A monodromy criterion for existence of Neron models and a result on semi-factoriality

Orecchia, G.

Author: Orecchia, G.
Title: A monodromy criterion for existence of Neron models and a result on semi-factoriality
Issue Date: 2018-02-27

The restriction $\mathcal{E}_U/U$ is an elliptic curve, which is canonically identified with its jacobian $\mathrm{Pic}^0_{\mathcal{E}_U/U}$; the smooth locus $\mathcal{E}^{sm}/S$ has a unique S-group scheme structure extending the one of $\mathcal{E}_U/U$, and is a semi-abelian scheme.

Let ζ_1, ζ_2 be the generic points of $D_1 = \{u = 0\}$ and $D_2 = \{v = 0\}$ respectively, and let s be the closed point $\{u = 0, v = 0\}$. The fibres of $\mathcal{E}^{sm}$ over ζ_1, ζ_2, s are all tori of dimension 1. It follows that $\mathcal{E}^{sm}$ is not toric-additive.

Example 3.13. Consider the nodal projective curve $\mathcal{E}' \subset \mathbb{P}^2_S$ given by the equation

$$Y^2Z = X^3 - X^2Z - uZ^3.$$

Again, $\mathcal{E}'_U = \mathrm{Pic}^0_{\mathcal{E}'_U/U}$; and the smooth locus $\mathcal{E}'^{sm}/S$ is a semi-abelian scheme. In this case, the fibre of $\mathcal{E}'$ over ζ_2 is smooth; so $\mu(\zeta_1) = 1, \mu(\zeta_2) = 0, \mu(s) = 1$. Thus $\mathcal{E}'$ is toric-additive.

4 Neron models of jacobians of stable curves

4.1 Generalities

Nodal curves

Definition 4.1. A *curve* C over an algebraically closed field k is a proper morphism of schemes $C \to \mathrm{Spec}\, k$, such that C is connected and its irreducible components have dimension 1. A curve C/k is called *nodal* if for every non-smooth point $p \in C$ there is an isomorphism of k-algebras $\widehat{\mathcal{O}}_{C,p} \to k[[x,y]]/xy$.

For a general base scheme S, a *nodal curve* $f\colon \mathcal{C} \to S$ is a proper, flat morphism of finite presentation, such that for each geometric point s of S the fibre $\mathcal{C}_s$ is a nodal curve.

We will denote by $\mathcal{C}^{ns}$ the subset of $\mathcal{C}$ of points at which f is not smooth. Seeing $\mathcal{C}^{ns}$ as the closed subscheme defined by the first Fitting ideal of $\Omega^1_{\mathcal{C}/S}$, we have for a nodal curve $\mathcal{C}/S$ that $\mathcal{C}^{ns}/S$ is finite, unramified and of finite presentation.

We report a lemma from [Hol17b].

Lemma 4.2 ([Hol17b], Prop.2.5)**.** *Let S be locally noetherian, $f\colon \mathcal{C} \to S$ be nodal, and p a geometric point of $\mathcal{C}^{ns}$ lying over $s \in S$. We have:*

i) there is an isomorphism

$$\widehat{\mathcal{O}}^{sh}_{\mathcal{C},p} \cong \frac{\widehat{\mathcal{O}}^{sh}_{S,s}[[x,y]]}{xy - \alpha}$$

for some element α in the maximal ideal of the completion $\widehat{\mathcal{O}}^{sh}_{S,s}$;

ii) the element α is in general not unique, but the ideal $(\alpha) \subset \widehat{\mathcal{O}}^{sh}_{S,s}$ is. Moreover, the ideal is the image in $\widehat{\mathcal{O}}^{sh}_{S,s}$ of a unique principal ideal of $\mathcal{O}^{sh}_{S,s}$, which we call thickness of p.

We remark that, if S is regular at s, then $\mathcal{C}$ is regular at p if and only if α is generated by a regular parameter of the regular ring $\mathcal{O}^{sh}_{S,\overline{s}}$.

Split singularities

Let k be a field (not necessarily algebraically closed), C/k a nodal curve, $n\colon C' \to C$ its normalization. Following [Liu02, 10.3.8], we say that $p \in C^{ns}$ is a *split ordinary double point* if its preimage $n^{-1}(p)$ consists of k-valued points. This implies in particular that p is k-valued. Moreover, if p belongs to two or more components of C, then it belongs to exactly two components Z_1, Z_2; these are smooth at p and meet transversally ([Liu02, 10.3.11]). We say that C/k has *split singularities* if every $p \in C^{ns}$ is a split ordinary double point.

A nodal curve C/k attains split singularities after a finite separable extension $k \to k'$. We also remark that a nodal curve with split singularities has irreducible components that are geometrically irreducible. Indeed, either C/k is smooth, in which case it is geometrically connected and therefore geometrically irreducible; or every irreducible component of the normalization of C contains a k-rational point and is therefore geometrically irreducible.

Lemma 4.3. *Let $\mathcal{C} \to S$ be a nodal curve and $s \in S$ such that $\mathcal{C}_s$ has split singularities. Let p be a geometric point of $\mathcal{C}_s$. Then the thickness (α) of p is generated by an element of the Zariski-local ring $\mathcal{O}_{S,s}$.*

Proof. The morphism $f\colon \mathcal{C}^{ns} \to S$ is finite unramified. Because $\mathcal{C}_s$ has split singularities, we see by [Sta16]TAG 04DG, that there exists an open neighbourhood U of s such that $f^{-1}(U) \to U$ is a disjoint union of closed immersions. In particular, $\mathcal{C}^{ns} \to S$ is a closed immersion at p, and to it we can associate an ideal I in the Zariski-local ring $\mathcal{O}_{S,s}$. We see (for example by [Hol17b, proof of part 2 of Prop. 2.5]) that α is the image of I in $\mathcal{O}^{sh}_{S,s}$; and moreover, since $\mathcal{O}_{S,s} \to \mathcal{O}^{sh}_{S,s}$ is faithfully flat, I is principal, which completes the proof. $\square$

Lemma 4.4. *Let $\mathcal{C} \to S$ be a nodal curve over a noetherian, regular, strictly local scheme. Let η be the generic point of S. The generic fibre $\mathcal{C}_\eta$ has split singularities.*

Proof. The non-smooth locus $\mathcal{C}^{ns}$ is finite unramified over S, hence a disjoint union of closed subschemes of S. Let $X \subseteq \mathcal{C}^{ns}$ be the part consisting of sections $S \to \mathcal{C}$.

