A monodromy criterion for existence of Neron models and a result on semi-factoriality

Orecchia, G.

Author: Orecchia, G.
Title: A monodromy criterion for existence of Neron models and a result on semi-factoriality
Issue Date: 2018-02-27

A monodromy criterion for existence of Néron models and a result on semi-factoriality

Proefschrift
ter verkrijging van
de graad van Doctor aan de Universiteit Leiden
op gezag van Rector Magnificus prof. mr. C.J.J.M. Stolker,
volgens besluit van het College voor Promoties
te verdedigen op dinsdag 27 februari 2018
klokke 15:00 uur

door

Giulio Orecchia
geboren te Rapallo, Italië,
in 1990

Promotor: Prof. dr. Sebastiaan J. Edixhoven

Promotor: Prof. dr. Qing Liu (Université de Bordeaux)

Co-promotor: Dr. David Holmes

Samenstelling van de promotiecommissie:

Prof. dr. Christian Liedtke (TU München)

Prof. dr. Johannes Nicaise (Imperial college London / KU Leuven)

Prof. dr. Bart de Smit

Prof. dr. Adrianus W. van der Vaart

This work was funded by Erasmus Mundus Algant-Doc and was carried out at Universiteit Leiden and Université de Bordeaux

THÈSE EN COTUTELLE PRÉSENTÉE

POUR OBTENIR LE GRADE DE

DOCTEUR

DE L'UNIVERSITÉ DE BORDEAUX

ET DE L'UNIVERSITÉ DE LEYDE

ÉCOLE DOCTORALE DE MATHÉMATIQUES ET INFORMATIQUE

INSTITUT DES MATHÉMATIQUES DE L'UNIVERSITÉ DE LEYDE

SPÉCIALITÉ Mathématiques Pures

Par Giulio ORECCHIA

Un critère de monodromie pour l'existence des modèles de Néron et un résultat sur la semi-factorialité

Sous la direction de David HOLMES et Qing LIU

Soutenue le 27 février 2018

Membres du jury :

Christian Liedtke	Professor, TU München	Prsidente
David HOLMES	Assistant professor, Universiteit Leiden	Directeur
Qing LIU	Professeur, Universit de Bordeaux,	Directeur
Johannes NICAISE	Hoofddocent, KU Leuven	Rapporteur

Chi vuole guardare bene la terra
deve tenersi alla distanza
necessaria.

Italo Calvino, *Il barone rampante*

General introduction

Reduction of elliptic curves

Consider an elliptic curve E defined over the field of rational numbers $\mathbb{Q}$, given by some polynomial equation in $\mathbb{P}^2_{\mathbb{Q}}$

$$f(x,y) = y^2 + a_1xy + a_3y - x^3 - a_2x^2 - a_4x - a_5 = 0$$

with $a_1, a_2, a_3, a_4, a_5 \in \mathbb{Q}$. A fundamental technique for studying E (for example, for finding its group of rational points) is to study the reduction of E modulo different primes numbers. As the rational coefficients $a_1, \dots, a_5$ need not be integers, it is a priori not clear how one should reduce the equation modulo a prime p.

One way to do this, is to apply a linear change of coordinates so that the denominators of the coefficients $a_1, \dots, a_5$ are not divisible by p; and then reduce the resulting equation f' modulo p to obtain a polynomial with coefficients in $\mathbb{F}_p$. There is no unique choice of linear change of coordinates: however, one can be picked that minimizes the maximal power of p that divides the discriminant of f'. The polynomial f' is called a *minimal Weierstrass model* for E at the prime p. In fact, it turns out that the curve $\overline{E}_p/\mathbb{F}_p$ defined by the reduction of f' modulo p does not depend on which minimal Weierstrass model we choose.

One significant advantage of this approach is that we obtain a reduction map modulo p on the $\mathbb{Q}$-valued points of E; namely, there is a well defined reduction function

$$\mathrm{red}_p \colon E(\mathbb{Q}) \to \overline{E}_p(\mathbb{F}_p). \tag{1}$$

Thus, minimal Weierstrass models give a good notion of reduction modulo p. Their drawback is that the curve $\overline{E}_p/\mathbb{F}_p$ need not be smooth: for every prime p dividing the discriminant of $f(x,y)$, the reduction $\overline{E}_p$ has a singular point. In this case, $\overline{E}_p$ is not an elliptic curve, and does not admit a group structure.

A first remedy to this issue is to remove the singular point from $\overline{E}_p$. The resulting subcurve $\overline{E}_p^{sm}$ is smooth, and admits a unique group structure compatible with the one on E. In other words, we have gained back the group structure and smoothness at the expenses of projectivity. However, we have lost something else along the way, that is, the reduction map red_p. Indeed the function $E(\mathbb{Q}) \to \overline{E}_p(\mathbb{F}_p)$ does not in general factor via $\overline{E}_p^{sm}(\mathbb{F}_p)$.

Néron models

Néron models, introduced in 1964 by Andrè Néron in his paper [Nér64], provide a canonical way of reducing E modulo a prime p, while preserving smoothness, group structure, and reduction map red_p. In fact, the definition makes sense in the more general setting of an abelian variety A defined over the fraction field K of a connected Dedekind scheme S of dimension 1. By definition, a Néron model for A_K is a smooth, separated scheme $\mathcal{N}/S$ restricting to A over K, satisfying a universal property: for every smooth scheme $T \to S$ and morphism $\varphi_K \colon T_K \to A_K$, there exists a unique morphism $\varphi\colon T \to \mathcal{N}$ extending φ_K.

There is a good reason for asking that the extension property applies to smooth points $T_K \to A_K$ and not only, say, to K-valued points of A_K: namely, the property ensures that Néron model are unique up to unique isomorphism, and inherit a group structure from A_K.

It is a theorem that Néron models of abelian varieties exist. On the other hand, although the definiton of Néron model makes sense for arbitrary smooth schemes of finite type over K, even reasonable schemes like $\mathbb{P}^1_K$ do not admit a Néron model.

In the special case of an elliptic curve E over the fraction field K of a discrete valuation ring R, the Néron model $\mathcal{N}$ over $S = \operatorname{Spec} R$ has a very concrete description: one first constructs the minimal Weierstrass model $\mathcal{W}/S$; its minimal desingularization $\mathcal{E}/S$ is the minimal regular model of E_K over S. In turn, its smooth locus $\mathcal{E}^{sm}/S$ is the Néron model of E_K. The identity component of $\mathcal{N}/S$ is the smooth locus of the minimal Weierstrass model W/S.

Among the numerous applications of Néron models in arithmetic geometry, the first we want to mention Serre and Tate's "Néron-Ogg-Shafarevich criterion" for good reduction of abelian varieties: an abelian variety A_K admits a proper, smooth (hence abelian) model $\mathcal{A}/S$ if and only if for some (equivalently, any) prime l different from the residue characteristic of S, the l-adic Tate module $T_l A(K^{sep})$ is unramified (i.e., the inertia group acts trivially on it). Another important application of Néron models is the semi-stable reduction theorem, stating that an abelian variety A/K admits a semi-abelian model after some finite extension $K \to K'$.

Néron models of jacobians

One particular class of abelian varieties are jacobians of smooth curves. Given a smooth curve C over K, we indicate by J_K/K its jacobian, an abelian variety of dimension equal to the genus of C_K.

In the case when C_K is an elliptic curve, the Abel-Jacobi map $C_K \to J_K$ is an isomorphism, and we have seen how the Néron model of $C_K = J_K$ has an easy description in terms of the minimal regular model $\mathcal{C}/S$ of C_K.

When C_K is a curve of higher genus, Raynaud has shown that it is still possible to describe the Néron model $\mathcal{N}/S$ of J_K by means of any regular model $\mathcal{C}/S$. Namely, one considers the relative Picard sheaf $\mathrm{Pic}^{[0]}_{\mathcal{C}/S}$ that parametrizes line bundles of total degree zero on each fibre, and takes the quotient by the étale group scheme given by the closure $E \subset \mathrm{Pic}^{[0]}_{\mathcal{C}/S}$ of the unit section $e\colon \operatorname{Spec} K \to \mathrm{Pic}^0_{C_K/K}$. The quotient sheaf is representable by a smooth separated scheme of finite type, which is the Néron model of J_K. A detailed explanation can be found in [BLR90, 9.5].

In recent years, the question has arisen of whether Néron models of jacobians of curves exist also when the base scheme S has arbitrary dimension. Holmes showed in [Hol17b] that the answer is in general no: he related the existence of Néron models to a combinatorial condition, called *alignment*, on the dual graphs of the fibres, that is automatically satisfied if $\dim S = 1$. The construction of a stack $\widetilde{\mathcal{M}}_{g,n}$ of aligned n-pointed stable curves and the related techniques have been fundamental in tackling problems such as resolving the Abel-Jacobi map $\overline{\mathcal{M}}_{g,n} \dashrightarrow \mathcal{J}$, where $\mathcal{J}$ is the unique semi-abelian extension of the universal Jacobian (see [Hol17a] for details).

This thesis

This document is divided in two parts, each one with its own introduction placed at the beginning of the relative part (sections 1 and 6):

- **Part I**: A monodromy criterion for existence of Néron models;
- **Part II**: Semi-factorial nodal curves and Néron lft-models.

In part I, I consider the problem of existence of Néron models of abelian schemes over bases of arbitrarily high dimension. For an abelian scheme degenerating to a semi-abelian scheme over a normal crossing divisor, I introduce a condition, called *toric-additivity*, on the action of monodromy on the l-adic

Tate module (for a prime l invertible on S). I show that toric-additivity is closely related to the property of existence of a Néron model.

In part II, I go back to the case of a base S of dimension 1, and I try to generalize Raynaud's construction of the Néron model of a jacobian to the case of a nodal curve C/K admitting a nodal model $\mathcal{C}/S$. The content of Part II appears in the paper [Ore17].

The two parts are the result of two distinct projects I pursued during my doctorate, and as such, they can be read independently one from another. In order to accommodate the reader, I repeated some of the definitions in the two parts, to ensure that each of them constitutes a self-contained document. The definitions are in any case consistent throughout the thesis; however, to avoid possible confusion, it must be pointed out that in part II I define a *circuit* of a graph to be what I called a *cycle* in part I. The latter is more appropriate terminology; however, the term *circuit-coprime* was coined in [Ore17] and I preferred not to change it since it already appears in a published manuscript.

Contents

Part I

A monodromy criterion for existence of Néron models

1 Introduction

We study the existence of Néron models of abelian varieties over a regular base of dimension possibly greater than 1. The question of their existence has first been raised in [Hol17b]: he considered the case of a nodal curve $\mathcal{C}/S$, smooth over an open dense $U \subset S$, and asked whether the jacobian $J := \mathrm{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model over S. The answer to this question turned out to be related to a restrictive combinatorial condition on the dual graphs of the fibres of $\mathcal{C}/S$, called *alignment*. More precisely, one has

Theorem 1.1 ([Hol17b], theorem 5.16, theorem 5.2)**.** *Suppose S is regular.*

i) if J/U admits a Néron model over S, then $\mathcal{C}/S$ is aligned;

ii) if moreover the total space $\mathcal{C}$ is regular, and $\mathcal{C}/S$ is aligned, then J/U admits a Néron model over S.

As the existence of a Néron model only depends on S and on the generic fibre J_K, the question arises naturally of whether alignment can be read only in terms of J_K and S. This is what we try to achieve in this paper, in the case where the degeneracy locus of $\mathcal{C}/S$ is a normal crossing divisor, by studying the Galois action on the Tate module $T_lJ(K^{sep})$ of the generic fibre, for a prime l invertible on S. We introduce a new condition, called *toric-additivity*, on $T_lJ(K^{sep})$, which is necessary and sufficient for the existence of a Néron model of J/U over S (the necessity is subject to restrictions on the base characteristic, though).

Toric-additivity is in general neither stronger nor weaker than alignment; however, it is equivalent to it in the case where the total space $\mathcal{C}$ is regular. Its advantage is that it allows us to treat also the case where $\mathcal{C}$ is not regular and does not admit a desingularization. Moreover, as it is a condition on the Tate module of the generic fibre J_K, it behaves well with respect to various types of base change, with respect to blowing-ups of $\mathcal{C}$, and with respect to isogenies.

Another upshot of toric-additivity is that it can be formulated as a property of a general abelian scheme A/U admitting a semi-abelian prolongation $\mathcal{A}/S$. We obtain a partial generalization of the results for jacobians of curves to this more general setting: we show that, if the base S has everywhere characteristic zero (i.e. it is a $\mathbb{Q}$-scheme), toric-additivity is a sufficient condition for the existence of a Néron model for A over S. The converse is still an open question.

1.1 Toric-additivity

We consider a connected, regular, locally noetherian base scheme S with a normal crossing divisor $D = \cup_{i\in\mathcal{I}} D_i$ and an abelian scheme A over the open complement $U = S \setminus D$, admitting a semi-abelian prolongation $\mathcal{A}$ over S. We introduce a condition on $\mathcal{A}/S$ called *toric-additivity* (definitions 3.5 and 3.7), which is defined étale locally on S, and can be expressed in two equivalent ways (see theorem 3.4) when S is the étale local ring at a geometric point:

- by imposing a strict condition on how the toric rank of the fibres of $\mathcal{A}/S$ varies on D. Roughly, if $D_1, \dots, D_n$ are irreducible components of D, the toric rank at the generic point of $D_1 \cap \ldots \cap D_n$ should be the sum of the toric ranks at the generic points of the D_i's.
- by asking that, for some prime l invertible on S (equivalently, for all such primes), the biggest pro-l quotient of the tame fundamental group, $\pi_1^{t,l}(U) = \bigoplus_{i\in\mathcal{I}} \mathbb{Z}_l(1)$, acts in a certain way on the Tate module $T_l A(K^{sep})$ of the generic fibre. Namely, there should be a decomposition $T_l A(K^{sep}) = \bigoplus_{i\in\mathcal{I}} V_i$ such that the i-th direct summand of $\pi_1^{t,l}(U)$ acts trivially on all V_j with $j \neq i$.

1.2 Results

We first consider the case of a nodal curve $\mathcal{C}/S$, smooth over U. In this case, the abelian scheme A/U is the jacobian $J := \mathrm{Pic}^0_{\mathcal{C}_U/U}$, and its semi-abelian prolongation $\mathcal{A}/S$ is the scheme $\mathrm{Pic}^0_{\mathcal{C}/S}$ representing the fppf-sheaf on S of invertible sheaves on $\mathcal{C}$ that have degree zero on each component of the fibres of $\mathcal{C}/S$.

Theorem 1.2 (theorem 4.17)**.**

a) If $\mathrm{Pic}^0_{\mathcal{C}/S}$ is toric-additive, then J admits a Néron model over S.

b) If moreover S is an excellent $\mathbb{Q}$-scheme, the converse is also true.

The strategy of proof follows these lines: we show that if the hypotheses of a) or b) are satisfied, there exists a blow-up $\mathcal{C}' \to \mathcal{C}$, such that $\mathcal{C}'/S$ is still a nodal curve, smooth over U, and $\mathcal{C}'$ is *regular*. As the properties of admitting a Néron model or of being toric-additive are not affected by the desingularization, we have reduced to the case where the relative curve has regular total space. In this case, it can be shown that alignment and toric-additivity are equivalent, and we apply theorem 1.1.

We partially extend these results to the general case of an abelian scheme A/U admitting a semi-abelian prolongation on S.

Theorem 1.3 (theorem 5.8)**.** *Assume S is a $\mathbb{Q}$-scheme. If $\mathcal{A}/S$ is toric-additive, then A/U admits a Néron model over S.*

The theorem is proved by explicitly constructing a Néron model for A. The construction is carried out by means of an auxiliary object, a *test-Néron model* $\mathcal{N}/S$. This is, roughly, defined to be a smooth, separated group-space, which is a model for A, and such that, for every strictly henselian discrete valuation ring R and morphism $\operatorname{Spec} R \to S$ meeting D transversally, the pullback $\mathcal{N}_Z/Z$ is a Néron model of its generic fibre. We show that toric-additivity implies the existence of test-Néron models; and that test-Néron models are Néron models. We remark that for this last fact, it is crucial that test-Néron are defined to be group objects; there are examples of objects that are similar to test-Néron models, in that they satisfy a similar property with respect to transversal traits, but fail to be a Néron model because they do not admit a group structure: an example is the *balanced Picard stack* $\mathcal{P}_{d,g} \to \overline{\mathcal{M}}_g$ constructed by Caporaso in [Cap08].

Whether toric-additivity is also a necessary condition for the existence of a Néron model is still an open question; the main obstacle is showing that a Néron model is always a test-Néron model.

1.3 Outline

In section 2, we first recall the definition of a Néron model (definition 2.7) and state a number of properties regarding the behaviour of Néron models under different sorts of base change. In the rest of the section, we follow closely Exposé IX of [GRR72], titled *Modèles de Néron et monodromie*, where the authors investigate the relation between the reduction type of the Néron model and the Galois action on the Tate module; we show that a number of results that are proved there stay true when the base has dimension higher than 1. Among these there is the characterization of the l-primary part of the group of components of a Néron model in terms of the Tate module $T_lA(K^{sep})$ of the generic fibre, for a prime l invertible on S (section 2.4).

In section 3, we introduce the condition of toric-additivity (definition 3.5). We show in theorem 3.4 that it can be equivalently stated as a condition on the Tate module $T_l A(K^{sep})$, for any l invertible on S, or as a condition on the toric ranks of the fibres of the semi-abelian scheme $\mathcal{A}/S$.

Section 4 is devoted to the case of jacobians of curves. After recalling the results of [Hol17b], we establish the relation between toric-additivity and the property of existence of a Néron model for the jacobian (theorem 4.17).

In section 5, we work under the assumption that the base S is a $\mathbb{Q}$-scheme; we attempt to relate toric-additivity and the property of existence of Néron models in the case of abelian schemes. We introduce test-Néron models and prove that they exist and are unique if $\mathcal{A}/S$ is toric-additive (proposition 5.5 and theorem 5.6). After a result on descent of test-Néron models (proposition 5.7), we conclude the section by showing that test-Néron models are Néron models, under the assumption of toric-additivity (proposition 5.9).

2 Generalities

2.1 Normal crossing divisors and tame fundamental group

We work over a connected, regular, locally noetherian, base scheme S.

Definition 2.1. Given a regular, noetherian local ring R, a *regular system of parameters* is a minimal subset $\{r_1, \dots, r_d\} \subset R$ of generators for the maximal ideal $\mathfrak{m} \subset R$.

Definition 2.2. A *strict normal crossing divisor* D on S is a closed subscheme $D \subset S$ such that, for every point $s \in S$, the preimage of D in the local ring $\mathcal{O}_{S,s}$ is the zero locus of a product $r_1 \cdot \ldots \cdot r_n$, where $\{r_1, \dots, r_n\}$ is a subset of a regular system of parameters $\{r_1, \dots, r_d\}$ of $\mathcal{O}_{S,s}$.

Write $\{D_i\}_{i \in \mathcal{I}}$ for the set of irreducible components of D. Then each D_i, seen as a reduced closed subscheme of S, is regular and of codimension 1 in S; moreover, for every finite subset $\mathcal{J} \subset \mathcal{I}$, the intersection $\bigcap_{j \in \mathcal{J}} D_j$ is regular, and each of its irreducible components has codimension $|\mathcal{J}|$.

Definition 2.3. A *normal crossing divisor* D on S is a closed subscheme $D \subset S$ for which there exists an étale surjective morphism $S' \to S$ such that the base change $D \times_S S'$ is a strict normal crossing divisor on S'.

Notice that for every geometric point s of S, the pullback of a normal crossing divisor D to the spectrum of the strict henselization $\mathcal{O}^{sh}_{S,s}$ is a strict normal crossing divisor.

Definition 2.4. A *trait* Z is an affine scheme with $\mathcal{O}(Z)$ a discrete valuation ring. Suppose we are given a morphism $f\colon Z \to S$ and a normal crossing divisor D on S; we say that f is *transversal to* D if for every component D_i of D, $D_i \times_S Z$ is a reduced point or is empty.

We can now introduce the hypotheses with which we will work for most of this part:

Situation 2.5. Let S be a regular, locally noetherian connected scheme, $D = \bigcup_{i\in\mathcal{I}} D_i$ a normal crossing divisor on S. We will denote by U the open $S \setminus D$, by η the generic point of S and by K the residue field $k(\eta)$. A separable closure of K will be denoted by K^s. Finally, we write ζ_i for the generic point of the irreducible component D_i of D.

Situation 2.6. In situation 2.5, we will often reduce to the simpler case where S is the spectrum of a strictly henselian local ring R. In this case, we say that it is a *strictly local* scheme. We write s for its closed point and $p \geq 0$ for its residue characterstic. We can write the normal crossing divisor D as a union $\bigcup_{i=1}^n \mathrm{div}(r_i)$ where $r_1, \ldots, r_n \in R$ form a subset of a regular system of parameters for R.

Suppose we are in situation 2.6. It is a consequence of Abhyankar's Lemma ([Gro71, XIII, 5.2]) that every finite etale morphism $V \to U$, tamely ramified over D ([Gro71, XIII, 3.2.c)]), with V connected, is dominated by a finite étale W/U given by

$$\mathcal{O}(W) = \frac{\mathcal{O}(U)[T_1, \ldots, T_n]}{T_1 - r_1^{m_1}, \ldots, T_n - r_n^{m_n}}$$

where the integers $m_1, \ldots, m_n$ are coprime to p. Denoting by $\mu_{r,U}$ the group-scheme of r-roots of unity, it follows that $\underline{\mathrm{Aut}}_U(W) = \prod_{i=1}^n \mu_{m_i,U}$. Then, the *tame fundamental group* of U is

$$\pi_1^t(U) = \prod_{l \neq p} \mathbb{Z}_l(1)^n.$$

Here $\mathbb{Z}_l(1) = \lim \mu_{l^r}(U)$ is non-canonically isomorphic to $\mathbb{Z}_l$, an isomorphism being given by a choice of a compatible system $(z_{l^r})_{r\geq 1}$ of primitive l^r-roots of unity. We will sometimes write $\widehat{\mathbb{Z}}'(1)$ in place of $\prod_{l\neq p} \mathbb{Z}_l(1)$.

For a prime $l \neq p$, the factor $\mathbb{Z}_l(1)^n$ of $\pi_1^t(U))$ is the biggest pro-l quotient of $\pi_1^t(U)$ and will be denoted by $\pi_1^{t,l}(U)$. It is the automorphism group of the fibre functor of finite étale morphisms $V \to U$ of degree a power of l.

2.2 Néron models of abelian schemes

The definition of Néron model

Let now S be any scheme, $U \subset S$ an open and A/U an abelian scheme.

Definition 2.7. A *Néron model* for A over S is a smooth, separated algebraic space [1] $\mathcal{N}/S$ of finite type, together with an isomorphism $\mathcal{N} \times_S U \to A$, satisfying the following universal property: for every smooth morphism of schemes $T \to S$ and U-morphism $f\colon T_U \to A$, there exists a unique morphism $g\colon T \to \mathcal{N}$ such that $g_{|U} = f$.

It follows immediately from the definition that a Néron model is unique up to unique isomorphism; moreover, applying its defining universal property to the morphisms $m\colon A \times_U A \to A, i\colon A \to A$, and $0_A\colon U \to A$ defining the group structure of A, we see that $\mathcal{N}/S$ inherits from A a unique S-group-space structure.

We also introduce a similar object, which satisfies a weaker universal property:

Definition 2.8. A *weak Néron model* for A over S is a smooth, separated algebraic space $\mathcal{N}/S$ of finite type, together with an isomorphism $\mathcal{N} \times_S U \to A$, satisfying the following universal property: every section $U \to A$ extends uniquely to a section $S \to \mathcal{N}$.

In particular, a Néron model is a weak Néron model. Notice that in the case of weak Néron models, we do not have any uniqueness statement, and they need not inherit a group structure from A.

We point out that our definition 2.8 of weak Néron model differs slightly from the one normally found in the literature: the latter requires that the universal property is satisfied for all $T \to S$ finite étale.

Base change properties

We proceed to analyse how Néron models behave under different types of base change. In general, the property of being a Néron model is not stable under arbitrary base change. However, we have that:

Lemma 2.9. *Let $\mathcal{N}/S$ be a Néron model of A/U; let $S' \to S$ be a smooth morphism and $U' = U \times_S S'$. Then the base change $\mathcal{N} \times_S S'$ is a Néron model of $A_{U'}$.*

[1] defined as in [Sta16]TAG 025Y.)

Proof. Let $X \to S'$ be a smooth scheme with a morphism $f\colon X_{U'} \to \mathcal{A}_{U'}$; by composition with the smooth morphism $S' \to S$ we obtain a smooth scheme $X \to S$ and a map $X \times_S U \to A_U$, which extends uniquely to an S-morphism $X \to \mathcal{N}$. This is the datum of an S'-morphism $X \to \mathcal{N} \times_S S'$ extending f. □

Lemma 2.10. *Let $\mathcal{N}/S$ be a smooth, separated algebraic space of finite type with an isomorphism $\mathcal{N} \times_S U \to A$. Let $S' \to S$ be a faithfully flat morphism and write $U' = U \times_S S'$. If $\mathcal{N} \times_S S'$ is a Néron model of $A \times_U U'$, then $\mathcal{N}/S$ is a Néron model of A.*

Proof. We first show that $\mathcal{N}/S$ satisfies the universal property of Néron models when the smooth morphism $T \to S$ is the identity. So, let $f\colon U \to A$ be a section of A/U. To show that f extends to a section $S \to \mathcal{N}$ we only need to check that the schematic closure X of $f(U)$ inside $\mathcal{N}$ is faithfully flat over S: indeed, $X \to S$ is birational and separated; if it is also flat and surjective it is automatically an isomorphism. Now, by base change of f we get a closed immersion $f'\colon U' \to A \times_U U'$, which extends to a section $g'\colon S' \to \mathcal{N} \times_S S'$ by hypothesis. The schematic image $g'(S')$ is necessarily the schematic closure of $f'(U')$ inside $\mathcal{N} \times_S S'$; since taking the schematic closure commutes with faithfully flat base change, we have $g'(S') = X \times_S S'$. We deduce that $X \to S$ is faithfully flat, as its base change via $S' \to S$ is such. Hence $f\colon U \to A$ extends to a section $g\colon S \to \mathcal{N}$. The uniqueness of the extension is a consequence of the separatedness of $\mathcal{N}$.

Next, let $T \to S$ be smooth and let $f\colon T_U \to A$. In order to extend f to a morphism $g\colon T \to \mathcal{N}$, it is enough to show that $\mathcal{N} \times_S T$ satisfies the extension property for sections $T_U \to A \times_U T_U$. By the previous paragraph, it is enough to know that $(\mathcal{N} \times_S T) \times_S S' = (\mathcal{N} \times_S S') \times_S T$ is a Néron model of $(A \times_U T_U) \times_U U'$. This is true by lemma 2.9, concluding the proof. □

Lemma 2.11. *Let A/U be abelian, $f\colon S' \to S$ a smooth surjective morphism, $U' = U \times_S S'$, and $\mathcal{N}'/S'$ a Néron model of $A \times_S S'$. Then there exists a Néron model $\mathcal{N}/S$ for A.*

Proof. Write $S'' := S' \times_S S'$, $p_1, p_2\colon S'' \to S'$ for the two projections and $q\colon S'' \to S$ for $f \circ p_1 = f \circ p_2$. By lemma 2.9, both $p_1^*\mathcal{N}$ and $p_2^*\mathcal{N}$ are Néron models of q^*A. By the uniqueness of Néron models, we obtain a descent datum for $\mathcal{N}'$ along $S' \to S$. Effectiveness of descent data for algebraic spaces ([Sta16]TAG 0ADV) yields a smooth, separated algebraic space $\mathcal{N}/S$ of finite type. By lemma 2.10, this is a Néron model for A/U. □

Although Néron models are not stable under base change (not even flat), they are preserved by localizations, as we see in the following lemma:

Lemma 2.12. *Assume S is locally noetherian. Let s be a point (resp. geometric point) of S and $\widetilde{S}$ the spectrum of the localization (resp. strict henselization) at s. Suppose that $\mathcal{N}/S$ is a Néron model for A/U. Then $\mathcal{N} \times_S \widetilde{S}$ is a Néron model for $A \times_U \widetilde{U}$, where $\widetilde{U} = \widetilde{S} \times_S U$.*

Proof. Let $\widetilde{Y} \to \widetilde{S}$ be a smooth scheme and $\widetilde{f} \colon \widetilde{Y}_{\widetilde{U}} \to A_{\widetilde{U}}$ a morphism. We may assume that $\widetilde{Y}$ is of finite type over $\widetilde{S}$, hence of finite presentation. By [GD67, 3, 8.8.2] there exist an open neighbourhood (resp. étale neighbourhood) S' of s, a scheme $Y' \to S'$ restricting to $\widetilde{Y}$ over $\widetilde{S}$, and a $(U \times_S S')$-morphism $f' \colon Y' \times_{S'} (U \times_S S') \to \mathcal{N} \times_S (U \times_S S')$ restricting to $\widetilde{f}$ on $\widetilde{U}$. By lemma 2.9, $\mathcal{N} \times_S S'$ is a Néron model of $\mathcal{N} \times_S (U \times_S S')$, hence we get a unique extension $g' \colon Y' \to \mathcal{N} \times_S S'$ of f'. The base-change of g' via $\widetilde{S} \to S'$ gives us the required unique extension of $\widetilde{f}$. □

Proposition 2.13. *Assume that S is regular. If $\mathcal{A}/S$ is a an abelian algebraic space, then it is a Néron model of its restriction $\mathcal{A} \times_S U$.*

Proof. Using lemma 2.10, we may assume that S is strictly local and that $\mathcal{A}/S$ is a scheme. We identify $\mathcal{A}$ with its double dual $\mathcal{A}'' = \operatorname{Pic}^0_{\mathcal{A}'/S}$. Now let $T \to S$ be smooth and $f \colon T_U \to \mathcal{A}_U$. Then f corresponds to an element of $A_U(T_U) = \operatorname{Pic}^0_{\mathcal{A}'/S}(T_U) = \operatorname{Pic}^0(\mathcal{A}'_{T_U}) / \operatorname{Pic}^0(T_U)$. Let $\mathcal{L}_U$ be an invertible sheaf with fibres of degree 0 on $\mathcal{A}'_{T_U}$ mapping to f in $\mathcal{A}_U(T_U)$. As $\mathcal{A}'_T$ is regular, $\mathcal{L}_U$ extends to an invertible sheaf of degree 0 on $\mathcal{A}'_T$, which yields a T-point of $\mathcal{A}'' = \mathcal{A}$ extending f. The uniqueness of the extension follows from the separatedness of $\mathcal{A}/S$. □

We conclude the subsection by stating the main theorem about Néron models in the case where the base S is of dimension 1.

Theorem 2.14 ([BLR90], 1.4/3)**.** *Let S be a connected Dedekind scheme with fraction field K and let A/K be an abelian variety. Then there exists a Néron model $\mathcal{N}$ over S for A/K.*

2.3 Semi-abelian models and the action of inertia

Semi-abelian schemes

Definition 2.15. Let κ be a field and G/κ a smooth, commutative κ-group scheme of finite type. We say that G/κ is *semi-abelian* if it fits into an exact sequence of fppf-sheaves over κ

$$0 \to T \to G \to B \to 0 \tag{2}$$

where T/κ is a torus and B/κ an abelian variety. We call $\mu := \dim T$ the *toric rank* of G and $\alpha := \dim B$ its *abelian rank.* These two numbers do not depend on the choice of exact sequence (2), and are stable under base field extensions. Notice that G is automatically geometrically connected.

For a general base scheme S, a smooth commutative S-group scheme $\mathcal{G}/S$ of finite type is *semi-abelian* if for all points $s \in S$, the fibre $\mathcal{G}_s/k(s)$ is semi-abelian.

Given a semi-abelian scheme $\mathcal{G}/S$, we define for later use a function

$$\mu\colon S \to \mathbb{Z}_{\geq 0} \tag{3}$$

which associates to a point $s \in S$ the toric rank of $\mathcal{G}_s$. It can be shown that it is an upper semi-continuous function.

Analogously we can define

$$\alpha\colon S \to \mathbb{Z}_{\geq 0} \tag{4}$$

for the abelian rank of fibres. The sum $\mu + \alpha$ is the locally constant function with value the relative dimension of $\mathcal{G}/S$.

Situation 2.16. For the rest of part I, we assume that we are in situation 2.5 and that we are also given

- an abelian scheme A/U of relative dimension $d \geq 0$;
- a smooth, separated S-group scheme of finite presentation $\mathcal{A}/S$, together with an isomorphism $\mathcal{A} \times_S U \to A$, such that the fibrewise-connected component of identity $\mathcal{A}^0/S$ is semi-abelian.

The assumption that such a semi-abelian extension of A exists tells us a lot about the structure of a Néron model $\mathcal{N}/S$ of A (provided that it exists):

Lemma 2.17. *Suppose A/U admits a Néron model $\mathcal{N}/S$. Then the canonical morphism $\mathcal{A} \to \mathcal{N}$ is an open immersion, and induces an isomorphism from $\mathcal{A}^0$ to the fibrewise-connected component of identity $\mathcal{N}^0$.*

Proof. The fact that $\mathcal{A} \to \mathcal{N}$ is an open immersion follows from [GRR72, IX, Prop. 3.1.e]. For every point $s \in S$ of codimension 1, the restriction of $\mathcal{N}^0$ to the local ring $\mathcal{O}_{S,s}$ is the Néron model of its generic fibre, by lemma 2.12. It follows by [Ray70b, XI, 1.15] that the induced morphism $\mathcal{A}^0 \to \mathcal{N}^0$ is an isomorphism. □

In particular, the fibrewise-connected component of $\mathcal{N}^0/S$ is semi-abelian.

The Tate module

For the rest of section 2, we will assume that S is strictly local, with closed point s and residue field $k = k(s)$ of characteristic $p \geq 0$.

Let l be a prime different from p and $r \geq 0$ an integer; we denote by $\mathcal{A}[l^r]$ the kernel of the multiplication map

$$l^r : \mathcal{A} \to \mathcal{A}.$$

It is a closed subgroup scheme of $\mathcal{A}$, étale and quasi-finite over S. Its restriction $\mathcal{A}[l^r]_U/U$ is a finite, étale U-group scheme of order l^{2rd}. Because its order is coprime to p, $\mathcal{A}[l^r]_U/U$ is tamely ramified over D. It follows that the action of $\mathrm{Gal}(K^s|K)$ on $A[l^r](K^s)$ factors via the quotient map

$$\mathrm{Gal}(K^s|K) \to \pi_1^t(U) = \widehat{\mathbb{Z}}'(1)^n.$$

We write G for $\pi_1^t(U)$ and I_i for the i-th copy of $\widehat{\mathbb{Z}}'(1)$, so that $G = \bigoplus_{i=1}^n I_i$.

Let $T_l\mathcal{A}$ be the l-adic sheaf $\lim_r \mathcal{A}[l^r]$ on S. The group of K^s-valued points of its generic fibre is the *Tate module*

$$T_l A(K^s) = \lim A[l^r](K^s),$$

a free $\mathbb{Z}_l$-module of rank $2d$, which inherits a continuous action of $\pi_1^t(U)$.

