A monodromy criterion for existence of Neron models and a result on semi-factoriality

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3 Toric-additivity

We work with the hypotheses of situation 2.5; we suppose that we are given an abelian scheme A/U of relative dimension d, and a semi-abelian scheme $\mathcal{A}/S$ with an isomorphism $\mathcal{A} \times_S U \to A$.

3.1 Definition of toric-additivity in the strictly local case

Assume that S is strictly local, with closed point s and residue field $k = k(s)$ of characteristic $p \geq 0$. The divisor D has finitely many irreducible components $D_1, \ldots, D_n$ for some $n \geq 0$.

We fix a prime $l \neq p$ and consider the Tate module $T_l A(K^s)$; we recall that it is a free $\mathbb{Z}_l$-module of rank $2d$ with an action of $\mathrm{Gal}(K^s|K)$, which factors via the surjection $\mathrm{Gal}(K^s|K) \to G := \pi_1^{t,l}(U) = \bigoplus_{i=1}^n I_i$, where $I_i = \mathbb{Z}_l(1)$ for each i.

Definition 3.1. Let $l \neq p$ be a prime. We say that the semi-abelian scheme $\mathcal{A}/S$ satisfies condition $\bigstar(l)$ if

$$T_l A(K^s) = \sum_{i=1}^n T_l A(K^s)^{\oplus_{j\neq i} I_j} \text{ or if } n = 0. \tag{20}$$

Remark 3.2.

- Whether $\mathcal{A}/S$ satisfies condition $\bigstar(l)$ depends only on the generic fibre A_K/K, and on the base S;
- suppose that $\mathcal{A}/S$ satisfies condition $\bigstar(l)$; let t be another geometric point of S, belonging to $D_1, D_2, \ldots, D_m$ for some $m \leq n$, and consider the strict henselization S' at t. Then the morphism

  $$\pi_1^{t,l}(U \times_S S') \to \pi_1^{t,l}(U)$$

  induced by $S' \to S$ is the natural inclusion

  $$\bigoplus_{i=1}^m I_j \to \bigoplus_{i=1}^n I_i.$$

  It can be easily seen that $\sum_{i=1}^n T_l A(K^s)^{\oplus_{j\neq i} I_j} \subseteq \sum_{i=1}^m T_l A(K^s)^{\oplus_{j\neq i} I_j}$; hence $\mathcal{A}_{S'}/S'$ also satisfies condition $\bigstar(l)$.
- Condition $\bigstar(l)$ is automatically satisfied if $n = 1$.

We are going to show that the validity of condition $\bigstar(l)$ is independent of the chosen prime $l \neq p$. We first need an auxiliary lemma, which we recommend to skip, as its only utility is to show that some specific submodules of the Tate module are direct summands. This simplifies some later proofs.

Lemma 3.3. *The Tate module $T_l A(K^s)$ satisfies the following properties:*

i) There exists a decomposition of $T := T_l A(K^s)$ into a direct sum

$$T \cong \bigoplus_{J \subseteq \{1,\ldots,n\}} T_J$$

where, for every $J \subseteq \{1, \ldots, n\}$, the submodule of invariants $T^{\bigoplus_{j\in J} I_j}$ is equal to $\bigoplus_{J' \supseteq J} T_{J'}$.

ii) The submodule $\sum_{i=1}^n T_l A(K^s)^{\oplus_{j\neq i} I_j}$ is a direct summand of $T_l A(K^s)$.