We claim that the open subscheme $\mathcal{C} \setminus X$ is normal. We will show it by using Serre's criterion for normality ([Liu02, 8.2.23]). First, as X has been removed, $\mathcal{C} \setminus X$ is regular at its points of codimension 1. Condition S_2 follows from the fact that $\mathcal{C} \setminus X$ is locally complete intersection over a regular, noetherian base, hence Cohen-Macaulay by [Liu02, 8.2.18]. This proves the claim.

Our next claim is that the normalization $\pi \colon \mathcal{C}' \to \mathcal{C}$ is finite and unramified. Since these are properties fpqc-local on the target, and since we already know that π induces an isomorphism over $\mathcal{C} \setminus X$, it is enough to check the claim over the completion of the strict henselization of points of X. Let x be such a point and s its image in S. Then $\widehat{\mathcal{O}}^{sh}_{\mathcal{C},x} \cong \frac{\widehat{\mathcal{O}}^{sh}_{S,s}[[u,v]]}{uv}$. Its integral closure is the inclusion

$$\frac{\widehat{\mathcal{O}}^{sh}_{S,s}[[u,v]]}{uv} \to \widehat{\mathcal{O}}^{sh}_{S,s}[[u]] \times \widehat{\mathcal{O}}^{sh}_{S,s}[[v]];$$

the corresponding morphism of spectra is indeed finite and unramified, proving the claim.

Now, let Y be the preimage of X via $\pi \colon \mathcal{C}' \to \mathcal{C}$. We have that Y is finite, unramified over X, and in particular finite étale over S. Hence Y is a disjoint union of sections $S \to \mathcal{C}'$. The restriction of π to the generic fibre $\mathcal{C}'_\eta \mathcal{C}_\eta$ is a normalization morphism, and we see that the preimage Y_η of $X_\eta = (\mathcal{C}_\eta)^{ns}$ consists of $k(\eta)$-valued points, as we wished to show.

□

The relative Picard scheme

Given a nodal curve $\mathcal{C} \to S$ we denote by $\mathrm{Pic}^0_{\mathcal{C}/S}$ the *degree-zero relative Picard* functor; it is constructed as the fppf-sheaf associated to the functor

$$\begin{array}{rccl} P^0_{\mathcal{C}/S} \colon & \mathbf{Sch}/S & \to & \mathbf{Ab} \\ & T \to S & \mapsto & \mathrm{Pic}^0(\mathcal{C} \times_S T) \end{array}$$

where by definition $\mathrm{Pic}^0(\mathcal{C} \times_S T)$ is the group of isomorphism classes of invertible sheaves $\mathcal{L}$ on $\mathcal{C} \times_S T$ such that, for every geometric point t of T and irreducible component X of the fibre $\mathcal{C}_t$, $\deg \mathcal{L}_{|X} = 0$.

It turns out that the degree-zero Picard functor $\mathrm{Pic}^0_{\mathcal{C}/S}$ of a nodal curve has an easy description if $\mathcal{C}/S$ admits a section. In this case, it is given by

$$\begin{array}{rcl} \mathrm{Pic}^0_{\mathcal{C}/S} \colon \mathbf{Sch}/S & \to & \mathbf{Ab} \\ T \to S & \mapsto & \dfrac{\mathrm{Pic}^0(\mathcal{C} \times_S T)}{\mathrm{Pic}(T)} \end{array}$$

If $\mathcal{C}/S$ is a smooth curve, it is well known that $\mathrm{Pic}^0_{\mathcal{C}/S}$ is represented by an abelian scheme, called the *jacobian* of $\mathcal{C}/S$. If $\mathcal{C}/S$ is only nodal, then $\mathrm{Pic}^0_{\mathcal{C}/S}$ is represented by a semi-abelian scheme ([BLR90, 9.4/1]).

Generalities on graphs

We use this subsection to list some graph-theoretical notions, since we are going to work with dual graphs of nodal curves. In what follows, we will simply use the word *graph* to refer to a finite, connected, undirected graph $G = (V, E)$.

A *path* on G is a walk on G in which all edges are distinct, and that never goes twice through the same vertex, except possibly for the first and last; a *cycle* is a path that starts and ends at the same vertex. A *loop* is a cycle consisting of only one edge.

A *tree* is a subgraph of G that does not contain cycles. We say that a tree $T \subset G$ is a *spanning tree* if it contains all vertices of G, in which case it is a maximal tree. Given a spanning tree $T \subset G$, the edges of G that are not contained in T are called *links* with respect to T. The number of links of G is independent of the chosen spanning tree and is equal to the first Betti number $h^1(G, \mathbb{Z})$. If a spanning tree T is fixed, for each of the links $e_1, \ldots, e_r$ with respect to T, the subgraph $T \cup e_i$ contains only one cycle C_i. The cycles $C_1, \ldots, C_r$ are called *fundamental cycles* with respect to T.

The dual graph of a curve

Let C be a curve with split singularities over a field k. We define the *dual graph* of C as the graph $\Gamma = (V, E)$ with $V = \{$ irreducible components of $C\}$, $E = \{p \in C^{ns}\}$; the extremal vertices of an edge p are the components containing p, which are indeed either one or two.

The following well-known fact gives a geometric interpretation to the first Betti number of the dual graph of a curve.