Now, over the closed point $s \in S$ there exists an exact sequence

$$0 \to T \to \mathcal{A}_s^0 \to B \to 0$$

as in (2); for a prime $l \neq p$, $\mathcal{A}_s^0$ is l-divisible and it follows that we have an exact sequence of l-adic sheaves

$$0 \to T_l T \to T_l\mathcal{A}_s^0 \to T_l B \to 0$$

which in turn gives an exact sequence of $\mathbb{Z}_l$-modules

$$0 \to T_l T(k) \to T_l\mathcal{A}_s^0(k) \to T_l B(k) \to 0 \tag{5}$$

Write μ and α for $\mu(s)$ and $\alpha(s)$. Taking ranks in the exact sequence (5), we have

- $\mathrm{rk}\, T_l T(k) = \mu$,
- $\mathrm{rk}\, T_l B(k) = 2\alpha$,
- $\mathrm{rk}\, T_l\mathcal{A}_s^0(k) = \mu + 2\alpha = 2d - \mu$.

The following lemma is particularly useful:

Lemma 2.18. *The inclusion of l-adic sheaves $T_l\mathcal{A}^0 \hookrightarrow T_l\mathcal{A}$ restricts to an equality over the closed point s; that is,*

$$(T_l\mathcal{A})_s = (T_l\mathcal{A}^0)_s \tag{6}$$

Proof. To prove this, it is enough to check that $T_l\mathcal{A}_s(k) = T_l\mathcal{A}_s^0(k)$. If $(x_v)_v$ is an element of the left-hand side, each x_v is a l^v-torsion element of $\mathcal{A}_s(k)$ infinitely divisible by l. Let Φ be the group of components of $\mathcal{A}_s$; it is a finite abelian group, by the assumption that $\mathcal{A}$ is of finite presentation. Let φ_v be the image of x_v in Φ; then φ_v belongs to the l^v-torsion subgroup of Φ. Moreover φ_v is infinitely divisible by l; it follows that $\varphi_v = 0$, and that x_v lies in $\mathcal{A}_s^0(k)$. □

The fixed part of the Tate module

Consider again the l^r-torsion subscheme $\mathcal{A}[l^r]/S$. As S is henselian, there is a canonical decomposition

$$\mathcal{A}[l^r] = \mathcal{A}[l^r]^f \sqcup \mathcal{A}[l^r]'$$

where $\mathcal{A}[l^r]^f/S$, called *the fixed part* of $\mathcal{A}[l^r]$, is finite over S and $\mathcal{A}[l^r]'_s = \emptyset$. It can be shown that $\mathcal{A}[l^r]^f$ is a subgroup-scheme of $\mathcal{A}[l^r]$, étale over S. As S is strictly-henselian, it is a disjoint union of copies of S, and we find

$$\mathcal{A}[l^r]^f(K^s) = \mathcal{A}[l^r]^f(K) = \mathcal{A}[l^r]^f(S) = \mathcal{A}[l^r]^f_s(k) = \mathcal{A}[l^r]_s(k). \tag{7}$$

We define the *fixed part* of $T_l\mathcal{A}$ as the limit $T_l\mathcal{A}^f = \lim \mathcal{A}[l^r]^f$; this is a free l-adic sheaf, whose group of K^s-valued point is a submodule of the Tate module

$$T_l\mathcal{A}^f(K^s) =: T_lA(K^s)^f \subseteq T_lA(K^s).$$

By taking the limit in (7) and applying lemma 2.18, we find

$$T_lA(K^s)^f = T_l\mathcal{A}^f(S) = T_l\mathcal{A}_s(k) = T_l\mathcal{A}_s^0(k). \tag{8}$$

This last equality enables us to determine the rank of the fixed submodule of the Tate module,

$$\operatorname{rk} T_lA^f(K^s) = 2d - \mu = \operatorname{rk} T_lA(K^s) - \mu \tag{9}$$

Moreover, we have that

$$T_l\mathcal{A}_s^0(k) \otimes \mathbb{Z}/l^r\mathbb{Z} = \mathcal{A}_s^0[l^r](k) \tag{10}$$

since $\mathcal{A}_s^0(k)$ is l-divisible. Hence,

$$T_lA(K^s)^f \otimes_{\mathbb{Z}_l} \mathbb{Z}/l^r\mathbb{Z} = \mathcal{A}_s^0[l^r](k). \tag{11}$$

In other words, $T_lA^f(K^s) \otimes_{\mathbb{Z}_l} \mathbb{Z}/l^r\mathbb{Z}$ is the submodule of $A[l^r](K^s)$ consisting of those points that extend to sections of the fibrewise-connected component of identity $\mathcal{A}^0$.

The following proposition gives us an alternative interpretation of the fixed part of $T_lA(K^s)$:

Proposition 2.19. *The submodule $T_lA(K^s)^f$ is the submodule $T_lA(K^s)^G \subseteq T_lA(K^s)$ of elements fixed by $G = \pi_1^t(U)$.*

Proof. We treat first the case $\dim S = 1$; so S is the spectrum of a discrete valuation ring. In this case, A/K admits a Néron model, $\mathcal{N}/S$. By assumption, the fibrewise-connected component of identity $\mathcal{A}^0$ is semi-abelian, and we have an identification $\mathcal{N}^0 = \mathcal{A}^0$ (lemma 2.17).

Now, equality (8) and lemma 2.18 tell us that

$$T_lA(K^s)^f = T_l\mathcal{A}_s^0(k) = T_l\mathcal{N}_s^0(k) = T_l\mathcal{N}_s(k).$$

By Hensel's lemma, $\mathcal{N}_s[l^r](k) = \mathcal{N}[l^r](S)$ and by the definition of Néron model the latter is equal to $\mathcal{N}_K[l^r](K) = A[l^r](K^s)^G$. Hence, $T_lA(K^s)^G = \lim A[l^r](K^s)^G$ is equal to $T_l\mathcal{N}_s(k)$ and we are done.

Let now S have dimension $\dim S \geq 2$. First, observe that $T_lA(K^s)^f \subseteq T_lA(K^s)^G$: indeed, as $T_l\mathcal{A}^f$ is free, its K^s-valued point are actually K-valued. We show the reverse inclusion. We start by claiming that there exists a closed subscheme $Z \subset S$, regular and of dimension 1, such that $Z \not\subseteq D$. For this, let $\{t_1, \ldots, t_n\}$ be a system of regular parameters of $\mathcal{O}(S)$, cutting out the divisor D. We complete the above set to a maximal system $\{t_1, \ldots, t_n, t_{n+1}, \ldots, t_{\dim S}\}$ of regular parameters and let $Z = Z(t_1 - t_2, t_2 - t_3, \ldots, t_{n-1} - t_n, t_{n+1}, t_{n+2}, \ldots, t_{dimS})$. Now, $\mathcal{O}(Z)$ is a strictly henselian discrete valuation ring, and the generic point ζ of Z lies in U. We let $L = k(\zeta)$ and $H = \mathrm{Gal}(L^s|L)$ for some separable closure $L \hookrightarrow L^s$. Since $\mathcal{A}[l^r]$ is finite étale over U, we have $\mathcal{A}[l^r](K) \subseteq \mathcal{A}[l^r](L)$ and by passing to the limit we obtain $T_lA(K^s)^G \subseteq T_lA(L^s)^H$. Moreover, by the dimension 1 case, $T_lA(L^s)^H = T_l(\mathcal{A}_Z)(L^s)^f = T_l\mathcal{A}_s(k)$; the latter is equal to $T_lA(K^s)^f$, concluding the proof. $\square$

The toric part of the Tate module

Denote by $\mathcal{T}_s$ the biggest subtorus of the semiabelian scheme $\mathcal{A}^0_s$; we have an inclusion of the l^r-torsion

$$\mathcal{T}_s[l^r] \subseteq \mathcal{A}^0_s[l^r].$$

As the restriction functor between the category of finite étale S-schemes and the category of finite étale k-schemes is an equivalence of categories, we obtain a canonical finite étale S-subscheme of $\mathcal{A}^0[l^r]$, called the *toric part* of $\mathcal{A}^0[l^r]$,

$$\mathcal{A}^0[l^r]^t \hookrightarrow \mathcal{A}^0[l^r]^f \hookrightarrow \mathcal{A}^0[l^r]$$

such that $A^0[l^r]^t \otimes_S k = \mathcal{T}_s[l^r]$.

Taking the limit, we find a free subsheaf $T_l\mathcal{A}^t$ of $\lim \mathcal{A}^0[l^r]^f = \lim \mathcal{A}[l^r]^f = T_l\mathcal{A}^f$. Then, passing to the generic fibre, we obtain a submodule $T_lA(K^s)^t$ of $T_lA(K^s)^f = T_lA(K^s)^G \subseteq T_lA(K^s)$, which we call *toric part* of $T_lA(K^s)$. Its rank is of course the rank of the $\mathbb{Z}_l$-module $T_l\mathcal{T}_s(k)$, that is

$$\operatorname{rk} T_lA(K^s)^t = \mu. \tag{12}$$

To summarize, we have a filtration of the Tate module

$$0 \xrightarrow{\mu} T_lA(K^s)^t \xrightarrow{2\alpha} T_lA(K^s)^f \xrightarrow{\mu} T_lA(K^s)$$

where the numbers on top of the arrows are the ranks of the successive quotients in the filtration.

The dual abelian variety and Weil pairing

We will now only work with the semi-abelian scheme $\mathcal{A}^0 \subset \mathcal{A}$; for this reason, we will write simply $\mathcal{A}$ for it, rather than $\mathcal{A}^0$. Consider the dual abelian variety A'_K of A_K. By [MB85, IV, 7.1], there exists a unique semi-abelian scheme $\mathcal{A}'/S$ extending A'_K. Let $\varphi\colon A_K \to A'_K$ be an isogeny; it extends uniquely to an isogeny $\mathcal{A} \to \mathcal{A}'$, inducing isogenies

$$\mathcal{T}_s \to \mathcal{T}'_s, \quad \mathcal{B}_s \to \mathcal{B}'_s$$

between the toric and abelian parts of $\mathcal{A}_s$ and $\mathcal{A}'_s$. We deduce the equality between the toric and abelian ranks

$$\mu = \mu' \quad \alpha = \alpha'.$$

By [MB85, II, 3.6] the natural functor

$$BIEXT(\mathcal{A}, \mathcal{A}'; \mathbb{G}_{m,S}) \to BIEXT(\mathcal{A}_K, \mathcal{A}'_K; \mathbb{G}_{m,K})$$

is an equivalence of categories; thus the Poincaré biextension on $A_K \times_K A'_K$ extends uniquely to a biextension on $\mathcal{A} \times_S \mathcal{A}'$, and we obtain for $l \neq p$ a perfect pairing

$$T_l\mathcal{A} \times T_l\mathcal{A}' \to T_l(\mathbb{G}_m) = \mathbb{Z}_l(1) \tag{13}$$

of l-adic sheaves on S extending the Weil pairing $\chi\colon T_lA(K^s) \times T_lA'(K^s) \to \mathbb{Z}_l(1)$.

Lemma 2.20 (Orthogonality theorem)**.** *The toric part $T_lA(K^s)^t$ is the orthogonal of the fixed part $T_lA'(K^s)^f = T_lA'(K^s)^G$ via the pairing χ.*

Proof. The proof follows the one given in [GRR72, IX, 2.4]: notice that, by comparing the ranks, we only need to check that $T_lA(K^s)^t \subseteq (T_lA'(K^s)^f)^\perp$.

We obtain $T_lA(K^s)^t$ and $T_lA'(K^s)^f$ by passing to the K^s-valued points of the generic fibres of $T_l\mathcal{A}^t$ and $T_l\mathcal{A}^f$. Therefore, we only need that the restriction of the pairing (13),

$$T_l\mathcal{A}^t \times T_l\mathcal{A}'^f \to T_l(\mathbb{G}_{m,S})$$

is the zero pairing. As $T_l\mathcal{A}^t$ and $T_l\mathcal{A}^f$ are constant l-adic sheaves, we may check this by restricting to the closed fibre. Now, the pairing

$$T_l\mathcal{T}_s \times T_l\mathcal{A}'_s \to T_l(\mathbb{G}_{m,k})$$

is identically zero by [GRR72, VIII, 4.10]. □

For each generic point ζ_i of the irreducible components $D_1, \dots, D_n$ of the divisor D, we can consider a strict henselization $S_i \to S$ at some geometric point $\overline{\zeta}_i$ lying over ζ_i. Over S_i, we can define the l-adic shaves

$$T_l(A_{S_i})^t \hookrightarrow T_l(A_{S_i})^f \hookrightarrow T_l\mathcal{A}_{S_i}.$$

We define $T_lA(K^s)^{t_i}$ and $T_lA(K^s)^{f_i}$ to be the groups of K^s-valued points of the generic fibre of $T_l(A_{S_i})^t$ and $T_l(A_{S_i})^f$ respectively; they are submodules of $T_lA(K^s)$, of rank μ_i and $2d - \mu_i$ respectively; moreover, $T_lA(K^s)^{t_i}$ and $T_lA'(K^s)^{f_i}$ are orthogonal to each other with respect to the pairing χ. By proposition 2.19, we have $T_lA'(K^s)^{f_i} = T_lA'(K^s)^{I_i}$; indeed, $I_i = \pi_1^t(S_i \setminus \{\overline{\zeta}_i\})$. Now, as we evidently have $T_lA'(K^s)^G = \bigcap_{i=1}^n T_lA'(K^s)^{I_i}$, by taking orthogonals with respect to χ we find the relation between toric parts

$$T_lA(K^s)^t = \sum_{i=1}^n T_lA(K^s)^{t_i}. \tag{14}$$

Taking ranks and using (12), we find that the function $\mu\colon S \to \mathbb{Z}_{\geq 0}$ satisfies the relation

$$\mu(s) \leq \sum_{i=1}^n \mu(\zeta_i). \tag{15}$$

The action of G on the Tate module is unipotent

We use the orthogonality lemma 2.20 to describe more explicitly the action of G on the Tate module.

Proposition 2.21. *There exists a submodule V of $T = T_l A(K^s)$ such that G acts trivially on V and on the quotient T/V.*

Proof. Clearly, G acts trivially on $V = T^G$. Now, as T^G is orthogonal to T'^t (where $T' = T_l A'(K^s)$) via the pairing χ, we obtain a perfect pairing $T/T^G \times T'^t \to \mathbb{Z}_l(1)$ which identifies T/T^G with $\operatorname{Hom}_{\mathbb{Z}_l}(T'^t, \mathbb{Z}_l(1))$. As $T'^t \subset T'^G$, we conclude that G acts trivially on T/T^G. $\square$

It follows from the above proposition that the action of G on $T_l A(K^s)$ is unipotent of level 2: that is, writing

$$\rho \colon G \to \operatorname{Aut}(T_l A(K^s) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l),$$

we have for every $g \in G$

$$(\rho(g) - \mathrm{id})^2 = 0.$$

Because the profinite group G acts on a $\mathbb{Q}_l$-vector space unipotently and continuously, the image of ρ is a pro-l-group. Thus, the action of G factors via its biggest pro-l-quotient

$$G = \widehat{\mathbb{Z}}(1)^n = \pi_1^t(U) \to \pi_1^{t,l}(U) = \mathbb{Z}_l(1)^n.$$

2.4 The group of components of a Néron model

Our objective now is to give an explicit description of the group of components of a Néron model, in terms of the Tate modules of A. In [GRR72, IX, 11] a description is provided of the full group of components in the case where the base has dimension 1. The case of a regular base of higher dimension is completely analogous; we will follow closely [GRR72], but will restrict our attention only to the prime-to-p part of the group of components; the description of the p-primary part involves more complicated theory.

We are still under the hypotheses of situation 2.16, with S strictly local, and now we assume that the abelian scheme A/U admits a Néron model $\mathcal{N}/S$. We denote by $\underline{\Phi}$ the étale S-group scheme of connected components of $\mathcal{N}/S$. It fits into an exact sequence of fppf-sheaves

$$0 \to \mathcal{N}^0 \to \mathcal{N} \to \underline{\Phi} \to 0.$$

Clearly, the restriction of $\underline{\Phi}$ to U is the trivial group scheme.

We are interested in the fibre of $\underline{\Phi}$ over the closed point $s \in S$; this is the étale k-group scheme of finite type

$$\underline{\Phi}_s = \frac{\mathcal{N}_s}{\mathcal{N}_s^0}.$$

As k is algebraically closed, $\underline{\Phi}_s$ is determined by its group of k-rational points $\Phi := \underline{\Phi}_s(k)$, which is a finite abelian group. We have

$$\Phi = \frac{\mathcal{N}_s(k)}{\mathcal{N}_s^0(k)} = \frac{\mathcal{N}(S)}{\mathcal{N}^0(S)} = \frac{A(U)}{A(U)^0} \tag{16}$$

where by $A(U)^0$ we denote the subset of $A(U)$ of U-points specializing to S-points of the identity component $\mathcal{N}^0$. Notice that the second equality is a consequence of Hensel's lemma and the third of the defining property of Néron models.

We let l be a prime different from the residue characteristic $p = \operatorname{char} k(s)$ and $n \geq 0$ be an integer. Taking l^n-torsion in the exact sequence

$$0 \to A(U)^0 \to A(U) \to \Phi \to 0$$

gives a long exact sequence

$$0 \to A[l^n](U)^0 \to A[l^n](U) \to \Phi[l^n] \to A(U)^0/l^n A(U)^0 \to \dots.$$

Multiplication by l^n on the fiberwise-connected component of identity $\mathcal{N}^0$ is an étale and surjective morphism; it follows that $\mathcal{N}^0(S) = A(U)^0$ is l-divisible; hence

$$\Phi[l^n] = \frac{A[l^n](U)}{A[l^n](U)^0} \tag{17}$$

Now, because $A[l^n]$ is finite étale over U, we have $A[l^n](U) = A[l^n](K)$. Writing T for $T_l A(K^s)$, we see that $A[l^n](U) = (T \otimes \mathbb{Z}/l^n\mathbb{Z})^G$, so we have an expression for the part of eq. (17) above the fraction line.

Let's turn to study $A[l^n](U)^0$. This is equal to $A[l^n](K)^0$ (again, the exponent 0 denotes those elements specializing to the identity component $\mathcal{N}^0$). The latter, by the defining property of Néron models, is simply $\mathcal{N}^0[l^n](S)$. Now, every section of a quasi-finite separated scheme over S factors via its fixed part, so $\mathcal{N}^0[l^n](S) = \mathcal{N}^0[l^n]^f(S)$. Because $\mathcal{N}^0(S)$ is l-divisible, we have $\mathcal{N}^0[l^n]^f(S) = (T_l\mathcal{N}^0)^f(S) \otimes_{\mathbb{Z}_l} \mathbb{Z}/l^n\mathbb{Z}$. Now, $(T_l\mathcal{N}^0)^f(S)$ is equal to $T_l A(K^s)^f$, which in turn is equal to $T_l A(K^s)^G$ by proposition 2.19.

This shows that $A[l^n](K)^0 = T_l A(K^s)^G \otimes_{\mathbb{Z}_l} \mathbb{Z}/l^n\mathbb{Z}$. We have found the relation

$$\Phi[l^n] = \frac{(T \otimes \mathbb{Z}/l^n\mathbb{Z})^G}{T^G \otimes \mathbb{Z}/l^n\mathbb{Z}}. \tag{18}$$

By taking the colimit over the powers of l we find that the l-primary part ${}_l\Phi$ of the group of components is given by

$${}_l\Phi = \mathrm{colim}_n\, \Phi[l^n] = \frac{(T \otimes \mathbb{Q}_l/\mathbb{Z}_l)^G}{T^G \otimes \mathbb{Q}_l/\mathbb{Z}_l}. \tag{19}$$

Example 2.22. We give an example in which we compute the 2-torsion of the group of components of the Néron model of an elliptic curve, in the case of $\dim S = 1$. Let $R = k[[t]]$ for some algebraically closed field k of characteristic zero. Let $S = \operatorname{Spec} R$, K be the fraction field of R, and consider the elliptic curve E/K given in $\mathbb{P}^2_K$ by

$$Y^2Z = (X - Z)(X^2 - t^2Z^2).$$

The same equation gives a nodal model $\mathcal{E}/S$; it follows that $E = \mathrm{Pic}^0_{E/K}$ admits a semi-abelian prolongation, given by $\mathcal{E}^{sm} = \mathrm{Pic}^0_{\mathcal{E}/S}$.

Let $\mathcal{N}/S$ be the Néron model of E over S; the open immersion $\mathcal{E}^{sm} \to \mathcal{N}$ identifies $\mathcal{E}^{sm}$ with $\mathcal{N}^0$. Let Φ be the group of components of the closed fibre of $\mathcal{N}$.

We work with the prime $l = 2$; by equation (18), the 2-torsion of Φ is given by

$$\Phi[2] = \frac{E[2](K)}{T_2E(K) \otimes \mathbb{Z}/2\mathbb{Z}}.$$

The Tate module $T_2E(\overline{K})$ is a free $\mathbb{Z}_2$-module of rank 2. Its K-rational part $T_2E(K) \subset T_2E(\overline{K})$ is free, of rank $2\dim E - \mu = 2 - 1 = 1$. Thus, $T_2E(K) \otimes \mathbb{Z}/2\mathbb{Z}$ is a free $\mathbb{Z}/2\mathbb{Z}$-submodule of rank 1 of $T_2E(\overline{K}) \otimes \mathbb{Z}/2\mathbb{Z} = E[2](\overline{K})$.

On the other hand, $E[2](K) = \{(0,1,0),(1,0,1),(t,0,1),(-t,0,1)\} = E[2](\overline{K}) \cong (\mathbb{Z}/2\mathbb{Z})^2$. We deduce that

$$\Phi[2] \cong \mathbb{Z}/2\mathbb{Z}.$$

Moreover, the points $(0,1,0)$ and $(1,0,1)$ extend to S-points of $\mathcal{E}^{sm} = \mathcal{N}^0$; while $(t,0,1)$ and $(-t,0,1)$ do not, as they specialize to the non-smooth point of $\mathcal{E}/S$. They do extend to S-points of $\mathcal{N}$ though, whose restriction to the closed fibre is contained in the only 2-torsion component different from the identity component.

3 Toric-additivity

We work with the hypotheses of situation 2.5; we suppose that we are given an abelian scheme A/U of relative dimension d, and a semi-abelian scheme $\mathcal{A}/S$ with an isomorphism $\mathcal{A} \times_S U \to A$.

3.1 Definition of toric-additivity in the strictly local case

Assume that S is strictly local, with closed point s and residue field $k = k(s)$ of characteristic $p \geq 0$. The divisor D has finitely many irreducible components $D_1, \ldots, D_n$ for some $n \geq 0$.

We fix a prime $l \neq p$ and consider the Tate module $T_l A(K^s)$; we recall that it is a free $\mathbb{Z}_l$-module of rank $2d$ with an action of $\mathrm{Gal}(K^s|K)$, which factors via the surjection $\mathrm{Gal}(K^s|K) \to G := \pi_1^{t,l}(U) = \bigoplus_{i=1}^n I_i$, where $I_i = \mathbb{Z}_l(1)$ for each i.

Definition 3.1. Let $l \neq p$ be a prime. We say that the semi-abelian scheme $\mathcal{A}/S$ satisfies condition $\bigstar(l)$ if

$$T_l A(K^s) = \sum_{i=1}^n T_l A(K^s)^{\oplus_{j\neq i} I_j} \text{ or if } n = 0. \tag{20}$$

Remark 3.2.

- Whether $\mathcal{A}/S$ satisfies condition $\bigstar(l)$ depends only on the generic fibre A_K/K, and on the base S;
- suppose that $\mathcal{A}/S$ satisfies condition $\bigstar(l)$; let t be another geometric point of S, belonging to $D_1, D_2, \ldots, D_m$ for some $m \leq n$, and consider the strict henselization S' at t. Then the morphism

  $$\pi_1^{t,l}(U \times_S S') \to \pi_1^{t,l}(U)$$

  induced by $S' \to S$ is the natural inclusion

  $$\bigoplus_{i=1}^m I_j \to \bigoplus_{i=1}^n I_i.$$

  It can be easily seen that $\sum_{i=1}^n T_l A(K^s)^{\oplus_{j\neq i} I_j} \subseteq \sum_{i=1}^m T_l A(K^s)^{\oplus_{j\neq i} I_j}$; hence $\mathcal{A}_{S'}/S'$ also satisfies condition $\bigstar(l)$.
- Condition $\bigstar(l)$ is automatically satisfied if $n = 1$.

We are going to show that the validity of condition $\bigstar(l)$ is independent of the chosen prime $l \neq p$. We first need an auxiliary lemma, which we recommend to skip, as its only utility is to show that some specific submodules of the Tate module are direct summands. This simplifies some later proofs.

Lemma 3.3. *The Tate module $T_lA(K^s)$ satisfies the following properties:*

i) There exists a decomposition of $T := T_lA(K^s)$ into a direct sum

$$T \cong \bigoplus_{J \subseteq \{1,\ldots,n\}} T_J$$

where, for every $J \subseteq \{1,\ldots,n\}$, the submodule of invariants $T^{\bigoplus_{j\in J} I_j}$ is equal to $\bigoplus_{J' \supseteq J} T_{J'}$.

ii) The submodule $\sum_{i=1}^n T_lA(K^s)^{\oplus_{j\neq i} I_j}$ is a direct summand of $T_lA(K^s)$.

Proof. We start with the proof of i). Notice first that, for any submodule $V \subseteq T$ and any subgroup $H \subseteq G$, the submodule of invariants V^H is a direct summand of V; indeed, the quotient V/V^H is torsion-free. Now we proceed by induction on n. If $n = 1$, write $T_{\{1\}} := T^{I_1}$, and $T_\emptyset = T/T^{I_1}$. In this case we have $T \cong T_{\{1\}} \oplus T_\emptyset$ as wished. Now let $m \geq 2$, assume that the statement is true for $n = m - 1$, and let $n = m$. By inductive hypothesis, we can write

$$T \cong \bigoplus_{J \subseteq \{1,\ldots,m-1\}} T_J \tag{21}$$

as in the statement. Define, for every $J \subseteq \{1,\ldots,m\}$,

$$V_J = \begin{cases} (T_{J\cap\{1,\ldots,m-1\}})^{I_m} & \text{if } m \in J; \\ T_J/(T_J)^{I_m} & \text{if } m \notin J. \end{cases}$$

It is easy to show that $T \cong \bigoplus_{J\subseteq\{1,\ldots,m\}} V_J$. Now, let $J \subseteq \{1,\ldots,m\}$. Suppose first that $m \notin J$. Then we have

$$\begin{aligned} T^{\bigoplus_{j\in J} I_j} &\cong \bigoplus_{J\subseteq J'\subseteq\{1,\ldots,m-1\}} T_{J'} \cong \bigoplus_{J\subseteq J'\subseteq\{1,\ldots,m-1\}} T_{J'}/(T_{J'})^{I_m} \oplus (T_{J'})^{I_m} \cong \\ &\cong \bigoplus_{J\subseteq J'\subseteq\{1,\ldots,m-1\}} V_{J'} \oplus V_{J'\cup\{m\}} \cong \bigoplus_{J\subseteq J'\subseteq\{1,\ldots,m\}} V_{J'}. \end{aligned}$$

If instead $m \in J$, then

$$T^{\bigoplus_{j\in J} I_j} \cong (T^{\bigoplus_{j\in J\setminus\{m\}} I_j})^{I_m} \cong \bigoplus_{J\setminus\{m\}\subseteq J'\subseteq\{1,\ldots,m\}} (V_{J'})^{I_m}.$$

Now, for a subset $J' \subseteq \{1, \dots, m\}$, $(V_{J'})^{I_m} \neq 0$ only if $m \in J'$; in this case, $(V_{J'})^{I_m} = V_{J'}$. It follows that

$$T^{\bigoplus_{j \in J} I_j} \cong \bigoplus_{J \subseteq J' \subseteq \{1,\dots,m\}} V_{J'}.$$

This proves i).

For ii), notice that for all $i = 1, \dots, n$, we have

$$T^{\bigoplus_{j \neq i} I_j} \cong T_{\{1,\dots,n\}} \oplus T_{\{1,\dots,n\} \setminus \{i\}}$$

and

$$\sum_{i=1}^{n} T^{\bigoplus_{j \neq i} I_j} \cong T_{\{1,\dots,n\}} \oplus \bigoplus_{i=1}^{n} T_{\{1,\dots,n\} \setminus \{i\}}.$$

Because of the decomposition of part i), we see that $\sum_{i=1}^{n} T^{\bigoplus_{j \neq i} I_j}$ is indeed a direct summand of T. □

Recall the upper semi-continuous function (3) $\mu\colon S \to \mathbb{Z}_{\geq 0}$. It takes the value $\mu(s)$ at the closed point of S, and the value $\mu(\zeta_i)$ at each generic point ζ_i of D_i.

Recall the inequality (15),

$$\mu(s) \leq \sum_{i=1}^{n} \mu(\zeta_i). \tag{22}$$

Theorem 3.4. *Let S be a regular, strictly local scheme, with closed point s of residue characteristic $p \geq 0$, $D = \bigcup_{i=1}^{n} D_i$ a normal crossing divisor on S. Let A be an abelian scheme over $U = S \setminus D$, of relative dimension d, admitting a semi-abelian prolongation $\mathcal{A}/S$. Let $l \neq p$ be a prime.*

The following conditions are equivalent:

- *a) $\mathcal{A}/S$ satisfies condition $\bigstar(l)$.*
- *b) For $i = 1, \dots, n$, let ζ_i be the generic point of D_i. The function $\mu\colon S \to \mathbb{Z}_{\geq 0}$ satisfies*

$$\mu(s) = \sum_{i=1}^{n} \mu(\zeta_i).$$

- *c) Let $G = \pi_1^t(U) = \bigoplus I_i$ with $I_i = \widehat{\mathbb{Z}}'(1)$. The Tate module $T_l A(K^s)$ decomposes as a direct sum*

$$T_l A(K^s) = V_1 \oplus V_2 \oplus \ldots \oplus V_n$$

of G-invariant submodules, such that for each $i = 1, \ldots, n$ and each $j \neq i$, $I_i \subset G$ acts trivially on V_j.

Proof. We will write shorthand T for $T_l A(K^s)$, μ for $\mu(s)$ and μ_i for $\mu(\zeta_i)$. Let us start with the equivalence $a) \Leftrightarrow b)$; we are going to proceed by induction on the number n. The case $n = 0$ being trivial, let first $n = 1$: in this case, condition $\bigstar(l)$ is automatically satisfied. We have to check that $\mu = \mu_1$, i.e. the toric rank at the closed point s is the same as the toric rank at the generic point of the (irreducible) divisor D. We know by eq. (22) that $\mu \leq \mu_1$; since $\mu\colon S \to \mathbb{Z}_{\geq 0}$ is upper-semicontinuous, we have the equality.

Let now N be an integer ≥ 2 and assume that the equivalence $a) \Leftrightarrow b)$ is true when $n = N - 1$; we show that it is true for $n = N$. In general we have $T \supseteq \sum_{i=1}^{n} T^{\oplus_{j \neq i} I_j}$, with equality if condition $\bigstar(l)$ is satisfied. We compare the ranks of the two sides. On the one hand, the rank of T is $2d$. Now write T_i for $T^{\oplus_{j \neq i} I_j}$. We have

$$\operatorname{rk} \sum_{i=1}^{n} T_i = \sum_{i=1}^{n} \operatorname{rk} T_i + \sum_{k=2}^{n} (-1)^k \sum_{J \subset \{1,\ldots,n\}, \#J = k} \operatorname{rk} \bigcap_{j \in J} T_j$$

by an inclusion-exclusion argument. However, for every $J \subset \{1, \ldots, n\}$ with $\#J \geq 2$, $\bigcap_{j \in J} T_j = T^G$. The equality above becomes

$$\operatorname{rk} \sum_{i=1}^{n} T_i = \sum_{i=1}^{n} \operatorname{rk} T_i + \sum_{k=2}^{n} (-1)^k \binom{n}{k} \operatorname{rk} T^G.$$

For every $i = 1, \ldots, n$, $\operatorname{rk} T_i = 2d - \mu(t_i) \geq 2d - \sum_{j \neq i} \mu_j$, where t_i is the generic point of $\bigcap_{j \neq i} D_j$. Also, $\sum_{k=2}^{n} (-1)^k \binom{n}{k} = 1 - n$, and $\operatorname{rk} T^G = 2d - \mu$. We obtain

$$\operatorname{rk} \sum_{i=1}^{n} T_i \geq 2nd - (n-1) \sum_{i=1}^{n} \mu_i + (1-n)(2d-\mu) = 2d + (n-1)(\mu - \sum_{i=1}^{n} \mu_i). \quad (23)$$

We have previously remarked that if condition $\bigstar(l)$ is satisfied, then it is satisfied also over S_i, the strict henselization at a geometric point lying over t_i; in this case, we can apply the inductive hypothesis: the inequality $\mu(t_i) \leq \sum_{j \neq i} \mu_j$ is an equality and thus eq. (23) is an equality as well.

We have obtained a chain of inequalities

$$\operatorname{rk} T = 2d \geq \operatorname{rk} \sum_{i=1}^{n} T_i \geq 2d + (n-1)(\mu - \sum_{i=1}^{n} \mu_i).$$

If condition $\bigstar(l)$ is satisfied, both $\geq$ signs in the line above are equalities, and therefore $(n-1)(\mu - \sum_{i=1}^n \mu_i) = 0$. Since $n-1>0$, we have indeed $\mu = \sum_{i=1}^n \mu_i$. Conversely, if $\mu = \sum_{i=1}^n \mu_i$, both inequalities are forced to be equalities; in particular $\operatorname{rk} T = \operatorname{rk} \sum_{i=1}^n T_i$. By lemma 3.3, $\sum_{i=1}^n T_i$ is a direct summand of T; hence $T = \sum_{i=1}^n T_i$, which is condition $\bigstar(l)$. This proves $a) \Leftrightarrow b)$.

Next, assume that a decomposition of T as in c) exists. Then, for every $1 \leq i \leq n$, $V_i \subseteq T^{\bigoplus_{j\neq i} I_j}$, and condition $\bigstar(l)$ is evidently satisfied; so we have $c) \Rightarrow a)$.

Finally, we prove $b) \Rightarrow c)$. Consider the canonical maps

$$\alpha\colon \bigoplus_{i=1}^n T^{t_i} \to \sum_{i=1}^n T^{t_i} = T^t; \quad \beta\colon T/T^G \to \bigoplus_{i=1}^n T/T^{I_i}.$$

Clearly, α is surjective and β is injective. However, because we have $\mu = \sum_{i=1}^n \mu_i$, comparing ranks we see that α is an isomorphism. The same is true for the analogous map

$$\alpha'\colon \bigoplus_{i=1}^n T'^{t_i} \to T'^t,$$

where $T' = T_l A'(K^s)$ and A'_K is the dual abelian variety. By lemma 2.20, T^G (resp. T^{I_i}) is orthogonal to T'^t (resp. T'^{t_i}) with respect to the pairing χ. Hence, β is obtained from α' by applying the functor $\operatorname{Hom}_{\mathbb{Z}_l}(\,\cdot\,, \mathbb{Z}_l(1))$. It follows that β is an isomorphism as well.