Proof. We start with the proof of i). Notice first that, for any submodule $V \subseteq T$ and any subgroup $H \subseteq G$, the submodule of invariants V^H is a direct summand of V; indeed, the quotient V/V^H is torsion-free. Now we proceed by induction on n. If $n = 1$, write $T_{\{1\}} := T^{I_1}$, and $T_\emptyset = T/T^{I_1}$. In this case we have $T \cong T_{\{1\}} \oplus T_\emptyset$ as wished. Now let $m \geq 2$, assume that the statement is true for $n = m - 1$, and let $n = m$. By inductive hypothesis, we can write

$$T \cong \bigoplus_{J \subseteq \{1,\ldots,m-1\}} T_J \tag{21}$$

as in the statement. Define, for every $J \subseteq \{1, \ldots, m\}$,

$$V_J = \begin{cases} (T_{J\cap\{1,\ldots,m-1\}})^{I_m} & \text{if } m \in J; \\ T_J/(T_J)^{I_m} & \text{if } m \notin J. \end{cases}$$

It is easy to show that $T \cong \bigoplus_{J\subseteq\{1,\ldots,m\}} V_J$. Now, let $J \subseteq \{1, \ldots, m\}$. Suppose first that $m \notin J$. Then we have

$$\begin{aligned} T^{\bigoplus_{j\in J} I_j} &\cong \bigoplus_{J\subseteq J'\subseteq\{1,\ldots,m-1\}} T_{J'} \cong \bigoplus_{J\subseteq J'\subseteq\{1,\ldots,m-1\}} T_{J'}/(T_{J'})^{I_m} \oplus (T_{J'})^{I_m} \cong \\ &\cong \bigoplus_{J\subseteq J'\subseteq\{1,\ldots,m-1\}} V_{J'} \oplus V_{J'\cup\{m\}} \cong \bigoplus_{J\subseteq J'\subseteq\{1,\ldots,m\}} V_{J'}. \end{aligned}$$

If instead $m \in J$, then

$$T^{\bigoplus_{j\in J} I_j} \cong (T^{\bigoplus_{j\in J\setminus\{m\}} I_j})^{I_m} \cong \bigoplus_{J\setminus\{m\}\subseteq J'\subseteq\{1,\ldots,m\}} (V_{J'})^{I_m}.$$

Now, for a subset $J' \subseteq \{1, \ldots, m\}$, $(V_{J'})^{I_m} \neq 0$ only if $m \in J'$; in this case, $(V_{J'})^{I_m} = V_{J'}$. It follows that

$$T^{\bigoplus_{j \in J} I_j} \cong \bigoplus_{J \subseteq J' \subseteq \{1,\ldots,m\}} V_{J'}.$$

This proves i).

For ii), notice that for all $i = 1, \ldots, n$, we have

$$T^{\bigoplus_{j \neq i} I_j} \cong T_{\{1,\ldots,n\}} \oplus T_{\{1,\ldots,n\} \setminus \{i\}}$$

and

$$\sum_{i=1}^{n} T^{\bigoplus_{j \neq i} I_j} \cong T_{\{1,\ldots,n\}} \oplus \bigoplus_{i=1}^{n} T_{\{1,\ldots,n\} \setminus \{i\}}.$$

Because of the decomposition of part i), we see that $\sum_{i=1}^{n} T^{\bigoplus_{j \neq i} I_j}$ is indeed a direct summand of T. $\square$

Recall the upper semi-continuous function (3) $\mu \colon S \to \mathbb{Z}_{\geq 0}$. It takes the value $\mu(s)$ at the closed point of S, and the value $\mu(\zeta_i)$ at each generic point ζ_i of D_i.

Recall the inequality (15),

$$\mu(s) \leq \sum_{i=1}^{n} \mu(\zeta_i). \tag{22}$$

Theorem 3.4. *Let S be a regular, strictly local scheme, with closed point s of residue characteristic $p \geq 0$, $D = \bigcup_{i=1}^{n} D_i$ a normal crossing divisor on S. Let A be an abelian scheme over $U = S \setminus D$, of relative dimension d, admitting a semi-abelian prolongation $\mathcal{A}/S$. Let $l \neq p$ be a prime.*

The following conditions are equivalent:

- *a) $\mathcal{A}/S$ satisfies condition $\bigstar(l)$.*
- *b) For $i = 1, \ldots, n$, let ζ_i be the generic point of D_i. The function $\mu \colon S \to \mathbb{Z}_{\geq 0}$ satisfies*

$$\mu(s) = \sum_{i=1}^{n} \mu(\zeta_i).$$

- *c) Let $G = \pi_1^t(U) = \bigoplus I_i$ with $I_i = \widehat{\mathbb{Z}}'(1)$. The Tate module $T_l A(K^s)$ decomposes as a direct sum*

$$T_l A(K^s) = V_1 \oplus V_2 \oplus \ldots \oplus V_n$$

of G-invariant submodules, such that for each $i = 1, \ldots, n$ and each $j \neq i$, $I_i \subset G$ acts trivially on V_j.