Lemma 4.5 ([BLR90], 9.2/8)**.** *Let C/k be a nodal curve over a field, Γ the* *dual graph of* $C \times_k \overline{k}$, $h^1(\Gamma, \mathbb{Z})$ *its first Betti number. Then*

$$h^1(\Gamma, \mathbb{Z}) = \mu := \textit{toric rank of } \operatorname{Pic}^0_{C/k}.$$

Labelled dual graphs

Given a nodal curve $f\colon \mathcal{C} \to S$ and a point s of S such that $\mathcal{C}_s$ has split singularities, we write $\Gamma_s = (V_s, E_s)$ for the dual graph associated to the fibre $\mathcal{C}_s$. Using the notation of [Hol17b], we write L_s for the monoid of principal ideals of the (Zariski-)local ring $\mathcal{O}_{S,s}$; then we let $l_s\colon E_s \to L_s$ be the function associating to each edge of Γ_s the thickness of the corresponding singular point of $\mathcal{C}_s$ (which indeed is an ideal of $\mathcal{O}_{S,s}$, by lemma 4.3). The pair (Γ_s, l_s) is the *labelled graph* of $\mathcal{C} \to S$ at the geometric point s.

Let now ζ, s be two points of S, such that s is contained in the closure $\overline{\{\zeta\}} \subset S$, and such that the fibres $\mathcal{C}_\zeta, \mathcal{C}_s$ have split singularities. Then the labelled graph (Γ_ζ, l_ζ) of $\mathcal{C}_\zeta$ is obtained from the labelled graph (Γ_s, l_s) of $\mathcal{C}_s$ by: 1)contracting all edges of Γ_s that are labelled by an ideal of $\mathcal{O}_{S,s}$ whose image in $\mathcal{O}_{S,\zeta}$ is the unit ideal; 2)for every edge e of Γ_s that does not get contracted, we label the corresponding edge of Γ_ζ by the image in $\mathcal{O}_{S,\zeta}$ of the label of e.

4.2 Holmes' condition of alignment

Definition 4.6 ([Hol17b], definition 2.11)**.** Let $\mathcal{C} \to S$ be a nodal curve and s a geometric point of S. We say that $\mathcal{C}/S$ is *aligned at s* if for every cycle $\gamma \subset \Gamma_s$ and every pair of edges e, e' of γ, there exist integers n, n' such that

$$l(e)^n = l(e')^{n'}.$$

We say that C/S is *aligned* if it is aligned at every geometric point of S.

Theorem 4.7 ([Hol17b], theorem 5.16, theorem 5.2)**.** *Let S be regular, $U \subset S$ a dense open, $f\colon \mathcal{C} \to S$ a nodal curve, with $f_U\colon \mathcal{C}_U \to U$ smooth.*

i) If the jacobian $\operatorname{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model over S, then $\mathcal{C}/S$ is aligned;

ii) if $\mathcal{C}$ is regular and $\mathcal{C}/S$ is aligned, then $\operatorname{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model over S.

We are soon going to show how the condition of alignment is closely related to toric-additivity of $\operatorname{Pic}^0_{\mathcal{C}_K/K}$. For the moment, we will consider a graph $\Gamma =$

(V, E), a set of $n \geq 1$ different colours $\mathfrak{C} := \{\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n\}$, and a colouring of the edges $\chi\colon E \to \mathfrak{C}$. We say that (Γ, χ) is *aligned* if for every cycle $\gamma \subset \Gamma$, the restriction of χ to γ is constant; in other words, if every cycle is monochromatic.

The following lemma gives us a criterion for alignment that is easier to check. The proof is due to Raymond van Bommel.

Lemma 4.8. *Let $(\Gamma, \chi\colon E \to \mathfrak{C} = \{\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n\})$ be a graph with a colouring of the edges. Fix a spanning tree T. Then (Γ, χ) is aligned if and only if every fundamental cycle is monochromatic.*

Proof. What we have to prove is that if every fundamental cycle is monochromatic, then (Γ, χ) is aligned, as the converse statement is obvious. We show that we can reduce to the case $n = 2$ (two colours). If there is only one colour the statement is clearly true. Suppose now the statement is false for some $n > 2$: that is, (Γ, χ) is not aligned but all fundamental cycles are monochromatic. Then there is some cycle γ in Γ that takes at least two distinct colours, $\mathfrak{c}_1$ and $\mathfrak{c}_2$. We can now pretend that $\mathfrak{c}_2, \mathfrak{c}_3, \ldots, \mathfrak{c}_n$ are different hues of one colour $\mathfrak{c}'$, and that our graph Γ is coloured with only two colours, $\mathfrak{c}_1$ and $\mathfrak{c}'$. Then, γ still takes two distinct colours, and all fundamental cycles are still monochromatic; this implies that the statement is false for $n = 2$. Thus we have reduced to proving the statement for $n = 2$ colours.

Let $(\Gamma, \chi\colon E \to \{\text{yellow, pink}\})$ be a coloured graph, and assume all fundamental cycles are monochromatic. We construct a new graph, which we call G, in the following way: the set of vertices of G consists of the disjoint union of two copies, V_y and V_p, of the set of vertices V of Γ. We connect the vertices with edges as follows: first, if v is a vertex of Γ, we create an edge e_v linking the corresponding vertices v_y and v_p in V_y and V_p. Next, if e is an yellow edge of Γ linking vertices v and w, we create an edge e_y between v_y and w_y; if instead e is pink, we create an edge e_p between v_p and w_p. This completes the construction of G.

We call G_y and G_p the subgraphs of G with underlying set of vertices V_y and V_p respectively. We will call the edges e_v linking G_y and G_p *vertical edges*, and the others *horizontal edges*. Now, consider the subgraph W of G given by the union of all vertical edges, and of all horizontal edges corresponding to edges of the spanning tree T. Clearly, W spans G, is connected, and does not contain cycles, otherwise T would itself contain a cycle. Hence W is a spanning tree for G; it follows that the links of G with respect to W are in bijection with the links of Γ with respect to T. Since the fundamental cycles of Γ are monochromatic, the fundamental cycles of G consist only of horizontal edges.