Notice that for every $i = 1, \dots, n$, the inverse morphism β^{-1} identifies $\bigoplus_{j\neq i} T/T^{I_j}$ with the submodule T^{I_i}/T^G of T/T^G. Moreover, since T^G is a direct factor of T, we can choose a section $h\colon T/T^G \to T$. As h maps T^{I_i}/T^G into T^{I_i}, we see that the image of $\bigoplus_{j\neq i} T/T^{I_j}$ via $h \circ \beta^{-1}$ is contained in T^{I_i}.

Write $T^G = T^t \oplus W$ for some submodule W; and write $W_i := T^{t_i} \oplus (h \circ \beta^{-1})(T/T^{I_i})$ for each i. Then $T = W \oplus W_1 \oplus W_2 \oplus \ldots \oplus W_n$. For each i, I_i acts trivially on T^G, hence on W and T^{t_j} for all j. Moreover, we have shown that for $j \neq i$, I_i acts trivially on $(h \circ \beta^{-1})(T/T^{I_j})$. Therefore I_i acts trivially on W_j for $j \neq i$.

Now, we may write $V_1 = W \oplus W_1$, and $V_i = W_i$ for all $i \geq 2$. It remains only to show that V_i is I_i-invariant. For this, let e_i be a topological generator of I_i. For every $x \in T$, $y \in T^{I_i}$, we have

$$\chi(e_i x - x, y) = \chi(x, e_i y - y) = \chi(x, 0) = 1.$$

Therefore $e_i x - x \in (T^{I_i})^\perp = T^{t_i}$ for every $x \in T$. In particular, for every $x \in V_i$, $e_i x \in V_i + T^{t_i} = V_i$, as we wished to show.

□

A consequence of theorem 3.4 $a) \Leftrightarrow b)$, is that the validity of condition $\bigstar(l)$ is independent of the choice of prime $l \neq p$. It is sensible to introduce a new name for the condition:

Definition 3.5. We say that the semi-abelian scheme $\mathcal{A}/S$ is *toric-additive* if the three equivalent conditions of theorem 3.4 are satisfied for some prime number $l \neq p$ (equivalently, for all such primes l).

Notice that, although we talk of "toric-additivity of the semi-abelian scheme $\mathcal{A}/S$", toric-additivity depends only on the generic fibre A_K (in fact, on its torsion K^s-points) and on S. This is a consequence of theorem 3.4, but follows also from the fact that a semi-abelian extension $\mathcal{A}/S$ of A_K is unique up to unique isomorphism ([Del85, Théorème pag.132]).

Lemma 3.6. *Let $m_1, \dots, m_n$ be positive integers and B be the $\Gamma(S, \mathcal{O}_S)$-algebra*

$$B = \frac{\Gamma(S, \mathcal{O}_S)[T_1, \dots, T_n]}{T_1^{m_1} - r_1, \dots, T_n^{m_n} - r_n} \tag{24}$$

Write $T = \operatorname{Spec} B$ and let $f \colon T \to S$ be the induced morphism of schemes. Then $\mathcal{A}/S$ is toric-additive if and only if $\mathcal{A}_T/T$ is toric-additive.

Proof. Notice that T is a regular strictly local scheme, so it makes sense to say that $\mathcal{A}_T/T$ is toric-additive. Now, clearly $f^{-1}(D) \to D$ is a homeomorphism, thus $\mathcal{A}/S$ satisfies condition ii) of theorem 3.4 if and only if $\mathcal{A}_T/T$ does. □

3.2 Global definition of toric additivity

We have defined toric-additivity over a strictly local base. We now remove this hypotheses and consider the more general case of situation 2.5.

Definition 3.7. We say that $\mathcal{A}/S$ is *toric-additive* at a geometric point s of S, if the base change $\mathcal{A} \otimes_S \operatorname{Spec} \mathcal{O}_{S,s}^{sh}$ to the strict henselization at s is toric-additive as in definition 3.5. We say that $\mathcal{A}/S$ is *toric-additive* if it is so at all geometric points s of S.

It is evident that toric-additivity is a property étale-local on the target. We actually have the stronger statement:

Lemma 3.8. *Toric-additivity is local on the target for the smooth topology.*

Proof. Given $f\colon T \to S$ smooth and surjective, the base change $D \times_S T$ is still a normal crossing divisor. Let x be a geometric point of T and call f_x the induced morphism

$$X := \operatorname{Spec} \mathcal{O}^{sh}_{T,x} \to Y := \operatorname{Spec} \mathcal{O}^{sh}_{S,f(x)}.$$

The image of a generic point ζ_i of $D_i \times_S X$ via f_x is a generic point of $D_i \times_S Y$; moreover the function $\mu\colon X \to \mathbb{Z}_{\geq 0}$ factors via Y. Thus it is clear that $\mathcal{A} \times_S X/X$ is toric-additive if and only if $\mathcal{A} \times_S Y/Y$ is. We deduce that $\mathcal{A} \times_S T/T$ is toric-additive if and only if $\mathcal{A}/S$ is. □

Lemma 3.9. *Toric-additivity of $\mathcal{A}/S$ is an open condition on S.*

Proof. Suppose that $\mathcal{A}/S$ is toric-additive at a geometric point s. It is enough to show that $\mathcal{A}/S$ is toric-additive on an étale neighbourhood of s, since étale morphisms are open. We choose an étale neighbourhood of finite type $W \to S$ of s such that $D_W = D \times_S W$ is a strict normal crossing divisor and such that s belongs to all irreducible components $D_1, \dots, D_n$ of D_W. Let t be another geometric point of W; we want to show that $\mathcal{A}_W/W$ is toric-additive at t. This is true if $t \notin D_W$, so we may assume without loss of generality that t belongs to $D_1, \dots, D_m$ for some $1 \leq m \leq n$. Let ζ be a geometric point lying over the generic point of $D_1 \cap D_2 \cap \ldots \cap D_m$; write W_ζ, W_t, W_s for the spectra of the strict henselizations of W at ζ, t, s respectively. The morphism $W_\zeta \to W$ factors via W_s; hence, by remark 3.2, $\mathcal{A}_W/W$ is toric-additive at ζ. We also have a natural map $W_\zeta \to W_t$. Choose a prime l different from the residue characteristics at t. The induced morphism $\pi_1^{t,l}(W_\zeta \cap U) = \mathbb{Z}_l(1)^m \to \pi_1^{t,l}(W_t \cap U) = \mathbb{Z}_l(1)^m$ is the identity. Because $\mathcal{A}_W/W$ is toric-additive at ζ, it follows that it is also at t, as we wished to show. □

Lemma 3.10. *Let A and B be two abelian schemes over U, admitting semi-abelian prolongations $\mathcal{A}/S$ and $\mathcal{B}/S$ respectively. Suppose that over the generic fibre of S, there exists an isogeny $f\colon A_K \to B_K$. Then $\mathcal{A}/S$ is toric-additive if and only if $\mathcal{B}/S$ is so.*

Proof. We may assume that the base S is strictly local of residue characteristic $p \geq 0$. For a prime $l \neq p$ not dividing the degree of f, f induces an isomorphism of Galois modules $T_l A(K^s) \to T_l B(K^s)$. □

Lemma 3.11. *Let*

$$0 \to \mathcal{A}' \to \mathcal{A} \to \mathcal{A}'' \to 0$$

be an exact sequence of semi-abelian schemes over S, whose restriction to U is abelian. Then $\mathcal{A}'$ and $\mathcal{A}''$ are toric-additive if and only if $\mathcal{A}$ is so.

Proof. We may assume that S is the spectrum of a strictly henselian local ring, with closed point s of residue characteristic $p \geq 0$. Let $l \neq p$ be a prime and T', T, T'' be the l-adic Tate modules $T_l A'(K^s), T_l A(K^s), T_l A''(K^s)$, endowed with a natural action of $G = \pi^{t,l}(U)$. As $A'(K^s)$ is l-divisible, we obtain an exact sequence of G-modules

$$0 \to T' \to T \to T'' \to 0.$$

Consider the induced map $\varphi\colon H^1(G, T') \to H^1(G, T)$; we claim that it is injective. An element of $H^1(G, T')$ is represented by a crossed homomorphism $f\colon G \to T'$ in $Z^1(G, T')$. Suppose that its image in $Z^1(G, T)$ is a coboundary; then there exists a $t \in T$ with $f(\sigma) = \sigma t - t$ for all $\sigma \in G$. Now, $\sigma t - t$ belongs to T^G, because $(\sigma - 1)^2 = 0$ for all $\sigma \in G$. It follows that $\ker \varphi \subset H^1(G, T'^G) = \mathrm{Hom}(G, T'^G)$. As the map $\mathrm{Hom}(G, T'^G) \to \mathrm{Hom}(G, T^G)$ is injective, we have $\ker \varphi = 0$, which proves the claim.

It follows that we have an exact sequence of G-invariant submodules,

$$0 \to T'^G \to T^G \to T''^G \to 0.$$

Taking ranks, we find that $\mu(s) = \mu'(s) + \mu''(s)$, where $\mu, \mu', \mu''\colon S \to \mathbb{Z}_{\geq 0}$ are the toric rank functions for $\mathcal{A}, \mathcal{A}', \mathcal{A}''$ respectively. Thus, these functions satisfy $\mu = \mu' + \mu''$.

Let now $\zeta_1, \ldots, \zeta_n$ be the generic points of the components $D_1, \ldots, D_n$ of D. If $\mathcal{A}'$ and $\mathcal{A}''$ are toric-additive, we have $\mu(s) = \mu'(s) + \mu''(s) = \sum_{i=1}^n \mu'(\zeta_i) + \sum_{i=1}^n \mu(\zeta_i) = \sum_{i=1}^n \mu(\zeta_i)$, which implies that $\mathcal{A}$ is toric-additive.

Conversely, if $\mathcal{A}$ is toric-additive, then $\mu(s) = \sum_{i=1}^n \mu(\zeta_i)$. Hence, $\mu'(s) + \mu''(s) = \sum_{i=1}^n \mu'(\zeta_i) + \sum_{i=1}^n \mu''(\zeta_i)$. This can be rewritten as $\mu'(s) - \sum_{i=1}^n \mu'(\zeta_i) = \sum_{i=1}^n \mu''(\zeta_i) - \mu''(s)$; here, eq. (15) tells us that the left-hand side is non-positive and that the right-hand side is non-negative; hence they are both zero, and the proof is complete.

□

3.3 Two examples

We give two examples, one of a semi-abelian scheme that is toric-additive, and one of one that is not. Let k be an algebraically closed field of characteristic zero, $S = \operatorname{Spec} k[[u, v]]$, and let D be the vanishing locus of uv.

Example 3.12. Consider the nodal projective curve $\mathcal{E} \subset \mathbb{P}^2_S$ given by the equation

$$Y^2 Z = X^3 - X^2 Z - uvZ^3.$$

The restriction $\mathcal{E}_U/U$ is an elliptic curve, which is canonically identified with its jacobian $\mathrm{Pic}^0_{\mathcal{E}_U/U}$; the smooth locus $\mathcal{E}^{sm}/S$ has a unique S-group scheme structure extending the one of $\mathcal{E}_U/U$, and is a semi-abelian scheme.

Let ζ_1, ζ_2 be the generic points of $D_1 = \{u = 0\}$ and $D_2 = \{v = 0\}$ respectively, and let s be the closed point $\{u = 0, v = 0\}$. The fibres of $\mathcal{E}^{sm}$ over ζ_1, ζ_2, s are all tori of dimension 1. It follows that $\mathcal{E}^{sm}$ is not toric-additive.

Example 3.13. Consider the nodal projective curve $\mathcal{E}' \subset \mathbb{P}^2_S$ given by the equation

$$Y^2Z = X^3 - X^2Z - uZ^3.$$

Again, $\mathcal{E}'_U = \mathrm{Pic}^0_{\mathcal{E}'_U/U}$; and the smooth locus $\mathcal{E}'^{sm}/S$ is a semi-abelian scheme. In this case, the fibre of $\mathcal{E}'$ over ζ_2 is smooth; so $\mu(\zeta_1) = 1, \mu(\zeta_2) = 0, \mu(s) = 1$. Thus $\mathcal{E}'$ is toric-additive.

4 Neron models of jacobians of stable curves

4.1 Generalities

Nodal curves

Definition 4.1. A *curve* C over an algebraically closed field k is a proper morphism of schemes $C \to \mathrm{Spec}\, k$, such that C is connected and its irreducible components have dimension 1. A curve C/k is called *nodal* if for every non-smooth point $p \in C$ there is an isomorphism of k-algebras $\widehat{\mathcal{O}}_{C,p} \to k[[x,y]]/xy$.

For a general base scheme S, a *nodal curve* $f\colon \mathcal{C} \to S$ is a proper, flat morphism of finite presentation, such that for each geometric point s of S the fibre $\mathcal{C}_s$ is a nodal curve.

We will denote by $\mathcal{C}^{ns}$ the subset of $\mathcal{C}$ of points at which f is not smooth. Seeing $\mathcal{C}^{ns}$ as the closed subscheme defined by the first Fitting ideal of $\Omega^1_{\mathcal{C}/S}$, we have for a nodal curve $\mathcal{C}/S$ that $\mathcal{C}^{ns}/S$ is finite, unramified and of finite presentation.

We report a lemma from [Hol17b].

Lemma 4.2 ([Hol17b], Prop.2.5)**.** *Let S be locally noetherian, $f\colon \mathcal{C} \to S$ be nodal, and p a geometric point of $\mathcal{C}^{ns}$ lying over $s \in S$. We have:*

i) there is an isomorphism

$$\widehat{\mathcal{O}}^{sh}_{\mathcal{C},p} \cong \frac{\widehat{\mathcal{O}}^{sh}_{S,s}[[x,y]]}{xy - \alpha}$$

for some element α in the maximal ideal of the completion $\widehat{\mathcal{O}}^{sh}_{S,s}$;

ii) the element α is in general not unique, but the ideal $(\alpha) \subset \widehat{\mathcal{O}}^{sh}_{S,s}$ is. Moreover, the ideal is the image in $\widehat{\mathcal{O}}^{sh}_{S,s}$ of a unique principal ideal of $\mathcal{O}^{sh}_{S,s}$, which we call thickness of p.

We remark that, if S is regular at s, then $\mathcal{C}$ is regular at p if and only if α is generated by a regular parameter of the regular ring $\mathcal{O}^{sh}_{S,\bar{s}}$.

Split singularities

Let k be a field (not necessarily algebraically closed), C/k a nodal curve, $n\colon C' \to C$ its normalization. Following [Liu02, 10.3.8], we say that $p \in C^{ns}$ is a *split ordinary double point* if its preimage $n^{-1}(p)$ consists of k-valued points. This implies in particular that p is k-valued. Moreover, if p belongs to two or more components of C, then it belongs to exactly two components Z_1, Z_2; these are smooth at p and meet transversally ([Liu02, 10.3.11]). We say that C/k has *split singularities* if every $p \in C^{ns}$ is a split ordinary double point.

A nodal curve C/k attains split singularities after a finite separable extension $k \to k'$. We also remark that a nodal curve with split singularities has irreducible components that are geometrically irreducible. Indeed, either C/k is smooth, in which case it is geometrically connected and therefore geometrically irreducible; or every irreducible component of the normalization of C contains a k-rational point and is therefore geometrically irreducible.

Lemma 4.3. *Let $\mathcal{C} \to S$ be a nodal curve and $s \in S$ such that $\mathcal{C}_s$ has split singularities. Let p be a geometric point of $\mathcal{C}_s$. Then the thickness (α) of p is generated by an element of the Zariski-local ring $\mathcal{O}_{S,s}$.*

Proof. The morphism $f\colon \mathcal{C}^{ns} \to S$ is finite unramified. Because $\mathcal{C}_s$ has split singularities, we see by [Sta16]TAG 04DG, that there exists an open neighbourhood U of s such that $f^{-1}(U) \to U$ is a disjoint union of closed immersions. In particular, $\mathcal{C}^{ns} \to S$ is a closed immersion at p, and to it we can associate an ideal I in the Zariski-local ring $\mathcal{O}_{S,s}$. We see (for example by [Hol17b, proof of part 2 of Prop. 2.5]) that α is the image of I in $\mathcal{O}^{sh}_{S,s}$; and moreover, since $\mathcal{O}_{S,s} \to \mathcal{O}^{sh}_{S,s}$ is faithfully flat, I is principal, which completes the proof. $\square$

Lemma 4.4. *Let $\mathcal{C} \to S$ be a nodal curve over a noetherian, regular, strictly local scheme. Let η be the generic point of S. The generic fibre $\mathcal{C}_\eta$ has split singularities.*

Proof. The non-smooth locus $\mathcal{C}^{ns}$ is finite unramified over S, hence a disjoint union of closed subschemes of S. Let $X \subseteq \mathcal{C}^{ns}$ be the part consisting of sections $S \to \mathcal{C}$.

We claim that the open subscheme $\mathcal{C} \setminus X$ is normal. We will show it by using Serre's criterion for normality ([Liu02, 8.2.23]). First, as X has been removed, $\mathcal{C} \setminus X$ is regular at its points of codimension 1. Condition S_2 follows from the fact that $\mathcal{C} \setminus X$ is locally complete intersection over a regular, noetherian base, hence Cohen-Macaulay by [Liu02, 8.2.18]. This proves the claim.

Our next claim is that the normalization $\pi\colon \mathcal{C}' \to \mathcal{C}$ is finite and unramified. Since these are properties fpqc-local on the target, and since we already know that π induces an isomorphism over $\mathcal{C} \setminus X$, it is enough to check the claim over the completion of the strict henselization of points of X. Let x be such a point and s its image in S. Then $\widehat{\mathcal{O}}^{sh}_{\mathcal{C},x} \cong \frac{\widehat{\mathcal{O}}^{sh}_{S,s}[[u,v]]}{uv}$. Its integral closure is the inclusion

$$\frac{\widehat{\mathcal{O}}^{sh}_{S,s}[[u,v]]}{uv} \to \widehat{\mathcal{O}}^{sh}_{S,s}[[u]] \times \widehat{\mathcal{O}}^{sh}_{S,s}[[v]];$$

the corresponding morphism of spectra is indeed finite and unramified, proving the claim.

Now, let Y be the preimage of X via $\pi\colon \mathcal{C}' \to \mathcal{C}$. We have that Y is finite, unramified over X, and in particular finite étale over S. Hence Y is a disjoint union of sections $S \to \mathcal{C}'$. The restriction of π to the generic fibre $\mathcal{C}'_\eta \mathcal{C}_\eta$ is a normalization morphism, and we see that the preimage Y_η of $X_\eta = (\mathcal{C}_\eta)^{ns}$ consists of $k(\eta)$-valued points, as we wished to show.

□

The relative Picard scheme

Given a nodal curve $\mathcal{C} \to S$ we denote by $\mathrm{Pic}^0_{\mathcal{C}/S}$ the *degree-zero relative Picard* functor; it is constructed as the fppf-sheaf associated to the functor

$$\begin{aligned} P^0_{\mathcal{C}/S}\colon \mathbf{Sch}/S &\to \mathbf{Ab} \\ T \to S &\mapsto \mathrm{Pic}^0(\mathcal{C} \times_S T) \end{aligned}$$

where by definition $\mathrm{Pic}^0(\mathcal{C} \times_S T)$ is the group of isomorphism classes of invertible sheaves $\mathcal{L}$ on $\mathcal{C} \times_S T$ such that, for every geometric point t of T and irreducible component X of the fibre $\mathcal{C}_t$, $\deg \mathcal{L}_{|X} = 0$.

It turns out that the degree-zero Picard functor $\mathrm{Pic}^0_{\mathcal{C}/S}$ of a nodal curve has an easy description if $\mathcal{C}/S$ admits a section. In this case, it is given by

$$\begin{array}{rcl} \mathrm{Pic}^0_{\mathcal{C}/S} \colon \mathbf{Sch}/S & \to & \mathbf{Ab} \\ T \to S & \mapsto & \dfrac{\mathrm{Pic}^0(\mathcal{C} \times_S T)}{\mathrm{Pic}(T)} \end{array}$$

If $\mathcal{C}/S$ is a smooth curve, it is well known that $\mathrm{Pic}^0_{\mathcal{C}/S}$ is represented by an abelian scheme, called the *jacobian* of $\mathcal{C}/S$. If $\mathcal{C}/S$ is only nodal, then $\mathrm{Pic}^0_{\mathcal{C}/S}$ is represented by a semi-abelian scheme ([BLR90, 9.4/1]).

Generalities on graphs

We use this subsection to list some graph-theoretical notions, since we are going to work with dual graphs of nodal curves. In what follows, we will simply use the word *graph* to refer to a finite, connected, undirected graph $G = (V, E)$.

A *path* on G is a walk on G in which all edges are distinct, and that never goes twice through the same vertex, except possibly for the first and last; a *cycle* is a path that starts and ends at the same vertex. A *loop* is a cycle consisting of only one edge.

A *tree* is a subgraph of G that does not contain cycles. We say that a tree $T \subset G$ is a *spanning tree* if it contains all vertices of G, in which case it is a maximal tree. Given a spanning tree $T \subset G$, the edges of G that are not contained in T are called *links* with respect to T. The number of links of G is independent of the chosen spanning tree and is equal to the first Betti number $h^1(G, \mathbb{Z})$. If a spanning tree T is fixed, for each of the links $e_1, \dots, e_r$ with respect to T, the subgraph $T \cup e_i$ contains only one cycle C_i. The cycles $C_1, \dots, C_r$ are called *fundamental cycles* with respect to T.

The dual graph of a curve

Let C be a curve with split singularities over a field k. We define the *dual graph* of C as the graph $\Gamma = (V, E)$ with $V = \{$ irreducible components of $C\}$, $E = \{p \in C^{ns}\}$; the extremal vertices of an edge p are the components containing p, which are indeed either one or two.

The following well-known fact gives a geometric interpretation to the first Betti number of the dual graph of a curve.

Lemma 4.5 ([BLR90], 9.2/8)**.** *Let C/k be a nodal curve over a field, Γ the* *dual graph of* $C \times_k \overline{k}$, $h^1(\Gamma, \mathbb{Z})$ *its first Betti number. Then*

$$h^1(\Gamma, \mathbb{Z}) = \mu := \textit{toric rank of } \operatorname{Pic}^0_{C/k}.$$

Labelled dual graphs

Given a nodal curve $f\colon \mathcal{C} \to S$ and a point s of S such that $\mathcal{C}_s$ has split singularities, we write $\Gamma_s = (V_s, E_s)$ for the dual graph associated to the fibre $\mathcal{C}_s$. Using the notation of [Hol17b], we write L_s for the monoid of principal ideals of the (Zariski-)local ring $\mathcal{O}_{S,s}$; then we let $l_s\colon E_s \to L_s$ be the function associating to each edge of Γ_s the thickness of the corresponding singular point of $\mathcal{C}_s$ (which indeed is an ideal of $\mathcal{O}_{S,s}$, by lemma 4.3). The pair (Γ_s, l_s) is the *labelled graph* of $\mathcal{C} \to S$ at the geometric point s.

Let now ζ, s be two points of S, such that s is contained in the closure $\overline{\{\zeta\}} \subset S$, and such that the fibres $\mathcal{C}_\zeta, \mathcal{C}_s$ have split singularities. Then the labelled graph (Γ_ζ, l_ζ) of $\mathcal{C}_\zeta$ is obtained from the labelled graph (Γ_s, l_s) of $\mathcal{C}_s$ by: 1)contracting all edges of Γ_s that are labelled by an ideal of $\mathcal{O}_{S,s}$ whose image in $\mathcal{O}_{S,\zeta}$ is the unit ideal; 2)for every edge e of Γ_s that does not get contracted, we label the corresponding edge of Γ_ζ by the image in $\mathcal{O}_{S,\zeta}$ of the label of e.

4.2 Holmes' condition of alignment

Definition 4.6 ([Hol17b], definition 2.11)**.** Let $\mathcal{C} \to S$ be a nodal curve and s a geometric point of S. We say that $\mathcal{C}/S$ is *aligned at s* if for every cycle $\gamma \subset \Gamma_s$ and every pair of edges e, e' of γ, there exist integers n, n' such that

$$l(e)^n = l(e')^{n'}.$$

We say that C/S is *aligned* if it is aligned at every geometric point of S.

Theorem 4.7 ([Hol17b], theorem 5.16, theorem 5.2)**.** *Let S be regular, $U \subset S$ a dense open, $f\colon \mathcal{C} \to S$ a nodal curve, with $f_U\colon \mathcal{C}_U \to U$ smooth.*

i) If the jacobian $\operatorname{Pic}^0_{\mathcal{C}_U/U}$ *admits a Néron model over S, then $\mathcal{C}/S$ is aligned;*

ii) if $\mathcal{C}$ is regular and $\mathcal{C}/S$ is aligned, then $\operatorname{Pic}^0_{\mathcal{C}_U/U}$ *admits a Néron model over S.*

We are soon going to show how the condition of alignment is closely related to toric-additivity of $\operatorname{Pic}^0_{\mathcal{C}_K/K}$. For the moment, we will consider a graph $\Gamma =$

(V, E), a set of $n \geq 1$ different colours $\mathfrak{C} := \{\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n\}$, and a colouring of the edges $\chi\colon E \to \mathfrak{C}$. We say that (Γ, χ) is *aligned* if for every cycle $\gamma \subset \Gamma$, the restriction of χ to γ is constant; in other words, if every cycle is monochromatic.

The following lemma gives us a criterion for alignment that is easier to check. The proof is due to Raymond van Bommel.

Lemma 4.8. *Let $(\Gamma, \chi\colon E \to \mathfrak{C} = \{\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n\})$ be a graph with a colouring of the edges. Fix a spanning tree T. Then (Γ, χ) is aligned if and only if every fundamental cycle is monochromatic.*

Proof. What we have to prove is that if every fundamental cycle is monochromatic, then (Γ, χ) is aligned, as the converse statement is obvious. We show that we can reduce to the case $n = 2$ (two colours). If there is only one colour the statement is clearly true. Suppose now the statement is false for some $n > 2$: that is, (Γ, χ) is not aligned but all fundamental cycles are monochromatic. Then there is some cycle γ in Γ that takes at least two distinct colours, $\mathfrak{c}_1$ and $\mathfrak{c}_2$. We can now pretend that $\mathfrak{c}_2, \mathfrak{c}_3, \ldots, \mathfrak{c}_n$ are different hues of one colour $\mathfrak{c}'$, and that our graph Γ is coloured with only two colours, $\mathfrak{c}_1$ and $\mathfrak{c}'$. Then, γ still takes two distinct colours, and all fundamental cycles are still monochromatic; this implies that the statement is false for $n = 2$. Thus we have reduced to proving the statement for $n = 2$ colours.

Let $(\Gamma, \chi\colon E \to \{\text{yellow, pink}\})$ be a coloured graph, and assume all fundamental cycles are monochromatic. We construct a new graph, which we call G, in the following way: the set of vertices of G consists of the disjoint union of two copies, V_y and V_p, of the set of vertices V of Γ. We connect the vertices with edges as follows: first, if v is a vertex of Γ, we create an edge e_v linking the corresponding vertices v_y and v_p in V_y and V_p. Next, if e is an yellow edge of Γ linking vertices v and w, we create an edge e_y between v_y and w_y; if instead e is pink, we create an edge e_p between v_p and w_p. This completes the construction of G.

We call G_y and G_p the subgraphs of G with underlying set of vertices V_y and V_p respectively. We will call the edges e_v linking G_y and G_p *vertical edges*, and the others *horizontal edges*. Now, consider the subgraph W of G given by the union of all vertical edges, and of all horizontal edges corresponding to edges of the spanning tree T. Clearly, W spans G, is connected, and does not contain cycles, otherwise T would itself contain a cycle. Hence W is a spanning tree for G; it follows that the links of G with respect to W are in bijection with the links of Γ with respect to T. Since the fundamental cycles of Γ are monochromatic, the fundamental cycles of G consist only of horizontal edges.

Now, suppose by contradiction that Γ contains a non-monochromatic cycle γ. Then γ defines a unique cycle γ' on G, and γ' necessarily contains some vertical edge, as γ is not monochromatic. However, by [Die05, 1.9.6], fundamental cycles form a basis of the cycle space (i.e. every cycle is a composition of fundamental cycles). As fundamental cycles of G do not contain vertical edges, γ' cannot contain vertical edges. This is a contradiction and the lemma is proved. $\square$

Lemma 4.9. *Let Γ be a graph with a colouring of the edges $\chi\colon E \to \mathfrak{C} = \{\mathfrak{c}_1, \mathfrak{c}_2, \ldots, \mathfrak{c}_n\}$. For every $1 \leq i \leq n$ let Γ_i be the graph obtained by contracting every edge whose colour is not $\mathfrak{c}_i$. Then*

$$h^1(\Gamma, \mathbb{Z}) \leq \sum_{i=1}^{n} h^1(\Gamma_i, \mathbb{Z})$$

with equality if and only if (Γ, χ) is aligned.

Proof. Fix a spanning tree T for Γ; we write T_i for the image of T in the contraction Γ_i. Notice that T_i need not be a tree; however, it is a subgraph of Γ_i containing every vertex of Γ_i; therefore, if T_i is a tree it is also spanning; and in any case T_i contains a spanning tree for Γ_i.

Claim 4.10. *(Γ, χ) is aligned if and only if for all $i = 1, \ldots, n$, T_i is a (spanning) tree in Γ_i.*

Suppose that (Γ, χ) is aligned and fix some i. We want to show that T_i is a tree in the contraction Γ_i. We can contract one edge at a time and see what happens to the image of T in the contraction. On the one hand, contracting an edge that is contained in T does not produce new cycles in the image of T. Now let $e \in E$ be a link with respect to T, and suppose that e gets contracted in Γ_i. Then $\chi(e) \neq \mathfrak{c}_i$. Let P be the unique path in T connecting the two extremal vertices of e. Then the union of P and e forms a cycle γ, which does not take the colour $\mathfrak{c}_i$ by the alignment hypothesis. Hence γ gets contracted to a point in Γ_i and once again no new cycle is produced in the image of T. Therefore T_i is a tree.

Conversely, suppose that (Γ, χ) is not aligned. By lemma 4.8, there is a fundamental cycle γ that takes two distinct colours, say $\mathfrak{c}_1$ and $\mathfrak{c}_2$. Let e be the only link contained in γ; we may assume $\chi(e) = \mathfrak{c}_1$. Thus, e is contracted in Γ_2. However, γ is not contracted to a point in Γ_2, since it contains some edge with colour $\mathfrak{c}_2$. It follows that T_2 is not a tree. This establishes the claim.

Now, $h^1(\Gamma, \mathbb{Z})$ is equal to the number of links with respect to T. We write $h^1(\Gamma, \mathbb{Z}) = b_1 + \ldots + b_n$, where b_i is the number of links of colour $\mathfrak{c}_i$. In the contraction Γ_i, the only links that are not contracted are those of colour $\mathfrak{c}_i$. Since T_i contains a spanning tree for Γ_i, we have $h^1(\Gamma_i, \mathbb{Z}) \geq b_i$, with equality if T_i is a tree. Hence $h^1(\Gamma, \mathbb{Z}) \leq h^1(\Gamma_1, \mathbb{Z}) + \ldots + h^1(\Gamma_n, \mathbb{Z})$. Moreover, by the claim, (Γ_i, χ) is aligned if and only if for every $i = 1, \ldots, n$, $h^1(\Gamma_i, \mathbb{Z}) = b_i$. This in turn is equivalent to $h^1(\Gamma, \mathbb{Z}) = h^1(\Gamma_1, \mathbb{Z}) + \ldots + h^1(\Gamma_n, \mathbb{Z})$, which completes the proof. □

4.3 Relation between toric-additivity and alignment

We now consider a connected, locally noetherian, regular base scheme S with a normal crossing divisor $D \subset S$, and a nodal curve $\mathcal{C}/S$, such that the base change $\mathcal{C}_U/U := S \setminus D$ is smooth.

If $S' \to S$ is a strict henselization at some geometric point s of S, and $D \cap S'$ is given by regular parameters $t_1, \ldots, t_n \in \mathcal{O}(S')$, then the thickness of any non-smooth point $p \in \mathcal{C}_s$ is generated by $t_1^{m_1} \cdot \ldots \cdot t_n^{m_n}$ for some non-negative integers $m_1, \ldots, m_n$. In particular, $\mathcal{C}$ is regular at p if and only if its thickness is generated by t_i for some $1 \leq i \leq n$.

Proposition 4.11. *Suppose that the total space $\mathcal{C}$ is regular. Then $\mathcal{C}/S$ is aligned if and only if $\mathrm{Pic}^0_{\mathcal{C}_K/K}$ is toric-additive.*

Proof. As both alignment and toric-additivity are checked over the strict henselizations at geometric points of S, we may assume that S is strictly local. Let $\Gamma_s = (V_s, E_s)$ be the dual graph of the fibre of $\mathcal{C}$ over the closed point $s \in S$, and $l_s \colon E_s \to L_s$ the labelling of the edges, taking value in the monoid L_s of principal ideals of $\mathcal{O}_S(S)$. We have already remarked that, since $\mathcal{C}$ is regular, the labels can only take the values $(t_1), \ldots, (t_n) \in L_s$. This means that $\mathcal{C}/S$ is aligned if and only if every cycle of Γ has edges with the same label.

Now, let $\{D_i\}_{i=1,\ldots,n}$ be the components of the divisor D. Each of them is cut out by a regular element $t_i \in \mathcal{O}_S(S)$ and is itself a regular, strictly local scheme. Let ζ_i be the generic point of D_i. By lemma 4.4, the curve $\mathcal{C}_{\zeta_i}$ has split singularities; its labelled graph $(\Gamma_{\zeta_i}, l_{\zeta_i})$ is obtained from (Γ_s, l_s) by contracting edges according to the procedure in section 4.1. Interpreting the different labels as colours, we can apply lemma 4.9 and conclude that $\mathcal{C}/S$ is aligned at s if and only if $h^1(\Gamma, \mathbb{Z}) = \sum_{i=1}^n h^1(\Gamma_i, \mathbb{Z})$. By lemma 4.5, we see that $\mu(s) = \sum_{i=1}^n \mu(\zeta_i)$, which is the condition for toric-additivity at s. This finishes the proof.

□

4.4 Toric-additivity and desingularization of curves

Let S be a connected, locally noetherian, regular base scheme S with a normal crossing divisor $D = D_1 \cup \ldots \cup D_n \subset S$, and let $\mathcal{C}/S$ be a nodal curve, such that the base change $\mathcal{C}_U/U := S \setminus D$ is smooth.

In [dJ96, 3.6], it is proven that if $\mathcal{C} \to S$ has *split* fibres, there exists a blow-up $\varphi\colon \mathcal{C}' \to \mathcal{C}$ such that $\mathcal{C}' \to S$ is still a nodal curve, and $\mathcal{C}'$ is *regular*. The condition of splitness implies that the irreducible components of the geometric fibres are smooth; or equivalently, that the dual graphs of the geometric fibres do not admit loops. We are going to introduce a condition on $\mathcal{C}/S$, weaker than splitness, and show that a statement analogous to the one in [dJ96, 3.6] holds for curves satisfying this condition.