Proof. We will write shorthand T for $T_l A(K^s)$, μ for $\mu(s)$ and μ_i for $\mu(\zeta_i)$. Let us start with the equivalence $a) \Leftrightarrow b)$; we are going to proceed by induction on the number n. The case $n = 0$ being trivial, let first $n = 1$: in this case, condition $\bigstar(l)$ is automatically satisfied. We have to check that $\mu = \mu_1$, i.e. the toric rank at the closed point s is the same as the toric rank at the generic point of the (irreducible) divisor D. We know by eq. (22) that $\mu \leq \mu_1$; since $\mu\colon S \to \mathbb{Z}_{\geq 0}$ is upper-semicontinuous, we have the equality.

Let now N be an integer ≥ 2 and assume that the equivalence $a) \Leftrightarrow b)$ is true when $n = N - 1$; we show that it is true for $n = N$. In general we have $T \supseteq \sum_{i=1}^n T^{\oplus_{j \neq i} I_j}$, with equality if condition $\bigstar(l)$ is satisfied. We compare the ranks of the two sides. On the one hand, the rank of T is $2d$. Now write T_i for $T^{\oplus_{j \neq i} I_j}$. We have

$$\operatorname{rk} \sum_{i=1}^n T_i = \sum_{i=1}^n \operatorname{rk} T_i + \sum_{k=2}^n (-1)^k \sum_{J \subset \{1,\ldots,n\}, \#J = k} \operatorname{rk} \bigcap_{j \in J} T_j$$

by an inclusion-exclusion argument. However, for every $J \subset \{1, \ldots, n\}$ with $\#J \geq 2$, $\bigcap_{j \in J} T_j = T^G$. The equality above becomes

$$\operatorname{rk} \sum_{i=1}^n T_i = \sum_{i=1}^n \operatorname{rk} T_i + \sum_{k=2}^n (-1)^k \binom{n}{k} \operatorname{rk} T^G.$$

For every $i = 1, \ldots, n$, $\operatorname{rk} T_i = 2d - \mu(t_i) \geq 2d - \sum_{j \neq i} \mu_j$, where t_i is the generic point of $\bigcap_{j \neq i} D_j$. Also, $\sum_{k=2}^n (-1)^k \binom{n}{k} = 1 - n$, and $\operatorname{rk} T^G = 2d - \mu$. We obtain

$$\operatorname{rk} \sum_{i=1}^n T_i \geq 2nd - (n-1) \sum_{i=1}^n \mu_i + (1-n)(2d - \mu) = 2d + (n-1)(\mu - \sum_{i=1}^n \mu_i). \quad (23)$$

We have previously remarked that if condition $\bigstar(l)$ is satisfied, then it is satisfied also over S_i, the strict henselization at a geometric point lying over t_i; in this case, we can apply the inductive hypothesis: the inequality $\mu(t_i) \leq \sum_{j \neq i} \mu_j$ is an equality and thus eq. (23) is an equality as well.

