A monodromy criterion for existence of Neron models and a result on semi-factoriality

Orecchia, G.

Author: Orecchia, G.
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Part I

A monodromy criterion for existence of Néron models

1 Introduction

We study the existence of Néron models of abelian varieties over a regular base of dimension possibly greater than 1. The question of their existence has first been raised in [Hol17b]: he considered the case of a nodal curve $\mathcal{C}/S$, smooth over an open dense $U \subset S$, and asked whether the jacobian $J := \mathrm{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model over S. The answer to this question turned out to be related to a restrictive combinatorial condition on the dual graphs of the fibres of $\mathcal{C}/S$, called *alignment*. More precisely, one has

Theorem 1.1 ([Hol17b], theorem 5.16, theorem 5.2)**.** *Suppose S is regular.*

i) if J/U admits a Néron model over S, then $\mathcal{C}/S$ is aligned;

ii) if moreover the total space $\mathcal{C}$ is regular, and $\mathcal{C}/S$ is aligned, then J/U admits a Néron model over S.

As the existence of a Néron model only depends on S and on the generic fibre J_K, the question arises naturally of whether alignment can be read only in terms of J_K and S. This is what we try to achieve in this paper, in the case where the degeneracy locus of $\mathcal{C}/S$ is a normal crossing divisor, by studying the Galois action on the Tate module $T_lJ(K^{sep})$ of the generic fibre, for a prime l invertible on S. We introduce a new condition, called *toric-additivity*, on $T_lJ(K^{sep})$, which is necessary and sufficient for the existence of a Néron model of J/U over S (the necessity is subject to restrictions on the base characteristic, though).

Toric-additivity is in general neither stronger nor weaker than alignment; however, it is equivalent to it in the case where the total space $\mathcal{C}$ is regular. Its advantage is that it allows us to treat also the case where $\mathcal{C}$ is not regular and does not admit a desingularization. Moreover, as it is a condition on the Tate module of the generic fibre J_K, it behaves well with respect to various types of base change, with respect to blowing-ups of $\mathcal{C}$, and with respect to isogenies.

Another upshot of toric-additivity is that it can be formulated as a property of a general abelian scheme A/U admitting a semi-abelian prolongation $\mathcal{A}/S$. We obtain a partial generalization of the results for jacobians of curves to this more general setting: we show that, if the base S has everywhere characteristic zero (i.e. it is a $\mathbb{Q}$-scheme), toric-additivity is a sufficient condition for the existence of a Néron model for A over S. The converse is still an open question.

1.1 Toric-additivity

We consider a connected, regular, locally noetherian base scheme S with a normal crossing divisor $D = \cup_{i \in \mathcal{I}} D_i$ and an abelian scheme A over the open complement $U = S \setminus D$, admitting a semi-abelian prolongation $\mathcal{A}$ over S. We introduce a condition on $\mathcal{A}/S$ called *toric-additivity* (definitions 3.5 and 3.7), which is defined étale locally on S, and can be expressed in two equivalent ways (see theorem 3.4) when S is the étale local ring at a geometric point:

- by imposing a strict condition on how the toric rank of the fibres of $\mathcal{A}/S$ varies on D. Roughly, if $D_1, \ldots, D_n$ are irreducible components of D, the toric rank at the generic point of $D_1 \cap \ldots \cap D_n$ should be the sum of the toric ranks at the generic points of the D_i's.
- by asking that, for some prime l invertible on S (equivalently, for all such primes), the biggest pro-l quotient of the tame fundamental group, $\pi_1^{t,l}(U) = \bigoplus_{i \in \mathcal{I}} \mathbb{Z}_l(1)$, acts in a certain way on the Tate module $T_l A(K^{sep})$ of the generic fibre. Namely, there should be a decomposition $T_l A(K^{sep}) = \bigoplus_{i \in \mathcal{I}} V_i$ such that the i-th direct summand of $\pi_1^{t,l}(U)$ acts trivially on all V_j with $j \neq i$.

1.2 Results

We first consider the case of a nodal curve $\mathcal{C}/S$, smooth over U. In this case, the abelian scheme A/U is the jacobian $J := \mathrm{Pic}^0_{\mathcal{C}_U/U}$, and its semi-abelian prolongation $\mathcal{A}/S$ is the scheme $\mathrm{Pic}^0_{\mathcal{C}/S}$ representing the fppf-sheaf on S of invertible sheaves on $\mathcal{C}$ that have degree zero on each component of the fibres of $\mathcal{C}/S$.

Theorem 1.2 (theorem 4.17)**.**

a) If $\mathrm{Pic}^0_{\mathcal{C}/S}$ is toric-additive, then J admits a Néron model over S.

b) If moreover S is an excellent $\mathbb{Q}$-scheme, the converse is also true.