Definition 4.12. Let $\mathcal{C} \to S$ be a nodal curve. We say that $\mathcal{C}/S$ is *disciplined* if, for every geometric point $\overline{s}$ of S, and $p \in \mathcal{C}_{\overline{s}}^{ns}$, at least one of the following is satisfied:

i) p belongs to two irreducible components of $\mathcal{C}_{\overline{s}}$;

ii) the thickness of p is a power of a regular parameter of $\mathcal{O}^{sh}_{S,\overline{s}}$.

We give first an auxiliary lemma:

Lemma 4.13. *Hypothesis as in the beginning of the subsection; suppose also that S is strictly local and that $\mathcal{C}/S$ is disciplined. Let $p \in \mathcal{C}_s^{ns}$ be a non-smooth point of the fibre over the closed point, such that p does not satisfy condition ii) of definition 4.12. Let X_1, X_2 be the distinct irreducible components of the closed fibre $\mathcal{C}_s$ containing p. Then there exists $i \in \{1, \dots, n\}$ and Y_1, Y_2 irreducible components of $\mathcal{C}_{\zeta_i}$ such that $X_1 \not\subset \overline{Y}_2 \supset X_2$ and $X_2 \not\subset \overline{Y}_1 \supset Y_1$.*

Proof. Let (Γ_s, l_s) be the labelled graph of $\mathcal{C}_s$. By hypothesis, the edge $e(p)$ corresponding to p has distinct extremal vertices, v_1 and v_2, and label $t_1^{m_1} \cdot \ldots \cdot t_l^{m_l}$, with $2 \leq l \leq n$ and $m_1, \dots, m_l \geq 1$. The fibres over the generic points $\zeta_1, \dots, \zeta_n$ have split singularities by lemma 4.4, so we can consider their labelled graphs (Γ_i, l_i). What we want to prove is that there exists $i \in \{1, \dots, l\}$ such that v_1 and v_2 are mapped to distinct vertices of (Γ_i, l_i) via the procedure described in section 4.1.

Suppose the contrary; as $e(p)$ is not contracted in any Γ_i, there exists a cycle γ in Γ_s, containing $e(p)$, such that for all $1 \leq i \leq l$, all edges $e \neq e(p)$ of γ are contracted in Γ_i. Let ζ_{12} be the generic point of $D_1 \cap D_2$; all edges $e \neq e(p)$ of γ are contracted in Γ_{12}, the labelled graph of $\mathcal{C}_{\zeta_{12}}$, and in particular v_1 and v_2 are mapped to the same vertex. The edge $e(p)$ is therefore mapped to a loop, with label $t_1^{m_1} t_2^{m_2}$. However, this contradicts the fact that $\mathcal{C} \to S$ is disciplined at ζ_{12}.

□

We introduce now some notation: given a scheme X, we will denote by $\mathrm{Sing}(X) \subseteq X$ the set of points that are not regular. We say that the *center* of a blow-up $\pi\colon Y \to X$ is the complement of the largest open $U \subset X$ such that $\pi^{-1}(U) \to U$ is an isomorphism.

Lemma 4.14. *Hypotheses as in the beginning of the subsection. Suppose $f\colon \mathcal{C} \to S$ is disciplined. Then there is an étale surjective $g\colon S' \to S$ and a blow-up $\varphi\colon \mathcal{C}' \to \mathcal{C} \times_S S'$ such that*

- *the center of φ is contained in Sing($\mathcal{C} \times_S S'$);*
- *$\mathcal{C}'$ is a nodal curve over S', smooth over $g^{-1}(U)$;*
- *$\mathcal{C}'$ is regular.*

Proof. First, notice that the order in which the blow-ups of the curve and the étale covers of the base are taken does not matter, as blowing-up commutes with étale base change. After replacing S by a suitable étale cover, we may assume that D is a strict normal crossing divisor. We can now apply [dJ96, 3.3.2] and assume that $\mathrm{Sing}(\mathcal{C}) \subset \mathcal{C}$ has codimension at least 3. As a consequence of lemma 4.4, after a further étale covering, we may assume that for every generic point ζ of D, the fibre $\mathcal{C}_\zeta$ has split singularities.

Now, let E be an irreducible component of $\mathcal{C}_D = \mathcal{C} \times_S D$ and let $\pi\colon \mathcal{C}' \to \mathcal{C}$ be the blow-up of $\mathcal{C}$ along E. If $p \in E$ is a regular point of $\mathcal{C}$, f is an isomorphism at p, because E is cut out by one equation. Otherwise, the completion of the strict henselization at (a geometric point lying over) p is of the form

$$\widehat{\mathcal{O}}^{sh}_{\mathcal{C},\overline{p}} \cong \frac{\widehat{\mathcal{O}}^{sh}_{S,\overline{f(p)}}[[x,y]]}{xy - t_1^{m_1} \cdot \ldots \cdot t_l^{m_l}}$$

with $t_1, \ldots, t_n$ regular parameters cutting out D, $1 \leq l \leq n$ and positive integers $m_1, \ldots, m_l$. In fact, because the singular locus has codimension at least three, we have $l \geq 2$, and $m_1 = \ldots = m_l = 1$.

The ideal of the pullback of E to $\widehat{\mathcal{O}}^{sh}_{\mathcal{C},\overline{p}}$ is either (t_i) for some $1 \leq i \leq l$, or one between (x, t_i) and (y, t_i) for some $1 \leq i \leq l$. In the first case, π is an isomorphism at p. In the second case, one can compute explicitly the blowing up of $\operatorname{Spec} \mathcal{O}^{sh}_{\mathcal{C},\overline{p}}$ at the ideal (x, t_i) (or (y, t_i)) and find that $f'\colon \mathcal{C}' \to S$ is still a nodal curve, disciplined, with $\mathrm{Sing}(\mathcal{C})$ of codimension at least three, and such that for every generic point ζ of D the fibre $\mathcal{C}_\zeta$ has split singularities. We omit the explicit computations.

Let $Y \subset \mathcal{C}$ be the center of $\pi\colon \mathcal{C}' \to \mathcal{C}$. Then Y consists only of non-regular points, hence it has codimension at least 3. As $f\colon \mathcal{C}' \to S$ is a curve, the fibres of π have dimension at most 1, hence $\pi^{-1}(Y)$ has codimension at least 2 in

$\mathcal{C}$. It follows that there is a bijection between the irreducible components of $\mathcal{C}_D$ and $\mathcal{C}'_D$, given by taking the preimage under π. Now, $\pi^{-1}(E)$ is a divisor, and for any other irreducible component E' of $\mathcal{C}_D$ that is a divisor, $\pi^{-1}(E')$ is also a divisor. We conclude that $\pi^*\colon \mathcal{C}^* \to \mathcal{C}$, the composition of the blowing-ups of all irreducible component of $\mathcal{C}_D$, is such that every component of $\mathcal{C}^*_D$ is a divisor. Besides, as previously noticed, $f^*\colon \mathcal{C}^* \to S$ is a nodal curve, disciplined, and $\mathrm{Sing}(\mathcal{C}^*)$ has codimension at least three.

Assume now by contradiction that $\mathrm{Sing}(\mathcal{C}^*) \neq \emptyset$, and let $p \in \mathrm{Sing}(\mathcal{C}^*)$. Then without loss of generality the thickness at p is $(t_1 \cdot \ldots \cdot t_l)$ for some $2 \leq l \leq n$. Consider the base change $\mathcal{C}^*_T/T$, where T is the spectrum of some strict henselization at $s = f^*(p)$. For every i let ξ_i be the generic point of $D_i \cap T$. By lemma 4.13, for some $i \in \{1, \ldots, l\}$, there are distinct components Y_1, Y_2 of $\mathcal{C}^*_{\xi_i}$ whose closure in $\mathcal{C}^*_{T \cap D_i}$ contain p. Because the fibre $\mathcal{C}^*_{\zeta_i}$ has split singularities, we deduce that there are components X_1, X_2 of $\mathcal{C}^*_{\zeta_i}$ whose closures E_1, E_2 in $\mathcal{C}^*_{D_i}$ contain p. But then E_1 and E_2 are given by (x, t_1) and (y, t_1) in $\widehat{\mathcal{O}}^{sh}_{\mathcal{C}^*, \overline{p}}$. In particular, they are not divisors. This is a contradiction, and therefore $\mathrm{Sing}(\mathcal{C}^*) = \emptyset$. □

Lemma 4.15. *Hypotheses as in the beginning of the subsection. Suppose that $f\colon \mathcal{C} \to S$ is such that $\mathrm{Pic}^0_{\mathcal{C}/S}$ is toric-additive. Then $\mathcal{C}/S$ is disciplined.*

Proof. We may assume that S is strictly local, with closed point s, and with D given by a system of regular parameters $t_1 \ldots, t_n$. Let $p \in \mathcal{C}^{ns}_s$, with thickness $t_1^{m_1} \cdot \ldots \cdot t_l^{m_l}$ for some $1 \leq l \leq n$ and $m_1, \ldots, m_l \geq 1$. We have to show that if $l \geq 2$ then p lies on two components of $\mathcal{C}_s$.

Suppose by contradiction that $l \geq 2$ and that p lies on only one component of $\mathcal{C}_s$. The dual graph Γ over s has a loop L corresponding to p, with label $t_1^{m_1} \cdot \ldots \cdot t_l^{m_l}$. For $1 \leq i \leq n$ call Γ_i the dual graph of the fibre $\mathcal{C}_{\zeta_i}$ over the generic point of D_i. The loop L is preserved in the dual graphs Γ_i for $1 \leq i \leq l$. Let Γ' be the graph obtained by Γ by removing the loop L, and define similarly Γ'_i, $1 \leq i \leq l$. We have that

$$h^1(\Gamma', \mathbb{Z}) \leq \sum_{i=1}^{l} h^1(\Gamma'_i, \mathbb{Z}) + \sum_{j=l+1}^{n} h^1(\Gamma_j, \mathbb{Z}).$$

This inequality follows from the identification of the first Betti number with the toric rank of the corresponding fibre of $\mathrm{Pic}^0_{\mathcal{C}/S}$; and from eq. (15).

For every $1 \leq i \leq l$, $h^1(\Gamma_i, \mathbb{Z}) = h^1(\Gamma'_i, \mathbb{Z}) + 1$. Since $l \geq 2$, we find that $h^1(\Gamma, \mathbb{Z}) = h^1(\Gamma', \mathbb{Z}) + 1 < \sum_{i=1}^{n} h^1(\Gamma_i, \mathbb{Z})$. In terms of toric ranks of fibres of $\mathrm{Pic}^0_{\mathcal{C}/S}$, the same inequality reads $\mu(s) < \sum_{i=1}^{n} \mu(\zeta_i)$. This contradicts the fact that $\mathrm{Pic}^0_{\mathcal{C}_K/K}$ is toric-additive. □

4.5 Toric-additivity and Néron models

We consider again a base S and a nodal curve $\mathcal{C}/S$ as in the previous subsection. Theorem 1.1 tells us that if $\mathrm{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model over S, then $\mathcal{C}/S$ is aligned. However, not all aligned curves admit a Néron model for their jacobian; in this subsection we show that curves that are not disciplined do not admit one.

Lemma 4.16. *Assume that S is an excellent $\mathbb{Q}$-scheme. Suppose that $\mathcal{C}/S$ is such that $\mathrm{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model $\mathcal{N}$ over S. Then $\mathcal{C}/S$ is disciplined.*

Proof. We may assume that S is strictly henselian, with closed point s and residue field $k = k(s)$. Assume by contradiction that $\mathcal{C}/S$ is not disciplined. Then there is some $p \in \mathcal{C}_s^{ns}$ that belongs to only one component X of $\mathcal{C}_s$, and such that its thickness is $t_1^{m_1} \cdot \ldots \cdot t_l^{m_l}$ with $m_i \geq 1$ and $2 \leq l \leq n$. Let $q \in \mathcal{C}_s(k)$ be a smooth k-rational point belonging to the same component as p. By Hensel's lemma, there exists a section $\sigma_q \colon S \to \mathcal{C}$ through q. We claim that the same is true for p: let $\widehat{S}$ be the spectrum of the completion of $\mathcal{O}(S)$ at its maximal ideal and consider the morphism

$$W := \mathrm{Spec}\, \widehat{\mathcal{O}}^{sh}_{\mathcal{C},p} \cong \mathrm{Spec}\, \frac{\mathcal{O}(\widehat{S})[[x,y]]}{xy - t_1^{m_1} \cdot \ldots \cdot t_l^{m_l}} \to \widehat{S}.$$

This has a section given by $x = t_1^{m_1}$, $y = t_2^{m_2} \cdot \ldots \cdot t_l^{m_l}$. Composing the section with the canonical morphism $W \to \mathcal{C}$, gives a morphism $\widehat{\sigma}_p \colon \widehat{S} \to \mathcal{C}$ going through p. Because S is excellent and henselian, it has the Artin approximation property, and there exists a section $\sigma_p \colon S \to \mathcal{C}$ which agrees with $\widehat{\sigma}_p$ when restricted to the closed point s, hence going through p.

We write $\mathcal{F} := \mathcal{I}(\sigma_p) \otimes_{\mathcal{O}_\mathcal{C}} \mathcal{O}(\sigma_q)$ for the coherent sheaf on $\mathcal{C}$ given by the tensor product of the ideal sheaf of σ_p with the invertible sheaf associated to the divisor σ_q. It is what is called a *torsion free, rank* 1 sheaf in the literature: it is S-flat, its fibres are of rank 1 at the generic points of fibres of $\mathcal{C}$, and have no embedded points. Notice that $\mathcal{F}$ is not an invertible sheaf, as $\dim_{k(p)} \mathcal{F} \otimes k(p) = 2$.

Let u_p and u_q be the restrictions of σ_p and σ_q to U. They are U-points of the smooth curve $\mathcal{C}_U/U$; the restriction of $\mathcal{F}$ to U is the invertible sheaf $\mathcal{F}_U = \mathcal{O}_{\mathcal{C}_U}(u_q - u_p)$. This is the datum of a U-point α of $\mathrm{Pic}^0_{\mathcal{C}_U/U}$: indeed, $\mathrm{Pic}(U) = 0$ because $\mathcal{O}(U)$ is a UFD, and C_U/U has a section, so $\mathrm{Pic}^0_{\mathcal{C}_U/U}(U) = \mathrm{Pic}^0(\mathcal{C}_U)$.

By the definition of Néron model, there is a unique section $\beta \colon S \to \mathcal{N}$ with $\beta_U = \alpha$. We write J for $\mathrm{Pic}^0_{\mathcal{C}/S}$. As J is semi-abelian, the canonical open immersion $J \to \mathcal{N}$ identifies J with the fibrewise-connected component of identity $\mathcal{N}^0$ (lemma 2.17). Write $\zeta_i, i = 1\dots,n$ for the generic points of the divisors D_i. Then $S_i := \operatorname{Spec}\mathcal{O}_{S,\zeta_i}$ is a trait, and the restriction $\mathcal{N}_{S_i}$ is a Néron model of its generic fibre. Therefore α_K extends uniquely to a section $\alpha_i\colon S_i \to \mathcal{N}_{S_i}$. As $\mathcal{F}_{S_i}$ is an invertible sheaf of degree 0 on every irreducible component of $\mathcal{C}_{\overline{\zeta}_i}$, $\mathcal{F}_{S_i}$ is a S_i-point of J_{S_i}, and α_i is given by $\mathcal{F}_{S_i}$. Therefore, the restriction of $\alpha\colon S \to \mathcal{N}$ to S_i factors through $J = \mathcal{N}^0$ for every $i = 1\dots,n$.

We denote now by Φ/S the étale group scheme of connected components of $\mathcal{N}$, and by $\Phi_{(l)}$ its l-primary part for a prime l. Lemma 5.2 tells us that, for every prime l different from the residue characteristic of S, the canonical morphism $\Phi_{(l)}(s) \to \bigoplus_{i=1}^n \Phi_{(l)}(\overline{\zeta}_i)$ is injective. By our assumption that S is a $\mathbb{Q}$-scheme, the canonical morphism

$$\Phi(s) \to \bigoplus_{i=1}^{n} \Phi(\overline{\zeta}_i)$$

is injective. This implies that α lands inside $J = \mathcal{N}^0$, or in other words that $\mathcal{F}_U$ extends to an invertible sheaf $\mathcal{L}$ on $\mathcal{C}$ such that $\mathcal{L}_s$ is of degree 0 on every component.

Now, let $Z \to S$ be a closed immersion, with Z a trait, such that the generic point ξ of Z lands into U (it is an easy check that such a closed immersion exists). As $\mathcal{F}_\xi$ and $\mathcal{L}_\xi$ define the same point of $\operatorname{Pic}^0_{\mathcal{C}_\xi/\xi}$, there are isomorphisms $\mu_\xi\colon \mathcal{F}_\xi \to \mathcal{L}_\xi$ and $\lambda_\xi\colon \mathcal{L}_\xi \to \mathcal{F}_\xi$. By the same argument as in [AK80, 7.8], μ_ξ and λ_ξ extend to morphisms $\mu\colon \mathcal{F}_Z \to \mathcal{L}_Z$ and $\lambda\colon \mathcal{L}_Z \to \mathcal{F}_Z$, which are non-zero on all fibres. Let's look at the restrictions to the closed fibre, $\mu_s\colon \mathcal{F}_s \to \mathcal{L}_s$, $\lambda_s\colon \mathcal{L}_s \to \mathcal{F}_s$. We know that $\mathcal{F}_s$ is trivial away from the component $X \subset \mathcal{C}_s$. So, if we write Y for the closure in $\mathcal{C}_s$ of the complement of X, we may restrict μ_s and λ_s to Y to get global sections l and l' of $\mathcal{L}_Y$ and $\mathcal{L}_Y^\vee$ respectively. Now, if $l = 0$, then the restriction μ_X of μ_s to X is non-zero, because μ_s is non-zero. If $l \neq 0$, as $\mathcal{L}_s$ is of degree zero on every component, we have $l(y) \notin \mathfrak{m}_y\mathcal{L}_y$ for every $y \in Y$, and in particular for $y \in Y \cap X$. It follows that also in this case $\mu_X \neq 0$. We can apply the same argument to l' and conclude that $\lambda_X \neq 0$. Then the compositions $\mu_X \circ \lambda_X\colon \mathcal{L}_X \to \mathcal{L}_X$ and $\lambda_X \circ \mu_X\colon \mathcal{F}_X \to \mathcal{F}_X$ are non-zero. As $\operatorname{End}_{\mathcal{O}_X}(\mathcal{F}_X) = k = \operatorname{End}_{\mathcal{O}_X}(\mathcal{L}_X)$, they are actually isomorphisms. It follows that $\mu_X\colon \mathcal{F}_X \to \mathcal{L}_X$ is an isomorphism. However, $\dim_{k(p)} \mathcal{F}_{k(p)} = 2$, while $\mathcal{L}_X$ is an invertible sheaf. This gives us the required contradiction.

□

Theorem 4.17. *Let S be a connected, locally noetherian, regular scheme, D a normal crossing divisor on S, $\mathcal{C} \to S$ a nodal curve smooth over $U = S \setminus D$.*

i) If $\operatorname{Pic}^0_{\mathcal{C}/S}$ is toric-additive, then $\operatorname{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model over S.

ii) If moreover S is an excellent $\mathbb{Q}$-scheme, the converse is also true.

Proof. Whether we are in the hypotheses of i) and ii), we know by lemmas 4.15 and 4.16 above that $\mathcal{C}/S$ is disciplined; hence by lemma 4.14 there exists an étale cover $g\colon S' \to S$ and a blow-up $\pi\colon \mathcal{C}' \to \mathcal{C}_{S'}$ which restricts to an isomorphism over $U' = U \times_S S'$, such that $\mathcal{C}'$ is regular.

Assume that $\mathrm{Pic}^0_{\mathcal{C}/S}$ is toric-additive. To show the existence of a Néron model over S, it is enough to show it over S'. The base change $\mathrm{Pic}^0_{\mathcal{C}_{S'}/S'}$ is toric-additive by lemma 3.8. The blow-up π does not affect $\mathcal{C}_{U'}$, so $\mathrm{Pic}^0_{\mathcal{C}'/S'}$ is still toric-additive. We can now apply proposition 4.11 and deduce that $\mathcal{C}'/S'$ is aligned. Hence by theorem 4.7, we find that $\mathrm{Pic}^0_{\mathcal{C}_{U'}/U'}$ admits a Néron model over S', proving i).

Now assume that S is a $\mathbb{Q}$-scheme and that $\mathrm{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model $\mathcal{N}$ over S. Then $\mathcal{N}' = \mathcal{N} \times_S S'$ is a Néron model for $\mathrm{Pic}^0_{\mathcal{C}'_{U'}/U'}$ over S'. Hence $\mathcal{C}'/S'$ is aligned by theorem 4.7; as $\mathcal{C}'$ is regular, we deduce by proposition 4.11 that $\mathrm{Pic}^0_{\mathcal{C}_{S'}/S'}$ is toric-additive. As toric-additivity descends along étale covers (lemma 3.8), $\mathrm{Pic}^0_{\mathcal{C}/S}$ is toric-additive. □

Corollary 4.18. *Let S be a connected, locally noetherian, regular, excellent $\mathbb{Q}$-scheme, D a normal crossing divisor on S, $\mathcal{C} \to S$ and $\mathcal{D} \to S$ two nodal curves, smooth over $U = S \setminus D$.*

Assume that over the generic point $\eta \in S$, there exists an isogeny

$$\mathrm{Pic}^0_{\mathcal{C}_\eta/\eta} \to \mathrm{Pic}^0_{\mathcal{D}_\eta/\eta}.$$

Then $\mathrm{Pic}^0_{\mathcal{C}_U/U}$ admits a Néron model over S if and only if $\mathrm{Pic}^0_{\mathcal{D}_U/U}$ does.

Proof. By lemma 3.10, $\mathrm{Pic}^0_{\mathcal{C}/S}$ is toric-additive if and only if $\mathrm{Pic}^0_{\mathcal{D}/S}$ is. By theorem 4.17, toric-additivity is equivalent to existence of a Néron model, and we conclude. □

5 Néron models of abelian schemes in characteristic zero

In this section, we consider a connected, locally noetherian, regular base scheme S, a normal crossing divisor D on S, an abelian scheme A/U of relative dimension d and a semi-abelian scheme $\mathcal{A}/S$ with a given isomorphism $\mathcal{A} \times_S U \to A$. We will retain the notation used in the previous sections.

5.1 Test-Néron models

Definition 5.1. Let $\mathcal{N}/S$ be a smooth, separated group algebraic space of finite type with an isomorphism $\mathcal{N} \times_S U \to A$; we say that it is a *test-Néron model* for A over S if, for every strictly henselian trait Z and morphism $Z \to S$ transversal to D (definition 2.4), the pullback $\mathcal{N} \times_S Z$ is the Néron model of its generic fibre.

It is clear that the property of being a test-Néron model is smooth-local on the base, and is also preserved by taking the localization at a point of the base, or the strict henselization at a geometric point.

We will start by working on a strictly local base. Recall that in this case, for a prime l different from the residue characteristic p at the closed point, the Tate module $T_lA(K^s)$ is acted on by $G = \bigoplus_{i=1}^n I_i = \bigoplus_{i=1}^n \widehat{\mathbb{Z}}'(1)$, the tame fundamental group of U.

Lemma 5.2. *For any subset $\mathcal{E} \subseteq \{1, \dots, n\}$ and any $m \in \mathbb{Z}$, there is a canonical injective group homomorphism*

$$\varphi_{\mathcal{E}} \colon \frac{A[m](K^s)^{\oplus_{i\in\mathcal{E}} I_i}}{T_lA(K^s)^{\oplus_{i\in\mathcal{E}} I_i} \otimes \mathbb{Z}/m\mathbb{Z}} \to \bigoplus_{i\in\mathcal{E}} \frac{A[m](K^s)^{I_i}}{T_lA(K^s)^{I_i} \otimes \mathbb{Z}/m\mathbb{Z}}. \tag{25}$$

If $\mathcal{A}/S$ is toric-additive, for any $\mathcal{E} \subseteq \{1, \dots, n\}$ the homomorphism $\varphi_{\mathcal{E}}$ is an isomorphism.

Remark 5.3. Recall the characterization of the group of components of Néron models in section 2.4. If S_i is a strict henselization at the generic point ζ_i of D_i, then there exists a Néron model $\mathcal{N}_i/S_i$ for $A \times_S S_i$. The i-th summand of the right-hand side of (5.2) is the group of components of $\mathcal{N}_i$ over the closed point of S_i. On the other hand, if ζ is the generic point of $\cap_{i\in\mathcal{E}} D_i$, and if A_K/K admits a Néron model over a strict henselization $\mathcal{O}^{sh}_{S,\overline{\zeta}}$, then the left hand side is its group of components over the closed point.

Proof. First, it follows easily from lemma 3.3 that $\left(T_l A(K^s)^{\bigoplus_{i\in\mathcal{E}} I_i}\right)\otimes\mathbb{Z}/m\mathbb{Z} = \bigcap_{i\in\mathcal{E}}\left(T_l A(K^s)^{I_i}\otimes\mathbb{Z}/m\mathbb{Z}\right)$. Given this, it is evident that the group homomorphism 25 is injective.

Let us assume that $\mathcal{A}/S$ is toric-additive. Then we have a decomposition of $T := T_l A(K^S)$ into a direct sum $V_1 \oplus \ldots \oplus V_n$ as in theorem 3.4. For a $\mathbb{Z}_l$-module M, we will write $M_{(m)}$ for $M \otimes_{\mathbb{Z}_l} \mathbb{Z}/m\mathbb{Z}$.

Now, if $\mathcal{E}$ is empty the statement of the lemma is obviously satisfied; otherwise, we can rename the components D_i, so that $\mathcal{E} = \{1, 2, \ldots, r\} \subseteq \{1, \ldots, n\}$ for some $1 \le r \le n$.

The left-hand side of eq. (25) is

$$\frac{(V_{1,(m)} \oplus \ldots \oplus V_{n,(m)})^{\bigoplus_{i\in\mathcal{E}} I_i}}{(V_1 \oplus \ldots \oplus V_n)^{\bigoplus_{i\in\mathcal{E}} I_i} \otimes \mathbb{Z}/m\mathbb{Z}} = \\ = \frac{(V_{1,(m)})^{I_1} \oplus \ldots \oplus (V_{r,(m)})^{I_r} \oplus V_{r+1,(m)} \oplus \ldots \oplus V_{n,(m)}}{(V_1)^{I_1}_{(m)} \oplus \ldots \oplus (V_r)^{I_r}_{(m)} \oplus V_{r+1,(m)} \oplus \ldots \oplus V_{n,(m)}} = \\ = \frac{(V_{1,(m)})^{I_1}}{(V_1)^{I_1}_{(m)}} \oplus \ldots \oplus \frac{(V_{r,(m)})^{I_r}}{(V_r)^{I_r}_{(m)}}.$$

The right hand side is

$$\bigoplus_{i=1}^{r} \frac{V_{1,(m)} \oplus \ldots \oplus (V_{i,(m)})^{I_i} \oplus \ldots \oplus V_{n,(m)}}{V_{1,(m)} \oplus \ldots \oplus (V_i)^{I_i}_{(m)} \oplus \ldots \oplus V_{n,(m)}} = \frac{(V_{1,(m)})^{I_1}}{(V_1)^{I_1}_{(m)}} \oplus \ldots \oplus \frac{(V_{r,(m)})^{I_r}}{(V_r)^{I_r}_{(m)}}.$$

So we have obtained the same expression on both sides, and $\varphi_{\mathcal{E}}$ induces the identity between them. □

Next, we make a choice of a compatible system of primitive roots of units; equivalently, we choose a topological generator for $\hat{\mathbb{Z}}'(1)$. This gives us, for each $i = 1, \ldots, n$, a topological generator e_i of I_i.

Lemma 5.4. *Assume that A is toric-additive. Then, for any subset $\mathcal{E} \subseteq \{1, \ldots, n\}$ and any $m \in \mathbb{Z}$, we have*

$$\frac{A[m](K^s)^{\oplus_{i\in\mathcal{E}} I_i}}{T_l A(K^s)^{\oplus_{i\in\mathcal{E}} I_i} \otimes \mathbb{Z}/m\mathbb{Z}} = \frac{A[m](K^s)^{\sum_{i\in\mathcal{E}} e_i}}{T_l A(K^s)^{\sum_{i\in\mathcal{E}} e_i} \otimes \mathbb{Z}/m\mathbb{Z}}$$

Proof. We have a decomposition

$$T_l A(K^s) = V_1 \oplus V_2 \oplus \ldots \oplus V_n$$

as in theorem 3.4. Again, for a $\mathbb{Z}_l$-module M, we write $M_{(m)} = M \otimes_{\mathbb{Z}_l} \mathbb{Z}/m\mathbb{Z}$; if $\mathcal{E} = \emptyset$ we are done, so we assume that $\mathcal{E} = \{1, \ldots, r\} \subseteq \{1, \ldots, n\}$ for some $1 \leq r \leq n$. The left hand side is

$$\frac{(V_{1,(m)})^{I_1} \oplus \ldots \oplus V_{r,(m)})^{I_r} \oplus V_{r+1,(m)} \oplus \ldots \oplus V_{n,(m)}}{(V_1)^{I_1}_{(m)} \oplus \ldots \oplus (V_r)^{I_r}_{(m)} \oplus V_{r+1,(m)} \oplus \ldots \oplus V_{n,(m)}}$$

The right hand side is

$$\frac{(V_{1,(m)})^{\sum_1^r e_i} \oplus \ldots \oplus (V_{n,(m)})^{\sum_1^r e_i}}{(V_1)^{\sum_1^r e_i}_{(m)} \oplus \ldots \oplus (V_n)^{\sum_1^r e_i}_{(m)}} =$$

$$= \frac{(V_{1,(m)})^{e_1} \oplus \ldots \oplus (V_{r,(m)})^{e_r} \oplus V_{r+1,(m)} \oplus \ldots \oplus V_{n,(m)}}{(V_1)^{e_1}_{(m)} \oplus \ldots \oplus (V_r)^{e_r}_{(m)} \oplus V_{r+1,(m)} \oplus \ldots \oplus V_{n,(m)}}$$

which concludes the proof. □

We now return to the hypotheses as in the beginning of the section, so S is not local anymore. From this moment, we will assume that S is a $\mathbb{Q}$-scheme, so it has residue characterstic 0 at every point. We will use the previous lemmas to prove existence and uniqueness of test-Néron models, under the hypothesis of toric-additivity of the base.

Proposition 5.5. *Suppose that S is a $\mathbb{Q}$-scheme and that $\mathcal{A}/S$ is toric-additive. If $\mathcal{N}/S$ and $\mathcal{N}'/S$ are two test-Néron models for A, there exists a unique isomorphism $\mathcal{N} \to \mathcal{N}'$ that restricts to the isomorphism $\mathcal{N}_U \to \mathcal{N}'_U$.*

Proof. The uniqueness is automatic, because $\mathcal{N}'$ is separated and $\mathcal{N}_U$ is schematically-dense in $\mathcal{N}$. For the existence part, we proceed by induction on the dimension of the base. In the case of $\dim S = 1$, let S^{sh} be a strict henselization of the trait S. The base change of a test-Néron model to S^{sh} is a Néron model. By lemma 2.10, $\mathcal{N}$ and $\mathcal{N}'$ are themselves Néron models over S, and therefore there exists an isomorphism $\mathcal{N} \to \mathcal{N}'$.

Now let $\dim S = M$ and assume the statement is true for $\dim S < M$. We claim that we can reduce to the case of a strictly local base S. Suppose that for every geometric point s of S we can construct an isomorphism $f_s \colon \mathcal{N}_{X_s} \to \mathcal{N}'_{X_s}$ where X_s is the spectrum of the strict henselization at s. Then we can spread out f_s to an isomorphism $f' \colon \mathcal{N}_{S'} \to \mathcal{N}'_{S'}$ for some étale cover S' of S. Let $S'' := S' \times_S S'$, $p_1, p_2 \colon S'' \to S'$ be the two projections and $q \colon S'' \to S$. Because test-Néron models are stable under étale base change, $q^*\mathcal{N}$ and $q^*\mathcal{N}'$ are test-Néron models. The two isomorphisms $p_1^* f, p_2^* f \colon q^*\mathcal{N} \to q^*\mathcal{N}'$ necessarily coincide, thus f descends to an isomorphism $\mathcal{N} \to \mathcal{N}'$, which proves our claim.

Let then S be strictly local, of dimension M, with closed point s. The open $V = S \setminus \{s\}$ has dimension $M-1$; since $\mathcal{A}_V/V$ is toric-additive, by inductive hypothesis there is a unique isomorphism $f_V : \mathcal{N}_V \to \mathcal{N}'_V$. We would like to extend it to the whole of S.

Let Z be a regular, closed subscheme of S of dimension 1, transversal to D. The existence of such $Z \subset S$ is easily checked. As Z is a strictly henselian trait, the pullbacks of $\mathcal{N}$ and $\mathcal{N}'$ to Z are Néron models of their generic fibre, hence there is a unique isomorphism $\alpha : \mathcal{N}_Z \to \mathcal{N}'_Z$. Now let $\underline{\Phi}$ and $\underline{\Phi'}$ be the étale S-group schemes of components of $\mathcal{N}$ and $\mathcal{N}'$; and let Φ and Φ' be the groups $\underline{\Phi}_s(k)$ and $\underline{\Phi}'_s(k)$ respectively. The restriction of α to the fibre over s induces an isomorphism $\Phi \to \Phi'$.

We show next that the isomorphism $\Phi \to \Phi'$ is independent of the choice of $Z \subset S$. Let's call L the fraction field of $\Gamma(Z, \mathcal{O}_Z)$. The morphism $Z \to S$ induces a group homomorphism

$$\pi_1(Z \setminus \{z\}) = \mathrm{Gal}(\overline{L}|L) = \widehat{\mathbb{Z}}(1) \to \pi_1(S \setminus D) = \bigoplus_{i=1}^{n} I_i = \bigoplus_{i=1}^{n} \widehat{\mathbb{Z}}(1) \qquad (26)$$

which sends a topological generator e of $\pi_1(Z \setminus \{z\})$ to a sum $\sum_{i=1}^{n} e_i$ of topological generators of the direct summands of $\pi_1(S \setminus D)$, since Z is transversal to D. By section 2.4, both Φ and Φ' are canonically isomorphic to

$$\bigoplus_{l \text{ prime}} \frac{(T_l A(\overline{L}) \otimes \mathbb{Q}_l/\mathbb{Z}_l)^{\mathrm{Gal}(\overline{L}|L)}}{T_l A(\overline{L})^{\mathrm{Gal}(\overline{L}|L)} \otimes \mathbb{Q}_l/\mathbb{Z}_l}.$$

We have a canonical isomorphism of $\mathbb{Z}_l$-modules $T_l A(\overline{K}) \to T_l A(\overline{L})$, compatible with the homomorphism 26, so that e acts on an element of $T_l A(\overline{L})$ as $\sum_{i=1}^{n} e_i$ acts on its image in $T_l A(\overline{K})$. Hence, writing G for $\pi_1(S \setminus D)$, Φ and Φ' are given by

$$\bigoplus_{l \text{ prime}} \frac{(T_l A(\overline{K}) \otimes \mathbb{Q}_l/\mathbb{Z}_l)^{\sum_{i=1}^{n} e_i}}{T_l A(\overline{K})^{\sum_{i=1}^{n} e_i} \otimes \mathbb{Q}_l/\mathbb{Z}_l} = \bigoplus_{l \text{ prime}} \frac{(T_l A(\overline{K}) \otimes \mathbb{Q}_l/\mathbb{Z}_l)^{G}}{T_l A(\overline{K})^{G} \otimes \mathbb{Q}_l/\mathbb{Z}_l}$$

the equality coming from the assumption of toric-additivity and lemma 5.4. This shows that the isomorphism $\Phi \to \Phi'$ is independent of the choice of $Z \subset S$. For this reason, we will write Φ for both groups Φ and Φ'.

