A monodromy criterion for existence of Neron models and a result on semi-factoriality

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Issue Date: 2018-02-27

show in theorem 3.4 that it can be equivalently stated as a condition on the Tate module $T_l A(K^{sep})$, for any l invertible on S, or as a condition on the toric ranks of the fibres of the semi-abelian scheme $\mathcal{A}/S$.

Section 4 is devoted to the case of jacobians of curves. After recalling the results of [Hol17b], we establish the relation between toric-additivity and the property of existence of a Néron model for the jacobian (theorem 4.17).

In section 5, we work under the assumption that the base S is a $\mathbb{Q}$-scheme; we attempt to relate toric-additivity and the property of existence of Néron models in the case of abelian schemes. We introduce test-Néron models and prove that they exist and are unique if $\mathcal{A}/S$ is toric-additive (proposition 5.5 and theorem 5.6). After a result on descent of test-Néron models (proposition 5.7), we conclude the section by showing that test-Néron models are Néron models, under the assumption of toric-additivity (proposition 5.9).

2 Generalities

2.1 Normal crossing divisors and tame fundamental group

We work over a connected, regular, locally noetherian, base scheme S.

Definition 2.1. Given a regular, noetherian local ring R, a *regular system of parameters* is a minimal subset $\{r_1, \dots, r_d\} \subset R$ of generators for the maximal ideal $\mathfrak{m} \subset R$.

Definition 2.2. A *strict normal crossing divisor* D on S is a closed subscheme $D \subset S$ such that, for every point $s \in S$, the preimage of D in the local ring $\mathcal{O}_{S,s}$ is the zero locus of a product $r_1 \cdot \ldots \cdot r_n$, where $\{r_1, \dots, r_n\}$ is a subset of a regular system of parameters $\{r_1, \dots, r_d\}$ of $\mathcal{O}_{S,s}$.

Write $\{D_i\}_{i \in \mathcal{I}}$ for the set of irreducible components of D. Then each D_i, seen as a reduced closed subscheme of S, is regular and of codimension 1 in S; moreover, for every finite subset $\mathcal{J} \subset \mathcal{I}$, the intersection $\bigcap_{j \in \mathcal{J}} D_j$ is regular, and each of its irreducible components has codimension $|\mathcal{J}|$.

Definition 2.3. A *normal crossing divisor* D on S is a closed subscheme $D \subset S$ for which there exists an étale surjective morphism $S' \to S$ such that the base change $D \times_S S'$ is a strict normal crossing divisor on S'.

Notice that for every geometric point s of S, the pullback of a normal crossing divisor D to the spectrum of the strict henselization $\mathcal{O}^{sh}_{S,s}$ is a strict normal crossing divisor.

Definition 2.4. A *trait* Z is an affine scheme with $\mathcal{O}(Z)$ a discrete valuation ring. Suppose we are given a morphism $f\colon Z \to S$ and a normal crossing divisor D on S; we say that f is *transversal to* D if for every component D_i of D, $D_i \times_S Z$ is a reduced point or is empty.

We can now introduce the hypotheses with which we will work for most of this part:

Situation 2.5. Let S be a regular, locally noetherian connected scheme, $D = \bigcup_{i\in\mathcal{I}} D_i$ a normal crossing divisor on S. We will denote by U the open $S \setminus D$, by η the generic point of S and by K the residue field $k(\eta)$. A separable closure of K will be denoted by K^s. Finally, we write ζ_i for the generic point of the irreducible component D_i of D.

Situation 2.6. In situation 2.5, we will often reduce to the simpler case where S is the spectrum of a strictly henselian local ring R. In this case, we say that it is a *strictly local* scheme. We write s for its closed point and $p \geq 0$ for its residue characterstic. We can write the normal crossing divisor D as a union $\bigcup_{i=1}^n \operatorname{div}(r_i)$ where $r_1, \ldots, r_n \in R$ form a subset of a regular system of parameters for R.

Suppose we are in situation 2.6. It is a consequence of Abhyankar's Lemma ([Gro71, XIII, 5.2]) that every finite etale morphism $V \to U$, tamely ramified over D ([Gro71, XIII, 3.2.c)]), with V connected, is dominated by a finite étale W/U given by

$$\mathcal{O}(W) = \frac{\mathcal{O}(U)[T_1, \ldots, T_n]}{T_1 - r_1^{m_1}, \ldots, T_n - r_n^{m_n}}$$

where the integers $m_1, \ldots, m_n$ are coprime to p. Denoting by $\mu_{r,U}$ the group-scheme of r-roots of unity, it follows that $\underline{\operatorname{Aut}}_U(W) = \prod_{i=1}^n \mu_{m_i,U}$. Then, the *tame fundamental group* of U is

$$\pi_1^t(U) = \prod_{l \neq p} \mathbb{Z}_l(1)^n.$$

Here $\mathbb{Z}_l(1) = \lim \mu_{l^r}(U)$ is non-canonically isomorphic to $\mathbb{Z}_l$, an isomorphism being given by a choice of a compatible system $(z_{l^r})_{r\geq 1}$ of primitive l^r-roots of unity. We will sometimes write $\widehat{\mathbb{Z}}'(1)$ in place of $\prod_{l\neq p} \mathbb{Z}_l(1)$.

For a prime $l \neq p$, the factor $\mathbb{Z}_l(1)^n$ of $\pi_1^t(U))$ is the biggest pro-l quotient of $\pi_1^t(U)$ and will be denoted by $\pi_1^{t,l}(U)$. It is the automorphism group of the fibre functor of finite étale morphisms $V \to U$ of degree a power of l.