Now, suppose by contradiction that Γ contains a non-monochromatic cycle γ. Then γ defines a unique cycle γ' on G, and γ' necessarily contains some vertical edge, as γ is not monochromatic. However, by [Die05, 1.9.6], fundamental cycles form a basis of the cycle space (i.e. every cycle is a composition of fundamental cycles). As fundamental cycles of G do not contain vertical edges, γ' cannot contain vertical edges. This is a contradiction and the lemma is proved. □

Lemma 4.9. *Let Γ be a graph with a colouring of the edges $\chi\colon E \to \mathfrak{C} = \{\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n\}$. For every $1 \le i \le n$ let Γ_i be the graph obtained by contracting every edge whose colour is not $\mathfrak{c}_i$. Then*

$$h^1(\Gamma, \mathbb{Z}) \le \sum_{i=1}^{n} h^1(\Gamma_i, \mathbb{Z})$$

with equality if and only if (Γ, χ) is aligned.

Proof. Fix a spanning tree T for Γ; we write T_i for the image of T in the contraction Γ_i. Notice that T_i need not be a tree; however, it is a subgraph of Γ_i containing every vertex of Γ_i; therefore, if T_i is a tree it is also spanning; and in any case T_i contains a spanning tree for Γ_i.

Claim 4.10. *(Γ, χ) is aligned if and only if for all $i = 1, \ldots, n$, T_i is a (spanning) tree in Γ_i.*

Suppose that (Γ, χ) is aligned and fix some i. We want to show that T_i is a tree in the contraction Γ_i. We can contract one edge at a time and see what happens to the image of T in the contraction. On the one hand, contracting an edge that is contained in T does not produce new cycles in the image of T. Now let $e \in E$ be a link with respect to T, and suppose that e gets contracted in Γ_i. Then $\chi(e) \neq \mathfrak{c}_i$. Let P be the unique path in T connecting the two extremal vertices of e. Then the union of P and e forms a cycle γ, which does not take the colour $\mathfrak{c}_i$ by the alignment hypothesis. Hence γ gets contracted to a point in Γ_i and once again no new cycle is produced in the image of T. Therefore T_i is a tree.

Conversely, suppose that (Γ, χ) is not aligned. By lemma 4.8, there is a fundamental cycle γ that takes two distinct colours, say $\mathfrak{c}_1$ and $\mathfrak{c}_2$. Let e be the only link contained in γ; we may assume $\chi(e) = \mathfrak{c}_1$. Thus, e is contracted in Γ_2. However, γ is not contracted to a point in Γ_2, since it contains some edge with colour $\mathfrak{c}_2$. It follows that T_2 is not a tree. This establishes the claim.

Now, $h^1(\Gamma, \mathbb{Z})$ is equal to the number of links with respect to T. We write $h^1(\Gamma, \mathbb{Z}) = b_1 + \ldots + b_n$, where b_i is the number of links of colour $\mathfrak{c}_i$. In the contraction Γ_i, the only links that are not contracted are those of colour $\mathfrak{c}_i$. Since T_i contains a spanning tree for Γ_i, we have $h^1(\Gamma_i, \mathbb{Z}) \ge b_i$, with equality if T_i is a tree. Hence $h^1(\Gamma, \mathbb{Z}) \le h^1(\Gamma_1, \mathbb{Z}) + \ldots + h^1(\Gamma_n, \mathbb{Z})$. Moreover, by the claim, (Γ_i, χ) is aligned if and only if for every $i = 1, \ldots, n$, $h^1(\Gamma_i, \mathbb{Z}) = b_i$. This in turn is equivalent to $h^1(\Gamma, \mathbb{Z}) = h^1(\Gamma_1, \mathbb{Z}) + \ldots + h^1(\Gamma_n, \mathbb{Z})$, which completes the proof. □

4.3 Relation between toric-additivity and alignment

We now consider a connected, locally noetherian, regular base scheme S with a normal crossing divisor $D \subset S$, and a nodal curve $\mathcal{C}/S$, such that the base change $\mathcal{C}_U/U := S \setminus D$ is smooth.

If $S' \to S$ is a strict henselization at some geometric point s of S, and $D \cap S'$ is given by regular parameters $t_1, \ldots, t_n \in \mathcal{O}(S')$, then the thickness of any non-smooth point $p \in \mathcal{C}_s$ is generated by $t_1^{m_1} \cdot \ldots \cdot t_n^{m_n}$ for some non-negative integers $m_1, \ldots, m_n$. In particular, $\mathcal{C}$ is regular at p if and only if its thickness is generated by t_i for some $1 \leq i \leq n$.

Proposition 4.11. *Suppose that the total space $\mathcal{C}$ is regular. Then $\mathcal{C}/S$ is aligned if and only if $\mathrm{Pic}^0_{\mathcal{C}_K/K}$ is toric-additive.*

Proof. As both alignment and toric-additivity are checked over the strict henselizations at geometric points of S, we may assume that S is strictly local. Let $\Gamma_s = (V_s, E_s)$ be the dual graph of the fibre of $\mathcal{C}$ over the closed point $s \in S$, and $l_s \colon E_s \to L_s$ the labelling of the edges, taking value in the monoid L_s of principal ideals of $\mathcal{O}_S(S)$. We have already remarked that, since $\mathcal{C}$ is regular, the labels can only take the values $(t_1), \ldots, (t_n) \in L_s$. This means that $\mathcal{C}/S$ is aligned if and only if every cycle of Γ has edges with the same label.

Now, let $\{D_i\}_{i=1,\ldots,n}$ be the components of the divisor D. Each of them is cut out by a regular element $t_i \in \mathcal{O}_S(S)$ and is itself a regular, strictly local scheme. Let ζ_i be the generic point of D_i. By lemma 4.4, the curve $\mathcal{C}_{\zeta_i}$ has split singularities; its labelled graph $(\Gamma_{\zeta_i}, l_{\zeta_i})$ is obtained from (Γ_s, l_s) by contracting edges according to the procedure in section 4.1. Interpreting the different labels as colours, we can apply lemma 4.9 and conclude that $\mathcal{C}/S$ is aligned at s if and only if $h^1(\Gamma, \mathbb{Z}) = \sum_{i=1}^n h^1(\Gamma_i, \mathbb{Z})$. By lemma 4.5, we see that $\mu(s) = \sum_{i=1}^n \mu(\zeta_i)$, which is the condition for toric-additivity at s. This finishes the proof.