Now, the surjective morphism

$$(T_l A(\overline{K}) \otimes \mathbb{Q}_l/\mathbb{Z}_l)^G \to \Phi$$

splits; letting N be the order of Φ, we obtain a surjective morphism between the N-torsion subgroups

$$A[N](K) = A[N](\overline{K})^G \to \Phi.$$

We pick a section $\Phi \to A[N](K)$ and denote by B its image. Consider the schematic closures $\mathcal{B}$ and $\mathcal{B}'$ of B inside $\mathcal{N}$ and $\mathcal{N}'$ respectively. Then $\mathcal{B}$ is a closed subgroup scheme of the étale S-group scheme $\mathcal{N}[N]$; in fact, it is the union $\sqcup_{\varphi\in\Phi} V_\varphi$ of some of its connected components. As $V_\varphi \to S$ is flat, separated and birational, it is an open immersion. As $\mathcal{N}[N]$ is finite over U, the restriction of $V_\varphi \to S$ to U is surjective, hence an isomorphism. In particular, it is given by some section $U \to A$, which restricts to a section $\operatorname{Spec} L \to A_{\operatorname{Spec} L}$ over the generic point of Z. As $\mathcal{N}_Z$ is a Néron model of its generic fibre, this section extends to a section $Z \to \mathcal{N}_Z$. This latter section is for sure contained in the schematic closure of V_φ, which is V_φ itself. This shows that $V_\varphi \to S$ is surjective, and in particular an isomorphism. Therefore, $\mathcal{B}$ is simply given by a disjoint union $\sqcup_{\varphi\in\Phi} b_\varphi$ of torsion sections $b_\varphi \colon S \to \mathcal{N}$, and the restriction $\mathcal{B}_s$ is canonically isomorphic to $\underline{\Phi}_s$. Similarly, we write $\mathcal{B}' = \sqcup_{\varphi\in\Phi} b'_\varphi$.

Let $\mathcal{A} \subset \mathcal{N}$ and $\mathcal{A}' \subset \mathcal{N}'$ be the fibrewise-connected components of identity. By uniqueness of semi-abelian extensions, there is a unique isomorphism $\mathcal{A} \to \mathcal{A}'$. Now let $\mathcal{M} = \bigcup_{\varphi\in\Phi}(b_\varphi + \mathcal{A}) \subseteq \mathcal{N}$. It is an open subgroup S-scheme of $\mathcal{N}$, and on the closed fibre we have $\mathcal{M}_s = \mathcal{N}_s$, since $\mathcal{B}_s = \underline{\Phi}_s$. In particular, $\mathcal{N} = \mathcal{N}_V \cup \mathcal{M}$. Writing similarly $\mathcal{M}' = \bigcup_{\varphi\in\Phi}(b'_\varphi + \mathcal{A}') \subseteq \mathcal{N}'$, we have $\mathcal{N}' = \mathcal{N}'_V \cup \mathcal{M}'$.

Now, we construct an isomorphism $\mathcal{M} \to \mathcal{M}'$ simply by sending b_φ to b'_φ and by means of the isomorphism $\mathcal{A} \to \mathcal{A}'$. To obtain an isomorphism $\mathcal{N} \to \mathcal{N}'$ it is enough to show that $\mathcal{N}_V \to \mathcal{N}'_V$ and $\mathcal{M} \to \mathcal{M}'$ agree on the intersection $\mathcal{N}_V \cap \mathcal{M} = \mathcal{M}_V$. This is clear: indeed, the isomorphism $\mathcal{N}_V \to \mathcal{N}'_V$ agrees with the restriction $\mathcal{A}_V \to \mathcal{A}'_V$, and it sends the schematic closure of B inside $\mathcal{N}_V$ to the schematic closure of B inside $\mathcal{N}'_V$; that is, it restricts to an isomorphism $\mathcal{B}_V \to \mathcal{B}'_V$ sending b_φ to b'_φ. □

Theorem 5.6. *Suppose that S is a $\mathbb{Q}$-scheme, and that $\mathcal{A}/S$ is toric-additive. Then there exists a test-Néron model $\mathcal{N}/S$ for A.*

Proof. Our proof is constructive; we subdivide it in steps.

Step 1: constructing the group Ψ. Let s be a geometric point of S, and write X_s for the spectrum of the strict henselization at s. Let K_s be the field of fractions of X_s, that is, the maximal extension of K unramified at s, and $\overline{K}$ an algebraic closure of K_s. Let $\mathcal{J}_s$ be the finite set of components of the strict normal crossing divisor $D \times_S X_s$.

For every prime l, the action of $\operatorname{Gal}(\overline{K}|K_s)$ factors via the quotient $\operatorname{Gal}(\overline{K}|K_s) \to G := \pi_1(U \times_S X_s) = \bigoplus_{i\in\mathcal{J}_s} I_i$ where $I_i = \widehat{\mathbb{Z}}(1)$.

We set

$$\Psi := \bigoplus_{l\ prime}\bigoplus_{i\in\mathcal{J}_s}\frac{\big(T_lA(\overline{K})\otimes\mathbb{Q}_l/\mathbb{Z}_l\big)^{I_i}}{T_lA(\overline{K})^{I_i}\otimes\mathbb{Q}_l/\mathbb{Z}_l} \tag{27}$$

The abelian group Ψ is finite, as each of its summands is the l-primary part of the group of components of the Néron model of A_{K_s} over the local ring at the generic point of D_i, which exists by theorem 2.14.

By lemma 5.2,

$$\Psi = \bigoplus_{l\ prime}\frac{\big(T_lA(\overline{K})\otimes\mathbb{Q}_l/\mathbb{Z}_l\big)^{\bigoplus_{i\in\mathcal{J}_s}I_i}}{T_lA(\overline{K})^{\bigoplus_{i\in\mathcal{J}_s}I_i}\otimes\mathbb{Q}_l/\mathbb{Z}_l}.$$

The surjective morphism

$$\bigoplus_l((T_lA(\overline{K})\otimes\mathbb{Q}_l/\mathbb{Z}_l)^{\bigoplus_{i\in\mathcal{J}_s}I_i})\to\Psi$$

splits; therefore, denoting by N the order of Ψ, we obtain a surjective morphism between the N-torsion subgroups

$$\pi\colon A[N](K_s)=A[N](\overline{K})^{\bigoplus_{i\in\mathcal{J}_s}I_i}\to\Psi.$$

We consider the set of sections $\mathcal{S}:=\{\alpha\colon\Psi\to A[N](K_s)\text{ such that }\pi\circ\alpha=\mathrm{id}\}$: it is a torsor under the finite group $\bigoplus_l(T_lA(K_s)\otimes\mathbb{Z}/N\mathbb{Z})$, and as such it is finite. As the group Ψ is finite as well, there exists a finite extension $K\to K'$, unramified over s, such that every section $\Psi\to A[N](K_s)$ factors via $A[N](K')$. Notice that $\mathcal{S}$ is non-empty, as the quotient map π splits; thus we can fix a section $\alpha\colon\Psi\to A[N](K')$.

Step 2: spreading out to an étale neighbourhood of s. The normalization of S inside K' is unramified over the image of s in S, hence étale over it ([Sta16]TAG 0BQK), so we obtain an étale neighbourhood S' of s, which we may assume to be connected, with fraction field K'. We write $\mathcal{J}'$ for the set of irreducible components of $D\times_S S'$. There is a natural function $\mathcal{J}_s\to\mathcal{J}'$: up to restricting S', we may assume that it is bijective. Indeed, its surjectivity corresponds to the fact that every component of $D\times_S S'$ contains (the image of) s; imposing also injectivity means asking that $D\times_S S'$ is a *strict* normal crossing divisor. Thus, we need not distinguish between $\mathcal{J}_s$ and $\mathcal{J}'$ and we will simply write $\mathcal{J}$ for this set.

Step 3: constructing the subgroup-scheme $\mathcal{H}\subseteq\mathcal{A}_{S'}\times_{S'}\Psi_{S'}$. We call $H\subseteq A[N](K')\times\Psi$ the image of Ψ via $(\alpha,\mathrm{id})\colon\Psi\to A[N](K')\times\Psi$; we let $\mathcal{H}/S'$ be the schematic closure of H inside $\mathcal{A}_{S'}\times_{S'}\Psi_{S'}$ (where $\Psi_{S'}$ denotes the constant group scheme over S' associated to the finite abelian group Ψ). It is a closed subgroup scheme of the étale S'-group scheme $\mathcal{A}_{S'}[N]\times_{S'}\Psi_{S'}$ and a disjoint union $\sqcup_{j\in\Psi}V_j$ of some of its connected components; moreover, over the generic point of S', each V_j restricts to a copy of $\operatorname{Spec} K'$. As $V_j \to S'$ is flat, separated and birational, it is an open immersion; thus $\mathcal{H} = \sqcup_{j\in\Psi}V_j \to S'$ is a disjoint union of open immersions. In fact, if we write $U' = U \times_S S'$, the base change $A_{U'}$ is an abelian scheme; therefore $A_{U'}[N] \times_{U'} \Psi_{U'}$ is finite, and each $V_j \to S'$ is an isomorphism over U'. This can be restated by saying that the composition

$$\mathcal{H}_{U'} \to \mathcal{A}_{U'} \times_{U'} \Psi_{U'} \to \Psi_{U'}$$

is an isomorphism.

Step 4: taking the quotient by $\mathcal{H}$. Consider now the fppf-quotient

$$\mathcal{N}^\alpha := \frac{\mathcal{A}_{S'} \times_{S'} \Psi_{S'}}{\mathcal{H}}.$$

First, we claim that its restriction $\mathcal{N}^\alpha_{U'}$ is canonically isomorphic to $A_{U'}$. Indeed, we observed that $\mathcal{H}_{U'} = \Psi_{U'}$, and the quotient morphism for $\Psi_{U'} \to \mathcal{A}_{U'} \times_{U'} \Psi_{U'}$, $\psi \mapsto (\alpha(\psi), \psi)$ is $\mathcal{A}_{U'} \times_{U'} \Psi_{U'} \to \mathcal{A}_{U'}$, $(a, \psi) \mapsto a - \alpha(\psi)$, which proves the claim.

Because $\mathcal{H}$ is étale, $\mathcal{N}^\alpha$ is automatically an algebraic space; we claim that it is actually representable by a scheme. As the quotient morphism $p\colon \mathcal{A}_{S'} \times_{S'} \Psi_{S'} \to \mathcal{N}^\alpha$ is an $\mathcal{H}$-torsor, p is étale. In particular the restriction of p to the connected component of identity, $\mathcal{A}_{S'} \times \{0\} \to \mathcal{N}^\alpha$, is étale; it is also separated, and an isomorphism over U. It follows that it is an open immersion. Hence, all other components $\mathcal{A}_{S'} \times \{\psi\}$ map to $\mathcal{N}^\alpha$ via an open immersion. The disjoint union $\bigsqcup_{\psi\in\Psi} \mathcal{A}_{S'} \times_{S'} \{\psi\}$ surjects onto $\mathcal{N}^\alpha$, and this gives us an open cover of $\mathcal{N}^\alpha$ by schemes.

In summary, we have obtained an S'-group scheme $\mathcal{N}^\alpha$, which restricts to A over U'; moreover, it is S'-smooth, of finite presentation, and separated, since $\mathcal{H}$ is closed in the separated scheme $\mathcal{A}_{S'} \times_{S'} \Psi_{S'}$.

Step 5: independence of the section α. We have used the notation $\mathcal{N}^\alpha$ as a reminder of our choice of section α done above. We show that $\mathcal{N}^\alpha$ does not depend on the choice of the section $\Psi \to A[N](K')$, or to put it better, we show that given two sections α, β we obtain a canonical isomorphism $\mathcal{N}^\alpha \to \mathcal{N}^\beta$. Actually, as soon as we prove that $\mathcal{N}^\alpha$ and $\mathcal{N}^\beta$ are test-Néron models (step 6), the existence of a canonical isomorphism between them is ensured by proposition 5.5; however, we still give an argument: suppose we choose another section $\beta\colon \Psi \to A[N](K')$ and let $H^\beta \subset A[N](K') \times \Psi$ be the image of Ψ via $(\beta, \mathrm{id})\colon \Psi \to A[N](K') \times \Psi$. Then the map $f_{\beta-\alpha}\colon H^\alpha \to H^\beta$ sending $(h, \psi) \in H^\alpha \subseteq A[N](K') \times \Psi$ to $(h + (\beta - \alpha)\psi, \psi)$ is an isomorphism. Moreover, $\beta - \alpha$ lands inside $\bigoplus_l T_l A(K') \otimes \mathbb{Z}/N\mathbb{Z}$, the subgroup of $A(K')$ consisting of those N-torsion points that extend to torsion sections of $\mathcal{A}_{S'}/S'$. Therefore $\beta - \alpha$ extends to a morphism of S'-group schemes $\Psi_{S'} \to \mathcal{A}_{S'}$. Now, the isomorphism

$$\mathcal{A}_{S'} \times_{S'} \Psi_{S'} \xrightarrow{\begin{pmatrix}1 & \beta-\alpha\\ 0 & 1\end{pmatrix}} \mathcal{A}_{S'} \times_{S'} \Psi_{S'}$$

restricts to $f_{\beta-\alpha}$ on H^α and therefore also restricts to an isomorphism $\mathcal{H}^\alpha \to \mathcal{H}^\beta$ between the schematic closures of H^α and H^β in $\mathcal{A}_{S'} \times_{S'} \Psi_{S'}$. Hence, we obtain an isomorphism $\mathcal{N}^\alpha = (\mathcal{A}_{S'} \times_{S'} \Psi_{S'})/\mathcal{H}^\alpha \to \mathcal{N}^\beta = (\mathcal{A}_{S'} \times_{S'} \Psi_{S'})/\mathcal{H}^\beta$ between the quotients, as wished. We can therefore forget about the choice of section and use the notation $\mathcal{N}/S'$ for the group-scheme just constructed.

Step 6: showing that $\mathcal{N}$ is a test-Néron model. To ease notation, let us write S in place of S', $D = \bigcup_{i\in\mathcal{J}} D_i$ for the strict normal crossing divisor $D \times_S S'$. Let Z be a strictly henselian trait, with closed point z, and $g\colon Z \to S$ a morphism transversal to D. Write T for the strict henselization of S at z and $\mathcal{E} \subseteq \mathcal{J}$ for the subset of indices of components D_i that contain z. Let also $\mathcal{M}/Z$ be the Néron model of $A \times_S Z$. The Néron mapping property gives a morphism $\mathcal{N}_Z \to \mathcal{M}$, which is an open immersion and induces an isomorphism between the fibrewise-connected components of identity, as they are both semi-abelian (lemma 2.17). Let Φ/S and Υ/Z be the étale group schemes of connected components of $\mathcal{N}/S$ and $\mathcal{M}/Z$ respectively. To show that $\mathcal{N}_Z \to \mathcal{M}$ is an isomorphism, we only need to check that the induced morphism $\Phi_{|Z} \to \Upsilon$ is an isomorphism. It is certainly an open immersion, so it suffices to show that $\Phi(z) \to \Upsilon(z)$ is an isomorphism.

We will fix a prime l and compare the l-primary parts of the two groups, which we denote ${}_l\Phi(z)$ and ${}_l\Upsilon(z)$. Let's start with ${}_l\Phi(z)$. The group scheme Φ/S being given by $\Psi_S/\mathcal{H}$, we have ${}_l\Phi(z) = {}_l\Psi(z)/{}_l\mathcal{H}(z)$. Recall that ${}_l\mathcal{H}$ is the schematic closure of ${}_lH$ inside $\mathcal{A} \times_S {}_l\Psi_S$. Hence, ${}_l\mathcal{H}(z)$ is identified with a subgroup of ${}_lH$ consisting of those elements $(a,\psi) \in {}_lH \subset A[N](K) \times {}_l\Psi$ such that a extends to a section of $\mathcal{A}_T/T$. These are exactly the pairs $(a,\psi) \in {}_lH$ such that $a \in T_lA(\overline{K})^{\oplus_{i\in\mathcal{E}} I_i} \otimes \mathbb{Z}/N\mathbb{Z}$. Therefore, ${}_l\mathcal{H}(z)$ is the kernel of the composition

$$\begin{aligned}{}_lH \xrightarrow{\sim} {}_l\Psi = \bigoplus_{i\in\mathcal{J}} \frac{\left(T_lA(\overline{K}) \otimes \mathbb{Z}/N\mathbb{Z}\right)^{I_i}}{T_lA(\overline{K})^{I_i} \otimes \mathbb{Z}/N\mathbb{Z}} \xrightarrow{pr} \\ \xrightarrow{pr} \bigoplus_{i\in\mathcal{E}} \frac{\left(T_lA(\overline{K}) \otimes \mathbb{Z}/N\mathbb{Z}\right)^{I_i}}{T_lA(\overline{K})^{I_i} \otimes \mathbb{Z}/N\mathbb{Z}} = \frac{\left(T_lA(\overline{K}) \otimes \mathbb{Z}/N\mathbb{Z}\right)^{\bigoplus_{i\in\mathcal{E}} I_i}}{T_lA(\overline{K})^{\bigoplus_{i\in\mathcal{E}} I_i} \otimes \mathbb{Z}/N\mathbb{Z}}\end{aligned}$$

from which it follows that

$${}_l\Phi(z) = \frac{{}_l\Psi(z)}{{}_l\mathcal{H}(z)} \cong \bigoplus_{i\in\mathcal{E}} \frac{\left(T_lA(\overline{K}) \otimes \mathbb{Z}/N\mathbb{Z}\right)^{I_i}}{T_lA(\overline{K})^{I_i} \otimes \mathbb{Z}/N\mathbb{Z}}.$$

Next, we look at ${}_l\Upsilon(z)$. Let's call K_Z the field of fractions of $\Gamma(Z, \mathcal{O}_Z)$. The morphism $Z \to T$ induces a group homomorphism

$$\pi_1(Z \setminus \{z\}) = \mathrm{Gal}(\overline{K}_Z|K_Z) = \widehat{\mathbb{Z}}(1) \to \pi_1(T \setminus D) = \bigoplus_{i \in \mathcal{E}} \widehat{\mathbb{Z}}(1)$$

which sends a topological generator e of $\widehat{\mathbb{Z}}(1)$ to a sum of topological generators $\sum_{i=1}^n e_i$, because Z meets D transversally.

Notice that there is a canonical identification $T_lA(\overline{K}_Z) = T_lA(\overline{K})$; the topological generator of $\mathrm{Gal}(\overline{K}_Z|K_Z)$ acts on the latter as $\sum_{i\in\mathcal{E}} e_i$ does. Therefore

$${}_l\Upsilon(z) = \frac{\left(T_lA(\overline{K}) \otimes \mathbb{Z}/N\mathbb{Z}\right)^{\sum_{i\in\mathcal{E}} e_i}}{T_lA(\overline{K})^{\sum_{i\in\mathcal{E}} e_i} \otimes \mathbb{Z}/N\mathbb{Z}}$$

By lemma 5.4 and lemma 5.2, ${}_l\Upsilon(z) \cong {}_l\Phi(z)$, as we wished to show. Hence $\mathcal{N}$ is a test-Néron model for $A_{U'}$ over S'.

Step 7: descending $\mathcal{N}$ along $S' \to S$. For every geometric point s of S, we have found an étale neighbourhood $S' \to S$ and a test-Néron model $\mathcal{N}/S'$ over S'. Using uniqueness up to unique isomorphism of test-Néron models, their stability under étale base change, and effectiveness of étale descent for algebraic spaces, we obtain a smooth separated algebraic space of finite type $\widetilde{\mathcal{N}}$ over S, and an isomorphism $\widetilde{\mathcal{N}} \times_S U \to A$. Because the property of being a test-Néron model is étale-local, $\widetilde{\mathcal{N}}$ is itself a test-Néron model for A over S. □

5.2 Test-Néron models and finite flat base change

In [Edi92], Edixhoven considers the case of an abelian variety A_K over the generic point of a trait S, and a tamely ramified extension of traits $\pi\colon S' \to S$ whose associated extension of fraction fields $K \to K'$ is Galois. He considers the Néron model $\mathcal{N}/S$ of A_K and the Néron model $\mathcal{N}'/S'$ of $A_{K'}$: after defining a certain equivariant action of $\mathrm{Gal}(K'|K)$ on the Weil restriction $\pi_*\mathcal{N}'$, he shows that $\mathcal{N}$ is naturally identified with the subgroup-scheme of $\mathrm{Gal}(K'/K)$-invariants of $\pi_*\mathcal{N}'$.

In this subsection, we aim to show an analogous statement for test-Néron models over a base of higher dimension and characteristic everywhere zero.

We let then S be a noetherian, regular, strictly local $\mathbb{Q}$-scheme, $D = \cup_{i=1}^n \mathrm{div}(t_i)$ a normal crossing divisor on S (thus the t_i are part of a system of regular parameters for $\mathcal{O}_S(S)$), A an abelian scheme over $U = S\setminus D$, $\mathcal{A}/S$ a toric-additive semi-abelian scheme extending A.

We can apply theorem 5.6 to construct a test-Néron model

$$\mathcal{N} = \frac{\mathcal{A} \times_S \Psi_S}{\mathcal{H}}.$$

Notice that the étale cover $S' \to S$ of the proof of theorem 5.6 is necessarily trivial in this case.

Consider now a finite flat cover $\pi\colon T \to S$ of the form

$$T = \operatorname{Spec} \frac{\mathcal{O}_S(S)[X_1, \ldots, X_n]}{X_1^{m_1} - t_1, \ldots, X_n^{m_n} - t_n}$$

for some positive integers $m_1, \ldots, m_n$. Then T is a regular strictly local scheme. We denote by K' its field of fractions. The morphism π is finite étale over U, and the preimage via $\pi\colon T \to S$ of D is the normal crossing divisor $\pi^{-1}(D) = \cup_{i=1}^n \operatorname{div} X_i$.

We have a commutative diagram

$$\begin{array}{ccc} \operatorname{Gal}(\overline{K}|K') & \twoheadrightarrow & \pi_1(U_T) = \bigoplus_{i=1}^n \widehat{\mathbb{Z}}(1) \\ \downarrow & & \downarrow \\ \operatorname{Gal}(\overline{K}|K) & \twoheadrightarrow & \pi_1(U) = \bigoplus_{i=1}^n \widehat{\mathbb{Z}}(1) \end{array}$$

where the right vertical arrow is given by multiplication by m_i on the i-th component. We will write $\pi_1(U) = \bigoplus_{i=1}^n I_i$ and identify $\pi_1(U_T)$ with its subgroup $\bigoplus_{i=1}^n m_i I_i$.

The fraction field K' of T is an extension of K of order $m_1 \cdot m_2 \cdot \ldots \cdot m_n$, and we write G for the Galois group $\operatorname{Gal}(K'|K) = \bigoplus_{i=1}^n I_i/m_i I_i = \bigoplus_{i=1}^n \mu_{m_i}$.

By lemma 3.6, $\mathcal{A} \times_S T$ is still toric-additive. We follow the construction carried out in the proof of theorem 5.6 to obtain a test-Néron model $\mathcal{M}/T$: to start with, we consider the finite abelian group

$$\Psi' = \bigoplus_{l\ prime} \bigoplus_{i=1}^n \frac{\left(T_l A(\overline{K}) \otimes \mathbb{Q}_l/\mathbb{Z}_l\right)^{m_i I_i}}{T_l A(\overline{K})^{m_i I_i} \otimes \mathbb{Q}_l/\mathbb{Z}_l} = \bigoplus_{l\ prime} \frac{\left(T_l A(\overline{K}) \otimes \mathbb{Q}_l/\mathbb{Z}_l\right)^{\oplus_{i=1}^n m_i I_i}}{T_l A(\overline{K})^{\bigoplus_{i=1}^n m_i I_i} \otimes \mathbb{Q}_l/\mathbb{Z}_l}$$

We claim that $T_l A(\overline{K})^{I_i} = T_l A(\overline{K})^{m_i I_i}$; indeed, letting e_i, be a topological generator of I_i, and denoting still by e_i the automorphism of $T_l A(\overline{K})$ induced by e_i, we know by section 2.3 that $(e_i - 1)^2 = 0$. Using this relation, we obtain

$$e_i^{m_i} - 1 = ((e_i - 1) + 1)^{m_i} - 1 = m(e_i - 1) + 1 - 1 = m(e_i - 1).$$

As $T_lA(\overline{K})$ is torsion-free, we see that $\ker(e_i^{m_i}-1)=\ker(e_i-1)$, which proves our claim. Hence, we actually have

$$\Psi' = \bigoplus_{l\ prime} \frac{\left(T_lA(\overline{K})\otimes \mathbb{Q}_l/\mathbb{Z}_l\right)^{\oplus_{i=1}^n m_i I_i}}{T_lA(\overline{K})^{\oplus_{i=1}^n I_i}\otimes \mathbb{Q}_l/\mathbb{Z}_l}$$

and it follows that Ψ' has a natural action of G.

Next, we let $N=\operatorname{ord}(\Psi')$ and choose a section $\alpha\colon \Psi'\to A[N](K')$. We write H' for the image of $\Psi' \xrightarrow{(\alpha,\mathrm{id})} A[N](K')\times\Psi'$ and $\mathcal{H}'$ for its schematic closure inside $\mathcal{A}_T\times_T\Psi'_T$. The fppf-quotient

$$\mathcal{M}=\frac{\mathcal{A}_T\times_T\Psi'_T}{\mathcal{H}'}$$

is represented by a test-Néron model for $A_{U'}$ over T.

In order to compare $\mathcal{M}$ and $\mathcal{N}$, we will consider the Weil restriction of $\mathcal{M}$ via $\pi\colon T\to S$, that is, the functor $\pi_*\mathcal{M}\colon (\mathbf{Sch}/S)\to\mathbf{Sets}$ given by $(Y\to S)\mapsto \mathcal{M}(Y\times_S T)$. Recall that we have an exact sequence of fppf-sheaves of abelian groups

$$0\to\mathcal{H}'\to\mathcal{A}_T\times_T\Psi'_T\to\mathcal{M}\to 0.$$

As π is a finite morphism, the higher direct images of π for the fppf-topology vanish, and we have an exact sequence of fppf-sheaves

$$0\to\pi_*\mathcal{H}'\to\pi_*\mathcal{A}_T\times_S\pi_*\Psi'_T\to\pi_*\mathcal{M}\to 0.$$

We claim that $\pi_*\mathcal{M}$ is representable by a scheme. By [Ray70b, XI, 1.16], semi-abelian schemes are quasi-projective, hence so is $\mathcal{A}_T\times_T\Psi'_T$. Clearly $\mathcal{H}'/T$ is quasi-projective as well. As $\pi\colon T\to S$ is finite and flat, $\pi_*\mathcal{H}'$ and $\pi_*\mathcal{A}_T\times_S\pi_*\Psi'_T$ are schemes (see for example [Edi92, 2.2]). Now, $\pi_*\mathcal{H}'/S$ is étale ([Sch94, 4.9]), and its intersection with the identity component of $\pi_*\mathcal{A}_T\times_S\pi_*\Psi'_T$ is trivial. Reasoning as in the proof of theorem 5.6, we conclude that $\pi_*\mathcal{M}$ has an open cover by schemes, hence it is a scheme.

We want to define an equivariant action of G on $\pi_*\mathcal{M}\to S$, where G acts trivially on S. To do this, we let first G act on $A_{K'}$ via the action of G on K'. By [Del85, 1.3 pag.132] the action of G extends uniquely to an equivariant action on $\mathcal{A}_T\to T$. We also have an obvious action of G on Ψ' which induces an equivariant action on $\Psi'_T\to T$. We put together these actions to find an equivariant action of G on $\mathcal{A}_T\times_T\Psi'_T\to T$: clearly H' is G-invariant, thus the same is true for its schematic closure $\mathcal{H}'$. Therefore the action of G descends to an equivariant action of G on $\mathcal{M}\to T$.

To define the action of G on $\pi_*\mathcal{M}$, we let $g\in G$ act on $\pi_*\mathcal{M}$ via the composition

$$\pi_*\mathcal{M} \times_S T \xrightarrow{(\mathrm{id},g)} \pi_*\mathcal{M} \times_S T \to \mathcal{M} \xrightarrow{g^{-1}} \mathcal{M}.$$

where the second arrow is given by the identity morphism $\pi_*\mathcal{M} \to \pi_*\mathcal{M}$. This defines the desired equivariant action of G on $\pi_*\mathcal{M} \to S$.

Consider the functor of fixed points $(\pi_*\mathcal{M})^G \colon \mathbf{Sch}/S \to \mathbf{Sets}$, $(Y \to S) \mapsto \pi_*\mathcal{M}(Y)^G$. Then $(\pi_*\mathcal{M})^G$ is represented by a closed subgroup-scheme of $\pi_*\mathcal{M}$, smooth over S by [Edi92, 3.1].

Proposition 5.7. *There is a canonical closed immersion $\iota \colon \mathcal{N} \to \pi_*\mathcal{M}$, which identifies $\mathcal{N}$ with the subgroup-scheme of fixed points $(\pi_*\mathcal{M})^G$.*

Proof. By generalities on the Weil restriction [BLR90, pag. 198], the canonical morphism $\mathcal{A} \to \pi_*\mathcal{A}_T$ is a closed immersion. The natural injection $\Psi \to \Psi'$ gives a closed immersion $\mathcal{A} \times_S \Psi_S \to \pi_*\mathcal{A}_T \times_S \Psi'_S = \pi_*(\mathcal{A}_T \times_T \Psi'_T)$. To show that it descends to a closed immersion $\mathcal{N} \to \pi_*\mathcal{M}$, it is enough to show that

$$\pi_*\mathcal{H}' \cap (\mathcal{A} \times_S \Psi_S) = \mathcal{H}. \tag{28}$$

We may assume that the section $\Psi \to A[N](K)$ used to construct H is obtained by restriction of the section $\Psi' \to A[N](K')$ used to construct H': indeed we know that it does not matter which section we choose. It follows that $H = H' \cap (A(K) \times_K \Psi)$, which realizes eq. (28) on the level of generic fibres. Now, $\pi_*\mathcal{H}'$ is étale over S, and it is a closed subscheme of $\pi_*\mathcal{A}_T \times_S \Psi'_S$. Hence, it is the schematic closure of its generic fibre, which is H'. Then, the intersection $\mathcal{H}^* := \pi_*\mathcal{H}' \cap (\pi_*\mathcal{A}_T \times_S \Psi_S)$ is clearly still étale over S, and has generic fibre H. Thus $\mathcal{H}^*$ is the schematic closure of H in $\pi_*\mathcal{A}_T \times_S \Psi_S$. On the other hand, $\mathcal{H} \to \mathcal{A} \times_S \Psi_S \to \pi_*\mathcal{A}_T \times_S \Psi_S$ is a closed immersion, and $\mathcal{H}$ is étale over S and has generic fibre H. As $\mathcal{H}$ and $\mathcal{H}^*$ are both étale over S, have same generic fibre and are both closed subschemes of $\pi_*\mathcal{A}_T \times_S \Psi_S$, they are equal. Since $\mathcal{H}$ is contained in $\mathcal{A} \times_S \Psi_S$, so is $\mathcal{H}^*$ and we obtain eq. (28). This proves that we have a closed immersion $\iota \colon \mathcal{N} \to \pi_*\mathcal{M}$.

Now, the restriction of ι to the generic fibre is the closed immersion $A \to \pi_*A_{K'}$, which identifies A with $(\pi_*A_{K'})^G$. Since $(\pi_*\mathcal{M})^G$ and $\mathcal{N}$ are both S-smooth closed subschemes of $\pi_*\mathcal{M}$ and they share the same generic fibre, they are equal. □

5.3 Test-Néron models are Néron models

The objective of this subsection is to prove the following:

Theorem 5.8. *Let S be a connected, locally noetherian, regular $\mathbb{Q}$-scheme, D a normal crossing divisor on S, A an abelian scheme over $U = S \setminus D$ extending* *to a toric-additive semi-abelian scheme $\mathcal{A}/S$. Then A admits a Néron model over S.*

In view of theorem 5.6, theorem 5.8 is an immediate corollary of the following proposition:

Proposition 5.9. *Hypotheses as in theorem 5.8. Let $\mathcal{N}/S$ be a test-Néron model for A over S. Then $\mathcal{N}/S$ is a Néron model.*

We will subdivide the proof of proposition 5.9 in two main steps (propositions 5.10 and 5.11).

Proposition 5.10. *In the hypotheses of proposition 5.9, assume S has dimension 2. Then $\mathcal{N}/S$ is a weak Néron model for A.*

Proof. Let $\sigma : U \to A$ be a section; we want to show that it extends to a section $S \to \mathcal{N}$, or equivalently, that the schematic closure $\overline{\sigma(U)} \subset \mathcal{N}$ is faithfully flat over S. The latter may be checked locally for the fpqc topology; hence, we may reduce to the case where S is the spectrum of a complete, strictly henselian local ring. The normal crossing divisor D has at most 2 components, and up to restricting U we may assume that it is given by the zero locus of uv, with u, v regular parameters for $\Gamma(S, \mathcal{O}_S)$.

Notice that the closure $\overline{\sigma(U)}$ may fail to be flat only over the closed points of S, as $S \setminus \{s\}$ is of dimension 1. By the flattening technique of Raynaud-Gruson ([GR71, 5.2.2]), there exists a blowing-up $\tilde{S} \to S$, centered at s, such that the schematic closure of $\sigma(U)$ inside $\mathcal{N}_{\tilde{S}}$ is flat over $\tilde{S}$. Because S has dimension 2, we can find a further blow-up $S' \to \tilde{S}$ such that the composition $S' \to S$ is a composition of finitely many blowing-ups, each given by blowing-up the ideal of a closed point with its reduced structure. It follows that the exceptional fibre $E \subset S'$ of $S' \to S$ is a chain of projective lines meeting transversally. Let $\Sigma \subset \mathcal{N}_{S'}$ be the schematic closure of $\sigma(U)$. The morphism $\Sigma \to S'$ is flat, but may a priori not be surjective. At this point we only know that the image of Σ contains $S' \setminus E$.