2.2 Néron models of abelian schemes

The definition of Néron model

Let now S be any scheme, $U \subset S$ an open and A/U an abelian scheme.

Definition 2.7. A *Néron model* for A over S is a smooth, separated algebraic space [1] $\mathcal{N}/S$ of finite type, together with an isomorphism $\mathcal{N} \times_S U \to A$, satisfying the following universal property: for every smooth morphism of schemes $T \to S$ and U-morphism $f \colon T_U \to A$, there exists a unique morphism $g \colon T \to \mathcal{N}$ such that $g_{|U} = f$.

It follows immediately from the definition that a Néron model is unique up to unique isomorphism; moreover, applying its defining universal property to the morphisms $m \colon A \times_U A \to A, i \colon A \to A$, and $0_A \colon U \to A$ defining the group structure of A, we see that $\mathcal{N}/S$ inherits from A a unique S-group-space structure.

We also introduce a similar object, which satisfies a weaker universal property:

Definition 2.8. A *weak Néron model* for A over S is a smooth, separated algebraic space $\mathcal{N}/S$ of finite type, together with an isomorphism $\mathcal{N} \times_S U \to A$, satisfying the following universal property: every section $U \to A$ extends uniquely to a section $S \to \mathcal{N}$.

In particular, a Néron model is a weak Néron model. Notice that in the case of weak Néron models, we do not have any uniqueness statement, and they need not inherit a group structure from A.

We point out that our definition 2.8 of weak Néron model differs slightly from the one normally found in the literature: the latter requires that the universal property is satisfied for all $T \to S$ finite étale.

Base change properties

We proceed to analyse how Néron models behave under different types of base change. In general, the property of being a Néron model is not stable under arbitrary base change. However, we have that:

Lemma 2.9. *Let $\mathcal{N}/S$ be a Néron model of A/U; let $S' \to S$ be a smooth morphism and $U' = U \times_S S'$. Then the base change $\mathcal{N} \times_S S'$ is a Néron model of $A_{U'}$.*

[1] defined as in [Sta16]TAG 025Y.)

Proof. Let $X \to S'$ be a smooth scheme with a morphism $f : X_{U'} \to \mathcal{A}_{U'}$; by composition with the smooth morphism $S' \to S$ we obtain a smooth scheme $X \to S$ and a map $X \times_S U \to A_U$, which extends uniquely to an S-morphism $X \to \mathcal{N}$. This is the datum of an S'-morphism $X \to \mathcal{N} \times_S S'$ extending f. □

Lemma 2.10. *Let $\mathcal{N}/S$ be a smooth, separated algebraic space of finite type with an isomorphism $\mathcal{N} \times_S U \to A$. Let $S' \to S$ be a faithfully flat morphism and write $U' = U \times_S S'$. If $\mathcal{N} \times_S S'$ is a Néron model of $A \times_U U'$, then $\mathcal{N}/S$ is a Néron model of A.*

Proof. We first show that $\mathcal{N}/S$ satisfies the universal property of Néron models when the smooth morphism $T \to S$ is the identity. So, let $f : U \to A$ be a section of A/U. To show that f extends to a section $S \to \mathcal{N}$ we only need to check that the schematic closure X of $f(U)$ inside $\mathcal{N}$ is faithfully flat over S: indeed, $X \to S$ is birational and separated; if it is also flat and surjective it is automatically an isomorphism. Now, by base change of f we get a closed immersion $f' : U' \to A \times_U U'$, which extends to a section $g' : S' \to \mathcal{N} \times_S S'$ by hypothesis. The schematic image $g'(S')$ is necessarily the schematic closure of $f'(U')$ inside $\mathcal{N} \times_S S'$; since taking the schematic closure commutes with faithfully flat base change, we have $g'(S') = X \times_S S'$. We deduce that $X \to S$ is faithfully flat, as its base change via $S' \to S$ is such. Hence $f : U \to A$ extends to a section $g : S \to \mathcal{N}$. The uniqueness of the extension is a consequence of the separatedness of $\mathcal{N}$.

Next, let $T \to S$ be smooth and let $f : T_U \to A$. In order to extend f to a morphism $g : T \to \mathcal{N}$, it is enough to show that $\mathcal{N} \times_S T$ satisfies the extension property for sections $T_U \to A \times_U T_U$. By the previous paragraph, it is enough to know that $(\mathcal{N} \times_S T) \times_S S' = (\mathcal{N} \times_S S') \times_S T$ is a Néron model of $(A \times_U T_U) \times_U U'$. This is true by lemma 2.9, concluding the proof. □

Lemma 2.11. *Let A/U be abelian, $f : S' \to S$ a smooth surjective morphism, $U' = U \times_S S'$, and $\mathcal{N}'/S'$ a Néron model of $A \times_S S'$. Then there exists a Néron model $\mathcal{N}/S$ for A.*

Proof. Write $S'' := S' \times_S S'$, $p_1, p_2 : S'' \to S'$ for the two projections and $q : S'' \to S$ for $f \circ p_1 = f \circ p_2$. By lemma 2.9, both $p_1^*\mathcal{N}$ and $p_2^*\mathcal{N}$ are Néron models of q^*A. By the uniqueness of Néron models, we obtain a descent datum for $\mathcal{N}'$ along $S' \to S$. Effectiveness of descent data for algebraic spaces ([Sta16]TAG 0ADV) yields a smooth, separated algebraic space $\mathcal{N}/S$ of finite type. By lemma 2.10, this is a Néron model for A/U. □