We have obtained a chain of inequalities

$$\operatorname{rk} T = 2d \geq \operatorname{rk} \sum_{i=1}^n T_i \geq 2d + (n-1)(\mu - \sum_{i=1}^n \mu_i).$$

If condition $\bigstar(l)$ is satisfied, both $\geq$ signs in the line above are equalities, and therefore $(n-1)(\mu - \sum_{i=1}^n \mu_i) = 0$. Since $n-1>0$, we have indeed $\mu = \sum_{i=1}^n \mu_i$. Conversely, if $\mu = \sum_{i=1}^n \mu_i$, both inequalities are forced to be equalities; in particular $\operatorname{rk} T = \operatorname{rk} \sum_{i=1}^n T_i$. By lemma 3.3, $\sum_{i=1}^n T_i$ is a direct summand of T; hence $T = \sum_{i=1}^n T_i$, which is condition $\bigstar(l)$. This proves $a) \Leftrightarrow b)$.

Next, assume that a decomposition of T as in c) exists. Then, for every $1 \leq i \leq n$, $V_i \subseteq T^{\bigoplus_{j \neq i} I_j}$, and condition $\bigstar(l)$ is evidently satisfied; so we have $c) \Rightarrow a)$.

Finally, we prove $b) \Rightarrow c)$. Consider the canonical maps

$$\alpha\colon \bigoplus_{i=1}^n T^{t_i} \to \sum_{i=1}^n T^{t_i} = T^t; \quad \beta\colon T/T^G \to \bigoplus_{i=1}^n T/T^{I_i}.$$

Clearly, α is surjective and β is injective. However, because we have $\mu = \sum_{i=1}^n \mu_i$, comparing ranks we see that α is an isomorphism. The same is true for the analogous map

$$\alpha'\colon \bigoplus_{i=1}^n T'^{t_i} \to T'^t,$$

where $T' = T_l A'(K^s)$ and A'_K is the dual abelian variety. By lemma 2.20, T^G (resp. T^{I_i}) is orthogonal to T'^t (resp. T'^{t_i}) with respect to the pairing χ. Hence, β is obtained from α' by applying the functor $\operatorname{Hom}_{\mathbb{Z}_l}(\,\cdot\,, \mathbb{Z}_l(1))$. It follows that β is an isomorphism as well.

Notice that for every $i = 1, \ldots, n$, the inverse morphism β^{-1} identifies $\bigoplus_{j \neq i} T/T^{I_j}$ with the submodule T^{I_i}/T^G of T/T^G. Moreover, since T^G is a direct factor of T, we can choose a section $h\colon T/T^G \to T$. As h maps T^{I_i}/T^G into T^{I_i}, we see that the image of $\bigoplus_{j \neq i} T/T^{I_j}$ via $h \circ \beta^{-1}$ is contained in T^{I_i}.

Write $T^G = T^t \oplus W$ for some submodule W; and write $W_i := T^{t_i} \oplus (h \circ \beta^{-1})(T/T^{I_i})$ for each i. Then $T = W \oplus W_1 \oplus W_2 \oplus \ldots \oplus W_n$. For each i, I_i acts trivially on T^G, hence on W and T^{t_j} for all j. Moreover, we have shown that for $j \neq i$, I_i acts trivially on $(h \circ \beta^{-1})(T/T^{I_j})$. Therefore I_i acts trivially on W_j for $j \neq i$.

Now, we may write $V_1 = W \oplus W_1$, and $V_i = W_i$ for all $i \geq 2$. It remains only to show that V_i is I_i-invariant. For this, let e_i be a topological generator of I_i. For every $x \in T$, $y \in T^{I_i}$, we have

$$\chi(e_i x - x, y) = \chi(x, e_i y - y) = \chi(x, 0) = 1.$$

Therefore $e_i x - x \in (T^{I_i})^\perp = T^{t_i}$ for every $x \in T$. In particular, for every $x \in V_i$, $e_i x \in V_i + T^{t_i} = V_i$, as we wished to show.

□

A consequence of theorem 3.4 $a) \Leftrightarrow b)$, is that the validity of condition $\bigstar(l)$ is independent of the choice of prime $l \neq p$. It is sensible to introduce a new name for the condition:

Definition 3.5. We say that the semi-abelian scheme $\mathcal{A}/S$ is *toric-additive* if the three equivalent conditions of theorem 3.4 are satisfied for some prime number $l \neq p$ (equivalently, for all such primes l).

