A monodromy criterion for existence of Neron models and a result on semi-factoriality

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5 Néron models of abelian schemes in characteristic zero

In this section, we consider a connected, locally noetherian, regular base scheme S, a normal crossing divisor D on S, an abelian scheme A/U of relative dimension d and a semi-abelian scheme $\mathcal{A}/S$ with a given isomorphism $\mathcal{A} \times_S U \to A$. We will retain the notation used in the previous sections.

5.1 Test-Néron models

Definition 5.1. Let $\mathcal{N}/S$ be a smooth, separated group algebraic space of finite type with an isomorphism $\mathcal{N} \times_S U \to A$; we say that it is a *test-Néron model* for A over S if, for every strictly henselian trait Z and morphism $Z \to S$ transversal to D (definition 2.4), the pullback $\mathcal{N} \times_S Z$ is the Néron model of its generic fibre.

It is clear that the property of being a test-Néron model is smooth-local on the base, and is also preserved by taking the localization at a point of the base, or the strict henselization at a geometric point.

We will start by working on a strictly local base. Recall that in this case, for a prime l different from the residue characteristic p at the closed point, the Tate module $T_l A(K^s)$ is acted on by $G = \bigoplus_{i=1}^n I_i = \bigoplus_{i=1}^n \widehat{\mathbb{Z}}'(1)$, the tame fundamental group of U.

Lemma 5.2. *For any subset $\mathcal{E} \subseteq \{1, \ldots, n\}$ and any $m \in \mathbb{Z}$, there is a canonical injective group homomorphism*

$$\varphi_{\mathcal{E}} \colon \frac{A[m](K^s)^{\oplus_{i\in\mathcal{E}} I_i}}{T_l A(K^s)^{\oplus_{i\in\mathcal{E}} I_i} \otimes \mathbb{Z}/m\mathbb{Z}} \to \bigoplus_{i\in\mathcal{E}} \frac{A[m](K^s)^{I_i}}{T_l A(K^s)^{I_i} \otimes \mathbb{Z}/m\mathbb{Z}}. \tag{25}$$

If $\mathcal{A}/S$ is toric-additive, for any $\mathcal{E} \subseteq \{1, \ldots, n\}$ the homomorphism $\varphi_{\mathcal{E}}$ is an isomorphism.

Remark 5.3. Recall the characterization of the group of components of Néron models in section 2.4. If S_i is a strict henselization at the generic point ζ_i of D_i, then there exists a Néron model $\mathcal{N}_i/S_i$ for $A \times_S S_i$. The i-th summand of the right-hand side of (5.2) is the group of components of $\mathcal{N}_i$ over the closed point of S_i. On the other hand, if ζ is the generic point of $\cap_{i\in\mathcal{E}} D_i$, and if A_K/K admits a Néron model over a strict henselization $\mathcal{O}^{sh}_{S,\overline{\zeta}}$, then the left hand side is its group of components over the closed point.

Proof. First, it follows easily from lemma 3.3 that $\left(T_l A(K^s)^{\bigoplus_{i\in\mathcal{E}} I_i}\right)\otimes \mathbb{Z}/m\mathbb{Z} = \bigcap_{i\in\mathcal{E}}\left(T_l A(K^s)^{I_i}\otimes\mathbb{Z}/m\mathbb{Z}\right)$. Given this, it is evident that the group homomorphism 25 is injective.

Let us assume that $\mathcal{A}/S$ is toric-additive. Then we have a decomposition of $T := T_l A(K^S)$ into a direct sum $V_1 \oplus \ldots \oplus V_n$ as in theorem 3.4. For a $\mathbb{Z}_l$-module M, we will write $M_{(m)}$ for $M \otimes_{\mathbb{Z}_l} \mathbb{Z}/m\mathbb{Z}$.

Now, if $\mathcal{E}$ is empty the statement of the lemma is obviously satisfied; otherwise, we can rename the components D_i, so that $\mathcal{E} = \{1, 2, \ldots, r\} \subseteq \{1, \ldots, n\}$ for some $1 \le r \le n$.

The left-hand side of eq. (25) is

$$\frac{(V_{1,(m)} \oplus \ldots \oplus V_{n,(m)})^{\bigoplus_{i\in\mathcal{E}} I_i}}{(V_1 \oplus \ldots \oplus V_n)^{\bigoplus_{i\in\mathcal{E}} I_i} \otimes \mathbb{Z}/m\mathbb{Z}} = \\ = \frac{(V_{1,(m)})^{I_1} \oplus \ldots \oplus (V_{r,(m)})^{I_r} \oplus V_{r+1,(m)} \oplus \ldots \oplus V_{n,(m)}}{(V_1)^{I_1}_{(m)} \oplus \ldots \oplus (V_r)^{I_r}_{(m)} \oplus V_{r+1,(m)} \oplus \ldots \oplus V_{n,(m)}} = \\ = \frac{(V_{1,(m)})^{I_1}}{(V_1)^{I_1}_{(m)}} \oplus \ldots \oplus \frac{(V_{r,(m)})^{I_r}}{(V_r)^{I_r}_{(m)}}.$$

The right hand side is

$$\bigoplus_{i=1}^{r} \frac{V_{1,(m)} \oplus \ldots \oplus (V_{i,(m)})^{I_i} \oplus \ldots \oplus V_{n,(m)}}{V_{1,(m)} \oplus \ldots \oplus (V_i)^{I_i}_{(m)} \oplus \ldots \oplus V_{n,(m)}} = \frac{(V_{1,(m)})^{I_1}}{(V_1)^{I_1}_{(m)}} \oplus \ldots \oplus \frac{(V_{r,(m)})^{I_r}}{(V_r)^{I_r}_{(m)}}.$$

So we have obtained the same expression on both sides, and $\varphi_{\mathcal{E}}$ induces the identity between them. □

Next, we make a choice of a compatible system of primitive roots of units; equivalently, we choose a topological generator for $\hat{\mathbb{Z}}'(1)$. This gives us, for each $i = 1, \ldots, n$, a topological generator e_i of I_i.

Lemma 5.4. *Assume that A is toric-additive. Then, for any subset $\mathcal{E} \subseteq \{1, \ldots, n\}$ and any $m \in \mathbb{Z}$, we have*

$$\frac{A[m](K^s)^{\oplus_{i\in\mathcal{E}} I_i}}{T_l A(K^s)^{\oplus_{i\in\mathcal{E}} I_i} \otimes \mathbb{Z}/m\mathbb{Z}} = \frac{A[m](K^s)^{\sum_{i\in\mathcal{E}} e_i}}{T_l A(K^s)^{\sum_{i\in\mathcal{E}} e_i} \otimes \mathbb{Z}/m\mathbb{Z}}$$

Proof. We have a decomposition

$$T_l A(K^s) = V_1 \oplus V_2 \oplus \ldots \oplus V_n$$

as in theorem 3.4. Again, for a $\mathbb{Z}_l$-module M, we write $M_{(m)} = M \otimes_{\mathbb{Z}_l} \mathbb{Z}/m\mathbb{Z}$; if $\mathcal{E} = \emptyset$ we are done, so we assume that $\mathcal{E} = \{1, \ldots, r\} \subseteq \{1, \ldots, n\}$ for some $1 \leq r \leq n$. The left hand side is

$$\frac{(V_{1,(m)})^{I_1} \oplus \ldots \oplus V_{r,(m)})^{I_r} \oplus V_{r+1,(m)} \oplus \ldots \oplus V_{n,(m)}}{(V_1)^{I_1}_{(m)} \oplus \ldots \oplus (V_r)^{I_r}_{(m)} \oplus V_{r+1,(m)} \oplus \ldots \oplus V_{n,(m)}}$$