□

4.4 Toric-additivity and desingularization of curves

Let S be a connected, locally noetherian, regular base scheme S with a normal crossing divisor $D = D_1 \cup \ldots \cup D_n \subset S$, and let $\mathcal{C}/S$ be a nodal curve, such that the base change $\mathcal{C}_U/U := S \setminus D$ is smooth.

In [dJ96, 3.6], it is proven that if $\mathcal{C} \to S$ has *split* fibres, there exists a blow-up $\varphi\colon \mathcal{C}' \to \mathcal{C}$ such that $\mathcal{C}' \to S$ is still a nodal curve, and $\mathcal{C}'$ is *regular*. The condition of splitness implies that the irreducible components of the geometric fibres are smooth; or equivalently, that the dual graphs of the geometric fibres do not admit loops. We are going to introduce a condition on $\mathcal{C}/S$, weaker than splitness, and show that a statement analogous to the one in [dJ96, 3.6] holds for curves satisfying this condition.

Definition 4.12. Let $\mathcal{C} \to S$ be a nodal curve. We say that $\mathcal{C}/S$ is *disciplined* if, for every geometric point $\overline{s}$ of S, and $p \in \mathcal{C}_{\overline{s}}^{ns}$, at least one of the following is satisfied:

i) p belongs to two irreducible components of $\mathcal{C}_{\overline{s}}$;

ii) the thickness of p is a power of a regular parameter of $\mathcal{O}^{sh}_{S,\overline{s}}$.

We give first an auxiliary lemma:

Lemma 4.13. *Hypothesis as in the beginning of the subsection; suppose also that S is strictly local and that $\mathcal{C}/S$ is disciplined. Let $p \in \mathcal{C}_s^{ns}$ be a non-smooth point of the fibre over the closed point, such that p does not satisfy condition ii) of definition 4.12. Let X_1, X_2 be the distinct irreducible components of the closed fibre $\mathcal{C}_s$ containing p. Then there exists $i \in \{1, \dots, n\}$ and Y_1, Y_2 irreducible components of $\mathcal{C}_{\zeta_i}$ such that $X_1 \not\subset \overline{Y}_2 \supset X_2$ and $X_2 \not\subset \overline{Y}_1 \supset Y_1$.*

Proof. Let (Γ_s, l_s) be the labelled graph of $\mathcal{C}_s$. By hypothesis, the edge $e(p)$ corresponding to p has distinct extremal vertices, v_1 and v_2, and label $t_1^{m_1} \cdot \ldots \cdot t_l^{m_l}$, with $2 \leq l \leq n$ and $m_1, \dots, m_l \geq 1$. The fibres over the generic points $\zeta_1, \dots, \zeta_n$ have split singularities by lemma 4.4, so we can consider their labelled graphs (Γ_i, l_i). What we want to prove is that there exists $i \in \{1, \dots, l\}$ such that v_1 and v_2 are mapped to distinct vertices of (Γ_i, l_i) via the procedure described in section 4.1.

Suppose the contrary; as $e(p)$ is not contracted in any Γ_i, there exists a cycle γ in Γ_s, containing $e(p)$, such that for all $1 \leq i \leq l$, all edges $e \neq e(p)$ of γ are contracted in Γ_i. Let ζ_{12} be the generic point of $D_1 \cap D_2$; all edges $e \neq e(p)$ of γ are contracted in Γ_{12}, the labelled graph of $\mathcal{C}_{\zeta_{12}}$, and in particular v_1 and v_2 are mapped to the same vertex. The edge $e(p)$ is therefore mapped to a loop, with label $t_1^{m_1} t_2^{m_2}$. However, this contradicts the fact that $\mathcal{C} \to S$ is disciplined at ζ_{12}.

□

We introduce now some notation: given a scheme X, we will denote by $\mathrm{Sing}(X) \subseteq X$ the set of points that are not regular. We say that the *center* of a blow-up $\pi\colon Y \to X$ is the complement of the largest open $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism.

Lemma 4.14. *Hypotheses as in the beginning of the subsection. Suppose $f\colon \mathcal{C} \to S$ is disciplined. Then there is an étale surjective $g\colon S' \to S$ and a blow-up $\varphi\colon \mathcal{C}' \to \mathcal{C} \times_S S'$ such that*

- *the center of φ is contained in Sing($\mathcal{C} \times_S S'$);*
- *$\mathcal{C}'$ is a nodal curve over S', smooth over $g^{-1}(U)$;*
- *$\mathcal{C}'$ is regular.*

Proof. First, notice that the order in which the blow-ups of the curve and the étale covers of the base are taken does not matter, as blowing-up commutes with étale base change. After replacing S by a suitable étale cover, we may assume that D is a strict normal crossing divisor. We can now apply [dJ96, 3.3.2] and assume that $\mathrm{Sing}(\mathcal{C}) \subset \mathcal{C}$ has codimension at least 3. As a consequence of lemma 4.4, after a further étale covering, we may assume that for every generic point ζ of D, the fibre $\mathcal{C}_\zeta$ has split singularities.

Now, let E be an irreducible component of $\mathcal{C}_D = \mathcal{C} \times_S D$ and let $\pi\colon \mathcal{C}' \to \mathcal{C}$ be the blow-up of $\mathcal{C}$ along E. If $p \in E$ is a regular point of $\mathcal{C}$, f is an isomorphism at p, because E is cut out by one equation. Otherwise, the completion of the strict henselization at (a geometric point lying over) p is of the form

$$\widehat{\mathcal{O}}^{sh}_{\mathcal{C},\overline{p}} \cong \frac{\widehat{\mathcal{O}}^{sh}_{S,\overline{f(p)}}[[x,y]]}{xy - t_1^{m_1} \cdot \ldots \cdot t_l^{m_l}}$$

with $t_1, \ldots, t_n$ regular parameters cutting out D, $1 \leq l \leq n$ and positive integers $m_1, \ldots, m_l$. In fact, because the singular locus has codimension at least three, we have $l \geq 2$, and $m_1 = \ldots = m_l = 1$.