Although Néron models are not stable under base change (not even flat), they are preserved by localizations, as we see in the following lemma:

Lemma 2.12. *Assume S is locally noetherian. Let s be a point (resp. geometric point) of S and $\widetilde{S}$ the spectrum of the localization (resp. strict henselization) at s. Suppose that $\mathcal{N}/S$ is a Néron model for A/U. Then $\mathcal{N} \times_S \widetilde{S}$ is a Néron model for $A \times_U \widetilde{U}$, where $\widetilde{U} = \widetilde{S} \times_S U$.*

Proof. Let $\widetilde{Y} \to \widetilde{S}$ be a smooth scheme and $\widetilde{f} \colon \widetilde{Y}_{\widetilde{U}} \to A_{\widetilde{U}}$ a morphism. We may assume that $\widetilde{Y}$ is of finite type over $\widetilde{S}$, hence of finite presentation. By [GD67, 3, 8.8.2] there exist an open neighbourhood (resp. étale neighbourhood) S' of s, a scheme $Y' \to S'$ restricting to $\widetilde{Y}$ over $\widetilde{S}$, and a $(U \times_S S')$-morphism $f' \colon Y' \times_{S'} (U \times_S S') \to \mathcal{N} \times_S (U \times_S S')$ restricting to $\widetilde{f}$ on $\widetilde{U}$. By lemma 2.9, $\mathcal{N} \times_S S'$ is a Néron model of $\mathcal{N} \times_S (U \times_S S')$, hence we get a unique extension $g' \colon Y' \to \mathcal{N} \times_S S'$ of f'. The base-change of g' via $\widetilde{S} \to S'$ gives us the required unique extension of $\widetilde{f}$. □

Proposition 2.13. *Assume that S is regular. If $\mathcal{A}/S$ is a an abelian algebraic space, then it is a Néron model of its restriction $\mathcal{A} \times_S U$.*

Proof. Using lemma 2.10, we may assume that S is strictly local and that $\mathcal{A}/S$ is a scheme. We identify $\mathcal{A}$ with its double dual $\mathcal{A}'' = \operatorname{Pic}^0_{\mathcal{A}'/S}$. Now let $T \to S$ be smooth and $f \colon T_U \to \mathcal{A}_U$. Then f corresponds to an element of $A_U(T_U) = \operatorname{Pic}^0_{\mathcal{A}'/S}(T_U) = \operatorname{Pic}^0(\mathcal{A}'_{T_U}) / \operatorname{Pic}^0(T_U)$. Let $\mathcal{L}_U$ be an invertible sheaf with fibres of degree 0 on $\mathcal{A}'_{T_U}$ mapping to f in $\mathcal{A}_U(T_U)$. As $\mathcal{A}'_T$ is regular, $\mathcal{L}_U$ extends to an invertible sheaf of degree 0 on $\mathcal{A}'_T$, which yields a T-point of $\mathcal{A}'' = \mathcal{A}$ extending f. The uniqueness of the extension follows from the separatedness of $\mathcal{A}/S$. □

We conclude the subsection by stating the main theorem about Néron models in the case where the base S is of dimension 1.

Theorem 2.14 ([BLR90], 1.4/3)**.** *Let S be a connected Dedekind scheme with fraction field K and let A/K be an abelian variety. Then there exists a Néron model $\mathcal{N}$ over S for A/K.*

2.3 Semi-abelian models and the action of inertia

Semi-abelian schemes

Definition 2.15. Let κ be a field and G/κ a smooth, commutative κ-group scheme of finite type. We say that G/κ is *semi-abelian* if it fits into an exact sequence of fppf-sheaves over κ

$$0 \to T \to G \to B \to 0 \tag{2}$$

where T/κ is a torus and B/κ an abelian variety. We call $\mu := \dim T$ the *toric rank* of G and $\alpha := \dim B$ its *abelian rank*. These two numbers do not depend on the choice of exact sequence (2), and are stable under base field extensions. Notice that G is automatically geometrically connected.

For a general base scheme S, a smooth commutative S-group scheme $\mathcal{G}/S$ of finite type is *semi-abelian* if for all points $s \in S$, the fibre $\mathcal{G}_s/k(s)$ is semi-abelian.

Given a semi-abelian scheme $\mathcal{G}/S$, we define for later use a function

$$\mu \colon S \to \mathbb{Z}_{\geq 0} \tag{3}$$

which associates to a point $s \in S$ the toric rank of $\mathcal{G}_s$. It can be shown that it is an upper semi-continuous function.

Analogously we can define

$$\alpha \colon S \to \mathbb{Z}_{\geq 0} \tag{4}$$

for the abelian rank of fibres. The sum $\mu + \alpha$ is the locally constant function with value the relative dimension of $\mathcal{G}/S$.

Situation 2.16. For the rest of part I, we assume that we are in situation 2.5 and that we are also given

- an abelian scheme A/U of relative dimension $d \geq 0$;
- a smooth, separated S-group scheme of finite presentation $\mathcal{A}/S$, together with an isomorphism $\mathcal{A} \times_S U \to A$, such that the fibrewise-connected component of identity $\mathcal{A}^0/S$ is semi-abelian.