The right hand side is

$$\frac{(V_{1,(m)})^{\sum_1^r e_i} \oplus \ldots \oplus (V_{n,(m)})^{\sum_1^r e_i}}{(V_1)^{\sum_1^r e_i}_{(m)} \oplus \ldots \oplus (V_n)^{\sum_1^r e_i}_{(m)}} =$$

$$= \frac{(V_{1,(m)})^{e_1} \oplus \ldots \oplus (V_{r,(m)})^{e_r} \oplus V_{r+1,(m)} \oplus \ldots \oplus V_{n,(m)}}{(V_1)^{e_1}_{(m)} \oplus \ldots \oplus (V_r)^{e_r}_{(m)} \oplus V_{r+1,(m)} \oplus \ldots \oplus V_{n,(m)}}$$

which concludes the proof. □

We now return to the hypotheses as in the beginning of the section, so S is not local anymore. From this moment, we will assume that S is a $\mathbb{Q}$-scheme, so it has residue characterstic 0 at every point. We will use the previous lemmas to prove existence and uniqueness of test-Néron models, under the hypothesis of toric-additivity of the base.

Proposition 5.5. *Suppose that S is a $\mathbb{Q}$-scheme and that $\mathcal{A}/S$ is toric-additive. If $\mathcal{N}/S$ and $\mathcal{N}'/S$ are two test-Néron models for A, there exists a unique isomorphism $\mathcal{N} \to \mathcal{N}'$ that restricts to the isomorphism $\mathcal{N}_U \to \mathcal{N}'_U$.*

Proof. The uniqueness is automatic, because $\mathcal{N}'$ is separated and $\mathcal{N}_U$ is schematically-dense in $\mathcal{N}$. For the existence part, we proceed by induction on the dimension of the base. In the case of $\dim S = 1$, let S^{sh} be a strict henselization of the trait S. The base change of a test-Néron model to S^{sh} is a Néron model. By lemma 2.10, $\mathcal{N}$ and $\mathcal{N}'$ are themselves Néron models over S, and therefore there exists an isomorphism $\mathcal{N} \to \mathcal{N}'$.

Now let $\dim S = M$ and assume the statement is true for $\dim S < M$. We claim that we can reduce to the case of a strictly local base S. Suppose that for every geometric point s of S we can construct an isomorphism $f_s \colon \mathcal{N}_{X_s} \to \mathcal{N}'_{X_s}$ where X_s is the spectrum of the strict henselization at s. Then we can spread out f_s to an isomorphism $f' \colon \mathcal{N}_{S'} \to \mathcal{N}'_{S'}$ for some étale cover S' of S. Let $S'' := S' \times_S S'$, $p_1, p_2 \colon S'' \to S'$ be the two projections and $q \colon S'' \to S$. Because test-Néron models are stable under étale base change, $q^*\mathcal{N}$ and $q^*\mathcal{N}'$ are test-Néron models. The two isomorphisms $p_1^* f, p_2^* f \colon q^*\mathcal{N} \to q^*\mathcal{N}'$ necessarily coincide, thus f descends to an isomorphism $\mathcal{N} \to \mathcal{N}'$, which proves our claim.

Let then S be strictly local, of dimension M, with closed point s. The open $V = S \setminus \{s\}$ has dimension $M-1$; since $\mathcal{A}_V/V$ is toric-additive, by inductive hypothesis there is a unique isomorphism $f_V\colon \mathcal{N}_V \to \mathcal{N}'_V$. We would like to extend it to the whole of S.

Let Z be a regular, closed subscheme of S of dimension 1, transversal to D. The existence of such $Z \subset S$ is easily checked. As Z is a strictly henselian trait, the pullbacks of $\mathcal{N}$ and $\mathcal{N}'$ to Z are Néron models of their generic fibre, hence there is a unique isomorphism $\alpha\colon \mathcal{N}_Z \to \mathcal{N}'_Z$. Now let $\underline{\Phi}$ and $\underline{\Phi'}$ be the étale S-group schemes of components of $\mathcal{N}$ and $\mathcal{N}'$; and let Φ and Φ' be the groups $\underline{\Phi}_s(k)$ and $\underline{\Phi}'_s(k)$ respectively. The restriction of α to the fibre over s induces an isomorphism $\Phi \to \Phi'$.

We show next that the isomorphism $\Phi \to \Phi'$ is independent of the choice of $Z \subset S$. Let's call L the fraction field of $\Gamma(Z, \mathcal{O}_Z)$. The morphism $Z \to S$ induces a group homomorphism

$$\pi_1(Z \setminus \{z\}) = \mathrm{Gal}(\overline{L}|L) = \widehat{\mathbb{Z}}(1) \to \pi_1(S \setminus D) = \bigoplus_{i=1}^{n} I_i = \bigoplus_{i=1}^{n} \widehat{\mathbb{Z}}(1) \qquad (26)$$

which sends a topological generator e of $\pi_1(Z \setminus \{z\})$ to a sum $\sum_{i=1}^n e_i$ of topological generators of the direct summands of $\pi_1(S \setminus D)$, since Z is transversal to D. By section 2.4, both Φ and Φ' are canonically isomorphic to

$$\bigoplus_{l \text{ prime}} \frac{(T_l A(\overline{L}) \otimes \mathbb{Q}_l/\mathbb{Z}_l)^{\mathrm{Gal}(\overline{L}|L)}}{T_l A(\overline{L})^{\mathrm{Gal}(\overline{L}|L)} \otimes \mathbb{Q}_l/\mathbb{Z}_l}.$$

We have a canonical isomorphism of $\mathbb{Z}_l$-modules $T_l A(\overline{K}) \to T_l A(\overline{L})$, compatible with the homomorphism 26, so that e acts on an element of $T_l A(\overline{L})$ as $\sum_{i=1}^n e_i$ acts on its image in $T_l A(\overline{K})$. Hence, writing G for $\pi_1(S \setminus D)$, Φ and Φ' are given by

$$\bigoplus_{l \text{ prime}} \frac{(T_l A(\overline{K}) \otimes \mathbb{Q}_l/\mathbb{Z}_l)^{\sum_{i=1}^n e_i}}{T_l A(\overline{K})^{\sum_{i=1}^n e_i} \otimes \mathbb{Q}_l/\mathbb{Z}_l} = \bigoplus_{l \text{ prime}} \frac{(T_l A(\overline{K}) \otimes \mathbb{Q}_l/\mathbb{Z}_l)^{G}}{T_l A(\overline{K})^{G} \otimes \mathbb{Q}_l/\mathbb{Z}_l}$$

the equality coming from the assumption of toric-additivity and lemma 5.4. This shows that the isomorphism $\Phi \to \Phi'$ is independent of the choice of $Z \subset S$. For this reason, we will write Φ for both groups Φ and Φ'.