Notice that, although we talk of "toric-additivity of the semi-abelian scheme $\mathcal{A}/S$", toric-additivity depends only on the generic fibre A_K (in fact, on its torsion K^s-points) and on S. This is a consequence of theorem 3.4, but follows also from the fact that a semi-abelian extension $\mathcal{A}/S$ of A_K is unique up to unique isomorphism ([Del85, Théorème pag.132]).

Lemma 3.6. *Let $m_1, \ldots, m_n$ be positive integers and B be the $\Gamma(S, \mathcal{O}_S)$-algebra*

$$B = \frac{\Gamma(S, \mathcal{O}_S)[T_1, \ldots, T_n]}{T_1^{m_1} - r_1, \ldots, T_n^{m_n} - r_n} \tag{24}$$

Write $T = \operatorname{Spec} B$ and let $f \colon T \to S$ be the induced morphism of schemes. Then $\mathcal{A}/S$ is toric-additive if and only if $\mathcal{A}_T/T$ is toric-additive.

Proof. Notice that T is a regular strictly local scheme, so it makes sense to say that $\mathcal{A}_T/T$ is toric-additive. Now, clearly $f^{-1}(D) \to D$ is a homeomorphism, thus $\mathcal{A}/S$ satisfies condition ii) of theorem 3.4 if and only if $\mathcal{A}_T/T$ does. □

3.2 Global definition of toric additivity

We have defined toric-additivity over a strictly local base. We now remove this hypotheses and consider the more general case of situation 2.5.

Definition 3.7. We say that $\mathcal{A}/S$ is *toric-additive* at a geometric point s of S, if the base change $\mathcal{A} \otimes_S \operatorname{Spec} \mathcal{O}_{S,s}^{sh}$ to the strict henselization at s is toric-additive as in definition 3.5. We say that $\mathcal{A}/S$ is *toric-additive* if it is so at all geometric points s of S.

It is evident that toric-additivity is a property étale-local on the target. We actually have the stronger statement:

Lemma 3.8. *Toric-additivity is local on the target for the smooth topology.*

Proof. Given $f\colon T \to S$ smooth and surjective, the base change $D \times_S T$ is still a normal crossing divisor. Let x be a geometric point of T and call f_x the induced morphism

$$X := \operatorname{Spec} \mathcal{O}^{sh}_{T,x} \to Y := \operatorname{Spec} \mathcal{O}^{sh}_{S,f(x)}.$$

The image of a generic point ζ_i of $D_i \times_S X$ via f_x is a generic point of $D_i \times_S Y$; moreover the function $\mu\colon X \to \mathbb{Z}_{\geq 0}$ factors via Y. Thus it is clear that $\mathcal{A} \times_S X/X$ is toric-additive if and only if $\mathcal{A} \times_S Y/Y$ is. We deduce that $\mathcal{A} \times_S T/T$ is toric-additive if and only if $\mathcal{A}/S$ is. $\square$

Lemma 3.9. *Toric-additivity of $\mathcal{A}/S$ is an open condition on S.*

Proof. Suppose that $\mathcal{A}/S$ is toric-additive at a geometric point s. It is enough to show that $\mathcal{A}/S$ is toric-additive on an étale neighbourhood of s, since étale morphisms are open. We choose an étale neighbourhood of finite type $W \to S$ of s such that $D_W = D \times_S W$ is a strict normal crossing divisor and such that s belongs to all irreducible components $D_1, \dots, D_n$ of D_W. Let t be another geometric point of W; we want to show that $\mathcal{A}_W/W$ is toric-additive at t. This is true if $t \notin D_W$, so we may assume without loss of generality that t belongs to $D_1, \dots, D_m$ for some $1 \leq m \leq n$. Let ζ be a geometric point lying over the generic point of $D_1 \cap D_2 \cap \dots \cap D_m$; write W_ζ, W_t, W_s for the spectra of the strict henselizations of W at ζ, t, s respectively. The morphism $W_\zeta \to W$ factors via W_s; hence, by remark 3.2, $\mathcal{A}_W/W$ is toric-additive at ζ. We also have a natural map $W_\zeta \to W_t$. Choose a prime l different from the residue characteristics at t. The induced morphism $\pi_1^{t,l}(W_\zeta \cap U) = \mathbb{Z}_l(1)^m \to \pi_1^{t,l}(W_t \cap U) = \mathbb{Z}_l(1)^m$ is the identity. Because $\mathcal{A}_W/W$ is toric-additive at ζ, it follows that it is also at t, as we wished to show. $\square$