The ideal of the pullback of E to $\widehat{\mathcal{O}}^{sh}_{\mathcal{C},\overline{p}}$ is either (t_i) for some $1 \leq i \leq l$, or one between (x, t_i) and (y, t_i) for some $1 \leq i \leq l$. In the first case, π is an isomorphism at p. In the second case, one can compute explicitly the blowing up of $\mathrm{Spec}\, \mathcal{O}^{sh}_{\mathcal{C},\overline{p}}$ at the ideal (x, t_i) (or (y, t_i)) and find that $f'\colon \mathcal{C}' \to S$ is still a nodal curve, disciplined, with $\mathrm{Sing}(\mathcal{C})$ of codimension at least three, and such that for every generic point ζ of D the fibre $\mathcal{C}_\zeta$ has split singularities. We omit the explicit computations.

Let $Y \subset \mathcal{C}$ be the center of $\pi\colon \mathcal{C}' \to \mathcal{C}$. Then Y consists only of non-regular points, hence it has codimension at least 3. As $f\colon \mathcal{C}' \to S$ is a curve, the fibres of π have dimension at most 1, hence $\pi^{-1}(Y)$ has codimension at least 2 in

$\mathcal{C}$. It follows that there is a bijection between the irreducible components of $\mathcal{C}_D$ and $\mathcal{C}'_D$, given by taking the preimage under π. Now, $\pi^{-1}(E)$ is a divisor, and for any other irreducible component E' of $\mathcal{C}_D$ that is a divisor, $\pi^{-1}(E')$ is also a divisor. We conclude that $\pi^*\colon \mathcal{C}^* \to \mathcal{C}$, the composition of the blowing-ups of all irreducible component of $\mathcal{C}_D$, is such that every component of $\mathcal{C}^*_D$ is a divisor. Besides, as previously noticed, $f^*\colon \mathcal{C}^* \to S$ is a nodal curve, disciplined, and $\mathrm{Sing}(\mathcal{C}^*)$ has codimension at least three.

Assume now by contradiction that $\mathrm{Sing}(\mathcal{C}^*) \neq \emptyset$, and let $p \in \mathrm{Sing}(\mathcal{C}^*)$. Then without loss of generality the thickness at p is $(t_1 \cdot \ldots \cdot t_l)$ for some $2 \leq l \leq n$. Consider the base change $\mathcal{C}^*_T/T$, where T is the spectrum of some strict henselization at $s = f^*(p)$. For every i let ξ_i be the generic point of $D_i \cap T$. By lemma 4.13, for some $i \in \{1, \ldots, l\}$, there are distinct components Y_1, Y_2 of $\mathcal{C}^*_{\xi_i}$ whose closure in $\mathcal{C}^*_{T \cap D_i}$ contain p. Because the fibre $\mathcal{C}^*_{\zeta_i}$ has split singularities, we deduce that there are components X_1, X_2 of $\mathcal{C}^*_{\zeta_i}$ whose closures E_1, E_2 in $\mathcal{C}^*_{D_i}$ contain p. But then E_1 and E_2 are given by (x, t_1) and (y, t_1) in $\widehat{\mathcal{O}}^{sh}_{\mathcal{C}^*, \overline{p}}$. In particular, they are not divisors. This is a contradiction, and therefore $\mathrm{Sing}(\mathcal{C}^*) = \emptyset$. □

Lemma 4.15. *Hypotheses as in the beginning of the subsection. Suppose that $f\colon \mathcal{C} \to S$ is such that $\mathrm{Pic}^0_{\mathcal{C}/S}$ is toric-additive. Then $\mathcal{C}/S$ is disciplined.*

Proof. We may assume that S is strictly local, with closed point s, and with D given by a system of regular parameters $t_1 \ldots, t_n$. Let $p \in \mathcal{C}^{ns}_s$, with thickness $t_1^{m_1} \cdot \ldots \cdot t_l^{m_l}$ for some $1 \leq l \leq n$ and $m_1, \ldots, m_l \geq 1$. We have to show that if $l \geq 2$ then p lies on two components of $\mathcal{C}_s$.

Suppose by contradiction that $l \geq 2$ and that p lies on only one component of $\mathcal{C}_s$. The dual graph Γ over s has a loop L corresponding to p, with label $t_1^{m_1} \cdot \ldots \cdot t_l^{m_l}$. For $1 \leq i \leq n$ call Γ_i the dual graph of the fibre $\mathcal{C}_{\zeta_i}$ over the generic point of D_i. The loop L is preserved in the dual graphs Γ_i for $1 \leq i \leq l$. Let Γ' be the graph obtained by Γ by removing the loop L, and define similarly Γ'_i, $1 \leq i \leq l$. We have that

$$h^1(\Gamma', \mathbb{Z}) \leq \sum_{i=1}^{l} h^1(\Gamma'_i, \mathbb{Z}) + \sum_{j=l+1}^{n} h^1(\Gamma_j, \mathbb{Z}).$$

This inequality follows from the identification of the first Betti number with the toric rank of the corresponding fibre of $\mathrm{Pic}^0_{\mathcal{C}/S}$; and from eq. (15).