The assumption that such a semi-abelian extension of A exists tells us a lot about the structure of a Néron model $\mathcal{N}/S$ of A (provided that it exists):

Lemma 2.17. *Suppose A/U admits a Néron model $\mathcal{N}/S$. Then the canonical morphism $\mathcal{A} \to \mathcal{N}$ is an open immersion, and induces an isomorphism from $\mathcal{A}^0$ to the fibrewise-connected component of identity $\mathcal{N}^0$.*

Proof. The fact that $\mathcal{A} \to \mathcal{N}$ is an open immersion follows from [GRR72, IX, Prop. 3.1.e]. For every point $s \in S$ of codimension 1, the restriction of $\mathcal{N}^0$ to the local ring $\mathcal{O}_{S,s}$ is the Néron model of its generic fibre, by lemma 2.12. It follows by [Ray70b, XI, 1.15] that the induced morphism $\mathcal{A}^0 \to \mathcal{N}^0$ is an isomorphism. □

In particular, the fibrewise-connected component of $\mathcal{N}^0/S$ is semi-abelian.

The Tate module

For the rest of section 2, we will assume that S is strictly local, with closed point s and residue field $k = k(s)$ of characteristic $p \geq 0$.

Let l be a prime different from p and $r \geq 0$ an integer; we denote by $\mathcal{A}[l^r]$ the kernel of the multiplication map

$$l^r \colon \mathcal{A} \to \mathcal{A}.$$

It is a closed subgroup scheme of $\mathcal{A}$, étale and quasi-finite over S. Its restriction $\mathcal{A}[l^r]_U/U$ is a finite, étale U-group scheme of order l^{2rd}. Because its order is coprime to p, $\mathcal{A}[l^r]_U/U$ is tamely ramified over D. It follows that the action of $\mathrm{Gal}(K^s|K)$ on $A[l^r](K^s)$ factors via the quotient map

$$\mathrm{Gal}(K^s|K) \to \pi_1^t(U) = \widehat{\mathbb{Z}}'(1)^n.$$

We write G for $\pi_1^t(U)$ and I_i for the i-th copy of $\widehat{\mathbb{Z}}'(1)$, so that $G = \bigoplus_{i=1}^n I_i$.

Let $T_l\mathcal{A}$ be the l-adic sheaf $\lim_r \mathcal{A}[l^r]$ on S. The group of K^s-valued points of its generic fibre is the *Tate module*

$$T_lA(K^s) = \lim A[l^r](K^s),$$

a free $\mathbb{Z}_l$-module of rank $2d$, which inherits a continuous action of $\pi_1^t(U)$.

Now, over the closed point $s \in S$ there exists an exact sequence

$$0 \to T \to \mathcal{A}_s^0 \to B \to 0$$

as in (2); for a prime $l \neq p$, $\mathcal{A}_s^0$ is l-divisible and it follows that we have an exact sequence of l-adic sheaves

$$0 \to T_lT \to T_l\mathcal{A}_s^0 \to T_lB \to 0$$

which in turn gives an exact sequence of $\mathbb{Z}_l$-modules

$$0 \to T_lT(k) \to T_l\mathcal{A}_s^0(k) \to T_lB(k) \to 0 \qquad (5)$$

Write μ and α for $\mu(s)$ and $\alpha(s)$. Taking ranks in the exact sequence (5), we have

- $\mathrm{rk}\, T_lT(k) = \mu$,
- $\mathrm{rk}\, T_lB(k) = 2\alpha$,
- $\mathrm{rk}\, T_l\mathcal{A}_s^0(k) = \mu + 2\alpha = 2d - \mu$.

The following lemma is particularly useful:

Lemma 2.18. *The inclusion of l-adic sheaves $T_l\mathcal{A}^0 \hookrightarrow T_l\mathcal{A}$ restricts to an equality over the closed point s; that is,*

$$(T_l\mathcal{A})_s = (T_l\mathcal{A}^0)_s \tag{6}$$

Proof. To prove this, it is enough to check that $T_l\mathcal{A}_s(k) = T_l\mathcal{A}_s^0(k)$. If $(x_v)_v$ is an element of the left-hand side, each x_v is a l^v-torsion element of $\mathcal{A}_s(k)$ infinitely divisible by l. Let Φ be the group of components of $\mathcal{A}_s$; it is a finite abelian group, by the assumption that $\mathcal{A}$ is of finite presentation. Let φ_v be the image of x_v in Φ; then φ_v belongs to the l^v-torsion subgroup of Φ. Moreover φ_v is infinitely divisible by l; it follows that $\varphi_v = 0$, and that x_v lies in $\mathcal{A}_s^0(k)$. □

The fixed part of the Tate module

Consider again the l^r-torsion subscheme $\mathcal{A}[l^r]/S$. As S is henselian, there is a canonical decomposition

$$\mathcal{A}[l^r] = \mathcal{A}[l^r]^f \sqcup \mathcal{A}[l^r]'$$

where $\mathcal{A}[l^r]^f/S$, called *the fixed part* of $\mathcal{A}[l^r]$, is finite over S and $\mathcal{A}[l^r]'_s = \emptyset$. It can be shown that $\mathcal{A}[l^r]^f$ is a subgroup-scheme of $\mathcal{A}[l^r]$, étale over S. As S is strictly-henselian, it is a disjoint union of copies of S, and we find

$$\mathcal{A}[l^r]^f(K^s) = \mathcal{A}[l^r]^f(K) = \mathcal{A}[l^r]^f(S) = \mathcal{A}[l^r]^f_s(k) = \mathcal{A}[l^r]_s(k). \tag{7}$$