Lemma 3.10. *Let A and B be two abelian schemes over U, admitting semi-abelian prolongations $\mathcal{A}/S$ and $\mathcal{B}/S$ respectively. Suppose that over the generic fibre of S, there exists an isogeny $f\colon A_K \to B_K$. Then $\mathcal{A}/S$ is toric-additive if and only if $\mathcal{B}/S$ is so.*

Proof. We may assume that the base S is strictly local of residue characteristic $p \geq 0$. For a prime $l \neq p$ not dividing the degree of f, f induces an isomorphism of Galois modules $T_l A(K^s) \to T_l B(K^s)$. $\square$

Lemma 3.11. *Let*

$$0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{A}'' \to 0$$

be an exact sequence of semi-abelian schemes over S, whose restriction to U is abelian. Then $\mathcal{A}'$ and $\mathcal{A}''$ are toric-additive if and only if $\mathcal{A}$ is so.

Proof. We may assume that S is the spectrum of a strictly henselian local ring, with closed point s of residue characteristic $p \geq 0$. Let $l \neq p$ be a prime and T', T, T'' be the l-adic Tate modules $T_l A'(K^s), T_l A(K^s), T_l A''(K^s)$, endowed with a natural action of $G = \pi^{t,l}(U)$. As $A'(K^s)$ is l-divisible, we obtain an exact sequence of G-modules

$$0 \to T' \to T \to T'' \to 0.$$

Consider the induced map $\varphi \colon H^1(G, T') \to H^1(G, T)$; we claim that it is injective. An element of $H^1(G, T')$ is represented by a crossed homomorphism $f \colon G \to T'$ in $Z^1(G, T')$. Suppose that its image in $Z^1(G, T)$ is a coboundary; then there exists a $t \in T$ with $f(\sigma) = \sigma t - t$ for all $\sigma \in G$. Now, $\sigma t - t$ belongs to T^G, because $(\sigma - 1)^2 = 0$ for all $\sigma \in G$. It follows that $\ker \varphi \subset H^1(G, T'^G) = \mathrm{Hom}(G, T'^G)$. As the map $\mathrm{Hom}(G, T'^G) \to \mathrm{Hom}(G, T^G)$ is injective, we have $\ker \varphi = 0$, which proves the claim.

It follows that we have an exact sequence of G-invariant submodules,

$$0 \to T'^G \to T^G \to T''^G \to 0.$$

Taking ranks, we find that $\mu(s) = \mu'(s) + \mu''(s)$, where $\mu, \mu', \mu'' \colon S \to \mathbb{Z}_{\geq 0}$ are the toric rank functions for $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ respectively. Thus, these functions satisfy $\mu = \mu' + \mu''$.

Let now $\zeta_1, \ldots, \zeta_n$ be the generic points of the components $D_1, \ldots, D_n$ of D. If $\mathcal{A}'$ and $\mathcal{A}''$ are toric-additive, we have $\mu(s) = \mu'(s) + \mu''(s) = \sum_{i=1}^n \mu'(\zeta_i) + \sum_{i=1}^n \mu(\zeta_i) = \sum_{i=1}^n \mu(\zeta_i)$, which implies that $\mathcal{A}$ is toric-additive.

