A monodromy criterion for existence of Neron models and a result on semi-factoriality

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Par Giulio ORECCHIA

Un critère de monodromie pour l'existence des modèles de Néron et un résultat sur la semi-factorialité

Sous la direction de David HOLMES et Qing LIU

Soutenue le 27 février 2018

Membres du jury :

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Chi vuole guardare bene la terra
deve tenersi alla distanza
necessaria.

Italo Calvino, *Il barone rampante*

General introduction

Reduction of elliptic curves

Consider an elliptic curve E defined over the field of rational numbers $\mathbb{Q}$, given by some polynomial equation in $\mathbb{P}^2_{\mathbb{Q}}$

$$f(x,y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_5 = 0$$

with $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Q}$. A fundamental technique for studying E (for example, for finding its group of rational points) is to study the reduction of E modulo different primes numbers. As the rational coefficients $a_1, \ldots, a_5$ need not be integers, it is a priori not clear how one should reduce the equation modulo a prime p.

One way to do this, is to apply a linear change of coordinates so that the denominators of the coefficients $a_1, \ldots, a_5$ are not divisible by p; and then reduce the resulting equation f' modulo p to obtain a polynomial with coefficients in $\mathbb{F}_p$. There is no unique choice of linear change of coordinates: however, one can be picked that minimizes the maximal power of p that divides the discriminant of f'. The polynomial f' is called a *minimal Weierstrass model* for E at the prime p. In fact, it turns out that the curve $\overline{E}_p/\mathbb{F}_p$ defined by the reduction of f' modulo p does not depend on which minimal Weierstrass model we choose.

One significant advantage of this approach is that we obtain a reduction map modulo p on the $\mathbb{Q}$-valued points of E; namely, there is a well defined reduction function

$$\mathrm{red}_p\colon E(\mathbb{Q}) \to \overline{E}_p(\mathbb{F}_p). \tag{1}$$

Thus, minimal Weierstrass models give a good notion of reduction modulo p. Their drawback is that the curve $\overline{E}_p/\mathbb{F}_p$ need not be smooth: for every prime p dividing the discriminant of $f(x,y)$, the reduction $\overline{E}_p$ has a singular point. In this case, $\overline{E}_p$ is not an elliptic curve, and does not admit a group structure.

A first remedy to this issue is to remove the singular point from $\overline{E}_p$. The resulting subcurve $\overline{E}_p^{sm}$ is smooth, and admits a unique group structure compatible with the one on E. In other words, we have gained back the group structure and smoothness at the expenses of projectivity. However, we have lost something else along the way, that is, the reduction map red_p. Indeed the function $E(\mathbb{Q}) \to \overline{E}_p(\mathbb{F}_p)$ does not in general factor via $\overline{E}_p^{sm}(\mathbb{F}_p)$.

Néron models

Néron models, introduced in 1964 by Andrè Néron in his paper [Nér64], provide a canonical way of reducing E modulo a prime p, while preserving smoothness, group structure, and reduction map red_p. In fact, the definition makes sense in the more general setting of an abelian variety A defined over the fraction field K of a connected Dedekind scheme S of dimension 1. By definition, a Néron model for A_K is a smooth, separated scheme $\mathcal{N}/S$ restricting to A over K, satisfying a universal property: for every smooth scheme $T \to S$ and morphism $\varphi_K \colon T_K \to A_K$, there exists a unique morphism $\varphi\colon T \to \mathcal{N}$ extending φ_K.

There is a good reason for asking that the extension property applies to smooth points $T_K \to A_K$ and not only, say, to K-valued points of A_K: namely, the property ensures that Néron model are unique up to unique isomorphism, and inherit a group structure from A_K.

It is a theorem that Néron models of abelian varieties exist. On the other hand, although the definiton of Néron model makes sense for arbitrary smooth schemes of finite type over K, even reasonable schemes like $\mathbb{P}^1_K$ do not admit a Néron model.

In the special case of an elliptic curve E over the fraction field K of a discrete valuation ring R, the Néron model $\mathcal{N}$ over $S = \operatorname{Spec} R$ has a very concrete description: one first constructs the minimal Weierstrass model $\mathcal{W}/S$; its minimal desingularization $\mathcal{E}/S$ is the minimal regular model of E_K over S. In turn, its smooth locus $\mathcal{E}^{sm}/S$ is the Néron model of E_K. The identity component of $\mathcal{N}/S$ is the smooth locus of the minimal Weierstrass model W/S.