We define the *fixed part* of $T_l\mathcal{A}$ as the limit $T_l\mathcal{A}^f = \lim \mathcal{A}[l^r]^f$; this is a free l-adic sheaf, whose group of K^s-valued point is a submodule of the Tate module

$$T_l\mathcal{A}^f(K^s) =: T_lA(K^s)^f \subseteq T_lA(K^s).$$

By taking the limit in (7) and applying lemma 2.18, we find

$$T_lA(K^s)^f = T_l\mathcal{A}^f(S) = T_l\mathcal{A}_s(k) = T_l\mathcal{A}_s^0(k). \tag{8}$$

This last equality enables us to determine the rank of the fixed submodule of the Tate module,

$$\operatorname{rk} T_lA^f(K^s) = 2d - \mu = \operatorname{rk} T_lA(K^s) - \mu \tag{9}$$

Moreover, we have that

$$T_l\mathcal{A}_s^0(k)\otimes\mathbb{Z}/l^r\mathbb{Z}=\mathcal{A}_s^0[l^r](k) \tag{10}$$

since $\mathcal{A}_s^0(k)$ is l-divisible. Hence,

$$T_lA(K^s)^f\otimes_{\mathbb{Z}_l}\mathbb{Z}/l^r\mathbb{Z}=\mathcal{A}_s^0[l^r](k). \tag{11}$$

In other words, $T_lA^f(K^s)\otimes_{\mathbb{Z}_l}\mathbb{Z}/l^r\mathbb{Z}$ is the submodule of $A[l^r](K^s)$ consisting of those points that extend to sections of the fibrewise-connected component of identity $\mathcal{A}^0$.

The following proposition gives us an alternative interpretation of the fixed part of $T_lA(K^s)$:

Proposition 2.19. *The submodule $T_lA(K^s)^f$ is the submodule $T_lA(K^s)^G\subseteq T_lA(K^s)$ of elements fixed by $G=\pi_1^t(U)$.*

Proof. We treat first the case $\dim S=1$; so S is the spectrum of a discrete valuation ring. In this case, A/K admits a Néron model, $\mathcal{N}/S$. By assumption, the fibrewise-connected component of identity $\mathcal{A}^0$ is semi-abelian, and we have an identification $\mathcal{N}^0=\mathcal{A}^0$ (lemma 2.17).

Now, equality (8) and lemma 2.18 tell us that

$$T_lA(K^s)^f=T_l\mathcal{A}_s^0(k)=T_l\mathcal{N}_s^0(k)=T_l\mathcal{N}_s(k).$$

By Hensel's lemma, $\mathcal{N}_s[l^r](k)=\mathcal{N}[l^r](S)$ and by the definition of Néron model the latter is equal to $\mathcal{N}_K[l^r](K)=A[l^r](K^s)^G$. Hence, $T_lA(K^s)^G=\lim A[l^r](K^s)^G$ is equal to $T_l\mathcal{N}_s(k)$ and we are done.

Let now S have dimension $\dim S\geq 2$. First, observe that $T_lA(K^s)^f\subseteq T_lA(K^s)^G$: indeed, as $T_l\mathcal{A}^f$ is free, its K^s-valued point are actually K-valued. We show the reverse inclusion. We start by claiming that there exists a closed subscheme $Z\subset S$, regular and of dimension 1, such that $Z\not\subseteq D$. For this, let $\{t_1,\dots,t_n\}$ be a system of regular parameters of $\mathcal{O}(S)$, cutting out the divisor D. We complete the above set to a maximal system $\{t_1,\dots,t_n,t_{n+1},\dots,t_{\dim S}\}$ of regular parameters and let $Z=Z(t_1-t_2,t_2-t_3,\dots,t_{n-1}-t_n,t_{n+1},t_{n+2},\dots,t_{dimS})$. Now, $\mathcal{O}(Z)$ is a strictly henselian discrete valuation ring, and the generic point ζ of Z lies in U. We let $L=k(\zeta)$ and $H=\mathrm{Gal}(L^s|L)$ for some separable closure $L\hookrightarrow L^s$. Since $\mathcal{A}[l^r]$ is finite étale over U, we have $\mathcal{A}[l^r](K)\subseteq\mathcal{A}[l^r](L)$ and by passing to the limit we obtain $T_lA(K^s)^G\subseteq T_lA(L^s)^H$. Moreover, by the dimension 1 case, $T_lA(L^s)^H=T_l(\mathcal{A}_Z)(L^s)^f=T_l\mathcal{A}_s(k)$; the latter is equal to $T_lA(K^s)^f$, concluding the proof. □

The toric part of the Tate module

Denote by $\mathcal{T}_s$ the biggest subtorus of the semiabelian scheme $\mathcal{A}_s^0$; we have an inclusion of the l^r-torsion

$$\mathcal{T}_s[l^r] \subseteq \mathcal{A}_s^0[l^r].$$

As the restriction functor between the category of finite étale S-schemes and the category of finite étale k-schemes is an equivalence of categories, we obtain a canonical finite étale S-subscheme of $\mathcal{A}^0[l^r]$, called the *toric part* of $\mathcal{A}^0[l^r]$,

$$\mathcal{A}^0[l^r]^t \hookrightarrow \mathcal{A}^0[l^r]^f \hookrightarrow \mathcal{A}^0[l^r]$$

such that $A^0[l^r]^t \otimes_S k = \mathcal{T}_s[l^r]$.