Conversely, if $\mathcal{A}$ is toric-additive, then $\mu(s) = \sum_{i=1}^n \mu(\zeta_i)$. Hence, $\mu'(s) + \mu''(s) = \sum_{i=1}^n \mu'(\zeta_i) + \sum_{i=1}^n \mu''(\zeta_i)$. This can be rewritten as $\mu'(s) - \sum_{i=1}^n \mu'(\zeta_i) = \sum_{i=1}^n \mu''(\zeta_i) - \mu''(s)$; here, eq. (15) tells us that the left-hand side is non-positive and that the right-hand side is non-negative; hence they are both zero, and the proof is complete.

□

3.3 Two examples

We give two examples, one of a semi-abelian scheme that is toric-additive, and one of one that is not. Let k be an algebraically closed field of characteristic zero, $S = \mathrm{Spec}\, k[[u, v]]$, and let D be the vanishing locus of uv.

Example 3.12. Consider the nodal projective curve $\mathcal{E} \subset \mathbb{P}^2_S$ given by the equation

$$Y^2 Z = X^3 - X^2 Z - uvZ^3.$$

The restriction $\mathcal{E}_U/U$ is an elliptic curve, which is canonically identified with its jacobian $\mathrm{Pic}^0_{\mathcal{E}_U/U}$; the smooth locus $\mathcal{E}^{sm}/S$ has a unique S-group scheme structure extending the one of $\mathcal{E}_U/U$, and is a semi-abelian scheme.

Let ζ_1, ζ_2 be the generic points of $D_1 = \{u = 0\}$ and $D_2 = \{v = 0\}$ respectively, and let s be the closed point $\{u = 0, v = 0\}$. The fibres of $\mathcal{E}^{sm}$ over ζ_1, ζ_2, s are all tori of dimension 1. It follows that $\mathcal{E}^{sm}$ is not toric-additive.

Example 3.13. Consider the nodal projective curve $\mathcal{E}' \subset \mathbb{P}^2_S$ given by the equation

$$Y^2Z = X^3 - X^2Z - uZ^3.$$

Again, $\mathcal{E}'_U = \mathrm{Pic}^0_{\mathcal{E}'_U/U}$; and the smooth locus $\mathcal{E}'^{sm}/S$ is a semi-abelian scheme. In this case, the fibre of $\mathcal{E}'$ over ζ_2 is smooth; so $\mu(\zeta_1) = 1, \mu(\zeta_2) = 0, \mu(s) = 1$. Thus $\mathcal{E}'$ is toric-additive.

4 Neron models of jacobians of stable curves

4.1 Generalities

Nodal curves

Definition 4.1. A *curve* C over an algebraically closed field k is a proper morphism of schemes $C \to \mathrm{Spec}\, k$, such that C is connected and its irreducible components have dimension 1. A curve C/k is called *nodal* if for every non-smooth point $p \in C$ there is an isomorphism of k-algebras $\widehat{\mathcal{O}}_{C,p} \to k[[x, y]]/xy$.

For a general base scheme S, a *nodal curve* $f : \mathcal{C} \to S$ is a proper, flat morphism of finite presentation, such that for each geometric point s of S the fibre $\mathcal{C}_s$ is a nodal curve.

We will denote by $\mathcal{C}^{ns}$ the subset of $\mathcal{C}$ of points at which f is not smooth. Seeing $\mathcal{C}^{ns}$ as the closed subscheme defined by the first Fitting ideal of $\Omega^1_{\mathcal{C}/S}$, we have for a nodal curve $\mathcal{C}/S$ that $\mathcal{C}^{ns}/S$ is finite, unramified and of finite presentation.

We report a lemma from [Hol17b].

Lemma 4.2 ([Hol17b], Prop.2.5)**.** *Let S be locally noetherian, $f : \mathcal{C} \to S$ be nodal, and p a geometric point of $\mathcal{C}^{ns}$ lying over $s \in S$. We have:*

i) there is an isomorphism

$$\widehat{\mathcal{O}}^{sh}_{\mathcal{C},p} \cong \frac{\widehat{\mathcal{O}}^{sh}_{S,s}[[x, y]]}{xy - \alpha}$$