For every $1 \leq i \leq l$, $h^1(\Gamma_i, \mathbb{Z}) = h^1(\Gamma'_i, \mathbb{Z}) + 1$. Since $l \geq 2$, we find that $h^1(\Gamma, \mathbb{Z}) = h^1(\Gamma', \mathbb{Z}) + 1 < \sum_{i=1}^{n} h^1(\Gamma_i, \mathbb{Z})$. In terms of toric ranks of fibres of $\mathrm{Pic}^0_{\mathcal{C}/S}$, the same inequality reads $\mu(s) < \sum_{i=1}^{n} \mu(\zeta_i)$. This contradicts the fact that $\mathrm{Pic}^0_{\mathcal{C}_K/K}$ is toric-additive. □

4.5 Toric-additivity and Néron models

We consider again a base S and a nodal curve $\mathcal{C}/S$ as in the previous subsection. Theorem 1.1 tells us that if $\mathrm{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model over S, then $\mathcal{C}/S$ is aligned. However, not all aligned curves admit a Néron model for their jacobian; in this subsection we show that curves that are not disciplined do not admit one.

Lemma 4.16. *Assume that S is an excellent $\mathbb{Q}$-scheme. Suppose that $\mathcal{C}/S$ is such that $\mathrm{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model $\mathcal{N}$ over S. Then $\mathcal{C}/S$ is disciplined.*

Proof. We may assume that S is strictly henselian, with closed point s and residue field $k = k(s)$. Assume by contradiction that $\mathcal{C}/S$ is not disciplined. Then there is some $p \in \mathcal{C}_s^{ns}$ that belongs to only one component X of $\mathcal{C}_s$, and such that its thickness is $t_1^{m_1} \cdot \ldots \cdot t_l^{m_l}$ with $m_i \geq 1$ and $2 \leq l \leq n$. Let $q \in \mathcal{C}_s(k)$ be a smooth k-rational point belonging to the same component as p. By Hensel's lemma, there exists a section $\sigma_q \colon S \to \mathcal{C}$ through q. We claim that the same is true for p: let $\widehat{S}$ be the spectrum of the completion of $\mathcal{O}(S)$ at its maximal ideal and consider the morphism

$$W := \mathrm{Spec}\, \widehat{\mathcal{O}}^{sh}_{\mathcal{C},p} \cong \mathrm{Spec}\, \frac{\mathcal{O}(\widehat{S})[[x,y]]}{xy - t_1^{m_1} \cdot \ldots \cdot t_l^{m_l}} \to \widehat{S}.$$

This has a section given by $x = t_1^{m_1}$, $y = t_2^{m_2} \cdot \ldots \cdot t_l^{m_l}$. Composing the section with the canonical morphism $W \to \mathcal{C}$, gives a morphism $\widehat{\sigma}_p \colon \widehat{S} \to \mathcal{C}$ going through p. Because S is excellent and henselian, it has the Artin approximation property, and there exists a section $\sigma_p \colon S \to \mathcal{C}$ which agrees with $\widehat{\sigma}_p$ when restricted to the closed point s, hence going through p.

We write $\mathcal{F} := \mathcal{I}(\sigma_p) \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{O}(\sigma_q)$ for the coherent sheaf on $\mathcal{C}$ given by the tensor product of the ideal sheaf of σ_p with the invertible sheaf associated to the divisor σ_q. It is what is called a *torsion free, rank* 1 sheaf in the literature: it is S-flat, its fibres are of rank 1 at the generic points of fibres of $\mathcal{C}$, and have no embedded points. Notice that $\mathcal{F}$ is not an invertible sheaf, as $\dim_{k(p)} \mathcal{F} \otimes k(p) = 2$.

Let u_p and u_q be the restrictions of σ_p and σ_q to U. They are U-points of the smooth curve $\mathcal{C}_U/U$; the restriction of $\mathcal{F}$ to U is the invertible sheaf $\mathcal{F}_U = \mathcal{O}_{\mathcal{C}_U}(u_q - u_p)$. This is the datum of a U-point α of $\mathrm{Pic}^0_{\mathcal{C}_U/U}$: indeed, $\mathrm{Pic}(U) = 0$ because $\mathcal{O}(U)$ is a UFD, and C_U/U has a section, so $\mathrm{Pic}^0_{\mathcal{C}_U/U}(U) = \mathrm{Pic}^0(\mathcal{C}_U)$.

By the definition of Néron model, there is a unique section $\beta \colon S \to \mathcal{N}$ with $\beta_U = \alpha$. We write J for $\mathrm{Pic}^0_{\mathcal{C}/S}$. As J is semi-abelian, the canonical open immersion $J \to \mathcal{N}$ identifies J with the fibrewise-connected component of identity $\mathcal{N}^0$ (lemma 2.17). Write $\zeta_i, i = 1\dots,n$ for the generic points of the divisors D_i. Then $S_i := \operatorname{Spec}\mathcal{O}_{S,\zeta_i}$ is a trait, and the restriction $\mathcal{N}_{S_i}$ is a Néron model of its generic fibre. Therefore α_K extends uniquely to a section $\alpha_i\colon S_i \to \mathcal{N}_{S_i}$. As $\mathcal{F}_{S_i}$ is an invertible sheaf of degree 0 on every irreducible component of $\mathcal{C}_{\overline{\zeta}_i}$, $\mathcal{F}_{S_i}$ is a S_i-point of J_{S_i}, and α_i is given by $\mathcal{F}_{S_i}$. Therefore, the restriction of $\alpha\colon S \to \mathcal{N}$ to S_i factors through $J = \mathcal{N}^0$ for every $i = 1\dots,n$.

We denote now by Φ/S the étale group scheme of connected components of $\mathcal{N}$, and by $\Phi_{(l)}$ its l-primary part for a prime l. Lemma 5.2 tells us that, for every prime l different from the residue characteristic of S, the canonical morphism $\Phi_{(l)}(s) \to \bigoplus_{i=1}^n \Phi_{(l)}(\overline{\zeta}_i)$ is injective. By our assumption that S is a $\mathbb{Q}$-scheme, the canonical morphism

$$\Phi(s) \to \bigoplus_{i=1}^n \Phi(\overline{\zeta}_i)$$

is injective. This implies that α lands inside $J = \mathcal{N}^0$, or in other words that $\mathcal{F}_U$ extends to an invertible sheaf $\mathcal{L}$ on $\mathcal{C}$ such that $\mathcal{L}_s$ is of degree 0 on every component.