We claim that $\Sigma \to S'$ is surjective. Let $p \in E$. It's easy to show that there exists some strictly henselian trait Z with closed point z and a closed immersion $Z \to S'$ mapping z to p and such that Z meets E transversally. We call L the field of fractions of $\mathcal{O}_Z(Z)$. The section $\sigma\colon U \to A$ restricts to a section $\sigma_L\colon \operatorname{Spec} L \to A_L$; to establish the claim, it suffices to show that σ_L extends to a section $Z \to \mathcal{N}_Z$. We consider the composition $\varphi : Z \to S' \to S$ and the pullbacks $\varphi^*(u), \varphi^*(v) \in \mathcal{O}_Z(Z)$. Let $m, n \in \mathbb{Z}_{\geq 1}$ be their respective valuations. Now let $\pi\colon T \to S$ be the finite flat morphism given by extracting an m-root of u and an n-root of v, that is,

$$T = \operatorname{Spec} \frac{\mathcal{O}_S(S)[x,y]}{x^m - u, y^n - v}.$$

Then T is itself the spectrum of a regular, strictly henselian local ring and the preimage $\pi^{-1}(D)$ is the zero locus of xy and hence a normal crossing divisor. The pullback of $\mathcal{A}$ via $T \to S$ is still toric-additive (lemma 3.6) and therefore we can construct a test Néron model $\mathcal{M}/T$. Writing $X = \pi_*\mathcal{M}$ for the Weil restriction along π and $G := \operatorname{Aut}_S(T) = \mu_m \oplus \mu_n$, we have by proposition 5.7 that $X^G = \mathcal{N}$.

Now, as Z is a strictly henselian, $\mathcal{O}(Z)$ contains all roots of elements of $\mathcal{O}(Z)^\times$, and we can find uniformizers $t_u, t_v \in \mathcal{O}_Z(Z)$ such that $t_u^m = \varphi^*(u)$ and $t_v^n = \varphi^*(v)$. These elements give us a lift of $\varphi\colon Z \to S$ to $\psi\colon Z \to T$. Then ψ is a closed immersion meeting $f^{-1}(D)$ transversally. This means that the base change $\mathcal{M}_Z/Z$ is a Néron model of its generic fibre. Consider the section $\sigma_L\colon \operatorname{Spec} L \to A_L$. Composing it with the closed immersion $A_L = (\pi_*\mathcal{M})_L^G \hookrightarrow (\pi_*\mathcal{M})_L$ gives, by definition of Weil restriction, a morphism $\operatorname{Spec} L \times_S T \to \mathcal{M}_L$. Precomposing with $(\mathrm{id}, \psi)\colon \operatorname{Spec} L \to \operatorname{Spec} L \times_S T$, we obtain a section $\widetilde{\sigma}_L\colon \operatorname{Spec} L \to \mathcal{M}_L$. As $\mathcal{M}_Z/Z$ is a Néron model of its generic fibre, $\widetilde{\sigma}_L$ extends uniquely to a section $Z \to \mathcal{M}_Z$. This gives us a morphism of T-schemes $Z \to \mathcal{M}$ and by composition a T-morphism $Z \times_S T \to Z \to \mathcal{M}$, that is, a section $m \in X(Z)$ of the Weil restriction. Notice that the generic fibre of m is σ_L, which lands in the part of X fixed by G; as $X^G = \mathcal{N}$ is a closed subscheme of X we deduce that m lands inside $\mathcal{N}$. So $m \in \mathcal{N}(Z)$ is the required extension of σ_L and we win.

As $\Sigma \to S'$ is faithfully flat, separated and birational, it is an isomorphism. Hence $\sigma\colon U \to A$ extends to a section $\sigma'\colon S' \to \mathcal{N}_{S'}$. We are going to show that σ' descends to a section $\theta : S \to \mathcal{N}$. The restriction of σ' to E maps a connected chain of projective lines to a connected component of $\mathcal{N}_s$ (where s is the closed point of S). Every connected component of $\mathcal{N}_s$ is isomorphic to the semi-abelian variety $\mathcal{N}_s^0$, hence does not contain projective lines. It follows that $\sigma'_{|E}$ is constant and that it descends to a morphism $\operatorname{Spec} k(s) \to \mathcal{N}_s$. Let $\mathcal{J}$ be the ideal sheaf of the exceptional fibre $E \subset S'$ and define $S'_n \subset S'$ to be the closed subscheme defined by $\mathcal{J}^{n+1}$ for every $n \geq 0$. Similarly let $S_n := \operatorname{Spec} \mathcal{O}_S(S)/\mathfrak{m}^{n+1}$, where $\mathfrak{m}$ is the maximal ideal of $\mathcal{O}_S(S)$. We have shown that $\sigma'_{|E} : S'_0 \to \mathcal{N}$ descends to a morphism $\theta_0\colon S_0 \to \mathcal{N}$. Now, by smoothness of $\mathcal{N}$, every morphism $S_{j-1} \to \mathcal{N}$ admits a lift $S_j \to \mathcal{N}$; the set of such lifts is given by $H^0(S_0, \Omega^1_{\mathcal{N}_{S_0}/S_0} \otimes_{\mathcal{O}_{S_0}} \mathfrak{m}^j/\mathfrak{m}^{j+1})$. The canonical morphism

$$H^0(S_0, \Omega^1_{\mathcal{N}_{S_0}/S_0} \otimes_{\mathcal{O}_{S_0}} \mathfrak{m}^j/\mathfrak{m}^{j+1}) \to H^0(S'_0, \Omega^1_{\mathcal{N}_{S'_0}/S'_0} \otimes_{\mathcal{O}_{S'_0}} \mathcal{J}^j/\mathcal{J}^{j+1})$$

is an isomorphism, due to the fact that the space of global sections of $\mathcal{J}^j/\mathcal{J}^{j+1} = \mathcal{O}_{S'_0}(j)$ is equal to $\mathfrak{m}^j/\mathfrak{m}^{j+1}$. Thus the set of liftings of $\alpha \in \operatorname{Hom}_S(S_j, \mathcal{N})$

to $\mathrm{Hom}_S(S_{j+1}, \mathcal{N})$ is naturally in bijection with the set of liftings of $\alpha_{|S'_j} \in \mathrm{Hom}_S(S'_j, \mathcal{N})$ to $\mathrm{Hom}_S(S'_{j+1}, \mathcal{N})$. The reductions modulo $\mathcal{J}^j$ of $\sigma' \colon S' \to \mathcal{N}_{S'}$ provides a compatible set of liftings of $\sigma'_{|S'_0}$, and therefore a compatible set of liftings of θ_0; which in turn by completeness of S yield the desired morphism $S \to \mathcal{N}$. □

The next step is extending the result to the case of $\dim S > 2$.

Proposition 5.11. *In the hypotheses of proposition 5.9, $\mathcal{N}/S$ is a weak Néron model.*

Proof. As in the proof of proposition 5.10, we may assume that S is the spectrum of a complete strictly henselian local ring. We proceed by induction on the dimension of S. If the dimension is 1, the statement is clearly true, and the case of dimension 2 is the statement of proposition 5.10. So we let $n \geq 3$ be the dimension of S and we suppose that the statement is true when S has dimension $n-1$. Let $\sigma \colon U \to A$ be a section. Because $V = S \setminus \{s\}$ has dimension $n-1$, and because $\mathcal{A}_V$ is still toric-additive (lemma 3.8), σ extends to $\sigma \colon V \to \mathcal{N}_V$.

Next, we cut S with a hyperplane H transversal to all the components of the normal crossing divisor D, but paying attention to choosing H so that $D \cap H$ (with its reduced structure) is still a normal crossing divisor on H. This is always possible: consider a system of regular parameters $u_1, u_2, \ldots, u_n$ for S such that D is the zero locus of $u_1 u_2 \cdots u_r$ for some $r \leq n$; then H can be chosen to be, for example, the hypersurface cut by $u_1 - u_n$. Because H is transversal to D, it is clear that the base change $\mathcal{N}_H/H$ is still a test-Néron model. By our inductive assumption on the dimension of the base, $\sigma_{|H} \colon H \cap U \to A$ extends to $\theta_0 \colon H \to \mathcal{N}$. Now we would like to put together the data of σ and θ_0 to extend $\sigma \colon V \to \mathcal{N}_V$ to a section $\theta \colon S \to \mathcal{N}$. Let $\mathcal{J} \subset \mathcal{O}_S$ be the ideal sheaf of H and for every $j \geq 1$ define S_j to be the closed subscheme cut by $\mathcal{J}^{j+1}$. We have a morphism $\theta_0 \colon H = S_0 \to \mathcal{N}$. By smoothness of $\mathcal{N}$, there exists for every $j \geq 0$ a lifting of θ_0 to an S-morphism $S_j \to \mathcal{N}$. The set of liftings of an S-morphism $S_{j-1} \to \mathcal{N}$ to an S-morphism $S_j \to \mathcal{N}$ is given by the global sections of the locally-free sheaf $\mathcal{F} := \Omega^1_{\mathcal{N}/S} \otimes \mathcal{J}^j/\mathcal{J}^{j+1}$ on S_0. Because $\dim S_0 \geq 2$ and $V = S \setminus \{s\}$, we have $H^0(S_0, \mathcal{F}) = H^0(V \cap S_0, \mathcal{F}_V)$, and the latter parametrizes liftings of morphisms $S_{j-1} \cap V \to \mathcal{N}$ to $S_j \cap V \to \mathcal{N}$. The section $\sigma \colon V \to \mathcal{N}_V$ gives a compatible choice of lifting for every $j \geq 0$, and we get by completeness of S a morphism $S \to \mathcal{N}$ agreeing with σ on V, as we wished. □

We can now conclude the proof of proposition 5.9.

Proof of proposition 5.9. Let $T \to S$ be a smooth morphism; then $\mathcal{A}_T/T$ is toric-additive by lemma 3.8 and the base change $\mathcal{N}_T/T$ is a test-Néron model. Now, given $\sigma_U : T_U \to \mathcal{A}$, we obtain a section $T_U \to \mathcal{A} \times_U T_U$, which by proposition 5.11 extends to a section $T \to \mathcal{N}_T$. The latter is the datum of an S-morphism $\sigma : T \to \mathcal{N}$ extending σ_U. □

We give a corollary of theorem 5.8.

Corollary 5.12. *Let S be a connected, locally noetherian, regular $\mathbb{Q}$-sccheme, D a regular divisor on S, A an abelian scheme over $U = S \setminus D$ extending to a semi-abelian scheme $\mathcal{A}/S$. Then A admits a Néron model over S.*

Proof. At every geometric point s of S, D has only one irreducible component. It follows that $\mathcal{A}/S$ is toric-additive and we conclude by theorem 5.8. □

Part II

Semi-factorial nodal curves and Néron lft-models

6 Introduction

Let S be the spectrum of a discrete valuation ring with fraction field K, and let $\mathcal{X} \to S$ be a scheme over S. Following [Pép13], we say that $\mathcal{X} \to S$ is *semi-factorial* if the restriction map

$$\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_K)$$

is surjective; namely, if every line bundle on the generic fibre $\mathcal{X}_K$ can be extended to a line bundle on $\mathcal{X}$.

We consider the case of a relative curve $\mathcal{X} \to S$. In [Pép13], Theorem 8.1, Pépin proved that given a geometrically reduced curve $\mathcal{X}_K/K$ with ordinary singularities and a proper flat model $\mathcal{X} \to S$, a semi-factorial flat model $\mathcal{X}' \to S$ can be obtained after a blowing-up $\mathcal{X}' \to \mathcal{X}$ with center in the special fibre.

The main result of this part is a necessary and sufficient condition for semi-factoriality in the case where $\mathcal{X} \to S$ is a proper, flat family of nodal curves, whose special fibre has split nodes. It turns out that in this case semi-factoriality is equivalent to a certain combinatorial condition involving the dual graph of the special fibre of $\mathcal{X}/S$ and a labelling of its edges, which we describe now. Let $t \in \Gamma(S, \mathcal{O}_S)$ be a uniformizer; every node of the special fibre is étale locally described by an equation of the form

a) $xy - t^n = 0$ for some $n \geq 1$, or

b) $xy = 0$ (if the node persists in the generic fibre).

Consider the dual graph $\Gamma = (V, E)$ associated to the special fibre of $\mathcal{X}/S$. We label its edges by the function $l \colon E \to \mathbb{Z}_{\geq 1} \cup \{\infty\}$

$$l(e) = \begin{cases} n & \text{if the node corresponding to } e \text{ is as in case a);} \\ \infty & \text{if the node corresponding to } e \text{ is as in case b).} \end{cases}$$

We say that the labelled graph (Γ, l) is *circuit-coprime* if, after contracting all edges with label ∞, every circuit of the graph has labels with greatest common divisor equal to 1. In particular, if Γ is a tree, (Γ, l) is automatically circuit-coprime.

The following theorem is our main result:

Theorem 6.1 (theorem 12.3)**.** *If the labelled graph* (Γ, l) *is circuit-coprime, the curve* $\mathcal{X} \to S$ *is semi-factorial. If moreover* $\Gamma(S, \mathcal{O}_S)$ *is strictly henselian, the converse holds as well.*

The proof (of the first statement) can be subdivided in three parts:

- we start by constructing a chain of proper birational morphisms of nodal curves over S

$$\ldots \to \mathcal{X}_n \to \mathcal{X}_{n-1} \to \ldots \to \mathcal{X}_1 \to \mathcal{X}_0 := \mathcal{X}$$

  where every arrow is the blowing-up at the reduced closed subscheme of non-regular closed points. A generalization (proposition 9.4) of the smoothening techniques developed in [BLR90], Chapter 3, allows us to show that given a line bundle L on $\mathcal{X}_K$ there exists a positive integer n such that L extends to a line bundle $\mathcal{L}$ on $\mathcal{X}_n$ (theorem 9.5).

- in the combinatorial heart of the proof, we provide a dictionary between geometry and graph theory to reduce the study of the blowing-ups $\mathcal{X}_n$ and line bundles on them to the study of their dual labelled graphs and integer labellings of their edges. We show that if the labelled graph (Γ, l) of $\mathcal{X}/S$ is circuit-coprime, there exists a generically trivial line bundle $\mathcal{E}$ on $\mathcal{X}_n$ such that $\mathcal{L} \otimes \mathcal{E}$ has degree 0 on each irreducible component of the exceptional fibre of $\pi_n \colon \mathcal{X}_n \to \mathcal{X}$.

- Finally, we show (proposition 10.2) that the direct image $\pi_{n*}(\mathcal{L} \otimes E)$ is a line bundle on $\mathcal{X}$ (which in particular extends L). This relies essentially on the fact that the exceptional fibre of π_n is a curve of genus zero.

As a corollary to the theorem, we refine Theorem 8.1 of [Pép13] in the case of nodal curves $\mathcal{X}/S$ with special fibre having split nodes, by explicitly describing a blowing-up with center in the special fibre that yields a semi-factorial model:

Corollary 6.2 (corollary 12.5)**.** *Let* $\mathcal{X}_1 \to \mathcal{X}$ *be the blowing-up centered at the reduced closed subscheme consisting of non-regular closed points of* $\mathcal{X}$*. Then the curve* $\mathcal{X}_1 \to S$ *is semi-factorial.*

This follows immediately, observing that $\mathcal{X}_1$ has circuit-coprime labelled graph.

Semi-factoriality is closely connected to Néron models of jacobians of curves. A famous construction of Raynaud ([Ray70a]) shows that if $\mathcal{X} \to S$ has regular total space, a Néron model over S for the jacobian $\mathrm{Pic}^0_{\mathcal{X}_K/K}$ is given by the S-group scheme $\mathrm{Pic}^{[0]}_{\mathcal{X}/S} / \mathrm{cl}(e)$, where $\mathrm{Pic}^{[0]}_{\mathcal{X}/S}$ represents line bundles of total degree zero on $\mathcal{X}$, and $\mathrm{cl}(e)$ is the schematic closure of the unit section $e\colon K \to \mathrm{Pic}^0_{\mathcal{X}_K/K}$. In [Pép13], Theorem 9.3., it is shown that the same construction works in the case of semi-factorial curves $\mathcal{X} \to S$ with smooth generic fibre. Our second main theorem is a corollary of theorem 6.1:

Theorem 6.3 (theorem 13.6)**.** *Let $\mathcal{X} \to S$ be a nodal curve over the spectrum of a discrete valuation ring. Then $\mathrm{Pic}_{\mathcal{X}/S} / \mathrm{cl}(e)$ is a Néron lft-model over S for $\mathrm{Pic}_{\mathcal{X}_K/K}$ if and only if the labelled graph (Γ, l) is circuit-coprime.*

Note that there are no smoothness assumptions on the generic fibre. The abbreviation "lft" stands for "locally of finite type", meaning that we do not require the model to be quasi-compact (even if we chose to impose degree restrictions on $\mathrm{Pic}_{\mathcal{X}_K/K}$, the resulting Néron lft-model may not be quasi-compact in general, as $\mathcal{X}_K/K$ may not be smooth).

6.1 Outline

In section 7 we introduce the basic definitions, including that of nodal curve with split singularities. In section 8 we define an infinite chain of blow-ups of a given nodal curve $\mathcal{X}/S$ and then show that every line bundle on the generic fibre $\mathcal{X}_K/K$ extends to a line bundle on one of these blow-ups (section 9). Section 10 contains an important technical lemma on descent of line bundles along blowing-ups. Section 11 is entirely graph-theoretic and contains the definition of circuit-coprime labelled graphs. The combinatorial results established in this section are then reinterpreted in section 12 in geometric terms in order to give a necessary and sufficient condition for semi-factoriality of nodal curves. In section 13, starting from a nodal curve $\mathcal{X}/S$, we construct a Néron model of the Picard scheme of its generic fibre.

7 Preliminaries

7.1 Nodal curves

Definition 7.1. A *curve* X over an algebraically closed field k is a proper morphism of schemes $X \to \operatorname{Spec} k$, such that X is connected and whose irreducible components have dimension 1. A curve X/k is called *nodal* if for every non-smooth point $x \in X$ there is an isomorphism of k-algebras $\widehat{\mathcal{O}}_{\mathcal{X},x} \to k[[x,y]]/xy$.

For a general base scheme S, a *nodal curve* $f : \mathcal{X} \to S$ is a proper, flat morphism of finite presentation, such that for each geometric point $\overline{s}$ of S the fibre $\mathcal{X}_{\overline{s}}$ is a nodal curve.

We are interested in the case where the base scheme S is a trait, that is, the spectrum of a discrete valuation ring. In what follows, whenever we have a trait S, unless otherwise specified we will denote by K the fraction field of $\Gamma(S, \mathcal{O}_S)$ and by k its residue field.

Definition 7.2. Let $X \to \operatorname{Spec} k$ be a nodal curve over a field and $n : X' \to X$ be the normalization morphism. A non-regular point $x \in X$ is a *split ordinary double point* if the points of $n^{-1}(x)$ are k-rational (in particular, x is k-rational). We say that $X \to \operatorname{Spec} k$ has *split singularities* if all non-regular points $x \in X$ are split ordinary double points.

It is clear that the base change of a curve with split singularities still has split singularities. Also, it follows from [Liu02], Corollary 10.3.22 that for any nodal curve $\mathcal{X} \to S$ over a trait there exists an étale base change of traits $S' \to S$ such that $\mathcal{X} \times_S S' \to S'$ has split singularities.

The following two lemmas are Corollary 10.3.22 b) and Lemma 10.3.11 of [Liu02]:

Lemma 7.3. *Let $f : \mathcal{X} \to S$ be a nodal curve over a trait and let $x \in \mathcal{X}$ be a split ordinary double point lying over the closed point $s \in S$. Write R for $\Gamma(S, \mathcal{O}_S)$ and $\mathfrak{m}$ for its maximal ideal. Then*

$$\widehat{\mathcal{O}}_{\mathcal{X},x} \cong \frac{\widehat{R}[[x,y]]}{xy - c}$$

for some $c \in \mathfrak{m}R$. The ideal generated by c does not depend on the choice of c.

We define an integer $\tau_x \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$, given by the valuation of c if $c \neq 0$ and by ∞ if $c = 0$. We call τ_x the *thickness* of x. The point x is non-regular if and only if $\tau_x \in \mathbb{Z}_{\geq 2} \cup \{\infty\}$; moreover, $\tau_x = \infty$ if and only if x is the specialization of a node of the generic fibre $\mathcal{X}_K$.

Remark 7.4. If the hypothesis that the special fibre has split singularities is dropped, the same result holds after replacing R and $\mathcal{O}_{\mathcal{X},x}$ by their strict henselizations.

Lemma 7.5. *Let X be a nodal curve over a field k, $x \in X$ a split ordinary double point such that at least two irreducible components of X pass through x. Then x belongs to exactly two irreducible components Z_1, Z_2 which are smooth at x and meet transversally.*

In view of lemma 7.5, if X/k is a nodal curve with split singularities, the *dual graph* G of X can be defined. The vertices of G correspond to the irreducible components of X, while every edge e between vertices v, w corresponds to a an ordinary double point contained in the components corresponding to v and w.

7.2 Semi-factoriality

Definition 7.6 ([Pép13] 1.1.)**.** Let $\mathcal{X} \to S$ be a scheme over a trait. We say that $\mathcal{X}$ is *semi-factorial* over S if the restriction map

$$\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_K)$$

is surjective.

8 Blowing-up nodal curves

Let $f\colon \mathcal{X} \to S$ be a nodal curve over a trait. In this section we study the effects of blowing-up non-regular points of $\mathcal{X}$ lying on the special fibre of $\mathcal{X} \to S$.

8.1 Blowing-up a closed non-regular point

Lemma 8.1. *Let $\mathcal{X} \to S$ be a nodal curve over a trait. Let x be a non-regular point lying on the special fibre of $\mathcal{X} \to S$. The blowing-up $\pi\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ centered at (the reduced closed subscheme given by) x gives by composition a nodal curve $\widetilde{\mathcal{X}} \to S$.*

Proof. The map $\pi\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ is proper, hence so is the composition $\widetilde{\mathcal{X}} \to S$. Let $\overline{x}$ be a geometric point of $\mathcal{X}$ lying over x. We write $A := \widehat{R}^{sh}$ for the completion at its maximal ideal of the strict henselization of R induced by $\overline{x}$. Similarly we let $B := \widehat{\mathcal{O}}^{\acute{e}t}_{\mathcal{X},\overline{x}}$ be the completion of the étale local ring of $\mathcal{X}$ at

$\overline{x}$. We have $B \cong A[[u,v]]/uv - c$ for some $c \in A$; we will assume that $c = 0$, as the reader can refer to [Liu02], Example 8.3.53 for the case $c \neq 0$.

The blowing-up $\mathcal{Z} \to \operatorname{Spec} B$ at the maximal ideal $\mathfrak{m} = (t,u,v) \subset B$ fits in a cartesian diagram

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \widetilde{\mathcal{X}} \\ \downarrow & & \downarrow \pi \\ \operatorname{Spec} B & \longrightarrow & \mathcal{X} \end{array}$$

with flat horizontal maps and is given by

$$\mathcal{Z} = \operatorname{Proj} \frac{B[S,U,V]}{I}$$

where I is the homogenous ideal

$$I = (uS - tU, vS - tV, uV, vU, UV).$$

The scheme $\mathcal{Z}$ is covered by three affine patches, given respectively by the loci where S, U, V are invertible. Namely we have:

$$D^+(S) \cong \operatorname{Spec} \frac{A[U,V]}{UV},\ D^+(U) \cong \operatorname{Spec} \frac{A[[u]][S]}{t - uS},\ D^+(V) \cong \operatorname{Spec} \frac{A[[v]][S]}{t - vS}.$$

To see that $\widetilde{\mathcal{X}}$ is S-flat, we check that the image of the uniformizer $t \in R$ is torsion-free in $\mathcal{O}_{\widetilde{\mathcal{X}}}$, which is immediate upon inspection of the coordinate rings of $D^+(S), D^+(U), D^+(V)$. Also, for all field valued points $y\colon \operatorname{Spec} L \to \operatorname{Spec} A$ lying over the closed point of $\operatorname{Spec} A$, the completed local rings at the singular points of $\mathcal{Z}_y$ are of the form $L[[x,y]]/xy$, as desired.

□

8.2 An infinite chain of blowing-ups

Write now $\mathcal{X}^{nreg}$ for the non-regular locus of $\mathcal{X}$. By the very definition of nodal curve, the locus $\mathcal{X}^{nreg}$ is a closed subset of $\mathcal{X}$, and in particular its intersection with the special fibre $\mathcal{X}_k \cap \mathcal{X}^{nreg}$ is a finite union of closed points. We inductively construct a chain of proper birational maps of nodal curves as follows.

Construction 8.2. Let Y_0 be the closed subscheme given by $\mathcal{X}_k \cap \mathcal{X}^{nreg}$ with its reduced structure. Blowing-up $\mathcal{Y}_0$ in $\mathcal{X}$ we obtain a proper birational morphism $\pi_1\colon \mathcal{X}_1 \to \mathcal{X}$, which restricts to an isomorphism on the dense open

$\mathcal{X} \setminus \mathcal{Y}_0$ and in particular over the generic fibre. For $i \in \mathbb{Z}_{\geq 1}$ we let $Y_i := (\mathcal{X}_i)_k \cap (\mathcal{X}_i)^{nreg}$ with its reduced structure, and define $\mathcal{X}_{i+1} \to \mathcal{X}_i$ to be the blowing-up at Y_i. We obtain a (possibly infinite) chain of proper birational S-morphisms between nodal curves,

$$(\pi_n \colon \mathcal{X}_n \to \mathcal{X}_{n-1})_{n \in \mathbb{Z}_{\geq 1}}, \quad \mathcal{X}_0 := \mathcal{X} \tag{29}$$

which eventually stabilizes if and only if the generic fibre $\mathcal{X}_K$ is regular.

8.3 The case of split singularities

From the calculations of the lemma 8.1 we deduce how blowing-up alters the special fibre of a nodal curve whose special fibre has split singularities. Let $\mathcal{X} \to S$ be such a curve and let $p \in \mathcal{X}$ be a non-regular point of the special fibre. We have $k(p) = k$. Let $\pi \colon \widetilde{\mathcal{X}} \to \mathcal{X}$ be the blow-up at p, $Y = \operatorname{Spec} \widehat{\mathcal{O}}_p$, and $\widetilde{Y} = Y \times_{\mathcal{X}} \widetilde{\mathcal{X}}$. Then $\pi_Y \colon \widetilde{Y} \to Y$ is the blowing-up at the closed point q of Y. Explicit calculations show that the exceptional fibre $\pi_Y^{-1}(q) = \pi^{-1}(p)$ is a chain of projective lines meeting transversally at nodes defined over k.

We now distinguish all possible cases:

- If $\tau_p = \infty$, so that p is the specialization of a node ζ of $\mathcal{X}_K$, $\pi^{-1}(p)$ is given by two copies of $\mathbb{P}^1_k$ meeting at a k-rational node p' with $\tau_{p'} = \infty$;
- if $\tau_p = 2$, $\pi^{-1}(p)$ consists of one $\mathbb{P}^1_k$;
- finally, if $\tau_p > 2$, then $\pi^{-1}(p)$ consists again of two copies of $\mathbb{P}^1_k$, meeting at a k-rational node p' with $\tau_{p'} = \tau_p - 2$.

In all cases, the intersection points between $\pi^{-1}(p)$ and the closure of its complement in $\widetilde{\mathcal{X}}_k$ are regular in $\widetilde{\mathcal{X}}$, that is, they have thickness 1, and are k-rational. Moreover, $\widetilde{\mathcal{X}} \to S$ has special fibre with split singularities.

9 Extending line bundles to blowing-ups of a nodal curve

Our first aim is to prove that for any line bundle L on the generic fibre $\mathcal{X}_K$, there exists an $n \geq 0$ such that L extends to a line bundle on the surface $\mathcal{X}_n$ of the chain of nodal curves (29). In order to do this, we recall and slightly generalize the definition of Néron's measure for the defect of smoothness presented in [BLR90], Chapter 3.

Definition 9.1. Let R be a discrete valuation ring and $\mathcal{Z}$ an R-scheme of finite type. Let $R \to R'$ be a local flat morphism of discrete valuation rings. Let $a \in \mathcal{Z}(R')$ and denote by $\Omega^1_{\mathcal{Z}/R}$ the $\mathcal{O}_{\mathcal{Z}}$-module of R-differentials. The pullback $a^*\Omega^1_{\mathcal{Z}/R}$ is a finitely-generated R'-module, thus a direct sum of a free and a torsion sub-module. We define *Néron's measure for the defect of smoothness* of $\mathcal{Z}$ along a as

$$\delta(a) := \text{length of the torsion part of } a^*\Omega^1_{\mathcal{Z}/R}$$

Remark 9.2. In [BLR90] 3.3, the measure for the defect of smoothness is defined for points with values in the strict henselization R^{sh} of R (which amounts to considering only local étale morphisms $R \to R'$). We allow more general maps because we will need them in the proof of theorem 9.5.

The following two lemmas generalize two analogous results in [BLR90] 3.3, concerning Néron's measure for the defect of smoothness to the case of points $a \in \mathcal{Z}(R')$ with R' a (possibly ramified) local flat extension of R. In the following lemma, we denote by $\mathcal{Z}^{sm}$ the S-smooth locus of $\mathcal{Z}$.

Lemma 9.3. *Let R be a discrete valuation ring and $\mathcal{Z}$ an R-scheme of finite type. Let $a \in \mathcal{Z}(R')$ for some local flat extension $R \to R'$ of discrete valuation rings. Assume that the restriction to the generic fibre $a_{K'} \colon \operatorname{Spec} K' \to \mathcal{Z}_{K'}$ factors through the smooth locus $\mathcal{Z}^{sm}_{K'}$ of $\mathcal{Z}_{K'}$. Then*

$$\delta(a) = 0 \Leftrightarrow a \in \mathcal{Z}^{sm}(R')$$

Proof. See [BLR90] 3.3/1, for a proof in the case of smooth generic fibre and $R \to R'$ a local étale map of discrete valuation rings. The same proof works for non-smooth generic fibre, as long as a_K factors through $\mathcal{Z}^{sm}$. Now notice that $a^*\Omega_{\mathcal{Z}/R} \cong (a')^*\Omega_{\mathcal{Z}_{R'}/R'}$, where $a' \colon \operatorname{Spec} R' \to \mathcal{Z}_{R'}$ is the section induced by a. We conclude by the fact that the smooth locus of $\mathcal{Z}/R$ is preserved under the faithfully flat base change $\operatorname{Spec} R' \to \operatorname{Spec} R$. □

Proposition 9.4. *Let R be a discrete valuation ring, $\mathcal{Z}/R$ a nodal curve, $f \colon R \to R'$ a finite locally free extension of discrete valuation rings with ramification index $r \in \mathbb{Z}_{\geq 1}$. Suppose $a \in \mathcal{Z}(R')$ is such that the restriction to the generic fibre $a_{K'}$ factors through the smooth locus of $\mathcal{Z}_K$, and that the restriction to the special fibre a_k is contained in the non-regular locus $\mathcal{Z}^{nreg}$. Let $\pi \colon \widetilde{\mathcal{Z}} \to \mathcal{Z}$ be the blowing-up at the closed point $p = a \cap \mathcal{Z}_k$ with its reduced structure and denote by $\widetilde{a} \in \widetilde{\mathcal{Z}}(R')$ the unique lifting of a to $\widetilde{\mathcal{Z}}$. Then, either $\widetilde{a}$ is contained in the regular locus of $\widetilde{\mathcal{Z}}$, or*

$$\delta(\widetilde{a}) \leq \max(\delta(a) - r, 0).$$

Proof. For $R' = R$, proposition 9.4 is a particular case of [BLR90] 3.6/3. The strategy of the proof is to reduce to this case.

Denote by t a uniformizer for R, and by u a uniformizer for R', with $u^r = t$ in R'. Since $\mathcal{Z}(R') = \mathcal{Z}_{R'}(R')$ the section a can be interpreted as a section $b \in \mathcal{Z}_{R'}(R')$. Because $\Omega^1_{\mathcal{Z}_{R'}/R'} \cong \Omega^1_{\mathcal{Z}/R} \otimes_R R'$, we have $\delta(a) = \delta(b)$. The flat map $f\colon R \to R'$ induces a cartesian diagram

$$\begin{array}{ccc} \widetilde{\mathcal{Z}}_{R'} & \longrightarrow & \widetilde{\mathcal{Z}} \\ \Big\downarrow{\scriptstyle \pi_{R'}} & & \Big\downarrow{\scriptstyle \pi} \\ \mathcal{Z}_{R'} & \xrightarrow{\ g\ } & \mathcal{Z} \end{array}$$

where $\pi_{R'}\colon \widetilde{\mathcal{Z}}_{R'} \to \mathcal{Z}_{R'}$ is the blowing-up of the preimage $g^{-1}(p)$ of p via $g\colon \mathcal{Z}_{R'} \to \mathcal{Z}$. Then the lifting $\widetilde{a} \in \mathcal{Z}(R')$ factors via the unique lifting of b to $\widetilde{b} \in \widetilde{\mathcal{Z}}_{R'}(R')$. All we need to prove is that $\delta(\widetilde{b}) \leq \max\{\delta(b) - r, 0\}$. We may work locally around p, and assume $\mathcal{Z} = \operatorname{Spec} A$ for some R-algebra A, and write $\mathcal{Z}_{R'} = \operatorname{Spec} B$ with $B = A \otimes_R R'$. By restricting $\mathcal{Z}$, we may also assume that p is the only non-smooth point of $\mathcal{Z}$. We let $(t, x_1, \ldots, x_n) \subset A$ be the maximal ideal corresponding to p. The ideal of the closed subscheme $g^{-1}(p) \subset \mathcal{Z}_{R'} = \operatorname{Spec} B$ is then $I = (u^r, x_1, \ldots, x_n) \subset B$, so in particular $g^{-1}(p)$ is a non-reduced point for $r > 1$.

We want to decompose the blowing-up $\pi_{R'}\colon \widetilde{\mathcal{Z}}_{R'} \to \mathcal{Z}_{R'}$ into a chain of r blowing-ups and then apply to each of these the known case described in the beginning. We construct the chain as follows: we first blow up the ideal $I_1 = (u, x_1, \ldots, x_n) \subset B$ and obtain a blowing-up map $\mathcal{Z}_1 \to \mathcal{Z}_{R'}$. The scheme $\mathcal{Z}_1$ is a closed subscheme of $\mathbb{P}^n_B$, whose defining homogeneous ideal is the kernel of the map of graded B-algebras

$$B[u^{(1)}, x_1^{(1)}, \ldots, x_n^{(1)}] \to \oplus_{d \geq 0} I_1^d$$

given by sending $u^{(1)}$ to u and $x_i^{(1)}$ to x_i for all $i = 1, \ldots, n$. The locus $D^+(u^{(1)}) \subset \mathcal{Z}_1$ where $u^{(1)}$ does not vanish is affine, and we denote it by $\mathcal{Y}_1$. We blow up its closed subscheme given by the ideal $(u, x_1^{(1)}/u^{(1)}, x_2^{(1)}/u^{(1)}, \ldots, x_n^{(1)}/u^{(1)})$, and obtain a map

$$\mathcal{Z}_2 \to \mathcal{Y}_1.$$

Next we consider the affine $\mathcal{Y}_2 := D^+(u^{(2)}) \subset \mathcal{Z}_2$ and reiterating the procedure r times, we end up with a chain of morphisms

$$\mathcal{Y}_r \to \mathcal{Y}_{r-1} \to \ldots \to \mathcal{Y}_1 \to \mathcal{Z}_{R'}$$

of affine schemes.

