A monodromy criterion for existence of Neron models and a result on semi-factoriality

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7 Preliminaries

7.1 Nodal curves

Definition 7.1. A *curve* X over an algebraically closed field k is a proper morphism of schemes $X \to \operatorname{Spec} k$, such that X is connected and whose irreducible components have dimension 1. A curve X/k is called *nodal* if for every non-smooth point $x \in X$ there is an isomorphism of k-algebras $\widehat{\mathcal{O}}_{\mathcal{X},x} \to k[[x,y]]/xy$.

For a general base scheme S, a *nodal curve* $f\colon \mathcal{X} \to S$ is a proper, flat morphism of finite presentation, such that for each geometric point $\overline{s}$ of S the fibre $\mathcal{X}_{\overline{s}}$ is a nodal curve.

We are interested in the case where the base scheme S is a trait, that is, the spectrum of a discrete valuation ring. In what follows, whenever we have a trait S, unless otherwise specified we will denote by K the fraction field of $\Gamma(S, \mathcal{O}_S)$ and by k its residue field.

Definition 7.2. Let $X \to \operatorname{Spec} k$ be a nodal curve over a field and $n\colon X' \to X$ be the normalization morphism. A non-regular point $x \in X$ is a *split ordinary double point* if the points of $n^{-1}(x)$ are k-rational (in particular, x is k-rational). We say that $X \to \operatorname{Spec} k$ has *split singularities* if all non-regular points $x \in X$ are split ordinary double points.

It is clear that the base change of a curve with split singularities still has split singularities. Also, it follows from [Liu02], Corollary 10.3.22 that for any nodal curve $\mathcal{X} \to S$ over a trait there exists an étale base change of traits $S' \to S$ such that $\mathcal{X} \times_S S' \to S'$ has split singularities.

The following two lemmas are Corollary 10.3.22 b) and Lemma 10.3.11 of [Liu02]:

Lemma 7.3. *Let $f\colon \mathcal{X} \to S$ be a nodal curve over a trait and let $x \in \mathcal{X}$ be a split ordinary double point lying over the closed point $s \in S$. Write R for $\Gamma(S, \mathcal{O}_S)$ and $\mathfrak{m}$ for its maximal ideal. Then*

$$\widehat{\mathcal{O}}_{\mathcal{X},x} \cong \frac{\widehat{R}[[x,y]]}{xy - c}$$

for some $c \in \mathfrak{m}R$. The ideal generated by c does not depend on the choice of c.

We define an integer $\tau_x \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, given by the valuation of c if $c \neq 0$ and by ∞ if $c = 0$. We call τ_x the *thickness* of x. The point x is non-regular if and only if $\tau_x \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$; moreover, $\tau_x = \infty$ if and only if x is the specialization of a node of the generic fibre $\mathcal{X}_K$.

Remark 7.4. If the hypothesis that the special fibre has split singularities is dropped, the same result holds after replacing R and $\mathcal{O}_{\mathcal{X},x}$ by their strict henselizations.

Lemma 7.5. *Let X be a nodal curve over a field k, $x \in X$ a split ordinary double point such that at least two irreducible components of X pass through x. Then x belongs to exactly two irreducible components Z_1, Z_2 which are smooth at x and meet transversally.*

In view of lemma 7.5, if X/k is a nodal curve with split singularities, the *dual graph* G of X can be defined. The vertices of G correspond to the irreducible components of X, while every edge e between vertices v, w corresponds to a an ordinary double point contained in the components corresponding to v and w.

7.2 Semi-factoriality

Definition 7.6 ([Pép13] 1.1.)**.** Let $\mathcal{X} \to S$ be a scheme over a trait. We say that $\mathcal{X}$ is *semi-factorial* over S if the restriction map

$$\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_K)$$

is surjective.

8 Blowing-up nodal curves

Let $f\colon \mathcal{X} \to S$ be a nodal curve over a trait. In this section we study the effects of blowing-up non-regular points of $\mathcal{X}$ lying on the special fibre of $\mathcal{X} \to S$.

8.1 Blowing-up a closed non-regular point

Lemma 8.1. *Let $\mathcal{X} \to S$ be a nodal curve over a trait. Let x be a non-regular point lying on the special fibre of $\mathcal{X} \to S$. The blowing-up $\pi\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ centered at (the reduced closed subscheme given by) x gives by composition a nodal curve $\widetilde{\mathcal{X}} \to S$.*

Proof. The map $\pi\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ is proper, hence so is the composition $\widetilde{\mathcal{X}} \to S$. Let $\overline{x}$ be a geometric point of $\mathcal{X}$ lying over x. We write $A := \widehat{R}^{sh}$ for the completion at its maximal ideal of the strict henselization of R induced by $\overline{x}$. Similarly we let $B := \widehat{\mathcal{O}}^{\acute{e}t}_{\mathcal{X},\overline{x}}$ be the completion of the étale local ring of $\mathcal{X}$ at