A monodromy criterion for existence of Neron models and a result on semi-factoriality

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Part II

Semi-factorial nodal curves and Néron lft-models

6 Introduction

Let S be the spectrum of a discrete valuation ring with fraction field K, and let $\mathcal{X} \to S$ be a scheme over S. Following [Pép13], we say that $\mathcal{X} \to S$ is *semi-factorial* if the restriction map

$$\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_K)$$

is surjective; namely, if every line bundle on the generic fibre $\mathcal{X}_K$ can be extended to a line bundle on $\mathcal{X}$.

We consider the case of a relative curve $\mathcal{X} \to S$. In [Pép13], Theorem 8.1, Pépin proved that given a geometrically reduced curve $\mathcal{X}_K/K$ with ordinary singularities and a proper flat model $\mathcal{X} \to S$, a semi-factorial flat model $\mathcal{X}' \to S$ can be obtained after a blowing-up $\mathcal{X}' \to \mathcal{X}$ with center in the special fibre.

The main result of this part is a necessary and sufficient condition for semi-factoriality in the case where $\mathcal{X} \to S$ is a proper, flat family of nodal curves, whose special fibre has split nodes. It turns out that in this case semi-factoriality is equivalent to a certain combinatorial condition involving the dual graph of the special fibre of $\mathcal{X}/S$ and a labelling of its edges, which we describe now. Let $t \in \Gamma(S, \mathcal{O}_S)$ be a uniformizer; every node of the special fibre is étale locally described by an equation of the form

a) $xy - t^n = 0$ for some $n \geq 1$, or

b) $xy = 0$ (if the node persists in the generic fibre).

Consider the dual graph $\Gamma = (V, E)$ associated to the special fibre of $\mathcal{X}/S$. We label its edges by the function $l\colon E \to \mathbb{Z}_{\geq 1} \cup \{\infty\}$

$$l(e) = \begin{cases} n & \text{if the node corresponding to } e \text{ is as in case a);} \\ \infty & \text{if the node corresponding to } e \text{ is as in case b).} \end{cases}$$

We say that the labelled graph (Γ, l) is *circuit-coprime* if, after contracting all edges with label ∞, every circuit of the graph has labels with greatest common divisor equal to 1. In particular, if Γ is a tree, (Γ, l) is automatically circuit-coprime.

The following theorem is our main result:

Theorem 6.1 (theorem 12.3)**.** *If the labelled graph* (Γ, l) *is circuit-coprime, the curve* $\mathcal{X} \to S$ *is semi-factorial. If moreover* $\Gamma(S, \mathcal{O}_S)$ *is strictly henselian, the converse holds as well.*

The proof (of the first statement) can be subdivided in three parts:

- we start by constructing a chain of proper birational morphisms of nodal curves over S

$$\ldots \to \mathcal{X}_n \to \mathcal{X}_{n-1} \to \ldots \to \mathcal{X}_1 \to \mathcal{X}_0 := \mathcal{X}$$

  where every arrow is the blowing-up at the reduced closed subscheme of non-regular closed points. A generalization (proposition 9.4) of the smoothening techniques developed in [BLR90], Chapter 3, allows us to show that given a line bundle L on $\mathcal{X}_K$ there exists a positive integer n such that L extends to a line bundle $\mathcal{L}$ on $\mathcal{X}_n$ (theorem 9.5).

- in the combinatorial heart of the proof, we provide a dictionary between geometry and graph theory to reduce the study of the blowing-ups $\mathcal{X}_n$ and line bundles on them to the study of their dual labelled graphs and integer labellings of their edges. We show that if the labelled graph (Γ, l) of $\mathcal{X}/S$ is circuit-coprime, there exists a generically trivial line bundle $\mathcal{E}$ on $\mathcal{X}_n$ such that $\mathcal{L} \otimes \mathcal{E}$ has degree 0 on each irreducible component of the exceptional fibre of $\pi_n \colon \mathcal{X}_n \to \mathcal{X}$.

- Finally, we show (proposition 10.2) that the direct image $\pi_{n*}(\mathcal{L} \otimes E)$ is a line bundle on $\mathcal{X}$ (which in particular extends L). This relies essentially on the fact that the exceptional fibre of π_n is a curve of genus zero.

As a corollary to the theorem, we refine Theorem 8.1 of [Pép13] in the case of nodal curves $\mathcal{X}/S$ with special fibre having split nodes, by explicitly describing a blowing-up with center in the special fibre that yields a semi-factorial model:

Corollary 6.2 (corollary 12.5)**.** *Let* $\mathcal{X}_1 \to \mathcal{X}$ *be the blowing-up centered at the reduced closed subscheme consisting of non-regular closed points of* $\mathcal{X}$*. Then the curve* $\mathcal{X}_1 \to S$ *is semi-factorial.*

This follows immediately, observing that $\mathcal{X}_1$ has circuit-coprime labelled graph.