Every blow-up $\mathcal{Y}_{i+1} \to \mathcal{Y}_i$ is the blow-up at a closed point, with reduced structure. Moreover, by the description in section 8.3, we can see that every $\mathcal{Y}_i$ has only one non-regular point p_i in the special fibre; working étale locally one sees that $\mathcal{Y}_{i+1} \to \mathcal{Y}_i$ is exactly the blowing-up at p_i.

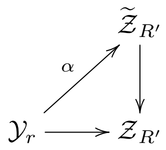
Let's now relate this chain of maps to the blowing-up $\widetilde{\mathcal{Z}}_{R'} \to \mathcal{Z}_{R'}$ given by the ideal $(u^r, x_1, \dots, x_n)$. Combining the relations

$$\frac{x_i^{(j-1)}}{u^{(j-1)}} u^{(j)} = u x_i^{(j)}$$

for all $j = 1, \dots, r$ (where we also set $x_i^{(0)} := x_i$ and $u^{(0)} := u$), we obtain in $\mathcal{Y}_r$ the equality

$$x_i = \frac{x_i^{(r)}}{u^{(r)}} u^r$$

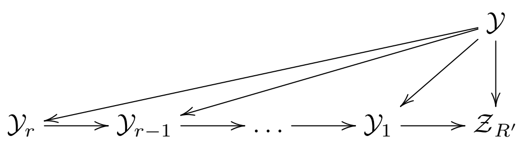
for all $i = 1, \dots, n$. Hence the ideal sheaf $(u^r, x_1, \dots, x_n)$ on $\mathcal{Z}_{R'}$ has preimage in $\mathcal{Y}_r$ which is free of rank 1, generated by u^r. By the universal property of blowing-up we obtain a unique map $\alpha\colon \mathcal{Y}_r \to \widetilde{\mathcal{Z}}_{R'}$ such that the diagram



commutes. Next, we focus on the blow-up map $\widetilde{\mathcal{Z}}_{R'} \to \mathcal{Z}_R$. The scheme $\widetilde{\mathcal{Z}}_{R'}$ is a closed subscheme of $\mathbb{P}^n_B$, whose defining homogeneous ideal is the kernel of the map of graded B-algebras

$$B[v, y_1, \dots, y_n] \to \oplus_{d \geq 0} I^d$$

given by sending v to u^r and y_i to x_i for all $i = 1, \dots, n$. So we have relations $v x_i = u^r y_i$ for all $i = 1, \dots, n$. Then the map $\alpha^*\colon \mathcal{O}_{\widetilde{\mathcal{X}}_{R'}} \to \mathcal{O}_{\mathcal{Y}_r}$ sends y_i to $x_i^{(r)}$ and v to $u^{(r)}$. We restrict our attention to the open affine $\mathcal{Y} \subset \widetilde{\mathcal{Z}}_{R'}$ where v does not vanish. Since v is mapped by α^* to $u^{(r)}$, which does not vanish on $\mathcal{Y}_r$, the map α factors as a map $\alpha'\colon \mathcal{Y}_r \to \mathcal{Y}$ followed by the inclusion $\mathcal{Y} \subset \widetilde{\mathcal{Z}}_{R'}$. Now we produce an inverse to α'. One checks that the ideal sheaf $(u, x_1, \dots, x_n)$ of $\mathcal{Z}_{R'}$ becomes free in $\mathcal{Y}$ (generated by u), hence we obtain a unique map $\mathcal{Y} \to \mathcal{Y}_1$ compatible with the maps to $\mathcal{Z}_{R'}$. Then the argument can be reiterated to produce a commutative diagram



In particular we obtain a map $\beta\colon \mathcal{Y} \to \mathcal{Y}_r$. It is an easy check that the maps α' and β produced between $\mathcal{Y}$ and $\mathcal{Y}_r$ are inverse one to another, hence they give an isomorphism $\mathcal{Y}_r \to \mathcal{Y}$.

If we let b_i be the unique lift to $\mathcal{Y}_i$ of $b_0 := b\colon R' \to \mathcal{Z}_{R'}$, wither b_i is in the regular locus of $\mathcal{Y}_i$, or $\mathcal{Y}_{i+1} \to \mathcal{Y}_i$ is the blowing-up at $b_i \cap (Y_i)_k$, in which case we obtain, by [BLR90] 3.3/5, that $\delta(b_{i+1}) \leq \max\{\delta(b_i) - 1, 0\}$. Now, if for some $1 \leq i \leq r$ the section b_i is contained in the regular locus of $\mathcal{Y}_i$, then also $\widetilde{b}$ is contained in the regular locus of $\mathcal{Y}$. Otherwise, $\delta(\widetilde{b}) \leq \max\{\delta(b) - r, 0\}$ as desired.

□

We now have the tools to prove our main result on extending line bundles to blowing-ups in the chain of morphisms (29).

Theorem 9.5. *Let S be a trait, with perfect fraction field K, $\mathcal{X}/S$ a nodal curve. Let L be a line bundle on $\mathcal{X}_K$. Let $(\pi_i\colon \mathcal{X}_i \to \mathcal{X}_{i-1})_i$ be the chain* (29) *of blow-ups. Then there exists $N \geq 0$ for which L extends to a line bundle $\mathcal{L}$ on $\mathcal{X}_N$.*

Proof. Let L be an invertible sheaf on $\mathcal{X}_K$, and D be a Cartier divisor with $\mathcal{O}_{\mathcal{X}_K}(D) \cong L$. We may take D to be supported on the smooth locus of $\mathcal{X}_K$ ([Sha13], Theorem 1.3.1) and see it as a Weil divisor. We may also assume that D is effective, since any Weil divisor is the difference of two effective Weil divisors.

The closed subscheme D_{red} given by the support of D with its reduced structure is a disjoint union of finitely many closed points of the smooth locus of $\mathcal{X}_K$. We write

$$D_{red} = \bigcup_{i=1}^{s} P_i$$

where $P_i \in \mathcal{X}_K^{sm}(K_i)$ for finite (separable) extensions $K \hookrightarrow K_i$, $i = 1, \ldots, s$. For each $i = 1, \ldots, s$, we let R_i be the localization at some prime of the integral closure of R in K_i, so that each R_i is a discrete valuation ring with fraction field K_i, and $R \to R_i$ is finite locally free. The curve $\mathcal{X}/R$ being proper, each P_i extends to $Q_i \in \mathcal{X}(R_i)$. Write $\mathcal{X}^{nsm}$ for the non-smooth locus of $\mathcal{X}/R$ and $\mathcal{X}^{nreg}$ for the non-regular locus of $\mathcal{X}$. Notice that $\delta(Q_i) > 0$ if and only if $Q_i \cap \mathcal{X}_k \in \mathcal{X}^{nsm}$, by lemma 9.3. Assume that the point $Q_i \cap \mathcal{X}_k$ lies in $\mathcal{X}^{nreg} \subset \mathcal{X}^{nsm}$. In this case, it is one of the closed points that are the center of the blowing-up $\mathcal{X}_1 \to \mathcal{X}$. By proposition 9.4, the unique lifting Q_i' of Q_i to $\mathcal{X}_1$ either is contained in the regular locus of $\mathcal{X}_1$, or it satisfies $\delta(Q_i') \leq \max(0, \delta(Q_i) - r_i)$, where $r_i \geq 1$ is the ramification index of $R \to R_i$. Applying repeatedly proposition 9.4, we see that there is $N > 0$ such that each of the points $P_i \in \mathcal{X}_K$ extends to $Q_i^{(N)} \in \mathcal{X}_N^{reg}(R_i)$. Therefore the Weil divisor D extends to a Weil divisor $\widetilde{D}$ on $\mathcal{X}_N$ that is supported on the union of the $Q_i^{(N)}$, hence on the regular locus of $\mathcal{X}_N$. This implies that $\widetilde{D}$ is a Cartier divisor, and the line bundle $\mathcal{O}_{\mathcal{X}_n}(\widetilde{D})$ restricts to $\mathcal{O}_{\mathcal{X}_K}(D) \cong L$ on $\mathcal{X}_K$. This completes the proof.

□

10 Descent of line bundles along blowing-ups

Lemma 10.1. *Let S be a trait and $\pi\colon \mathcal{Y} \to \mathcal{X}$ a proper morphism of flat S-schemes, which restricts to an isomorphism over the generic point of S. Assume that the special fibre $\mathcal{X}_k$ is reduced. Then $\pi_*\mathcal{O}_{\mathcal{Y}} \cong \mathcal{O}_{\mathcal{X}}$.*

Proof. Consider an affine open $W \subset \mathcal{X}$. The morphism $\mathcal{O}_{\mathcal{X}}(W) \to \pi_*\mathcal{O}_{\mathcal{Y}}(W)$ is integral ([Liu02], Prop.3.3.18). Denoting by t a uniformizer of $\Gamma(S, \mathcal{O}_S)$, we have a commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X}}(W) & \longrightarrow & \pi_*\mathcal{O}_{\mathcal{Y}}(W) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{X}}(W)[t^{-1}] & \xrightarrow{\cong} & (\pi_*\mathcal{O}_{\mathcal{Y}}(W))[t^{-1}] \end{array}$$

The two vertical arrows are injective because $\mathcal{X}$ and $\mathcal{Y}$ are S-flat; the lower arrow is an isomorphism because π is generically an isomorphism and $(\pi_*\mathcal{O}_{\mathcal{Y}}(W))[t^{-1}] = \pi_*(\mathcal{O}_{\mathcal{Y}}(W)[t^{-1}])$. It follows that the upper arrow is injective. We claim that $\mathcal{O}_{\mathcal{X}}(W)$ is integrally closed in $\mathcal{O}_{\mathcal{X}}(W)[t^{-1}]$, so that the upper arrow is an isomorphism, which proves the lemma. Take then $g \in \mathcal{O}_{\mathcal{X}}(W)[t^{-1}]$ satisfying a monic polynomial equation $g^m + a_1 g^{m-1} + \ldots + a_m = 0$ with coefficients in $\mathcal{O}_{\mathcal{X}}(W)$ and write $g = f/t^n$ with $f \in \mathcal{O}_{\mathcal{X}}(W)$ and $n \geq 0$ minimal. We want to show that n is zero. We have

$$\frac{f^m}{t^{nm}} + a_1 \frac{f^{m-1}}{t^{n(m-1)}} + \ldots + a_m = 0.$$

Suppose by contradiction $n \geq 1$. Upon multiplying by t^{nm} the above relation, we find that $f^m \in t\mathcal{O}_{\mathcal{X}}(W)$. Because the special fibre of $\mathcal{X}$ is reduced, the ring $\mathcal{O}_{\mathcal{X}}(W)/t\mathcal{O}_X(W)$ is reduced, hence $f \in t\mathcal{O}_{\mathcal{X}}(W)$. This violates the hypothesis of minimality of n and we have a contradiction. Hence $n = 0$ and $g \in \mathcal{O}_{\mathcal{X}}(W)$, proving the claim.

□

Proposition 10.2. *Let S be a trait, $\mathcal{X}/S$ a nodal curve, and $\mathcal{Y} \to \mathcal{X}$ the blowing-up of $\mathcal{X}$ at a closed point $p \in \mathcal{X}$. Let $\mathcal{L}$ be a line bundle on $\mathcal{Y}$ such that its restriction to every irreducible component of the exceptional locus of π has degree zero. Then $\pi_*\mathcal{L}$ is a line bundle on $\mathcal{X}$.*

Proof. We first consider the case where S is the spectrum of a strictly henselian discrete valuation ring. In this case, the special fibre of $\mathcal{X} \to S$ has split singularities, hence, as seen in section 8.3, the exceptional fibre E of $\mathcal{Y} \to \mathcal{X}$ consists either of a projective line, or of two projective lines meeting at a k-rational node.

The sheaf $\pi_*\mathcal{L}$ is a coherent $\mathcal{O}_\mathcal{X}$-module. Since the curve $\mathcal{X}$ is reduced, to show that $\pi_*\mathcal{L}$ is a line bundle it is enough to check that $\dim_{k(x)} \pi_*\mathcal{L} \otimes_{\mathcal{O}_\mathcal{X}} k(x) = 1$ for all $x \in \mathcal{X}$. This clearly holds for $x \in \mathcal{X}$ different from p. We remain with the case $x = p$. Denote by $\mathcal{O}_p$ the local ring of $\mathcal{X}$ at p. Let

$$\mathcal{Z} := \mathcal{Y} \times_\mathcal{X} \operatorname{Spec} \mathcal{O}_p$$

so that $\mathcal{Z}$ is the blow-up of $\operatorname{Spec} \mathcal{O}_p$ at its closed point. We write $\mathcal{I}$ for the ideal sheaf $\mathfrak{m}_p \mathcal{O}_\mathcal{Z} \subset \mathcal{O}_\mathcal{Z}$. For every $n \geq 1$ define

$$\mathcal{Z}_n := \mathcal{Y} \times_\mathcal{X} \operatorname{Spec} \mathcal{O}_p/\mathfrak{m}_p^n$$

so we have $\mathcal{O}_{\mathcal{Z}_n} = \mathcal{O}_\mathcal{Z}/\mathcal{I}^n$. In particular, $\mathcal{Z}_1$ is the exceptional fibre of the blowing-up $\mathcal{Z} \to \operatorname{Spec} \mathcal{O}_p$, which coincides with the exceptional fibre E of $\pi\colon \mathcal{Y} \to \mathcal{X}$.

The formal function theorem tells us that there is a natural isomorphism

$$\Phi\colon \lim_n (\pi_*\mathcal{L}) \otimes_{\mathcal{O}_\mathcal{X}} \mathcal{O}_p/\mathfrak{m}_p^n \to \lim_n H^0(\mathcal{Z}_n, \mathcal{L}_{|\mathcal{Z}_n}).$$

We claim that $\mathcal{L}_{|\mathcal{Z}_n}$ is trivial for all $n \geq 1$. We start with the case $n = 1$: the dual graph of the curve $\mathcal{Z}_1$ is a tree, hence $\operatorname{Pic}(\mathcal{Z}_1)$ is the product of the Picard groups of the components of $\mathcal{Z}_1$ (this can be checked via the Mayer-Vietoris sequence for $\mathcal{O}^\times$, for example). In other words, a line bundle on $\mathcal{Z}_1$ is determined by its restrictions to the components of $\mathcal{Z}_1$. As $\operatorname{Pic}(\mathbb{P}^1_k) = \mathbb{Z}$ via the degree map, we have $\mathcal{L}_{|\mathcal{Z}_1} = \mathcal{O}_{\mathcal{Z}_1}$. Now let $n \geq 1$ and assume that $\mathcal{L}_{|\mathcal{Z}_n}$ is trivial. There is an exact sequence of sheaves of groups on $\mathcal{Z}$

$$0 \to \mathcal{I}^n/\mathcal{I}^{n+1} \to (\mathcal{O}_\mathcal{Z}/\mathcal{I}^{n+1})^\times \to (\mathcal{O}_\mathcal{Z}/\mathcal{I}^n)^\times \to 1$$

with the first map sending α to $1 + \alpha$. The ideal sheaf $\mathcal{I}$ is canonically isomorphic to the invertible sheaf $\mathcal{O}_\mathcal{Z}(1)$. Hence $\mathcal{I}^n/\mathcal{I}^{n+1} = \mathcal{O}_{\mathcal{Z}_1}(n)$. Taking the long exact sequence of cohomology we obtain

$$H^1(\mathcal{Z}_1, \mathcal{O}_{\mathcal{Z}_1}(n)) \to H^1(\mathcal{Z}_{n+1}, \mathcal{O}^\times_{\mathcal{Z}_{n+1}}) \to H^1(\mathcal{Z}_n, \mathcal{O}^\times_{\mathcal{Z}_n}) \to 0.$$

We find that the term $H^1(\mathcal{Z}_1, \mathcal{O}_{\mathcal{Z}_1}(n))$ vanishes using Mayer-Vietoris exact sequence and the fact that $H^1(\mathbb{P}^1_k, \mathcal{O}_{\mathbb{P}^1_k}(n)) = 0$. It follows that the restriction map $\mathrm{Pic}(\mathcal{Z}_{n+1}) \to \mathrm{Pic}(\mathcal{Z}_n)$ is an isomorphism. Since the sheaf $\mathcal{L}_{|\mathcal{Z}_{n+1}}$ restricts to the trivial sheaf on $\mathcal{Z}_n$, it is itself trivial, establishing the claim.

We obtain

$$\lim_n H^0(\mathcal{Z}_n, \mathcal{L}_{|\mathcal{Z}_n}) \cong \lim_n H^0(\mathcal{Z}_n, \mathcal{O}_{\mathcal{Z}_n}) \cong \lim_n (\pi_* \mathcal{O}_{\mathcal{Y}}) \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_p/\mathfrak{m}_p^n \cong \widehat{\mathcal{O}}_p$$

the second isomorphism coming again from the formal function theorem applied to $\mathcal{O}_{\mathcal{Y}}$ and the third coming from lemma 10.1. Finally, we obtain by composition with Φ an isomorphism

$$\lim_n (\pi_* \mathcal{L}) \otimes_{\mathcal{O}} \mathcal{O}_p/\mathfrak{m}_p^n \to \widehat{\mathcal{O}}_p$$

which induces an isomorphism $\pi_*\mathcal{L} \otimes_{\mathcal{O}} \mathcal{O}_p/\mathfrak{m}_p \to \mathcal{O}_p/\mathfrak{m}_p = k(p)$, as desired.

Now we drop the assumption of strict henselianity on the base, so let S be the spectrum of a discrete valuation ring. Let S' be the étale local ring of S with respect to some separable closure of the residue field of S. The cartesian diagram

$$\begin{array}{ccc} \mathcal{Y}_{S'} & \xrightarrow{f} & \mathcal{Y} \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \mathcal{X}_{S'} & \xrightarrow{g} & \mathcal{X} \end{array}$$

has faithfully flat horizontal arrows, and $\mathcal{Y}_{S'} \to \mathcal{X}_{S'}$ is the blowing-up at $g^{-1}(p)$. Let $\mathcal{L}$ be a line bundle on $\mathcal{Y}$ as in the hypotheses. The restrictions of $f^*\mathcal{L}$ to the irreducible components of the exceptional fibre of π' have degree zero, hence $\pi'_* f^* \mathcal{L}$ is a line bundle. Moreover the canonical map

$$g^* \pi_* \mathcal{L} \to \pi'_* f^* \mathcal{L}$$

is an isomorphism, because g is flat. Hence $g^*\pi_*\mathcal{L}$ is a line bundle, and so is $\pi_*\mathcal{L}$ by faithful flatness of g. □

11 Graph theory

In this section we develop some graph-theoretic results that, together with the results of sections 9 and 10, will be needed to prove theorem 12.3.

11.1 Labelled graphs

Let $G = (V, E)$ be a connected, finite graph. For the whole of this section, we will just write "graph" to mean finite, connected graph. A *circuit* in G is a closed walk in G all of whose edges and vertices are distinct except for the first and last vertex. A *path* is an open walk all of whose edges and vertices are distinct.

A *tree* of G is a connected subgraph $T \subset G$ containing no circuit. A *spanning tree* of G is a tree of G containing all of the vertices of G, that is, a maximal tree of G. Given a spanning tree $T \subset G$, we call *links* the edges not belonging to T.

Let $n = |E|$, $m = |V|$. Given a spanning tree T, the number of links of T is easily seen to be $n - m + 1$. The number

$$r := n - m + 1$$

is called *nullity* of G and is equal to the first Betti number $\operatorname{rk} H^1(G, \mathbb{Z})$.

Fix a spanning tree $T \subset G$. For each link $c_1, \ldots, c_r$ of T, the subgraph $T \cup c_i$ contains exactly one circuit $C_i \subset G$. We call $C_1, \ldots, C_r$ *fundamental circuits* of G (with respect to T).

Let $(G, l) = (V, E, l)$ be the datum of a graph and of a labelling of the edges $l \colon E \to \mathbb{Z}_{\geq 1}$ by positive integers. We say that (G, l) is a $\mathbb{N}$-*labelled graph.*

11.2 Circuit matrices

Given a graph G, let $e_1, e_2, \ldots, e_n$ be its edges and $\gamma_1, \ldots, \gamma_s$ its circuits. Fix an arbitrary orientation of the edges of G, and an orientation of each circuit (that is, one of the two travelling directions on the closed walk).

Definition 11.1. The *circuit matrix* of G is the $s \times n$ matrix M_G whose entries a_{ij} are defined as follows:

$$a_{ij} = \begin{cases} 0 & \text{if the edge } e_j \text{ is not in } \gamma_i; \\ 1 & \text{if the edge } e_j \text{ is in } \gamma_i \text{ and its orientation agrees} \\ & \text{with the orientation of } \gamma_i; \\ -1 & \text{if the edge } e_j \text{ is in } \gamma_i \text{ and its orientation does not agree} \\ & \text{with the orientation of } \gamma_i. \end{cases}$$

Hence every row of M_G corresponds to a circuit of G and each column to an edge.

Now fix a spanning tree of G. Let $c_1, \dots, c_r$ be the corresponding links, where r is the nullity of G, and $C_1, \dots, C_r$ the associated fundamental circuits. Consider the $r \times n$ submatrix N_G of M_G given by singling out the rows corresponding to fundamental circuits. One can reorder edges and circuits so that the j-th column corresponds to the link c_j for $1 \leq j \leq r$ and that the i-th row corresponds to the circuit C_i. If we also choose the orientation of every fundamental circuit C_i so that it agrees with the orientation of the link c_i, the matrix N_G has the form

$$N_G = [\mathbb{I}_r | N']$$

where $\mathbb{I}_r$ is the identity $r \times r$-matrix and N' is an integer matrix.

Definition 11.2. The matrix N_G constructed above is called the *fundamental circuit matrix* of G (with respect to the spanning tree T).

It is clear that N_G has rank r.

Theorem 11.3 ([TS92], Theorem 6.7.)**.** *The rank of M_G is equal to the rank of N_G.*

Let now (G, l) be an $\mathbb{N}$-labelled graph. We generalize the definitions above to this case.

Definition 11.4. The *labelled circuit matrix* of (G, l) is the $s \times n$ matrix $M_{(G,l)}$ whose entries b_{ij} are defined as follows:

$$b_{ij} = \begin{cases} 0 & \text{if the edge } e_j \text{ is not in } \gamma_i; \\ l(e_j) & \text{if the edge } e_j \text{ is in } \gamma_i \text{ and its orientation agrees} \\ & \text{with the orientation of } \gamma_i; \\ -l(e_j) & \text{if the edge } e_j \text{ is in } \gamma_i \text{ and its orientation does not agree} \\ & \text{with the orientation of } \gamma_i. \end{cases}$$

The *labelled fundamental circuit (lfc) matrix* of (G, l) is the $r \times n$ matrix $N_{(G,l)}$ constructed from $M_{(G,l)}$ by taking only the rows corresponding to fundamental circuits with respect to a given spanning tree T.

We immediately see that

$$M_{(G,l)} = M_G \cdot L \text{ and } N_{(G,l)} = N_G \cdot L$$

where L is the diagonal square matrix of order n whose (i, i)-th entry is $l(e_i)$.

Example 11.5. Consider the $\mathbb{N}$-labelled graph (G, l) with oriented edges in fig. 1.

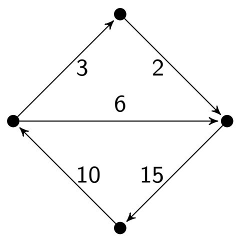


Figure 1: An oriented $\mathbb{N}$-labelled graph (G, l)

We assign to each of its three circuits the clockwise travelling direction. We obtain a circuit matrix of G and a labelled circuit matrix of (G, l):

$$M_G = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \end{bmatrix} \qquad M_{(G,l)} = \begin{bmatrix} 3 & 2 & -6 & 0 & 0 \\ 0 & 0 & 6 & 15 & 10 \\ 3 & 2 & 0 & 15 & 10 \end{bmatrix}$$

Choose the spanning tree with edges labelled by $3, 6$ and 10. The fundamental circuit matrix of G and lfc-matrix of (G, l) are obtained from M_G and $M_{(G,l)}$ by removing the third row:

$$N_G = \begin{bmatrix} 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad N_{(G,l)} = \begin{bmatrix} 3 & 2 & -6 & 0 & 0 \\ 0 & 0 & 6 & 15 & 10 \end{bmatrix}$$

Let M be an integer-valued matrix with a rows and b columns. There exist matrices $A \in \mathrm{GL}(a, \mathbb{Z})$ and $B \in \mathrm{GL}(b, \mathbb{Z})$ such that

$$AMB = \begin{bmatrix} d_1 & 0 & 0 & & \dots & & 0 \\ 0 & d_2 & 0 & & \dots & & 0 \\ 0 & 0 & \ddots & & & & 0 \\ \vdots & & & d_k & & & \vdots \\ & & & & 0 & & \\ & & & & & \ddots & \\ 0 & & & \dots & & & 0 \end{bmatrix}$$

where the diagonal entries satisfy $d_i | d_{i+1}$ for $i = 1, \dots, k-1$. This is the so-called *Smith normal form* of M and it is unique up to multiplication of the diagonal entries by units of $\mathbb{Z}$. For $1 \leq i \leq k$, the integer d_i is the quotient D_i/D_{i-1}, where D_i equals the greatest common divisor of all minors of order i of M.

Going back to the matrices $M_{(G,l)}$ and its submatrix $N_{(G,l)}$, it follows from theorem 11.3 that their Smith normal forms both have rank equal to the nullity r of the graph G. Besides, as any row of $M_{(G,l)}$ is a $\mathbb{Z}$-linear combination of rows of $N_{(G,l)}$, we see that the numbers D_i defined above are the same for the two matrices. It follows that $M_{(G,l)}$ and $N_{(G,l)}$ have the same non-zero numbers d_i appearing on the diagonal. Moreover, the numbers $d_1, \dots, d_r$ are defined up to multiplication by -1, hence do not depend on the choices of orientation of edges or circuits, but only on the $\mathbb{N}$-labelled graph (G, l).

11.3 Cartier labellings and blow-up graphs

Let (G, l) be an $\mathbb{N}$-labelled graph. Let $\mathbb{Z}^V$ be the free abelian group generated by the set of vertices V. Any element φ of $\mathbb{Z}^V$ can be interpreted as a vertex labelling $\varphi\colon V \to \mathbb{Z}$ of the graph G.

Definition 11.6. An element $\varphi \in \mathbb{Z}^V$ is a *Cartier vertex labelling* if for every edge $e \in E$ with endpoints $v, w \in V$, $l(e)$ divides $\varphi(v) - \varphi(w)$.

We denote by $\mathcal{C} \subset \mathbb{Z}^V$ the subgroup of Cartier vertex labellings.

Definition 11.7. We call *multidegree operator* the group homomorphism $\delta\colon \mathcal{C} \to \mathbb{Z}^V$ which sends $\varphi \in \mathcal{C}$ to

$$v \mapsto \sum_{\substack{\text{edges } e \\ \text{incident to } v}} \frac{\varphi(w) - \varphi(v)}{l(e)}$$

where w denotes the other endpoint of e (which is v itself if e is a loop).

Lemma 11.8. *The kernel of δ consists of the constant vertex labellings, hence there is an exact sequence*

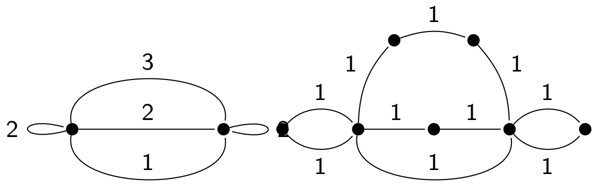
$$0 \to \mathbb{Z} \to \mathcal{C} \xrightarrow{\delta} \mathbb{Z}^V.$$

Proof. Any constant vertex labelling is in the kernel of δ. Conversely, let $\varphi \in \ker \delta$ and let $v \in V$ be a vertex where φ attains its maximum. Then for all the vertices w adjacent to v one has $\varphi(w) = \varphi(v)$. Since the graph is finite and connected, one can repeat the argument and find that φ is a constant labelling. □

Remark 11.9. When the edge-labelling $l\colon E \to \mathbb{Z}_{\geq 1}$ is constant with value 1, the multidegree operator δ coincides with the Laplacian operator of the graph G.

Definition 11.10. Given an $\mathbb{N}$-labelled graph $(G, l) = (V, E, l)$ we define the *total blow-up graph* $(\widetilde{G}, \widetilde{l}) = (\widetilde{V}, \widetilde{E}, \widetilde{l})$ to be the $\mathbb{N}$-labelled graph constructed as follows starting from (G, l): every edge $e \in E$ is replaced by a path consisting of $l(e)$ edges, and $\widetilde{l}\colon \widetilde{E} \to \mathbb{Z}$ is set to be the constant labelling with value 1.

Example 11.11. Figure 2 shows an $\mathbb{N}$-labelled graph (a) and its total blow-up graph (b).



(a) (b)

Figure 2: An $\mathbb{N}$-labelled graph G (a) and its total blow-up graph $\widetilde{G}$ (b).

We call *old vertices* the vertices in the image of the inclusion map $V \hookrightarrow \widetilde{V}$. We call *new vertices* the remaining vertices.

Notice that every new vertex is incident to exactly two edges, and belongs to a unique path (corresponding to some edge $e \in E$) connecting two old vertices of $\widetilde{V}$. Just as before we consider the group of Cartier vertex labellings $\widetilde{\mathcal{C}}$ of $(\widetilde{G}, \widetilde{l})$, and the multidegree operator $\widetilde{\delta}\colon \widetilde{\mathcal{C}} \to \mathbb{Z}^{\widetilde{V}}$.

We obtain a morphism of exact sequences

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{C} & \xrightarrow{\delta} & \mathbb{Z}^V \\ & & \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle \epsilon} \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \widetilde{\mathcal{C}} & \xrightarrow{\widetilde{\delta}} & \mathbb{Z}^{\widetilde{V}} \end{array} \tag{30}$$

The map $\epsilon\colon \mathbb{Z}^V \to \mathbb{Z}^{\widetilde{V}}$ is given by extending vertex-labellings by zero on the set of new vertices. The map $\iota\colon \mathcal{C} \to \widetilde{\mathcal{C}}$ sends a Cartier vertex labelling φ on G to the Cartier vertex labelling $\iota(\varphi)$ on $\widetilde{G}$ whose value at old vertices in inherited by φ, and extended by linear interpolation to the new vertices.

More precisely: if e is an edge of G with endpoints v, w which is replaced in $\widetilde{G}$ by a path consisting of vertices $v = v_0, v_1, \ldots, v_{l(e)} = w$, we set for each $k = 0, \ldots, l(e)$

$$\iota(\varphi)(v_k) = \frac{(l(e) - k)\varphi(v) + k\varphi(w)}{l(e)}.$$

The Cartier condition on φ implies that this labelling takes integer values.

Let $H = \operatorname{coker} \delta$, $\widetilde{H} = \operatorname{coker} \widetilde{\delta}$. The commutative diagram above yields a group homomorphism $\overline{\epsilon}\colon H \to \widetilde{H}$.

Lemma 11.12. *The group homomorphism $\overline{\epsilon}\colon H \to \widetilde{H}$ is injective.*

Proof. Let $\alpha \in \mathbb{Z}^V$ be a vertex labelling and let $\epsilon(\alpha) \in \mathbb{Z}^{\widetilde{V}}$ be its extension by zero. Assume that there exists a Cartier vertex labelling $\widetilde{\varphi} \in \widetilde{\mathcal{C}}$ such that $\epsilon(\alpha) = \widetilde{\delta}(\widetilde{\varphi})$. Then $\widetilde{\delta}(\widetilde{\varphi})$ takes value zero on all new vertices of $\widetilde{G}$. Hence, if v is a new vertex of $\widetilde{G}$ adjacent to two verteces v' and v'', we have $\widetilde{\varphi}(v') - \widetilde{\varphi}(v) = \widetilde{\varphi}(v) - \widetilde{\varphi}(v'')$. We immediately see that $\widetilde{\varphi}$ is an interpolation of a Cartier vertex labelling $\varphi \in \mathcal{C}$, i.e. $\widetilde{\varphi}$ is in the image of ι. Since $\epsilon\colon \mathbb{Z}^V \to \mathbb{Z}^{\widetilde{V}}$ is injective, $\alpha = \delta(\varphi)$. □

Our aim now is to give necessary and sufficient conditions on the $\mathbb{N}$-labelled graph (G, l) for the map $\overline{\epsilon}\colon H \to \widetilde{H}$ to be surjective (hence an isomorphism).

11.4 Circuit-coprime graphs

Definition 11.13. Let $(G, l) = (V, E, l)$ be an $\mathbb{N}$-labelled graph. We say that (G, l) is *circuit-coprime* if for every circuit $C \subset G$, $\gcd\{l(e) | e \text{ is an edge of } C\} = 1$.

Example 11.14. In fig. 3 the $\mathbb{N}$-labelled graph (a) is circuit-coprime, whereas the $\mathbb{N}$ -labelled graph (b) is not, as it contains a loop labelled by 3 in addition to a circuit labelled by $6, 10$ and 10.

Lemma 11.15. *Let $(G, l) = (V, E, l)$ be an $\mathbb{N}$-labelled graph. Denote by r its nullity. The Smith normal form of the matrix $M_{(G,l)}$ has diagonal entries $d_1 = d_2 = \ldots = d_r = 1$ if and only if (G, l) is circuit-coprime.*

Proof. Assume first that (G, l) is not circuit-coprime. Let C be a circuit whose labels have greatest common divisor $D \neq 1$. Pick an edge e of C. The subgraph $C \setminus e$ is a tree; let T be a spanning tree of G containing it. Then e is a link for T, and C is its associated fundamental circuit. The lfc-matrix $N_{(G,l)}$ has a row corresponding to the circuit C, hence all entries of this row are divisible

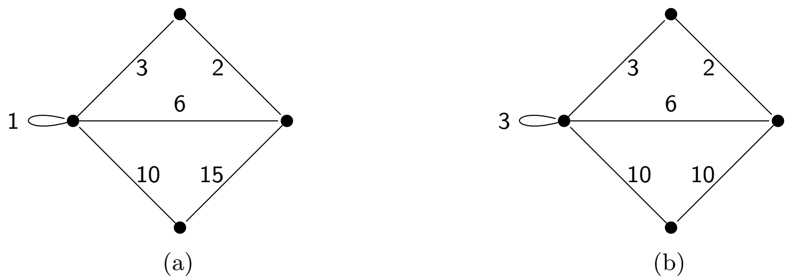


Figure 3: A circuit-coprime $\mathbb{N}$-labelled graph (a) and an $\mathbb{N}$ -labelled graph that is not circuit-coprime (b).

by D. Then the linear map $f\colon \mathbb{Z}^n \to \mathbb{Z}^r$ defined by $N_{(G,l)}$ is not surjective; hence the linear map associated to the Smith normal form of $N_{(G,l)}$ is not surjective either. Therefore, some (necessarily non-zero) diagonal entry of the Smith normal form of $N_{(G,l)}$ is different from ± 1. As previously remarked, the Smith normal forms of $M_{(G,l)}$ and $N_{(G,l)}$ have the same non-zero diagonal entries, hence $d_r \neq \pm 1$.

Conversely, assume that G is circuit-coprime. After fixing some spanning tree T, consider the lfc-matrix $N_{(G,l)}$. We only need to prove that the diagonal entries of the Smith normal form of $N_{(G,l)}$ are all 1, which amounts to proving that the greatest common divisor d of the minors of order r of the lfc-matrix $N_{(G,l)}$ is 1.

