mathbb{Z}_{\geq 1}}, \quad \mathcal{X}_0 := \mathcal{X} \tag{29}$$

which eventually stabilizes if and only if the generic fibre $\mathcal{X}_K$ is regular.

8.3 The case of split singularities

From the calculations of the lemma 8.1 we deduce how blowing-up alters the special fibre of a nodal curve whose special fibre has split singularities. Let $\mathcal{X} \to S$ be such a curve and let $p \in \mathcal{X}$ be a non-regular point of the special fibre. We have $k(p) = k$. Let $\pi \colon \widetilde{\mathcal{X}} \to \mathcal{X}$ be the blow-up at p, $Y = \operatorname{Spec} \widehat{\mathcal{O}}_p$, and $\widetilde{Y} = Y \times_{\mathcal{X}} \widetilde{\mathcal{X}}$. Then $\pi_Y \colon \widetilde{Y} \to Y$ is the blowing-up at the closed point q of Y. Explicit calculations show that the exceptional fibre $\pi_Y^{-1}(q) = \pi^{-1}(p)$ is a chain of projective lines meeting transversally at nodes defined over k.

We now distinguish all possible cases:

- If $\tau_p = \infty$, so that p is the specialization of a node ζ of $\mathcal{X}_K$, $\pi^{-1}(p)$ is given by two copies of $\mathbb{P}^1_k$ meeting at a k-rational node p' with $\tau_{p'} = \infty$;
- if $\tau_p = 2$, $\pi^{-1}(p)$ consists of one $\mathbb{P}^1_k$;
- finally, if $\tau_p > 2$, then $\pi^{-1}(p)$ consists again of two copies of $\mathbb{P}^1_k$, meeting at a k-rational node p' with $\tau_{p'} = \tau_p - 2$.

In all cases, the intersection points between $\pi^{-1}(p)$ and the closure of its complement in $\widetilde{\mathcal{X}}_k$ are regular in $\widetilde{\mathcal{X}}$, that is, they have thickness 1, and are k-rational. Moreover, $\widetilde{\mathcal{X}} \to S$ has special fibre with split singularities.

9 Extending line bundles to blowing-ups of a nodal curve

Our first aim is to prove that for any line bundle L on the generic fibre $\mathcal{X}_K$, there exists an $n \geq 0$ such that L extends to a line bundle on the surface $\mathcal{X}_n$ of the chain of nodal curves (29). In order to do this, we recall and slightly generalize the definition of Néron's measure for the defect of smoothness presented in [BLR90], Chapter 3.