Now, the surjective morphism

$$(T_l A(\overline{K}) \otimes \mathbb{Q}_l/\mathbb{Z}_l)^G \to \Phi$$

splits; letting N be the order of Φ, we obtain a surjective morphism between the N-torsion subgroups

$$A[N](K) = A[N](\overline{K})^G \to \Phi.$$

We pick a section $\Phi \to A[N](K)$ and denote by B its image. Consider the schematic closures $\mathcal{B}$ and $\mathcal{B}'$ of B inside $\mathcal{N}$ and $\mathcal{N}'$ respectively. Then $\mathcal{B}$ is a closed subgroup scheme of the étale S-group scheme $\mathcal{N}[N]$; in fact, it is the union $\sqcup_{\varphi\in\Phi} V_\varphi$ of some of its connected components. As $V_\varphi \to S$ is flat, separated and birational, it is an open immersion. As $\mathcal{N}[N]$ is finite over U, the restriction of $V_\varphi \to S$ to U is surjective, hence an isomorphism. In particular, it is given by some section $U \to A$, which restricts to a section $\operatorname{Spec} L \to A_{\operatorname{Spec} L}$ over the generic point of Z. As $\mathcal{N}_Z$ is a Néron model of its generic fibre, this section extends to a section $Z \to \mathcal{N}_Z$. This latter section is for sure contained in the schematic closure of V_φ, which is V_φ itself. This shows that $V_\varphi \to S$ is surjective, and in particular an isomorphism. Therefore, $\mathcal{B}$ is simply given by a disjoint union $\sqcup_{\varphi\in\Phi} b_\varphi$ of torsion sections $b_\varphi \colon S \to \mathcal{N}$, and the restriction $\mathcal{B}_s$ is canonically isomorphic to $\underline{\Phi}_s$. Similarly, we write $\mathcal{B}' = \sqcup_{\varphi\in\Phi} b'_\varphi$.

Let $\mathcal{A} \subset \mathcal{N}$ and $\mathcal{A}' \subset \mathcal{N}'$ be the fibrewise-connected components of identity. By uniqueness of semi-abelian extensions, there is a unique isomorphism $\mathcal{A} \to \mathcal{A}'$. Now let $\mathcal{M} = \bigcup_{\varphi\in\Phi}(b_\varphi + \mathcal{A}) \subseteq \mathcal{N}$. It is an open subgroup S-scheme of $\mathcal{N}$, and on the closed fibre we have $\mathcal{M}_s = \mathcal{N}_s$, since $\mathcal{B}_s = \underline{\Phi}_s$. In particular, $\mathcal{N} = \mathcal{N}_V \cup \mathcal{M}$. Writing similarly $\mathcal{M}' = \bigcup_{\varphi\in\Phi}(b'_\varphi + \mathcal{A}') \subseteq \mathcal{N}'$, we have $\mathcal{N}' = \mathcal{N}'_V \cup \mathcal{M}'$.

Now, we construct an isomorphism $\mathcal{M} \to \mathcal{M}'$ simply by sending b_φ to b'_φ and by means of the isomorphism $\mathcal{A} \to \mathcal{A}'$. To obtain an isomorphism $\mathcal{N} \to \mathcal{N}'$ it is enough to show that $\mathcal{N}_V \to \mathcal{N}'_V$ and $\mathcal{M} \to \mathcal{M}'$ agree on the intersection $\mathcal{N}_V \cap \mathcal{M} = \mathcal{M}_V$. This is clear: indeed, the isomorphism $\mathcal{N}_V \to \mathcal{N}'_V$ agrees with the restriction $\mathcal{A}_V \to \mathcal{A}'_V$, and it sends the schematic closure of B inside $\mathcal{N}_V$ to the schematic closure of B inside $\mathcal{N}'_V$; that is, it restricts to an isomorphism $\mathcal{B}_V \to \mathcal{B}'_V$ sending b_φ to b'_φ. □

Theorem 5.6. *Suppose that S is a $\mathbb{Q}$-scheme, and that $\mathcal{A}/S$ is toric-additive. Then there exists a test-Néron model $\mathcal{N}/S$ for A.*

Proof. Our proof is constructive; we subdivide it in steps.

Step 1: constructing the group Ψ. Let s be a geometric point of S, and write X_s for the spectrum of the strict henselization at s. Let K_s be the field of fractions of X_s, that is, the maximal extension of K unramified at s, and $\overline{K}$ an algebraic closure of K_s. Let $\mathcal{J}_s$ be the finite set of components of the strict normal crossing divisor $D \times_S X_s$.

For every prime l, the action of $\operatorname{Gal}(\overline{K}|K_s)$ factors via the quotient $\operatorname{Gal}(\overline{K}|K_s) \to G := \pi_1(U \times_S X_s) = \bigoplus_{i\in\mathcal{J}_s} I_i$ where $I_i = \widehat{\mathbb{Z}}(1)$.

We set

$$\Psi := \bigoplus_{l\ prime} \bigoplus_{i\in\mathcal{J}_s} \frac{\big(T_l A(\overline{K}) \otimes \mathbb{Q}_l/\mathbb{Z}_l\big)^{I_i}}{T_l A(\overline{K})^{I_i} \otimes \mathbb{Q}_l/\mathbb{Z}_l} \tag{27}$$

The abelian group Ψ is finite, as each of its summands is the l-primary part of the group of components of the Néron model of A_{K_s} over the local ring at the generic point of D_i, which exists by theorem 2.14.

By lemma 5.2,

$$\Psi = \bigoplus_{l\ prime} \frac{\big(T_l A(\overline{K}) \otimes \mathbb{Q}_l/\mathbb{Z}_l\big)^{\bigoplus_{i\in\mathcal{J}_s} I_i}}{T_l A(\overline{K})^{\bigoplus_{i\in\mathcal{J}_s} I_i} \otimes \mathbb{Q}_l/\mathbb{Z}_l}.$$

The surjective morphism

$$\bigoplus_l ((T_l A(\overline{K}) \otimes \mathbb{Q}_l/\mathbb{Z}_l)^{\bigoplus_{i\in\mathcal{J}_s} I_i}) \to \Psi$$

splits; therefore, denoting by N the order of Ψ, we obtain a surjective morphism between the N-torsion subgroups

$$\pi\colon A[N](K_s) = A[N](\overline{K})^{\bigoplus_{i\in\mathcal{J}_s} I_i} \to \Psi.$$

We consider the set of sections $\mathcal{S} := \{\alpha\colon \Psi \to A[N](K_s) \text{ such that } \pi\circ\alpha = \mathrm{id}\}$: it is a torsor under the finite group $\bigoplus_l (T_l A(K_s) \otimes \mathbb{Z}/N\mathbb{Z})$, and as such it is finite. As the group Ψ is finite as well, there exists a finite extension $K \to K'$, unramified over s, such that every section $\Psi \to A[N](K_s)$ factors via $A[N](K')$. Notice that $\mathcal{S}$ is non-empty, as the quotient map π splits; thus we can fix a section $\alpha\colon \Psi \to A[N](K')$.

Step 2: spreading out to an étale neighbourhood of s. The normalization of S inside K' is unramified over the image of s in S, hence étale over it ([Sta16]TAG 0BQK), so we obtain an étale neighbourhood S' of s, which we may assume to be connected, with fraction field K'. We write $\mathcal{J}'$ for the set of irreducible components of $D \times_S S'$. There is a natural function $\mathcal{J}_s \to \mathcal{J}'$: up to restricting S', we may assume that it is bijective. Indeed, its surjectivity corresponds to the fact that every component of $D \times_S S'$ contains (the image of) s; imposing also injectivity means asking that $D \times_S S'$ is a *strict* normal crossing divisor. Thus, we need not distinguish between $\mathcal{J}_s$ and $\mathcal{J}'$ and we will simply write $\mathcal{J}$ for this set.