Among the numerous applications of Néron models in arithmetic geometry, the first we want to mention Serre and Tate's "Néron-Ogg-Shafarevich criterion" for good reduction of abelian varieties: an abelian variety A_K admits a proper, smooth (hence abelian) model $\mathcal{A}/S$ if and only if for some (equivalently, any) prime l different from the residue characteristic of S, the l-adic Tate module $T_l A(K^{sep})$ is unramified (i.e., the inertia group acts trivially on it). Another important application of Néron models is the semi-stable reduction theorem, stating that an abelian variety A/K admits a semi-abelian model after some finite extension $K \to K'$.

Néron models of jacobians

One particular class of abelian varieties are jacobians of smooth curves. Given a smooth curve C over K, we indicate by J_K/K its jacobian, an abelian variety of dimension equal to the genus of C_K.

In the case when C_K is an elliptic curve, the Abel-Jacobi map $C_K \to J_K$ is an isomorphism, and we have seen how the Néron model of $C_K = J_K$ has an easy description in terms of the minimal regular model $\mathcal{C}/S$ of C_K.

When C_K is a curve of higher genus, Raynaud has shown that it is still possible to describe the Néron model $\mathcal{N}/S$ of J_K by means of any regular model $\mathcal{C}/S$. Namely, one considers the relative Picard sheaf $\mathrm{Pic}^{[0]}_{\mathcal{C}/S}$ that parametrizes line bundles of total degree zero on each fibre, and takes the quotient by the étale group scheme given by the closure $E \subset \mathrm{Pic}^{[0]}_{\mathcal{C}/S}$ of the unit section $e\colon \operatorname{Spec} K \to \mathrm{Pic}^0_{C_K/K}$. The quotient sheaf is representable by a smooth separated scheme of finite type, which is the Néron model of J_K. A detailed explanation can be found in [BLR90, 9.5].

In recent years, the question has arisen of whether Néron models of jacobians of curves exist also when the base scheme S has arbitrary dimension. Holmes showed in [Hol17b] that the answer is in general no: he related the existence of Néron models to a combinatorial condition, called *alignment*, on the dual graphs of the fibres, that is automatically satisfied if $\dim S = 1$. The construction of a stack $\widetilde{\mathcal{M}}_{g,n}$ of aligned n-pointed stable curves and the related techniques have been fundamental in tackling problems such as resolving the Abel-Jacobi map $\overline{\mathcal{M}}_{g,n} \dashrightarrow \mathcal{J}$, where $\mathcal{J}$ is the unique semi-abelian extension of the universal Jacobian (see [Hol17a] for details).

This thesis

This document is divided in two parts, each one with its own introduction placed at the beginning of the relative part (sections 1 and 6):

- **Part I**: A monodromy criterion for existence of Néron models;
- **Part II**: Semi-factorial nodal curves and Néron lft-models.

In part I, I consider the problem of existence of Néron models of abelian schemes over bases of arbitrarily high dimension. For an abelian scheme degenerating to a semi-abelian scheme over a normal crossing divisor, I introduce a condition, called *toric-additivity*, on the action of monodromy on the l-adic

Tate module (for a prime l invertible on S). I show that toric-additivity is closely related to the property of existence of a Néron model.

In part II, I go back to the case of a base S of dimension 1, and I try to generalize Raynaud's construction of the Néron model of a jacobian to the case of a nodal curve C/K admitting a nodal model $\mathcal{C}/S$. The content of Part II appears in the paper [Ore17].

The two parts are the result of two distinct projects I pursued during my doctorate, and as such, they can be read independently one from another. In order to accommodate the reader, I repeated some of the definitions in the two parts, to ensure that each of them constitutes a self-contained document. The definitions are in any case consistent throughout the thesis; however, to avoid possible confusion, it must be pointed out that in part II I define a *circuit* of a graph to be what I called a *cycle* in part I. The latter is more appropriate terminology; however, the term *circuit-coprime* was coined in [Ore17] and I preferred not to change it since it already appears in a published manuscript.

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