Taking the limit, we find a free subsheaf $T_l\mathcal{A}^t$ of $\lim \mathcal{A}^0[l^r]^f = \lim \mathcal{A}[l^r]^f = T_l\mathcal{A}^f$. Then, passing to the generic fibre, we obtain a submodule $T_lA(K^s)^t$ of $T_lA(K^s)^f = T_lA(K^s)^G \subseteq T_lA(K^s)$, which we call *toric part* of $T_lA(K^s)$. Its rank is of course the rank of the $\mathbb{Z}_l$-module $T_l\mathcal{T}_s(k)$, that is

$$\operatorname{rk} T_lA(K^s)^t = \mu. \tag{12}$$

To summarize, we have a filtration of the Tate module

$$0 \xrightarrow{\mu} T_lA(K^s)^t \xrightarrow{2\alpha} T_lA(K^s)^f \xrightarrow{\mu} T_lA(K^s)$$

where the numbers on top of the arrows are the ranks of the successive quotients in the filtration.

The dual abelian variety and Weil pairing

We will now only work with the semi-abelian scheme $\mathcal{A}^0 \subset \mathcal{A}$; for this reason, we will write simply $\mathcal{A}$ for it, rather than $\mathcal{A}^0$. Consider the dual abelian variety A'_K of A_K. By [MB85, IV, 7.1], there exists a unique semi-abelian scheme $\mathcal{A}'/S$ extending A'_K. Let $\varphi\colon A_K \to A'_K$ be an isogeny; it extends uniquely to an isogeny $\mathcal{A} \to \mathcal{A}'$, inducing isogenies

$$\mathcal{T}_s \to \mathcal{T}'_s, \quad \mathcal{B}_s \to \mathcal{B}'_s$$

between the toric and abelian parts of $\mathcal{A}_s$ and $\mathcal{A}'_s$. We deduce the equality between the toric and abelian ranks

$$\mu = \mu' \quad \alpha = \alpha'.$$

By [MB85, II, 3.6] the natural functor

$$BIEXT(\mathcal{A}, \mathcal{A}'; \mathbb{G}_{m,S}) \to BIEXT(\mathcal{A}_K, \mathcal{A}'_K; \mathbb{G}_{m,K})$$

is an equivalence of categories; thus the Poincaré biextension on $A_K \times_K A'_K$ extends uniquely to a biextension on $\mathcal{A} \times_S \mathcal{A}'$, and we obtain for $l \neq p$ a perfect pairing

$$T_l\mathcal{A} \times T_l\mathcal{A}' \to T_l(\mathbb{G}_m) = \mathbb{Z}_l(1) \tag{13}$$

of l-adic sheaves on S extending the Weil pairing $\chi\colon T_lA(K^s) \times T_lA'(K^s) \to \mathbb{Z}_l(1)$.

Lemma 2.20 (Orthogonality theorem)**.** *The toric part $T_lA(K^s)^t$ is the orthogonal of the fixed part $T_lA'(K^s)^f = T_lA'(K^s)^G$ via the pairing χ.*

Proof. The proof follows the one given in [GRR72, IX, 2.4]: notice that, by comparing the ranks, we only need to check that $T_lA(K^s)^t \subseteq (T_lA'(K^s)^f)^\perp$.

We obtain $T_lA(K^s)^t$ and $T_lA'(K^s)^f$ by passing to the K^s-valued points of the generic fibres of $T_l\mathcal{A}^t$ and $T_l\mathcal{A}^f$. Therefore, we only need that the restriction of the pairing (13),

$$T_l\mathcal{A}^t \times T_l\mathcal{A}'^f \to T_l(\mathbb{G}_{m,S})$$

is the zero pairing. As $T_l\mathcal{A}^t$ and $T_l\mathcal{A}^f$ are constant l-adic sheaves, we may check this by restricting to the closed fibre. Now, the pairing

$$T_l\mathcal{T}_s \times T_l\mathcal{A}'_s \to T_l(\mathbb{G}_{m,k})$$

is identically zero by [GRR72, VIII, 4.10]. □

For each generic point ζ_i of the irreducible components $D_1, \dots, D_n$ of the divisor D, we can consider a strict henselization $S_i \to S$ at some geometric point $\overline{\zeta}_i$ lying over ζ_i. Over S_i, we can define the l-adic shaves

$$T_l(A_{S_i})^t \hookrightarrow T_l(A_{S_i})^f \hookrightarrow T_l\mathcal{A}_{S_i}.$$

We define $T_lA(K^s)^{t_i}$ and $T_lA(K^s)^{f_i}$ to be the groups of K^s-valued points of the generic fibre of $T_l(A_{S_i})^t$ and $T_l(A_{S_i})^f$ respectively; they are submodules of $T_lA(K^s)$, of rank μ_i and $2d - \mu_i$ respectively; moreover, $T_lA(K^s)^{t_i}$ and $T_lA'(K^s)^{f_i}$ are orthogonal to each other with respect to the pairing χ. By proposition 2.19, we have $T_lA'(K^s)^{f_i} = T_lA'(K^s)^{I_i}$; indeed, $I_i = \pi_1^t(S_i \setminus \{\overline{\zeta}_i\})$. Now, as we evidently have $T_lA'(K^s)^G = \bigcap_{i=1}^n T_lA'(K^s)^{I_i}$, by taking orthogonals with respect to χ we find the relation between toric parts

$$T_lA(K^s)^t = \sum_{i=1}^n T_lA(K^s)^{t_i}. \tag{14}$$

Taking ranks and using (12), we find that the function $\mu\colon S \to \mathbb{Z}_{\geq 0}$ satisfies the relation

$$\mu(s) \leq \sum_{i=1}^n \mu(\zeta_i). \tag{15}$$

The action of G on the Tate module is unipotent

We use the orthogonality lemma 2.20 to describe more explicitly the action of G on the Tate module.