Step 3: constructing the subgroup-scheme $\mathcal{H} \subseteq \mathcal{A}_{S'} \times_{S'} \Psi_{S'}$. We call $H \subseteq A[N](K') \times \Psi$ the image of Ψ via $(\alpha, \mathrm{id})\colon \Psi \to A[N](K') \times \Psi$; we let $\mathcal{H}/S'$ be the schematic closure of H inside $\mathcal{A}_{S'} \times_{S'} \Psi_{S'}$ (where $\Psi_{S'}$ denotes the constant group scheme over S' associated to the finite abelian group Ψ). It is a closed subgroup scheme of the étale S'-group scheme $\mathcal{A}_{S'}[N] \times_{S'} \Psi_{S'}$ and a disjoint union $\sqcup_{j\in\Psi}V_j$ of some of its connected components; moreover, over the generic point of S', each V_j restricts to a copy of $\operatorname{Spec} K'$. As $V_j \to S'$ is flat, separated and birational, it is an open immersion; thus $\mathcal{H} = \sqcup_{j\in\Psi}V_j \to S'$ is a disjoint union of open immersions. In fact, if we write $U' = U \times_S S'$, the base change $A_{U'}$ is an abelian scheme; therefore $A_{U'}[N] \times_{U'} \Psi_{U'}$ is finite, and each $V_j \to S'$ is an isomorphism over U'. This can be restated by saying that the composition

$$\mathcal{H}_{U'} \to \mathcal{A}_{U'} \times_{U'} \Psi_{U'} \to \Psi_{U'}$$

is an isomorphism.

Step 4: taking the quotient by $\mathcal{H}$. Consider now the fppf-quotient

$$\mathcal{N}^\alpha := \frac{\mathcal{A}_{S'} \times_{S'} \Psi_{S'}}{\mathcal{H}}.$$

First, we claim that its restriction $\mathcal{N}^\alpha_{U'}$ is canonically isomorphic to $A_{U'}$. Indeed, we observed that $\mathcal{H}_{U'} = \Psi_{U'}$, and the quotient morphism for $\Psi_{U'} \to \mathcal{A}_{U'} \times_{U'} \Psi_{U'}$, $\psi \mapsto (\alpha(\psi), \psi)$ is $\mathcal{A}_{U'} \times_{U'} \Psi_{U'} \to \mathcal{A}_{U'}$, $(a, \psi) \mapsto a - \alpha(\psi)$, which proves the claim.

Because $\mathcal{H}$ is étale, $\mathcal{N}^\alpha$ is automatically an algebraic space; we claim that it is actually representable by a scheme. As the quotient morphism $p\colon \mathcal{A}_{S'} \times_{S'} \Psi_{S'} \to \mathcal{N}^\alpha$ is an $\mathcal{H}$-torsor, p is étale. In particular the restriction of p to the connected component of identity, $\mathcal{A}_{S'} \times \{0\} \to \mathcal{N}^\alpha$, is étale; it is also separated, and an isomorphism over U. It follows that it is an open immersion. Hence, all other components $\mathcal{A}_{S'} \times \{\psi\}$ map to $\mathcal{N}^\alpha$ via an open immersion. The disjoint union $\bigsqcup_{\psi\in\Psi} \mathcal{A}_{S'} \times_{S'} \{\psi\}$ surjects onto $\mathcal{N}^\alpha$, and this gives us an open cover of $\mathcal{N}^\alpha$ by schemes.

In summary, we have obtained an S'-group scheme $\mathcal{N}^\alpha$, which restricts to A over U'; moreover, it is S'-smooth, of finite presentation, and separated, since $\mathcal{H}$ is closed in the separated scheme $\mathcal{A}_{S'} \times_{S'} \Psi_{S'}$.

Step 5: independence of the section α. We have used the notation $\mathcal{N}^\alpha$ as a reminder of our choice of section α done above. We show that $\mathcal{N}^\alpha$ does not depend on the choice of the section $\Psi \to A[N](K')$, or to put it better, we show that given two sections α, β we obtain a canonical isomorphism $\mathcal{N}^\alpha \to \mathcal{N}^\beta$. Actually, as soon as we prove that $\mathcal{N}^\alpha$ and $\mathcal{N}^\beta$ are test-Néron models (step 6), the existence of a canonical isomorphism between them is ensured by proposition 5.5; however, we still give an argument: suppose we choose another section $\beta\colon \Psi \to A[N](K')$ and let $H^\beta \subset A[N](K') \times \Psi$ be the image of Ψ via $(\beta, \mathrm{id})\colon \Psi \to A[N](K') \times \Psi$. Then the map $f_{\beta-\alpha}\colon H^\alpha \to H^\beta$ sending $(h, \psi) \in H^\alpha \subseteq A[N](K') \times \Psi$ to $(h + (\beta - \alpha)\psi, \psi)$ is an isomorphism. Moreover, $\beta - \alpha$ lands inside $\bigoplus_l T_l A(K') \otimes \mathbb{Z}/N\mathbb{Z}$, the subgroup of $A(K')$ consisting of those N-torsion points that extend to torsion sections of $\mathcal{A}_{S'}/S'$. Therefore $\beta - \alpha$ extends to a morphism of S'-group schemes $\Psi_{S'} \to \mathcal{A}_{S'}$. Now, the isomorphism

$$\mathcal{A}_{S'} \times_{S'} \Psi_{S'} \xrightarrow{\begin{pmatrix} 1 & \beta-\alpha \\ 0 & 1 \end{pmatrix}} \mathcal{A}_{S'} \times_{S'} \Psi_{S'}$$

restricts to $f_{\beta-\alpha}$ on H^α and therefore also restricts to an isomorphism $\mathcal{H}^\alpha \to \mathcal{H}^\beta$ between the schematic closures of H^α and H^β in $\mathcal{A}_{S'} \times_{S'} \Psi_{S'}$. Hence, we obtain an isomorphism $\mathcal{N}^\alpha = (\mathcal{A}_{S'} \times_{S'} \Psi_{S'})/\mathcal{H}^\alpha \to \mathcal{N}^\beta = (\mathcal{A}_{S'} \times_{S'} \Psi_{S'})/\mathcal{H}^\beta$ between the quotients, as wished. We can therefore forget about the choice of section and use the notation $\mathcal{N}/S'$ for the group-scheme just constructed.

Step 6: showing that $\mathcal{N}$ is a test-Néron model. To ease notation, let us write S in place of S', $D = \bigcup_{i \in \mathcal{J}} D_i$ for the strict normal crossing divisor $D \times_S S'$. Let Z be a strictly henselian trait, with closed point z, and $g\colon Z \to S$ a morphism transversal to D. Write T for the strict henselization of S at z and $\mathcal{E} \subseteq \mathcal{J}$ for the subset of indices of components D_i that contain z. Let also $\mathcal{M}/Z$ be the Néron model of $A \times_S Z$. The Néron mapping property gives a morphism $\mathcal{N}_Z \to \mathcal{M}$, which is an open immersion and induces an isomorphism between the fibrewise-connected components of identity, as they are both semi-abelian (lemma 2.17). Let Φ/S and Υ/Z be the étale group schemes of connected components of $\mathcal{N}/S$ and $\mathcal{M}/Z$ respectively. To show that $\mathcal{N}_Z \to \mathcal{M}$ is an isomorphism, we only need to check that the induced morphism $\Phi_{|Z} \to \Upsilon$ is an isomorphism. It is certainly an open immersion, so it suffices to show that $\Phi(z) \to \Upsilon(z)$ is an isomorphism.