Semi-factoriality is closely connected to Néron models of jacobians of curves. A famous construction of Raynaud ([Ray70a]) shows that if $\mathcal{X} \to S$ has regular total space, a Néron model over S for the jacobian $\operatorname{Pic}^0_{\mathcal{X}_K/K}$ is given by the S-group scheme $\operatorname{Pic}^{[0]}_{\mathcal{X}/S} / \operatorname{cl}(e)$, where $\operatorname{Pic}^{[0]}_{\mathcal{X}/S}$ represents line bundles of total degree zero on $\mathcal{X}$, and $\operatorname{cl}(e)$ is the schematic closure of the unit section $e\colon K \to \operatorname{Pic}^0_{\mathcal{X}_K/K}$. In [Pép13], Theorem 9.3., it is shown that the same construction works in the case of semi-factorial curves $\mathcal{X} \to S$ with smooth generic fibre. Our second main theorem is a corollary of theorem 6.1:

Theorem 6.3 (theorem 13.6)**.** *Let $\mathcal{X} \to S$ be a nodal curve over the spectrum of a discrete valuation ring. Then $\operatorname{Pic}_{\mathcal{X}/S} / \operatorname{cl}(e)$ is a Néron lft-model over S for $\operatorname{Pic}_{\mathcal{X}_K/K}$ if and only if the labelled graph (Γ, l) is circuit-coprime.*

Note that there are no smoothness assumptions on the generic fibre. The abbreviation "lft" stands for "locally of finite type", meaning that we do not require the model to be quasi-compact (even if we chose to impose degree restrictions on $\operatorname{Pic}_{\mathcal{X}_K/K}$, the resulting Néron lft-model may not be quasi-compact in general, as $\mathcal{X}_K/K$ may not be smooth).

6.1 Outline

In section 7 we introduce the basic definitions, including that of nodal curve with split singularities. In section 8 we define an infinite chain of blow-ups of a given nodal curve $\mathcal{X}/S$ and then show that every line bundle on the generic fibre $\mathcal{X}_K/K$ extends to a line bundle on one of these blow-ups (section 9). Section 10 contains an important technical lemma on descent of line bundles along blowing-ups. Section 11 is entirely graph-theoretic and contains the definition of circuit-coprime labelled graphs. The combinatorial results established in this section are then reinterpreted in section 12 in geometric terms in order to give a necessary and sufficient condition for semi-factoriality of nodal curves. In section 13, starting from a nodal curve $\mathcal{X}/S$, we construct a Néron model of the Picard scheme of its generic fibre.

7 Preliminaries

7.1 Nodal curves

Definition 7.1. A *curve* X over an algebraically closed field k is a proper morphism of schemes $X \to \operatorname{Spec} k$, such that X is connected and whose irreducible components have dimension 1. A curve X/k is called *nodal* if for every non-smooth point $x \in X$ there is an isomorphism of k-algebras $\widehat{\mathcal{O}}_{\mathcal{X},x} \to k[[x,y]]/xy$.

For a general base scheme S, a *nodal curve* $f \colon \mathcal{X} \to S$ is a proper, flat morphism of finite presentation, such that for each geometric point $\overline{s}$ of S the fibre $\mathcal{X}_{\overline{s}}$ is a nodal curve.

We are interested in the case where the base scheme S is a trait, that is, the spectrum of a discrete valuation ring. In what follows, whenever we have a trait S, unless otherwise specified we will denote by K the fraction field of $\Gamma(S, \mathcal{O}_S)$ and by k its residue field.

Definition 7.2. Let $X \to \operatorname{Spec} k$ be a nodal curve over a field and $n \colon X' \to X$ be the normalization morphism. A non-regular point $x \in X$ is a *split ordinary double point* if the points of $n^{-1}(x)$ are k-rational (in particular, x is k-rational). We say that $X \to \operatorname{Spec} k$ has *split singularities* if all non-regular points $x \in X$ are split ordinary double points.

It is clear that the base change of a curve with split singularities still has split singularities. Also, it follows from [Liu02], Corollary 10.3.22 that for any nodal curve $\mathcal{X} \to S$ over a trait there exists an étale base change of traits $S' \to S$ such that $\mathcal{X} \times_S S' \to S'$ has split singularities.

The following two lemmas are Corollary 10.3.22 b) and Lemma 10.3.11 of [Liu02]:

Lemma 7.3. *Let $f \colon \mathcal{X} \to S$ be a nodal curve over a trait and let $x \in \mathcal{X}$ be a split ordinary double point lying over the closed point $s \in S$. Write R for $\Gamma(S, \mathcal{O}_S)$ and $\mathfrak{m}$ for its maximal ideal. Then*

$$\widehat{\mathcal{O}}_{\mathcal{X},x} \cong \frac{\widehat{R}[[x,y]]}{xy - c}$$

for some $c \in \mathfrak{m}R$. The ideal generated by c does not depend on the choice of c.

We define an integer $\tau_x \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, given by the valuation of c if $c \neq 0$ and by ∞ if $c = 0$. We call τ_x the *thickness* of x. The point x is non-regular if and only if $\tau_x \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$; moreover, $\tau_x = \infty$ if and only if x is the specialization of a node of the generic fibre $\mathcal{X}_K$.