Proposition 2.21. *There exists a submodule V of $T = T_l A(K^s)$ such that G acts trivially on V and on the quotient T/V.*

Proof. Clearly, G acts trivially on $V = T^G$. Now, as T^G is orthogonal to T'^t (where $T' = T_l A'(K^s)$) via the pairing χ, we obtain a perfect pairing $T/T^G \times T'^t \to \mathbb{Z}_l(1)$ which identifies T/T^G with $\mathrm{Hom}_{\mathbb{Z}_l}(T'^t, \mathbb{Z}_l(1))$. As $T'^t \subset T'^G$, we conclude that G acts trivially on T/T^G. □

It follows from the above proposition that the action of G on $T_l A(K^s)$ is unipotent of level 2: that is, writing

$$\rho\colon G \to \mathrm{Aut}(T_l A(K^s) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l),$$

we have for every $g \in G$

$$(\rho(g) - \mathrm{id})^2 = 0.$$

Because the profinite group G acts on a $\mathbb{Q}_l$-vector space unipotently and continuously, the image of ρ is a pro-l-group. Thus, the action of G factors via its biggest pro-l-quotient

$$G = \widehat{\mathbb{Z}}(1)^n = \pi_1^t(U) \to \pi_1^{t,l}(U) = \mathbb{Z}_l(1)^n.$$

2.4 The group of components of a Néron model

Our objective now is to give an explicit description of the group of components of a Néron model, in terms of the Tate modules of A. In [GRR72, IX, 11] a description is provided of the full group of components in the case where the base has dimension 1. The case of a regular base of higher dimension is completely analogous; we will follow closely [GRR72], but will restrict our attention only to the prime-to-p part of the group of components; the description of the p-primary part involves more complicated theory.

We are still under the hypotheses of situation 2.16, with S strictly local, and now we assume that the abelian scheme A/U admits a Néron model $\mathcal{N}/S$. We denote by $\underline{\Phi}$ the étale S-group scheme of connected components of $\mathcal{N}/S$. It fits into an exact sequence of fppf-sheaves

$$0 \to \mathcal{N}^0 \to \mathcal{N} \to \underline{\Phi} \to 0.$$

Clearly, the restriction of $\underline{\Phi}$ to U is the trivial group scheme.

We are interested in the fibre of $\underline{\Phi}$ over the closed point $s \in S$; this is the étale k-group scheme of finite type

$$\underline{\Phi}_s = \frac{\mathcal{N}_s}{\mathcal{N}_s^0}.$$

As k is algebraically closed, $\underline{\Phi}_s$ is determined by its group of k-rational points $\Phi := \underline{\Phi}_s(k)$, which is a finite abelian group. We have

$$\Phi = \frac{\mathcal{N}_s(k)}{\mathcal{N}_s^0(k)} = \frac{\mathcal{N}(S)}{\mathcal{N}^0(S)} = \frac{A(U)}{A(U)^0} \tag{16}$$

where by $A(U)^0$ we denote the subset of $A(U)$ of U-points specializing to S-points of the identity component $\mathcal{N}^0$. Notice that the second equality is a consequence of Hensel's lemma and the third of the defining property of Néron models.

We let l be a prime different from the residue characteristic $p = \operatorname{char} k(s)$ and $n \geq 0$ be an integer. Taking l^n-torsion in the exact sequence

$$0 \to A(U)^0 \to A(U) \to \Phi \to 0$$

gives a long exact sequence

$$0 \to A[l^n](U)^0 \to A[l^n](U) \to \Phi[l^n] \to A(U)^0/l^n A(U)^0 \to \dots.$$

Multiplication by l^n on the fiberwise-connected component of identity $\mathcal{N}^0$ is an étale and surjective morphism; it follows that $\mathcal{N}^0(S) = A(U)^0$ is l-divisible; hence

$$\Phi[l^n] = \frac{A[l^n](U)}{A[l^n](U)^0} \tag{17}$$

Now, because $A[l^n]$ is finite étale over U, we have $A[l^n](U) = A[l^n](K)$. Writing T for $T_l A(K^s)$, we see that $A[l^n](U) = (T \otimes \mathbb{Z}/l^n\mathbb{Z})^G$, so we have an expression for the part of eq. (17) above the fraction line.

Let's turn to study $A[l^n](U)^0$. This is equal to $A[l^n](K)^0$ (again, the exponent 0 denotes those elements specializing to the identity component $\mathcal{N}^0$). The latter, by the defining property of Néron models, is simply $\mathcal{N}^0[l^n](S)$. Now, every section of a quasi-finite separated scheme over S factors via its fixed part, so $\mathcal{N}^0[l^n](S) = \mathcal{N}^0[l^n]^f(S)$. Because $\mathcal{N}^0(S)$ is l-divisible, we have $\mathcal{N}^0[l^n]^f(S) = (T_l\mathcal{N}^0)^f(S) \otimes_{\mathbb{Z}_l} \mathbb{Z}/l^n\mathbb{Z}$. Now, $(T_l\mathcal{N}^0)^f(S)$ is equal to $T_l A(K^s)^f$, which in turn is equal to $T_l A(K^s)^G$ by proposition 2.19.