We will fix a prime l and compare the l-primary parts of the two groups, which we denote ${}_l\Phi(z)$ and ${}_l\Upsilon(z)$. Let's start with ${}_l\Phi(z)$. The group scheme Φ/S being given by $\Psi_S/\mathcal{H}$, we have ${}_l\Phi(z) = {}_l\Psi(z)/{}_l\mathcal{H}(z)$. Recall that ${}_l\mathcal{H}$ is the schematic closure of ${}_lH$ inside $\mathcal{A} \times_S {}_l\Psi_S$. Hence, ${}_l\mathcal{H}(z)$ is identified with a subgroup of ${}_lH$ consisting of those elements $(a, \psi) \in {}_lH \subset A[N](K) \times {}_l\Psi$ such that a extends to a section of $\mathcal{A}_T/T$. These are exactly the pairs $(a, \psi) \in {}_lH$ such that $a \in T_lA(\overline{K})^{\oplus_{i\in\mathcal{E}} I_i} \otimes \mathbb{Z}/N\mathbb{Z}$. Therefore, ${}_l\mathcal{H}(z)$ is the kernel of the composition

$$\begin{aligned} {}_lH \xrightarrow{\sim} {}_l\Psi = \bigoplus_{i \in \mathcal{J}} \frac{\left(T_lA(\overline{K}) \otimes \mathbb{Z}/N\mathbb{Z}\right)^{I_i}}{T_lA(\overline{K})^{I_i} \otimes \mathbb{Z}/N\mathbb{Z}} \xrightarrow{pr} \\ \xrightarrow{pr} \bigoplus_{i \in \mathcal{E}} \frac{\left(T_lA(\overline{K}) \otimes \mathbb{Z}/N\mathbb{Z}\right)^{I_i}}{T_lA(\overline{K})^{I_i} \otimes \mathbb{Z}/N\mathbb{Z}} = \frac{\left(T_lA(\overline{K}) \otimes \mathbb{Z}/N\mathbb{Z}\right)^{\bigoplus_{i\in\mathcal{E}} I_i}}{T_lA(\overline{K})^{\bigoplus_{i\in\mathcal{E}} I_i} \otimes \mathbb{Z}/N\mathbb{Z}} \end{aligned}$$

from which it follows that

$${}_l\Phi(z) = \frac{{}_l\Psi(z)}{{}_l\mathcal{H}(z)} \cong \bigoplus_{i \in \mathcal{E}} \frac{\left(T_lA(\overline{K}) \otimes \mathbb{Z}/N\mathbb{Z}\right)^{I_i}}{T_lA(\overline{K})^{I_i} \otimes \mathbb{Z}/N\mathbb{Z}}.$$

Next, we look at ${}_l\Upsilon(z)$. Let's call K_Z the field of fractions of $\Gamma(Z, \mathcal{O}_Z)$. The morphism $Z \to T$ induces a group homomorphism

$$\pi_1(Z \setminus \{z\}) = \mathrm{Gal}(\overline{K}_Z|K_Z) = \widehat{\mathbb{Z}}(1) \to \pi_1(T \setminus D) = \bigoplus_{i \in \mathcal{E}} \widehat{\mathbb{Z}}(1)$$

which sends a topological generator e of $\widehat{\mathbb{Z}}(1)$ to a sum of topological generators $\sum_{i=1}^n e_i$, because Z meets D transversally.

Notice that there is a canonical identification $T_l A(\overline{K}_Z) = T_l A(\overline{K})$; the topological generator of $\mathrm{Gal}(\overline{K}_Z|K_Z)$ acts on the latter as $\sum_{i \in \mathcal{E}} e_i$ does. Therefore

$${}_l\Upsilon(z) = \frac{\left(T_l A(\overline{K}) \otimes \mathbb{Z}/N\mathbb{Z}\right)^{\sum_{i\in\mathcal{E}} e_i}}{T_l A(\overline{K})^{\sum_{i\in\mathcal{E}} e_i} \otimes \mathbb{Z}/N\mathbb{Z}}$$

By lemma 5.4 and lemma 5.2, ${}_l\Upsilon(z) \cong {}_l\Phi(z)$, as we wished to show. Hence $\mathcal{N}$ is a test-Néron model for $A_{U'}$ over S'.

Step 7: descending $\mathcal{N}$ along $S' \to S$. For every geometric point s of S, we have found an étale neighbourhood $S' \to S$ and a test-Néron model $\mathcal{N}/S'$ over S'. Using uniqueness up to unique isomorphism of test-Néron models, their stability under étale base change, and effectiveness of étale descent for algebraic spaces, we obtain a smooth separated algebraic space of finite type $\widetilde{\mathcal{N}}$ over S, and an isomorphism $\widetilde{\mathcal{N}} \times_S U \to A$. Because the property of being a test-Néron model is étale-local, $\widetilde{\mathcal{N}}$ is itself a test-Néron model for A over S. □

5.2 Test-Néron models and finite flat base change

In [Edi92], Edixhoven considers the case of an abelian variety A_K over the generic point of a trait S, and a tamely ramified extension of traits $\pi \colon S' \to S$ whose associated extension of fraction fields $K \to K'$ is Galois. He considers the Néron model $\mathcal{N}/S$ of A_K and the Néron model $\mathcal{N}'/S'$ of $A_{K'}$: after defining a certain equivariant action of $\mathrm{Gal}(K'|K)$ on the Weil restriction $\pi_*\mathcal{N}'$, he shows that $\mathcal{N}$ is naturally identified with the subgroup-scheme of $\mathrm{Gal}(K'/K)$-invariants of $\pi_*\mathcal{N}'$.

In this subsection, we aim to show an analogous statement for test-Néron models over a base of higher dimension and characteristic everywhere zero.

We let then S be a noetherian, regular, strictly local $\mathbb{Q}$-scheme, $D = \cup_{i=1}^n \mathrm{div}(t_i)$ a normal crossing divisor on S (thus the t_i are part of a system of regular parameters for $\mathcal{O}_S(S)$), A an abelian scheme over $U = S \setminus D$, $\mathcal{A}/S$ a toric-additive semi-abelian scheme extending A.

We can apply theorem 5.6 to construct a test-Néron model

$$\mathcal{N} = \frac{\mathcal{A} \times_S \Psi_S}{\mathcal{H}}.$$

Notice that the étale cover $S' \to S$ of the proof of theorem 5.6 is necessarily trivial in this case.

Consider now a finite flat cover $\pi\colon T \to S$ of the form

$$T = \operatorname{Spec} \frac{\mathcal{O}_S(S)[X_1, \dots, X_n]}{X_1^{m_1} - t_1, \dots, X_n^{m_n} - t_n}$$

for some positive integers $m_1, \dots, m_n$. Then T is a regular strictly local scheme. We denote by K' its field of fractions. The morphism π is finite étale over U, and the preimage via $\pi\colon T \to S$ of D is the normal crossing divisor $\pi^{-1}(D) = \cup_{i=1}^n \operatorname{div} X_i$.

We have a commutative diagram

$$\begin{array}{ccc} \operatorname{Gal}(\overline{K}|K') & \twoheadrightarrow & \pi_1(U_T) = \bigoplus_{i=1}^n \widehat{\mathbb{Z}}(1) \\ \downarrow & & \downarrow \\ \operatorname{Gal}(\overline{K}|K) & \twoheadrightarrow & \pi_1(U) = \bigoplus_{i=1}^n \widehat{\mathbb{Z}}(1) \end{array}$$

where the right vertical arrow is given by multiplication by m_i on the i-th component. We will write $\pi_1(U) = \bigoplus_{i=1}^n I_i$ and identify $\pi_1(U_T)$ with its subgroup $\bigoplus_{i=1}^n m_i I_i$.