Now, let $Z \to S$ be a closed immersion, with Z a trait, such that the generic point ξ of Z lands into U (it is an easy check that such a closed immersion exists). As $\mathcal{F}_\xi$ and $\mathcal{L}_\xi$ define the same point of $\operatorname{Pic}^0_{\mathcal{C}_\xi/\xi}$, there are isomorphisms $\mu_\xi\colon \mathcal{F}_\xi \to \mathcal{L}_\xi$ and $\lambda_\xi\colon \mathcal{L}_\xi \to \mathcal{F}_\xi$. By the same argument as in [AK80, 7.8], μ_ξ and λ_ξ extend to morphisms $\mu\colon \mathcal{F}_Z \to \mathcal{L}_Z$ and $\lambda\colon \mathcal{L}_Z \to \mathcal{F}_Z$, which are non-zero on all fibres. Let's look at the restrictions to the closed fibre, $\mu_s\colon \mathcal{F}_s \to \mathcal{L}_s$, $\lambda_s\colon \mathcal{L}_s \to \mathcal{F}_s$. We know that $\mathcal{F}_s$ is trivial away from the component $X \subset \mathcal{C}_s$. So, if we write Y for the closure in $\mathcal{C}_s$ of the complement of X, we may restrict μ_s and λ_s to Y to get global sections l and l' of $\mathcal{L}_Y$ and $\mathcal{L}_Y^\vee$ respectively. Now, if $l = 0$, then the restriction μ_X of μ_s to X is non-zero, because μ_s is non-zero. If $l \neq 0$, as $\mathcal{L}_s$ is of degree zero on every component, we have $l(y) \notin \mathfrak{m}_y\mathcal{L}_y$ for every $y \in Y$, and in particular for $y \in Y \cap X$. It follows that also in this case $\mu_X \neq 0$. We can apply the same argument to l' and conclude that $\lambda_X \neq 0$. Then the compositions $\mu_X \circ \lambda_X\colon \mathcal{L}_X \to \mathcal{L}_X$ and $\lambda_X \circ \mu_X\colon \mathcal{F}_X \to \mathcal{F}_X$ are non-zero. As $\operatorname{End}_{\mathcal{O}_X}(\mathcal{F}_X) = k = \operatorname{End}_{\mathcal{O}_X}(\mathcal{L}_X)$, they are actually isomorphisms. It follows that $\mu_X\colon \mathcal{F}_X \to \mathcal{L}_X$ is an isomorphism. However, $\dim_{k(p)} \mathcal{F}_{k(p)} = 2$, while $\mathcal{L}_X$ is an invertible sheaf. This gives us the required contradiction.

□

Theorem 4.17. *Let S be a connected, locally noetherian, regular scheme, D a normal crossing divisor on S, $\mathcal{C} \to S$ a nodal curve smooth over $U = S \setminus D$.*

i) If $\operatorname{Pic}^0_{\mathcal{C}/S}$ is toric-additive, then $\operatorname{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model over S.

ii) If moreover S is an excellent $\mathbb{Q}$-scheme, the converse is also true.

Proof. Whether we are in the hypotheses of i) and ii), we know by lemmas 4.15 and 4.16 above that $\mathcal{C}/S$ is disciplined; hence by lemma 4.14 there exists an étale cover $g\colon S' \to S$ and a blow-up $\pi\colon \mathcal{C}' \to \mathcal{C}_{S'}$ which restricts to an isomorphism over $U' = U \times_S S'$, such that $\mathcal{C}'$ is regular.

Assume that $\mathrm{Pic}^0_{\mathcal{C}/S}$ is toric-additive. To show the existence of a Néron model over S, it is enough to show it over S'. The base change $\mathrm{Pic}^0_{\mathcal{C}_{S'}/S'}$ is toric-additive by lemma 3.8. The blow-up π does not affect $\mathcal{C}_{U'}$, so $\mathrm{Pic}^0_{\mathcal{C}'/S'}$ is still toric-additive. We can now apply proposition 4.11 and deduce that $\mathcal{C}'/S'$ is aligned. Hence by theorem 4.7, we find that $\mathrm{Pic}^0_{\mathcal{C}_{U'}/U'}$ admits a Néron model over S', proving i).

Now assume that S is a $\mathbb{Q}$-scheme and that $\mathrm{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model $\mathcal{N}$ over S. Then $\mathcal{N}' = \mathcal{N} \times_S S'$ is a Néron model for $\mathrm{Pic}^0_{\mathcal{C}'_{U'}/U'}$ over S'. Hence $\mathcal{C}'/S'$ is aligned by theorem 4.7; as $\mathcal{C}'$ is regular, we deduce by proposition 4.11 that $\mathrm{Pic}^0_{\mathcal{C}_{S'}/S'}$ is toric-additive. As toric-additivity descends along étale covers (lemma 3.8), $\mathrm{Pic}^0_{\mathcal{C}/S}$ is toric-additive. □

Corollary 4.18. *Let S be a connected, locally noetherian, regular, excellent $\mathbb{Q}$-scheme, D a normal crossing divisor on S, $\mathcal{C} \to S$ and $\mathcal{D} \to S$ two nodal curves, smooth over $U = S \setminus D$.*

Assume that over the generic point $\eta \in S$, there exists an isogeny

$$\mathrm{Pic}^0_{\mathcal{C}_\eta/\eta} \to \mathrm{Pic}^0_{\mathcal{D}_\eta/\eta}.$$

Then $\mathrm{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model over S if and only if $\mathrm{Pic}^0_{\mathcal{D}_U/U}$ does.

Proof. By lemma 3.10, $\mathrm{Pic}^0_{\mathcal{C}/S}$ is toric-additive if and only if $\mathrm{Pic}^0_{\mathcal{D}/S}$ is. By theorem 4.17, toric-additivity is equivalent to existence of a Néron model, and we conclude. □