This shows that $A[l^n](K)^0 = T_l A(K^s)^G \otimes_{\mathbb{Z}_l} \mathbb{Z}/l^n\mathbb{Z}$. We have found the relation

$$\Phi[l^n] = \frac{(T \otimes \mathbb{Z}/l^n\mathbb{Z})^G}{T^G \otimes \mathbb{Z}/l^n\mathbb{Z}}. \quad (18)$$

By taking the colimit over the powers of l we find that the l-primary part ${}_l\Phi$ of the group of components is given by

$${}_l\Phi = \operatorname{colim}_n \Phi[l^n] = \frac{(T \otimes \mathbb{Q}_l/\mathbb{Z}_l)^G}{T^G \otimes \mathbb{Q}_l/\mathbb{Z}_l}. \quad (19)$$

Example 2.22. We give an example in which we compute the 2-torsion of the group of components of the Néron model of an elliptic curve, in the case of $\dim S = 1$. Let $R = k[[t]]$ for some algebraically closed field k of characteristic zero. Let $S = \operatorname{Spec} R$, K be the fraction field of R, and consider the elliptic curve E/K given in $\mathbb{P}^2_K$ by

$$Y^2Z = (X - Z)(X^2 - t^2Z^2).$$

The same equation gives a nodal model $\mathcal{E}/S$; it follows that $E = \operatorname{Pic}^0_{E/K}$ admits a semi-abelian prolongation, given by $\mathcal{E}^{sm} = \operatorname{Pic}^0_{\mathcal{E}/S}$.

Let $\mathcal{N}/S$ be the Néron model of E over S; the open immersion $\mathcal{E}^{sm} \to \mathcal{N}$ identifies $\mathcal{E}^{sm}$ with $\mathcal{N}^0$. Let Φ be the group of components of the closed fibre of $\mathcal{N}$.

We work with the prime $l = 2$; by equation (18), the 2-torsion of Φ is given by

$$\Phi[2] = \frac{E[2](K)}{T_2E(K) \otimes \mathbb{Z}/2\mathbb{Z}}.$$

The Tate module $T_2E(\overline{K})$ is a free $\mathbb{Z}_2$-module of rank 2. Its K-rational part $T_2E(K) \subset T_2E(\overline{K})$ is free, of rank $2\dim E - \mu = 2 - 1 = 1$. Thus, $T_2E(K) \otimes \mathbb{Z}/2\mathbb{Z}$ is a free $\mathbb{Z}/2\mathbb{Z}$-submodule of rank 1 of $T_2E(\overline{K}) \otimes \mathbb{Z}/2\mathbb{Z} = E[2](\overline{K})$.

On the other hand, $E[2](K) = \{(0,1,0), (1,0,1), (t,0,1), (-t,0,1)\} = E[2](\overline{K}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. We deduce that

$$\Phi[2] \cong \mathbb{Z}/2\mathbb{Z}.$$

Moreover, the points $(0,1,0)$ and $(1,0,1)$ extend to S-points of $\mathcal{E}^{sm} = \mathcal{N}^0$; while $(t,0,1)$ and $(-t,0,1)$ do not, as they specialize to the non-smooth point of $\mathcal{E}/S$. They do extend to S-points of $\mathcal{N}$ though, whose restriction to the closed fibre is contained in the only 2-torsion component different from the identity component.

3 Toric-additivity

We work with the hypotheses of situation 2.5; we suppose that we are given an abelian scheme A/U of relative dimension d, and a semi-abelian scheme $\mathcal{A}/S$ with an isomorphism $\mathcal{A} \times_S U \to A$.

3.1 Definition of toric-additivity in the strictly local case

Assume that S is strictly local, with closed point s and residue field $k = k(s)$ of characteristic $p \geq 0$. The divisor D has finitely many irreducible components $D_1, \ldots, D_n$ for some $n \geq 0$.

We fix a prime $l \neq p$ and consider the Tate module $T_l A(K^s)$; we recall that it is a free $\mathbb{Z}_l$-module of rank $2d$ with an action of $\mathrm{Gal}(K^s|K)$, which factors via the surjection $\mathrm{Gal}(K^s|K) \to G := \pi_1^{t,l}(U) = \bigoplus_{i=1}^n I_i$, where $I_i = \mathbb{Z}_l(1)$ for each i.

Definition 3.1. Let $l \neq p$ be a prime. We say that the semi-abelian scheme $\mathcal{A}/S$ satisfies condition $\bigstar(l)$ if

$$T_l A(K^s) = \sum_{i=1}^{n} T_l A(K^s)^{\oplus_{j\neq i} I_j} \text{ or if } n = 0. \tag{20}$$

Remark 3.2.

- Whether $\mathcal{A}/S$ satisfies condition $\bigstar(l)$ depends only on the generic fibre A_K/K, and on the base S;
- suppose that $\mathcal{A}/S$ satisfies condition $\bigstar(l)$; let t be another geometric point of S, belonging to $D_1, D_2, \ldots, D_m$ for some $m \leq n$, and consider the strict henselization S' at t. Then the morphism

  $$\pi_1^{t,l}(U \times_S S') \to \pi_1^{t,l}(U)$$

  induced by $S' \to S$ is the natural inclusion

  $$\bigoplus_{i=1}^{m} I_j \to \bigoplus_{i=1}^{n} I_i.$$

  It can be easily seen that $\sum_{i=1}^n T_l A(K^s)^{\oplus_{j\neq i} I_j} \subseteq \sum_{i=1}^m T_l A(K^s)^{\oplus_{j\neq i} I_j}$; hence $\mathcal{A}_{S'}/S'$ also satisfies condition $\bigstar(l)$.
- Condition $\bigstar(l)$ is automatically satisfied if $n = 1$.