The fraction field K' of T is an extension of K of order $m_1 \cdot m_2 \cdot \ldots \cdot m_n$, and we write G for the Galois group $\operatorname{Gal}(K'|K) = \bigoplus_{i=1}^n I_i/m_i I_i = \bigoplus_{i=1}^n \mu_{m_i}$.

By lemma 3.6, $\mathcal{A} \times_S T$ is still toric-additive. We follow the construction carried out in the proof of theorem 5.6 to obtain a test-Néron model $\mathcal{M}/T$: to start with, we consider the finite abelian group

$$\Psi' = \bigoplus_{l\ prime} \bigoplus_{i=1}^n \frac{\left(T_l A(\overline{K}) \otimes \mathbb{Q}_l/\mathbb{Z}_l\right)^{m_i I_i}}{T_l A(\overline{K})^{m_i I_i} \otimes \mathbb{Q}_l/\mathbb{Z}_l} = \bigoplus_{l\ prime} \frac{\left(T_l A(\overline{K}) \otimes \mathbb{Q}_l/\mathbb{Z}_l\right)^{\oplus_{i=1}^n m_i I_i}}{T_l A(\overline{K})^{\bigoplus_{i=1}^n m_i I_i} \otimes \mathbb{Q}_l/\mathbb{Z}_l}$$

We claim that $T_l A(\overline{K})^{I_i} = T_l A(\overline{K})^{m_i I_i}$; indeed, letting e_i, be a topological generator of I_i, and denoting still by e_i the automorphism of $T_l A(\overline{K})$ induced by e_i, we know by section 2.3 that $(e_i - 1)^2 = 0$. Using this relation, we obtain

$$e_i^{m_i} - 1 = ((e_i - 1) + 1)^{m_i} - 1 = m(e_i - 1) + 1 - 1 = m(e_i - 1).$$

As $T_l A(\overline{K})$ is torsion-free, we see that $\ker(e_i^{m_i} - 1) = \ker(e_i - 1)$, which proves our claim. Hence, we actually have

$$\Psi' = \bigoplus_{l \; prime} \frac{\left(T_l A(\overline{K}) \otimes \mathbb{Q}_l/\mathbb{Z}_l\right)^{\oplus_{i=1}^n m_i I_i}}{T_l A(\overline{K})^{\oplus_{i=1}^n I_i} \otimes \mathbb{Q}_l/\mathbb{Z}_l}$$

and it follows that Ψ' has a natural action of G.

Next, we let $N = \mathrm{ord}(\Psi')$ and choose a section $\alpha\colon \Psi' \to A[N](K')$. We write H' for the image of $\Psi' \xrightarrow{(\alpha,\mathrm{id})} A[N](K') \times \Psi'$ and $\mathcal{H}'$ for its schematic closure inside $\mathcal{A}_T \times_T \Psi'_T$. The fppf-quotient

$$\mathcal{M} = \frac{\mathcal{A}_T \times_T \Psi'_T}{\mathcal{H}'}$$

is represented by a test-Néron model for $A_{U'}$ over T.

In order to compare $\mathcal{M}$ and $\mathcal{N}$, we will consider the Weil restriction of $\mathcal{M}$ via $\pi\colon T \to S$, that is, the functor $\pi_*\mathcal{M}\colon (\mathbf{Sch}/S) \to \mathbf{Sets}$ given by $(Y \to S) \mapsto \mathcal{M}(Y \times_S T)$. Recall that we have an exact sequence of fppf-sheaves of abelian groups

$$0 \to \mathcal{H}' \to \mathcal{A}_T \times_T \Psi'_T \to \mathcal{M} \to 0.$$

As π is a finite morphism, the higher direct images of π for the fppf-topology vanish, and we have an exact sequence of fppf-sheaves

$$0 \to \pi_*\mathcal{H}' \to \pi_*\mathcal{A}_T \times_S \pi_*\Psi'_T \to \pi_*\mathcal{M} \to 0.$$

We claim that $\pi_*\mathcal{M}$ is representable by a scheme. By [Ray70b, XI, 1.16], semi-abelian schemes are quasi-projective, hence so is $\mathcal{A}_T \times_T \Psi'_T$. Clearly $\mathcal{H}'/T$ is quasi-projective as well. As $\pi\colon T \to S$ is finite and flat, $\pi_*\mathcal{H}'$ and $\pi_*\mathcal{A}_T \times_S \pi_*\Psi'_T$ are schemes (see for example [Edi92, 2.2]). Now, $\pi_*\mathcal{H}'/S$ is étale ([Sch94, 4.9]), and its intersection with the identity component of $\pi_*\mathcal{A}_T \times_S \pi_*\Psi'_T$ is trivial. Reasoning as in the proof of theorem 5.6, we conclude that $\pi_*\mathcal{M}$ has an open cover by schemes, hence it is a scheme.

We want to define an equivariant action of G on $\pi_*\mathcal{M} \to S$, where G acts trivially on S. To do this, we let first G act on $A_{K'}$ via the action of G on K'. By [Del85, 1.3 pag.132] the action of G extends uniquely to an equivariant action on $\mathcal{A}_T \to T$. We also have an obvious action of G on Ψ' which induces an equivariant action on $\Psi'_T \to T$. We put together these actions to find an equivariant action of G on $\mathcal{A}_T \times_T \Psi'_T \to T$: clearly H' is G-invariant, thus the same is true for its schematic closure $\mathcal{H}'$. Therefore the action of G descends to an equivariant action of G on $\mathcal{M} \to T$.

To define the action of G on $\pi_*\mathcal{M}$, we let $g \in G$ act on $\pi_*\mathcal{M}$ via the composition

$$\pi_*\mathcal{M} \times_S T \xrightarrow{(\mathrm{id},g)} \pi_*\mathcal{M} \times_S T \to \mathcal{M} \xrightarrow{g^{-1}} \mathcal{M}.$$

where the second arrow is given by the identity morphism $\pi_*\mathcal{M} \to \pi_*\mathcal{M}$. This defines the desired equivariant action of G on $\pi_*\mathcal{M} \to S$.

Consider the functor of fixed points $(\pi_*\mathcal{M})^G \colon \mathbf{Sch}/S \to \mathbf{Sets}$, $(Y \to S) \mapsto \pi_*\mathcal{M}(Y)^G$. Then $(\pi_*\mathcal{M})^G$ is represented by a closed subgroup-scheme of $\pi_*\mathcal{M}$, smooth over S by [Edi92, 3.1].