The strategy of proof follows these lines: we show that if the hypotheses of a) or b) are satisfied, there exists a blow-up $\mathcal{C}' \to \mathcal{C}$, such that $\mathcal{C}'/S$ is still a nodal curve, smooth over U, and $\mathcal{C}'$ is *regular*. As the properties of admitting a Néron model or of being toric-additive are not affected by the desingularization, we have reduced to the case where the relative curve has regular total space. In this case, it can be shown that alignment and toric-additivity are equivalent, and we apply theorem 1.1.

We partially extend these results to the general case of an abelian scheme A/U admitting a semi-abelian prolongation on S.

Theorem 1.3 (theorem 5.8)**.** *Assume S is a $\mathbb{Q}$-scheme. If $\mathcal{A}/S$ is toric-additive, then A/U admits a Néron model over S.*

The theorem is proved by explicitly constructing a Néron model for A. The construction is carried out by means of an auxiliary object, a *test-Néron model* $\mathcal{N}/S$. This is, roughly, defined to be a smooth, separated group-space, which is a model for A, and such that, for every strictly henselian discrete valuation ring R and morphism $\operatorname{Spec} R \to S$ meeting D transversally, the pullback $\mathcal{N}_Z/Z$ is a Néron model of its generic fibre. We show that toric-additivity implies the existence of test-Néron models; and that test-Néron models are Néron models. We remark that for this last fact, it is crucial that test-Néron are defined to be group objects; there are examples of objects that are similar to test-Néron models, in that they satisfy a similar property with respect to transversal traits, but fail to be a Néron model because they do not admit a group structure: an example is the *balanced Picard stack* $\mathcal{P}_{d,g} \to \overline{\mathcal{M}}_g$ constructed by Caporaso in [Cap08].

Whether toric-additivity is also a necessary condition for the existence of a Néron model is still an open question; the main obstacle is showing that a Néron model is always a test-Néron model.

1.3 Outline

In section 2, we first recall the definition of a Néron model (definition 2.7) and state a number of properties regarding the behaviour of Néron models under different sorts of base change. In the rest of the section, we follow closely Exposé IX of [GRR72], titled *Modèles de Néron et monodromie*, where the authors investigate the relation between the reduction type of the Néron model and the Galois action on the Tate module; we show that a number of results that are proved there stay true when the base has dimension higher than 1. Among these there is the characterization of the l-primary part of the group of components of a Néron model in terms of the Tate module $T_l A(K^{sep})$ of the generic fibre, for a prime l invertible on S (section 2.4).

In section 3, we introduce the condition of toric-additivity (definition 3.5). We show in theorem 3.4 that it can be equivalently stated as a condition on the Tate module $T_l A(K^{sep})$, for any l invertible on S, or as a condition on the toric ranks of the fibres of the semi-abelian scheme $\mathcal{A}/S$.

Section 4 is devoted to the case of jacobians of curves. After recalling the results of [Hol17b], we establish the relation between toric-additivity and the property of existence of a Néron model for the jacobian (theorem 4.17).

In section 5, we work under the assumption that the base S is a $\mathbb{Q}$-scheme; we attempt to relate toric-additivity and the property of existence of Néron models in the case of abelian schemes. We introduce test-Néron models and prove that they exist and are unique if $\mathcal{A}/S$ is toric-additive (proposition 5.5 and theorem 5.6). After a result on descent of test-Néron models (proposition 5.7), we conclude the section by showing that test-Néron models are Néron models, under the assumption of toric-additivity (proposition 5.9).

2 Generalities

2.1 Normal crossing divisors and tame fundamental group

We work over a connected, regular, locally noetherian, base scheme S.

Definition 2.1. Given a regular, noetherian local ring R, a *regular system of parameters* is a minimal subset $\{r_1, \dots, r_d\} \subset R$ of generators for the maximal ideal $\mathfrak{m} \subset R$.

Definition 2.2. A *strict normal crossing divisor* D on S is a closed subscheme $D \subset S$ such that, for every point $s \in S$, the preimage of D in the local ring $\mathcal{O}_{S,s}$ is the zero locus of a product $r_1 \cdot \dots \cdot r_n$, where $\{r_1, \dots, r_n\}$ is a subset of a regular system of parameters $\{r_1, \dots, r_d\}$ of $\mathcal{O}_{S,s}$.

Write $\{D_i\}_{i \in \mathcal{I}}$ for the set of irreducible components of D. Then each D_i, seen as a reduced closed subscheme of S, is regular and of codimension 1 in S; moreover, for every finite subset $\mathcal{J} \subset \mathcal{I}$, the intersection $\bigcap_{j \in \mathcal{J}} D_j$ is regular, and each of its irreducible components has codimension $|\mathcal{J}|$.

Definition 2.3. A *normal crossing divisor* D on S is a closed subscheme $D \subset S$ for which there exists an étale surjective morphism $S' \to S$ such that the base change $D \times_S S'$ is a strict normal crossing divisor on S'.

Notice that for every geometric point s of S, the pullback of a normal crossing divisor D to the spectrum of the strict henselization $\mathcal{O}_{S,s}^{sh}$ is a strict normal crossing divisor.