Proposition 5.7. *There is a canonical closed immersion $\iota \colon \mathcal{N} \to \pi_*\mathcal{M}$, which identifies $\mathcal{N}$ with the subgroup-scheme of fixed points $(\pi_*\mathcal{M})^G$.*

Proof. By generalities on the Weil restriction [BLR90, pag. 198], the canonical morphism $\mathcal{A} \to \pi_*\mathcal{A}_T$ is a closed immersion. The natural injection $\Psi \to \Psi'$ gives a closed immersion $\mathcal{A} \times_S \Psi_S \to \pi_*\mathcal{A}_T \times_S \Psi'_S = \pi_*(\mathcal{A}_T \times_T \Psi'_T)$. To show that it descends to a closed immersion $\mathcal{N} \to \pi_*\mathcal{M}$, it is enough to show that

$$\pi_*\mathcal{H}' \cap (\mathcal{A} \times_S \Psi_S) = \mathcal{H}. \tag{28}$$

We may assume that the section $\Psi \to A[N](K)$ used to construct H is obtained by restriction of the section $\Psi' \to A[N](K')$ used to construct H': indeed we know that it does not matter which section we choose. It follows that $H = H' \cap (A(K) \times_K \Psi)$, which realizes eq. (28) on the level of generic fibres. Now, $\pi_*\mathcal{H}'$ is étale over S, and it is a closed subscheme of $\pi_*\mathcal{A}_T \times_S \Psi'_S$. Hence, it is the schematic closure of its generic fibre, which is H'. Then, the intersection $\mathcal{H}^* := \pi_*\mathcal{H}' \cap (\pi_*\mathcal{A}_T \times_S \Psi_S)$ is clearly still étale over S, and has generic fibre H. Thus $\mathcal{H}^*$ is the schematic closure of H in $\pi_*\mathcal{A}_T \times_S \Psi_S$. On the other hand, $\mathcal{H} \to \mathcal{A} \times_S \Psi_S \to \pi_*\mathcal{A}_T \times_S \Psi_S$ is a closed immersion, and $\mathcal{H}$ is étale over S and has generic fibre H. As $\mathcal{H}$ and $\mathcal{H}^*$ are both étale over S, have same generic fibre and are both closed subschemes of $\pi_*\mathcal{A}_T \times_S \Psi_S$, they are equal. Since $\mathcal{H}$ is contained in $\mathcal{A} \times_S \Psi_S$, so is $\mathcal{H}^*$ and we obtain eq. (28). This proves that we have a closed immersion $\iota \colon \mathcal{N} \to \pi_*\mathcal{M}$.

Now, the restriction of ι to the generic fibre is the closed immersion $A \to \pi_*A_{K'}$, which identifies A with $(\pi_*A_{K'})^G$. Since $(\pi_*\mathcal{M})^G$ and $\mathcal{N}$ are both S-smooth closed subschemes of $\pi_*\mathcal{M}$ and they share the same generic fibre, they are equal. □

5.3 Test-Néron models are Néron models

The objective of this subsection is to prove the following:

Theorem 5.8. *Let S be a connected, locally noetherian, regular $\mathbb{Q}$-scheme, D a normal crossing divisor on S, A an abelian scheme over $U = S \setminus D$ extending* *to a toric-additive semi-abelian scheme $\mathcal{A}/S$. Then A admits a Néron model over S.*

In view of theorem 5.6, theorem 5.8 is an immediate corollary of the following proposition:

Proposition 5.9. *Hypotheses as in theorem 5.8. Let $\mathcal{N}/S$ be a test-Néron model for A over S. Then $\mathcal{N}/S$ is a Néron model.*

We will subdivide the proof of proposition 5.9 in two main steps (propositions 5.10 and 5.11).

Proposition 5.10. *In the hypotheses of proposition 5.9, assume S has dimension 2. Then $\mathcal{N}/S$ is a weak Néron model for A.*

Proof. Let $\sigma : U \to A$ be a section; we want to show that it extends to a section $S \to \mathcal{N}$, or equivalently, that the schematic closure $\overline{\sigma(U)} \subset \mathcal{N}$ is faithfully flat over S. The latter may be checked locally for the fpqc topology; hence, we may reduce to the case where S is the spectrum of a complete, strictly henselian local ring. The normal crossing divisor D has at most 2 components, and up to restricting U we may assume that it is given by the zero locus of uv, with u, v regular parameters for $\Gamma(S, \mathcal{O}_S)$.

Notice that the closure $\overline{\sigma(U)}$ may fail to be flat only over the closed points of S, as $S \setminus \{s\}$ is of dimension 1. By the flattening technique of Raynaud-Gruson ([GR71, 5.2.2]), there exists a blowing-up $\tilde{S} \to S$, centered at s, such that the schematic closure of $\sigma(U)$ inside $\mathcal{N}_{\tilde{S}}$ is flat over $\tilde{S}$. Because S has dimension 2, we can find a further blow-up $S' \to \tilde{S}$ such that the composition $S' \to S$ is a composition of finitely many blowing-ups, each given by blowing-up the ideal of a closed point with its reduced structure. It follows that the exceptional fibre $E \subset S'$ of $S' \to S$ is a chain of projective lines meeting transversally. Let $\Sigma \subset \mathcal{N}_{S'}$ be the schematic closure of $\sigma(U)$. The morphism $\Sigma \to S'$ is flat, but may a priori not be surjective. At this point we only know that the image of Σ contains $S' \setminus E$.

We claim that $\Sigma \to S'$ is surjective. Let $p \in E$. It's easy to show that there exists some strictly henselian trait Z with closed point z and a closed immersion $Z \to S'$ mapping z to p and such that Z meets E transversally. We call L the field of fractions of $\mathcal{O}_Z(Z)$. The section $\sigma \colon U \to A$ restricts to a section $\sigma_L \colon \operatorname{Spec} L \to A_L$; to establish the claim, it suffices to show that σ_L extends to a section $Z \to \mathcal{N}_Z$. We consider the composition $\varphi : Z \to S' \to S$ and the pullbacks $\varphi^*(u), \varphi^*(v) \in \mathcal{O}_Z(Z)$. Let $m, n \in \mathbb{Z}_{\geq 1}$ be their respective valuations. Now let $\pi \colon T \to S$ be the finite flat morphism given by extracting an m-root of u and an n-root of v, that is,

$$T = \operatorname{Spec} \frac{\mathcal{O}_S(S)[x,y]}{x^m - u, y^n - v}.$$

Then T is itself the spectrum of a regular, strictly henselian local ring and the preimage $\pi^{-1}(D)$ is the zero locus of xy and hence a normal crossing divisor. The pullback of $\mathcal{A}$ via $T \to S$ is still toric-additive (lemma 3.6) and therefore we can construct a test Néron model $\mathcal{M}/T$. Writing $X = \pi_*\mathcal{M}$ for the Weil restriction along π and $G := \operatorname{Aut}_S(T) = \mu_m \oplus \mu_n$, we have by proposition 5.7 that $X^G = \mathcal{N}$.

Now, as Z is a strictly henselian, $\mathcal{O}(Z)$ contains all roots of elements of $\mathcal{O}(Z)^\times$, and we can find uniformizers $t_u, t_v \in \mathcal{O}_Z(Z)$ such that $t_u^m = \varphi^*(u)$ and $t_v^n = \varphi^*(v)$. These elements give us a lift of $\varphi\colon Z \to S$ to $\psi\colon Z \to T$. Then ψ is a closed immersion meeting $f^{-1}(D)$ transversally. This means that the base change $\mathcal{M}_Z/Z$ is a Néron model of its generic fibre. Consider the section $\sigma_L\colon \operatorname{Spec} L \to A_L$. Composing it with the closed immersion $A_L = (\pi_*\mathcal{M})_L^G \hookrightarrow (\pi_*\mathcal{M})_L$ gives, by definition of Weil restriction, a morphism $\operatorname{Spec} L \times_S T \to \mathcal{M}_L$. Precomposing with $(\mathrm{id}, \psi)\colon \operatorname{Spec} L \to \operatorname{Spec} L \times_S T$, we obtain a section $\widetilde{\sigma}_L\colon \operatorname{Spec} L \to \mathcal{M}_L$. As $\mathcal{M}_Z/Z$ is a Néron model of its generic fibre, $\widetilde{\sigma}_L$ extends uniquely to a section $Z \to \mathcal{M}_Z$. This gives us a morphism of T-schemes $Z \to \mathcal{M}$ and by composition a T-morphism $Z \times_S T \to Z \to \mathcal{M}$, that is, a section $m \in X(Z)$ of the Weil restriction. Notice that the generic fibre of m is σ_L, which lands in the part of X fixed by G; as $X^G = \mathcal{N}$ is a closed subscheme of X we deduce that m lands inside $\mathcal{N}$. So $m \in \mathcal{N}(Z)$ is the required extension of σ_L and we win.

As $\Sigma \to S'$ is faithfully flat, separated and birational, it is an isomorphism. Hence $\sigma\colon U \to A$ extends to a section $\sigma'\colon S' \to \mathcal{N}_{S'}$. We are going to show that σ' descends to a section $\theta : S \to \mathcal{N}$. The restriction of σ' to E maps a connected chain of projective lines to a connected component of $\mathcal{N}_s$ (where s is the closed point of S). Every connected component of $\mathcal{N}_s$ is isomorphic to the semi-abelian variety $\mathcal{N}_s^0$, hence does not contain projective lines. It follows that $\sigma'_{|E}$ is constant and that it descends to a morphism $\operatorname{Spec} k(s) \to \mathcal{N}_s$. Let $\mathcal{J}$ be the ideal sheaf of the exceptional fibre $E \subset S'$ and define $S'_n \subset S'$ to be the closed subscheme defined by $\mathcal{J}^{n+1}$ for every $n \geq 0$. Similarly let $S_n := \operatorname{Spec} \mathcal{O}_S(S)/\mathfrak{m}^{n+1}$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_S(S)$. We have shown that $\sigma'_{|E} : S'_0 \to \mathcal{N}$ descends to a morphism $\theta_0\colon S_0 \to \mathcal{N}$. Now, by smoothness of $\mathcal{N}$, every morphism $S_{j-1} \to \mathcal{N}$ admits a lift $S_j \to \mathcal{N}$; the set of such lifts is given by $H^0(S_0, \Omega^1_{\mathcal{N}_{S_0}/S_0} \otimes_{\mathcal{O}_{S_0}} \mathfrak{m}^j/\mathfrak{m}^{j+1})$. The canonical morphism

$$H^0(S_0, \Omega^1_{\mathcal{N}_{S_0}/S_0} \otimes_{\mathcal{O}_{S_0}} \mathfrak{m}^j/\mathfrak{m}^{j+1}) \to H^0(S'_0, \Omega^1_{\mathcal{N}_{S'_0}/S'_0} \otimes_{\mathcal{O}_{S'_0}} \mathcal{J}^j/\mathcal{J}^{j+1})$$

is an isomorphism, due to the fact that the space of global sections of $\mathcal{J}^j/\mathcal{J}^{j+1} = \mathcal{O}_{S'_0}(j)$ is equal to $\mathfrak{m}^j/\mathfrak{m}^{j+1}$. Thus the set of liftings of $\alpha \in \operatorname{Hom}_S(S_j, \mathcal{N})$

to $\mathrm{Hom}_S(S_{j+1}, \mathcal{N})$ is naturally in bijection with the set of liftings of $\alpha_{|S'_j} \in \mathrm{Hom}_S(S'_j, \mathcal{N})$ to $\mathrm{Hom}_S(S'_{j+1}, \mathcal{N})$. The reductions modulo $\mathcal{J}^j$ of $\sigma' \colon S' \to \mathcal{N}_{S'}$ provides a compatible set of liftings of $\sigma'_{|S'_0}$, and therefore a compatible set of liftings of θ_0; which in turn by completeness of S yield the desired morphism $S \to \mathcal{N}$. □

The next step is extending the result to the case of $\dim S > 2$.

Proposition 5.11. *In the hypotheses of proposition 5.9, $\mathcal{N}/S$ is a weak Néron model.*

Proof. As in the proof of proposition 5.10, we may assume that S is the spectrum of a complete strictly henselian local ring. We proceed by induction on the dimension of S. If the dimension is 1, the statement is clearly true, and the case of dimension 2 is the statement of proposition 5.10. So we let $n \geq 3$ be the dimension of S and we suppose that the statement is true when S has dimension $n-1$. Let $\sigma \colon U \to A$ be a section. Because $V = S \setminus \{s\}$ has dimension $n-1$, and because $\mathcal{A}_V$ is still toric-additive (lemma 3.8), σ extends to $\sigma \colon V \to \mathcal{N}_V$.

Next, we cut S with a hyperplane H transversal to all the components of the normal crossing divisor D, but paying attention to choosing H so that $D \cap H$ (with its reduced structure) is still a normal crossing divisor on H. This is always possible: consider a system of regular parameters $u_1, u_2, \ldots, u_n$ for S such that D is the zero locus of $u_1 u_2 \cdots u_r$ for some $r \leq n$; then H can be chosen to be, for example, the hypersurface cut by $u_1 - u_n$. Because H is transversal to D, it is clear that the base change $\mathcal{N}_H/H$ is still a test-Néron model. By our inductive assumption on the dimension of the base, $\sigma_{|H} \colon H \cap U \to A$ extends to $\theta_0 \colon H \to \mathcal{N}$. Now we would like to put together the data of σ and θ_0 to extend $\sigma \colon V \to \mathcal{N}_V$ to a section $\theta \colon S \to \mathcal{N}$. Let $\mathcal{J} \subset \mathcal{O}_S$ be the ideal sheaf of H and for every $j \geq 1$ define S_j to be the closed subscheme cut by $\mathcal{J}^{j+1}$. We have a morphism $\theta_0 \colon H = S_0 \to \mathcal{N}$. By smoothness of $\mathcal{N}$, there exists for every $j \geq 0$ a lifting of θ_0 to an S-morphism $S_j \to \mathcal{N}$. The set of liftings of an S-morphism $S_{j-1} \to \mathcal{N}$ to an S-morphism $S_j \to \mathcal{N}$ is given by the global sections of the locally-free sheaf $\mathcal{F} := \Omega^1_{\mathcal{N}/S} \otimes \mathcal{J}^j/\mathcal{J}^{j+1}$ on S_0. Because $\dim S_0 \geq 2$ and $V = S \setminus \{s\}$, we have $H^0(S_0, \mathcal{F}) = H^0(V \cap S_0, \mathcal{F}_V)$, and the latter parametrizes liftings of morphisms $S_{j-1} \cap V \to \mathcal{N}$ to $S_j \cap V \to \mathcal{N}$. The section $\sigma \colon V \to \mathcal{N}_V$ gives a compatible choice of lifting for every $j \geq 0$, and we get by completeness of S a morphism $S \to \mathcal{N}$ agreeing with σ on V, as we wished. □

We can now conclude the proof of proposition 5.9.

Proof of proposition 5.9. Let $T \to S$ be a smooth morphism; then $\mathcal{A}_T/T$ is toric-additive by lemma 3.8 and the base change $\mathcal{N}_T/T$ is a test-Néron model. Now, given $\sigma_U : T_U \to \mathcal{A}$, we obtain a section $T_U \to \mathcal{A} \times_U T_U$, which by proposition 5.11 extends to a section $T \to \mathcal{N}_T$. The latter is the datum of an S-morphism $\sigma : T \to \mathcal{N}$ extending σ_U. □

We give a corollary of theorem 5.8.

Corollary 5.12. *Let S be a connected, locally noetherian, regular $\mathbb{Q}$-sccheme, D a regular divisor on S, A an abelian scheme over $U = S \setminus D$ extending to a semi-abelian scheme $\mathcal{A}/S$. Then A admits a Néron model over S.*

Proof. At every geometric point s of S, D has only one irreducible component. It follows that $\mathcal{A}/S$ is toric-additive and we conclude by theorem 5.8. □