As we have seen in section 11.2, we have the relation

$$N_{(G,l)} = N_G \cdot L.$$

Let N' be a maximal square submatrix of $N_{(G,l)}$. Then N' corresponds to r edges of G, which we denote $e_{i_1}, e_{i_2}, \ldots, e_{i_r}$. Let N'' be the corresponding square submatrix of N_G. We have the relation

$$\det N' = \prod_{j=1}^{r} l(e_{i_j}) \det N''$$

By [TS92], Theorem 6.15, all minors of N_G are either $1, 0$ or -1, hence $\det N''$ is either $1, 0$ or -1. Moreover, by [TS92], Theorem 6.10, a square submatrix of order r of N_G has determinant ± 1 if and only if the corresponding r edges are the complement of a spanning tree. Hence $\det N' = \pm \prod_{i=1}^{r} l(e_{i_j})$ if the edges $e_{i_1}, e_{i_2}, \ldots, e_{i_r}$ form the complement of a spanning tree of G, otherwise $\det N' = 0$. We claim that

$$d := \gcd\{\det N' | N' \text{ is an } r \times r \text{ square submatrix of } N_{(G,l)}\} = 1.$$

Let p be a prime number and denote by E_p the set of edges e of G whose label $l(e)$ is divisible by p. Because (G, l) is circuit-coprime, E_p contains no circuit; hence E_p is contained in some spanning tree T of G. There are exactly r edges, $e_1, e_2, \ldots, e_r$, that do not belong to T. These give a square $r \times r$ submatrix of $N_{(G,l)}$ whose determinant is $\prod_{i=1}^{r} l(e_i) \not\equiv 0 \pmod{p}$, since $e_1, \ldots, e_r \notin E_p$. Hence $p \nmid d$. It follows that $d = 1$; since $d_i | d_{i+1}$ for all $i = 1, \ldots, r-1$ and $d_r | d$, we obtain the result. □

Proposition 11.16. *Let $(G, l) = (V, E, l)$ be an $\mathbb{N}$-labelled graph. The group homomorphism $\overline{\epsilon}\colon H \to \widetilde{H}$ is an isomorphism if and only if (G, l) is circuit-coprime.*

Proof. We already know that $\overline{\epsilon}\colon H \to \widetilde{H}$ is injective by lemma 11.12. It is surjective if and only if for every vertex-labelling $\alpha \in \mathbb{Z}^{\widetilde{V}}$, there exists $\widetilde{\varphi} \in \widetilde{\mathcal{C}}$ such that $\widetilde{\delta}(\widetilde{\varphi}) + \alpha$ is in the image of the extension-by-zero map $\epsilon\colon \mathbb{Z}^V \to \mathbb{Z}^{\widetilde{V}}$, i.e. $\widetilde{\delta}(\widetilde{\varphi}) + \alpha$ is supported on the set of old vertices. We may of course assume that α belongs to the canonical basis of $\mathbb{Z}^{\widetilde{V}}$. That is, $\alpha = \chi_v$ for some vertex v of $\widetilde{G}$, where

$$\chi_v(w) = \begin{cases} 1 & \text{if } w = v \\ 0 & \text{if } w \neq v. \end{cases}$$

If v is an old vertex of $\widetilde{G}$, χ_v is an extension by zero of a vertex-labelling on G, so we may assume that v is a new vertex. Then v belongs to some path $P \subset \widetilde{G}$ associated to some edge $\overline{e} \in E$. Denote by $w_0, w_1, \ldots, w_{l(\overline{e})}$ the vertices of the path P, so that w_0 and $w_{l(\overline{e})}$ are old vertices, and the numbering of the indices follows the order of the vertices on the path. For every $i = 1, \ldots, l(\overline{e})$, let $\alpha_i = \chi_{w_i} - \chi_{w_0} \in \mathbb{Z}^{\widetilde{V}}$ be the vertex-labelling that has value 1 at w_i, value -1 at w_0, and value 0 everywhere else. Then it is easy to check that the images $\overline{\alpha}_i$ of the α_i in $\widetilde{H}$ satisfy $k\overline{\alpha}_1 = \overline{\alpha}_k$ for all $k = 1, \ldots, l(\overline{e})$. Hence, if $\overline{\alpha_1}$ is in the image of $\overline{\epsilon}\colon H \to \widetilde{H}$, so are all the $\overline{\alpha_i}$ for $1 \leq i \leq l(e)$. This shows that we can take v to be equal to w_1; hence $\chi_v = \chi_{w_1}$ takes value 1 on a new vertex v adjacent to an old vertex, and value zero at all other vertices.

We ask whether an element $\widetilde{\varphi} \in \widetilde{\mathcal{C}}$ exists such that $\widetilde{\delta}(\widetilde{\varphi}) + \chi_{w_1}$ is supported only on the old vertices. In other words, $\widetilde{\delta}(\widetilde{\varphi})$ must be zero on all new vertices except for the vertex w_1, where it has to take the value -1. This is equivalent to asking that, for every new vertex z, adjacent to vertices z_1, z_2,

$$\begin{cases} \widetilde{\varphi}(z) - \widetilde{\varphi}(z_1) = \widetilde{\varphi}(z_2) - \widetilde{\varphi}(z) & \text{if } z \neq w_1 \\ (\widetilde{\varphi}(z_1) - \widetilde{\varphi}(z)) + (\widetilde{\varphi}(z_2) - \widetilde{\varphi}(z)) = -1 & \text{if } z = w_1 \end{cases} \tag{31}$$

holds.

We claim that such a $\widetilde{\varphi}$ exists if and only if there exists a vertex-labelling φ of the graph G, such that, for every edge $e \in E$ with endpoints v_0, v_1,

$$\begin{cases} \varphi(v_1) - \varphi(v_0) \equiv 0 \bmod l(e) & \text{if } e \neq \overline{e} \\ \varphi(v_1) - \varphi(v_0) \equiv 1 \bmod l(e) & \text{if } e = \overline{e}, v_0 = w_0, v_1 = w_{l(e)} \end{cases} \tag{32}$$

where we have identified the old vertices $w_0, w_{l(e)}$ with the corresponding vertices in G. Indeed, given $\widetilde{\varphi}$ one obtains φ simply by restriction to old vertices. Conversely, given a φ as in (37), $\widetilde{\varphi}$ is obtained as follows: for an edge $e \neq \overline{e}$, we define $\widetilde{\varphi}$ on the corresponding path $\{z_0 = v_0, z_1, z_2, \ldots, z_{l(e)} = v_1\}$ by:

$$\forall k = 0, 1, \ldots, l(e), \quad \widetilde{\varphi}(z_k) = \frac{k\varphi(v_1) + (l(e) - k)\varphi(v_0)}{l(e)}.$$

On the path $\{w_0 = v_0, w_1, \ldots, w_{l(\overline{e})} = v_1\}$ corresponding to the edge $\overline{e}$, we set instead

$$\widetilde{\varphi}(w_k) = \begin{cases} \frac{k\varphi(v_1) + (l(e) - k)(\varphi(v_0) + 1)}{l(e)} & \text{if } k \in \{1, 2, \ldots, l(\overline{e})\} \\ \widetilde{\varphi}(v_0) & \text{if } k = 0; \end{cases}$$

which establishes the claim.

If the graph G is a tree it is clear that such a φ can be found. If there are circuits in G, the existence of a solution φ depends of course on the labels of the circuits. Fix an orientation on G, so that we have source and target functions $s, t \colon E \to V$, and so that $s(\overline{e}) = w_0$, $t(\overline{e}) = w_{l(\overline{e})}$. Assume that a vertex-labelling φ of G satisfying the conditions (37) exists. In particular we have that $\varphi(t(\overline{e})) - \varphi(s(\overline{e})) \equiv 1 \bmod l(\overline{e})$. For every edge $e \in E$ let

$$x(e) := \begin{cases} \frac{\varphi(t(e)) - \varphi(s(e))}{l(e)} & \text{if } e \neq \overline{e} \\ \frac{\varphi(t(e)) - \varphi(s(e)) - 1}{l(e)} & \text{if } e = \overline{e} \end{cases}$$

Let $C \subset G$ be a circuit consisting of vertices $v_0, v_1, \ldots, v_s = v_0$ connected by edges $e_0, e_1, e_2, \ldots, e_s = e_0$, so that e_i connects v_i and v_{i+1} for every $i \in \mathbb{Z}/s\mathbb{Z}$. Notice that the increasing numbering gives an orientation to C. We have

$$(\varphi(v_s) - \varphi(v_{s-1})) + (\varphi(v_{s-1}) - \varphi(v_{s-2})) + \ldots + (\varphi(v_1) - \varphi(v_s)) = 0.$$

Setting

$$a_i = \begin{cases} 1 & \text{if } t(e_i) = v_{i+1}, s(e_i) = v_i \\ -1 & \text{if } t(e_i) = v_i, s(e_i) = v_{i+1} \end{cases} \tag{33}$$

for every $i \in \mathbb{Z}/s\mathbb{Z}$, we obtain

$$\sum a_i x_{e_i} l(e_i) = 0$$

if the edge $\overline{e}$ does not belong to the circuit C, whereas if $\overline{e} \in C$ we have

$$\sum a_i x_{e_i} l(e_i) = \begin{cases} -1 & \text{if the orientations of } C \text{ and } \overline{e} \text{ agree;} \\ 1 & \text{if the orientations of } C \text{ and } \overline{e} \text{ do not agree;} \end{cases}$$

Let $C_1, \ldots, C_m$ be the circuits of G. Choose an orientation for each circuit, so that we can form the labelled circuit matrix $M_{(G,l)}$ associated to G. We see that the vector $\underline{x} = (x_1, \ldots, x_n)$ is a solution of

$$M_{(G,l)} x = b(\overline{e})$$

where $b(\overline{e}) = (b_1, \ldots, b_m)$ with

$$b_i = \begin{cases} 0 & \text{if } \overline{e} \notin C_i; \\ -1 & \text{if } \overline{e} \in C_i \text{ and the orientation of } \overline{e} \text{ agrees with the} \\ & \text{orientation of } C_i; \\ 1 & \text{if } \overline{e} \in C_i \text{ and the orientation of } \overline{e} \text{ does not agree with the} \\ & \text{orientation of } C_i. \end{cases}$$

Conversely, a solution $x \in \mathbb{Z}^n$ to the system $M_{(G,l)} x = b(\overline{e})$ yields a vertex labelling φ as in (37). We conclude that the map $\overline{\epsilon} \colon H \to \widetilde{H}$ is surjective if and only if for every edge $e \in E$, there is a solution $x \in \mathbb{Z}^n$ to

$$M_{(G,l)} x = b(e).$$

After having chosen a spanning tree T and formed the lfc-matrix $N_{(G,l)}$, this is in turn equivalent to the map $\mathbb{Z}^n \to \mathbb{Z}^r$ defined by $N_{(G,l)}$ being surjective. Indeed, the set $\{b(e) | e \text{ is a link of } T\}$ is a basis for $\mathbb{Z}^r$. Now, $N_{(G,l)}$ is surjective if and only if its Smith normal form (or equivalently the one of $M_{(G,l)}$) has only 1's on the diagonal. By lemma 11.15, we conclude.

□

11.5 $\mathbb{N}_\infty$-labelled graphs

We want to generalize the results of the previous subsection to labelled graphs whose labels can attain the value ∞. Denote by $\mathbb{N}_\infty$ the set $\mathbb{Z}_{\geq 1} \cup \{\infty\}$. Let $(G, l) = (V, E, l)$ be the datum of a graph, with set of vertices V and set of edges E, and of a function $l \colon E \to \mathbb{N}_\infty$. We say that (G, l) is an $\mathbb{N}_\infty$-*labelled graph.*

The notions of Cartier vertex labelling 11.6 and multidegree operator 11.7 carry over to this setting without change, imposing that the only integer divisible by ∞ is 0, and setting $\frac{0}{\infty} = 0$ in the definition of multidegree operator.

In particular, if a vertex-labelling on (G, l) is Cartier, it attains the same value at the two extremal vertices of an edge with label ∞.

Definition 11.17. Given an $\mathbb{N}_\infty$-labelled graph $(G, l) = (V, E, l)$ we define the *first-blow-up graph* $G_1 = (V_1, E_1, l_1)$ to be the $\mathbb{N}_\infty$-labelled graph constructed as follows starting from (G, l): every edge $e \in E$ with $l(e) = 1$ is preserved unaltered; every edge $e \in E$ with $l(e) \geq 2$ is replaced by a path consisting of an edge labelled by 1, followed by an edge labelled by $l(e) - 2$ (which could equal 0 or ∞), followed by an edge labelled by 1.

We define inductively for every integer $n \geq 1$ the *n-th blow-up graph* $G_n = (V_n, E_n, l_n)$ as the first-blow-up graph of G_{n-1}.

Example 11.18. Figure 4 shows an $\mathbb{N}_\infty$-labelled graph (a) with its first (b) and second (c) blow-up graphs.

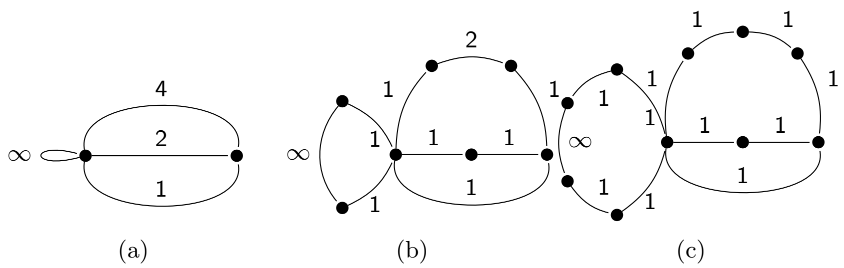


Figure 4: An $\mathbb{N}_\infty$-labelled graph (a) with its first (b) and second (c) blow-up graphs

Denote by $\mathcal{C}_n$ the group of Cartier vertex-labellings on (G_n, l_n). Just as in

(30), we obtain a commutative diagram

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{\delta} & \mathbb{Z}^V \\
\downarrow{\scriptstyle \iota_1} & & \downarrow{\scriptstyle \epsilon_1} \\
\mathcal{C}_1 & \xrightarrow{\delta_1} & \mathbb{Z}^{V_1} \\
\downarrow{\scriptstyle \iota_2} & & \downarrow{\scriptstyle \epsilon_2} \\
\vdots & & \vdots \\
\downarrow & & \downarrow \\
\mathcal{C}_n & \xrightarrow{\delta_n} & \mathbb{Z}^{V_n} \\
\downarrow{\scriptstyle \iota_n} & & \downarrow{\scriptstyle \epsilon_n} \\
\vdots & & \vdots
\end{array}$$

The vertical maps ϵ_j are once again extension by zero; the maps ι_j are defined as follows: if e is an edge of G_{j-1} which is replaced in G_j by a path consisting of vertices $v_0 = v, v_1, v_2, v_3 = w$ (with possibly $v_1 = v_2$, if $l_{j-1}(e) = 2$), and φ is Cartier vertex labelling on G_{j-1}, we set $\iota_j(\varphi)$ to take the value $\varphi(v)$ at v_0, $\frac{(l(e)-1)\varphi(v)+\varphi(w)}{l(e)}$ at v_1, $\frac{\varphi(v)+(l(e)-1)\varphi(w)}{l(e)}$ at v_2, $\varphi(w)$ at v_3. The diagram above gives rise to a chain of group homomorphisms

$$H \to H_1 \to H_2 \to \ldots \to H_n \to \ldots \tag{34}$$

between the cokernels of the rows. Each map of the chain (34) is injective; we ask whether they are all isomorphisms, i.e. under which conditions

$$H \to \operatorname{colim} H_i \tag{35}$$

is an isomorphism.

Definition 11.19. Let $(G, l) = (V, E, l)$ be an $\mathbb{N}_\infty$-labelled graph. We let $(G, l^\circ) = (V, E, l^\circ)$ be the $\mathbb{N}_0 := \mathbb{Z}_{\geq 0}$-labelled graph obtained from (G, l) by setting $l^\circ(e) = 0$ for all edges e with label $l(e) = \infty$.

We say that (G, l) is *circuit-coprime* if for every circuit $C \subset G$,

$$\gcd(l^\circ(e) | e \text{ is an edge of } C) = 1.$$

Here we define the gcd of a subset $S \subset \mathbb{Z}$ to be the non-negative generator of the ideal $\langle S \rangle \subset \mathbb{Z}$.

Remark 11.20. An $\mathbb{N}_\infty$-labelled graph containing a circuit whose labels are all ∞ is not circuit-coprime. Indeed, $\gcd(0) = 0$.

Proposition 11.21. *Let (G, l) be an $\mathbb{N}_\infty$-labelled graph. The map (35) is an isomorphism if and only if (G, l) is circuit-coprime.*

Proof. Instead of (G, l) and its blow-up graphs $(G_1, l_1), (G_2, l_2), \ldots$ we consider $(G, l^\circ), (G_1, l_1^\circ), (G_2, l_2^\circ), \ldots$. We keep the same notion of Cartier vertex labelling and multidegree operator, by imposing that the only integer divisible by 0 is 0, and that $0/0 = 0$. The chain of homomorphisms 34 is also preserved. To keep the notation light, we drop the $^\circ$'s. From now on, the proof is a readaptation of the content of section 11.4. First, for labelled graphs whose labels attain the value 0, we define the labelled circuit matrix $M_{(G,l)}$ and labelled fundamental circuit matrix $N_{(G,l)}$ in the same way as in section 11.2. Lemma 11.15 stays true in this setting, so we find that (G, l) is circuit-coprime if and only if $N_{(G,l)}$ is surjective.

To finish the proof we only need to readapt proposition 11.16 to our new setting. So, we want to show that $N_{(G,l)}$ is surjective if and only if $\epsilon_n \colon H \to H_n$ is surjective for all $n \geq 1$. We fix an integer n big enough, so that all labels of G_n are 1's or 0's. As in proposition 11.16, we let $\alpha \in \mathbb{Z}^{V_n}$; we may pick $\alpha = \chi_v$ for some vertex v belonging to some path $P \subset G_n$ associated to some edge $\overline{e} \in E$. Denote by $w_0, w_1, \ldots, w_r$ the vertices of the path P. We may still assume that $v = w_1$. Indeed, if there is no edge in P labelled by zero, one reasons as in proposition 11.16; otherwise, if there is an edge in P labelled by 0, then it has to be the edge connecting w_s and w_{s+1}, with $s = \frac{r-1}{2}$. We may assume without loss of generality that $v = w_k$ for $k \leq s$. We get that $\overline{\chi_{w_k}} = k\overline{\chi_{w_1}}$ in H_n (as always, compare with proposition 11.16).

An element $\tilde{\varphi}$ in $\mathcal{C}_n$ is such that $\delta_n(\varphi_n) + \chi_{w_1}$ is supported on the old vertices is a vertex-labelling $\tilde{\varphi} \in Z^{V_n}$ satisfying the following: for every new vertex z, adjacent to vertices z_1 and z_2,

$$\begin{cases} \widetilde{\varphi}(z) - \widetilde{\varphi}(z_1) = \widetilde{\varphi}(z_2) - \widetilde{\varphi}(z) & \text{if } z \neq w_1 \\ (\widetilde{\varphi}(z_1) - \widetilde{\varphi}(z)) + (\widetilde{\varphi}(z_2) - \widetilde{\varphi}(z)) = -1 & \text{if } z = w_1 \end{cases} \tag{36}$$

That such a Cartier vertex-labelling exists means that there is a vertex-labelling $\tilde{\varphi}$ satisfying the condition 36 above, plus the extra condition that $\tilde{\varphi}(z_1) = \tilde{\varphi}(z_2)$ for any two adjacent vertices z_1, z_2 connected by an edge labelled by zero.

In turn, such a $\widetilde{\varphi}$ exists if and only if there exists a vertex-labelling φ of G such that, for every edge $e \in E$ with endpoints v_0, v_1,

$$\begin{cases} \varphi(v_1) - \varphi(v_0) \equiv 0 \bmod l(e) & \text{if } e \neq \overline{e} \\ \varphi(v_1) - \varphi(v_0) \equiv 1 \bmod l(e) & \text{if } e = \overline{e}, v_0 = w_0, v_1 = w_r \end{cases} \tag{37}$$

where we have identified the old vertices w_0, w_r with the corresponding vertices in G. This is the same condition as condition 37 in proposition 11.16. From this point on, the rest of the proof coincides with the proof of proposition 11.16; we only mention that, at the point when $x(e)$ is defined, one can assign to it any value if $l(e) = 0$.

□

12 Semi-factoriality of nodal curves

Let S be the spectrum of a discrete valuation ring R having perfect fraction field K, residue field k and uniformizer t. Let $f\colon \mathcal{X} \to S$ be a nodal curve whose special fibre has split singularities, and $\Gamma = (V, E)$ be the dual graph of the special fibre $\mathcal{X}_k$. For any $v \in V$, we denote by X_v the corresponding irreducible component of the special fibre $\mathcal{X}_k$.

Definition 12.1. The *labelled graph* of $\mathcal{X} \to S$ is the $\mathbb{N}_\infty$-labelled graph (Γ, l) whose labelling l assigns to each edge of Γ the thickness (see section 7.1) of the corresponding singular point of $\mathcal{X}_k$.

Our aim is to relate the property of being circuit-coprime for the graph (Γ, l) to the semi-factoriality of $f\colon \mathcal{X} \to S$. To this end, we are going to provide a dictionary between the geometry of $\mathcal{X}/S$ and the combinatorial objects introduced in section 11.

Denote by $\mathrm{Div}_k(\mathcal{X})$ the group of Weil divisors on $\mathcal{X}$ supported on the special fibre $\mathcal{X}_k$. It is the free abelian group generated by the irreducible components of $\mathcal{X}_k$. Hence we obtain a natural isomorphism $\mathrm{Div}_k(\mathcal{X}) \to \mathbb{Z}^V$.

Let $\mathcal{C}(\mathcal{X})$ be the group of Cartier divisors on $\mathcal{X}$ whose restriction to the generic fibre $\mathcal{X}_K$ is trivial. We claim that the natural map $\mathcal{C}(\mathcal{X}) \to \mathrm{Div}_k(\mathcal{X})$ is injective. This follows from ([GD67], 21.6.9 (i)) under the assumption that $\mathcal{X}$ is normal, which is not satisfied if $\mathcal{X}/S$ has singular generic fibre. However, the proof only requires that for all $x \in \mathcal{X}_k$, $\mathrm{depth}(\mathcal{O}_{\mathcal{X},x}) = 1$ implies $\dim \mathcal{O}_{\mathcal{X},x} = 1$. This is immediately checked: let $x \in \mathcal{X}_k$ with $\dim \mathcal{O}_{\mathcal{X},x} \neq 1$; then x is a closed point of $\mathcal{X}_k$. By S-flatness of $\mathcal{X}$, the uniformizer t is not a zero divisor in $\mathcal{O}_{\mathcal{X},x}$; as $\mathcal{X}_k$ is reduced, $\mathcal{O}_{\mathcal{X},x}/t\mathcal{O}_{\mathcal{X},x}$ is reduced. Every reduced noetherian ring of dimension 1 is Cohen-Macaulay, hence $\mathrm{depth}(\mathcal{O}_{\mathcal{X},x}/t\mathcal{O}_{\mathcal{X},x}) = 1$, and we deduce by [Sta16]TAG 0AUI that $\mathrm{depth}(\mathcal{O}_{\mathcal{X},x}) = 2$, establishing the claim. Hence $\mathcal{C}(\mathcal{X})$ is in a natural way a subgroup of $\mathrm{Div}_k(\mathcal{X})$.

Finally, denote by $E(\mathcal{X})$ the kernel of the restriction map $\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_K)$, so that $E(\mathcal{X})$ is the group of isomorphism classes of line bundles on $\mathcal{X}$ that are generically trivial. We have an exact sequence of groups

$$0 \to \mathbb{Z} \to \mathcal{C}(\mathcal{X}) \to E(\mathcal{X}) \to 0$$

where the first map sends 1 to $\operatorname{div}(t)$ and the second map sends D to $\mathcal{O}_{\mathcal{X}}(D)$. Indeed, every principal Cartier divisor supported on the special fibre belongs to $\mathbb{Z}\operatorname{div}(t)$. For this we can reduce to showing that every regular function on $\mathcal{X}$ that is generically invertible is of the form $t^n u$ for some $n \in \mathbb{Z}_{\geq 0}$ and $u \in \mathcal{O}_{\mathcal{X}}(\mathcal{X})^\times$. By [Sta16]TAG 0AY8 we have $f_*\mathcal{O}_{\mathcal{X}} = \mathcal{O}_S$, from which the claim easily follows.

Lemma 12.2. *Hypotheses as in the beginning of this section.*

i) The natural isomorphism $\operatorname{Div}_k(\mathcal{X}) \to \mathbb{Z}^V$ *identifies* $\mathcal{C}(\mathcal{X}) \subset \operatorname{Div}_k(\mathcal{X})$ *with the subgroup* $\mathcal{C} \subset \mathbb{Z}^V$ *of Cartier vertex labellings (definition 11.6).*

Let

$$0 \to \mathbb{Z} \to \mathcal{C} \xrightarrow{\delta} \mathbb{Z}^V$$

be the exact sequence of lemma 11.8, where δ *is the multi-degree operator (definition 11.7).*

ii) The isomorphism $\mathcal{C}(\mathcal{X}) \to \mathcal{C}$ *induces an exact sequence*

$$0 \to \mathbb{Z} \to \mathcal{C}(\mathcal{X}) \xrightarrow{\delta_{\mathcal{X}}} \mathbb{Z}^V.$$

The first arrow is the map $1 \mapsto \operatorname{div}(t)$*; the map* $\delta_{\mathcal{X}}$ *factors via the map* $E(\mathcal{X}) \to \mathbb{Z}^V$*, which sends a line bundle* $\mathcal{L}$ *to the vertex labelling*

$$v \mapsto \deg \mathcal{L}_{|X_v}.$$

Let

$$\ldots \to \mathcal{X}_n \to \ldots \to \mathcal{X}_1 \to \mathcal{X}_0 = \mathcal{X}$$

be the chain of blowing-ups (29)*. Denote by* π_n *the composition* $\mathcal{X}_n \to \mathcal{X}$*.*

iii) For every $n \geq 0$ *the labelled graph of* $\mathcal{X}_n \to S$ *is the* n*-th blow-up graph* (Γ_n, l_n) *of* (Γ, l) *(definition 11.17). The new vertices of* (Γ_n, l_n) *correspond to the irreducible components of the exceptional fibre of* $\mathcal{X}_n \to \mathcal{X}$*.*

iv) Let $\mathcal{C}_n$ *be the group of Cartier vertex labellings on* $\mathcal{X}_n$*. The map* $\mathcal{C}(\mathcal{X}) \to \mathcal{C}(\mathcal{X}_n)$ *induced by* $\iota\colon \mathcal{C} \to \mathcal{C}_n$ *(section 11.3) descends to the pullback map* $\pi_n^*\colon E(\mathcal{X}) \to E(\mathcal{X}_n)$*.*

Proof.

i) Let $D = \sum_v n_v X_v \in \operatorname{Div}_k(\mathcal{X})$. We want to show the equivalence of the two conditions:

a) for every node $p \in \mathcal{X}_k$ lying on distinct components X_w, X_z of $\mathcal{X}_k$, the thickness τ_p divides $n_w - n_z$ (with the convention that ∞ divides only 0);

b) D is Cartier.

As every Weil divisor D is Cartier on the generic fibre and on the regular locus of $\mathcal{X}$, we may fix a node $p \in \mathcal{X}_k$ and reduce to work on the complete local ring $\widehat{\mathcal{O}}_{\mathcal{X},p}$. We identify $\widehat{\mathcal{O}}_{\mathcal{X},p}$ with $A = \widehat{R}[[x,y]]/xy - t^{\tau_p}$. Let X_w and X_z be the components of $\mathcal{X}_k$ through p, and let Y_w, Y_z be their preimages in $\operatorname{Spec} A$, which are given by the ideals (x,t) and (y,t) of A respectively.

Assume a) is true; we are going to deduce that D is Cartier at p. We may assume that the two components X_w and X_z are distinct, otherwise D is given by $\operatorname{div}(t^{n_w})$ locally at p and is automatically Cartier at p. As $\operatorname{div}(x) = \tau_p Y_w$, we have that $(n_w - n_z)Y_w = \operatorname{div}(x^{\frac{n_w - n_z}{\tau_p}})$ is Cartier. Therefore $D - \operatorname{div}(t^{n_z}) = \sum_v (n_v - n_z)X_v$ is Cartier at p, and also D is.

Assume now b) and that p lies on distinct components X_w, X_z of $\mathcal{X}_k$. We may assume that the restriction of D to $\operatorname{Spec} A$, $n_w Y_w + n_z Y_z$, is the divisor of some regular function $f \in A = \widehat{R}[[x,y]]/xy - t^{\tau_p}$. We first consider the case $\tau_p = \infty$. As f is a unit in $A[t^{-1}]$, there exists $g \in A$ and $n \geq 0$ such that $fg = t^n$. Now, let f_x be the image of f in A/xA. As the latter is a unique factorization domain, $f_x = t^{m_1}u_1$ for some unit $u_1 \in (A/xA)^\times$ and $m_1 \leq n$. Moreover, we have $m_1 = n_w$. Similarly, we write $f_y = t^{m_2}u_2 \in A/yA$, with $m_2 = n_z$. As the images of f_x and f_y in $A/(x,y)A = R$ coincide, we find that $m_1 = m_2$, that is, $n_w = n_z$, as desired. Now we remain with the case $\tau_p \neq \infty$. Replacing f by ft^{-n_z}, we get $\operatorname{div}(f) = (n_w - n_z)Y_w$. We want to show that τ_p divides $m := n_w - n_z$. Let $d = \gcd(m, \tau_p)$. As $\operatorname{div}(x) = \tau_p Y_w$, we may replace f by a product of powers of f and x and assume that $m = d$. Write $\tau_p = m\alpha$, for some $\alpha \in \mathbb{Z}$. We have $\operatorname{div}(f^\alpha/x) = 0$, hence, as $\operatorname{Spec} A$ is normal, f^α/x is a unit in A. Now, reducing modulo t, one can easily see that α has to be 1, so $m = \tau_p$ as desired.

ii) The composition $\mathbb{Z} \to \mathcal{C} \to \mathcal{C}(\mathcal{X})$ sends 1 to $\sum_v X_v = \mathcal{X}_k = \operatorname{div}(t)$. The map $\delta_{\mathcal{X}}$ factors via the cokernel of $\mathbb{Z} \to \mathcal{C}(\mathcal{X})$, which is indeed $E(\mathcal{X})$. For the characterization of the map $\delta_{\mathcal{X}}$, recall first that $\delta\colon \mathcal{C} \to \mathbb{Z}^V$ sends a Cartier vertex labelling φ to the vertex labelling

$$v \mapsto \sum_{\substack{\text{edges } e \\ \text{incident to } v}} \frac{\varphi(w) - \varphi(v)}{l(e)}$$

where w denotes the other endpoint of e. The composition $\delta_{\mathcal{X}}\colon \mathcal{C}(\mathcal{X}) \to$

$\mathcal{C} \to \mathbb{Z}^V$ sends a Cartier divisor $D = \sum_v n_v X_v$ to

$$v \mapsto \sum_{\substack{\text{nodes } p \\ \text{lying on } X_v}} \frac{n_w - n_v}{\tau_p}$$

with τ_p being the thickness of the node p, X_w the second component passing through p. We want to check that $\delta_{\mathcal{X}}(D)$ is the vertex labelling $v \mapsto \deg \mathcal{O}(D)_{|X_v}$. Fix a vertex z; multiplication by t^{n_z} gives an isomorphism $\mathcal{O}(D) \cong \mathcal{O}(D')$ where $D' = \sum_v (n_v - n_z) X_v$. We reduce to computing the contribution to $\deg \mathcal{O}(D')_{|X_z}$ coming from $(n_v - n_z) X_v$, where $v \in V$ is some vertex different from z. The contribution is zero if $\mathcal{X}_v$ and $\mathcal{X}_z$ do not intersect; otherwise, let $p \in X_v \cap X_z$, with thickness τ_p. Notice that $\tau_p | n_v - n_z$. Locally at p, the divisor $(n_v - n_z) X_v$ is given by the fractional ideal $I = (x^{(n_v - n_z)/\tau_p}, t^{n_v - n_z}) = (x^{(n_v - n_z)/\tau_p})$ of $\widehat{\mathcal{O}}_{\mathcal{X},p} \cong \widehat{R}[[x,y]]/xy - t^{\tau_p}$. Restricting to the branch $y = 0, t = 0$, we obtain the fractional ideal $I \otimes \widehat{\mathcal{O}}_{\mathcal{X},p}/y = (x^{(n_v - n_z)/\tau_p})$ of $k[[x]]$, hence a contribution of $(n_v - n_z)/\tau_p$ to the degree of $\mathcal{O}(D')_{|X_z}$. Summing over all the nodes in $X_v \cap X_z$, we recover the map $\delta_{\mathcal{X}}$.

iii) This can be read directly in the description of the effect of blowing-up on the special fibre provided in section 8.3.

iv) The commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{C}(\mathcal{X}) & \longrightarrow & E(\mathcal{X}) & \longrightarrow & 0 \\ & & \downarrow \scriptstyle{\mathrm{id}} & & \downarrow \scriptstyle{\iota} & & \downarrow \scriptstyle{\bar{\iota}} & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathcal{C}(\mathcal{X}_n) & \longrightarrow & E(\mathcal{X}_n) & \longrightarrow & 0 \end{array}$$

yields a map $\bar{\iota}\colon E(\mathcal{X}) \to E(\mathcal{X}_n)$. Such map fits into the commutative diagram

$$\begin{array}{ccc} E(\mathcal{X}) & \xrightarrow{\delta_{\mathcal{X}}} & \mathbb{Z}^V \\ \downarrow \scriptstyle{\bar{\iota}} & & \downarrow \scriptstyle{\epsilon} \\ E(\mathcal{X}_n) & \xrightarrow{\delta_{\mathcal{X}_n}} & \mathbb{Z}^{V_n} \end{array}$$

where $\epsilon\colon \mathbb{Z}^V \to \mathbb{Z}^{V_n}$ is the extension by zero map, and the two horizontal maps are induced by the exact sequences as in ii) for $\mathcal{X}$ and $\mathcal{X}_n$. They associate to a line bundle its multi-degree on the special fibre, and are injective. The pullback map $\pi_n^*\colon E(\mathcal{X}) \to E(\mathcal{X}_n)$ makes the diagram above commutative as well; it follows that it coincides with $\bar{\iota}$.

□

Theorem 12.3. *Let $\mathcal{X} \to S$ be a nodal curve over a trait with perfect fraction field K, and assume that the special fibre $\mathcal{X}_k$ has split singularities.*

i) If the labelled graph (Γ, l) is circuit-coprime then $\mathcal{X} \to S$ is semi-factorial.

ii) Suppose that $\Gamma(S, \mathcal{O}_S)$ is strictly-henselian. If $\mathcal{X}$ is semi-factorial over S, then the labelled graph (Γ, l) is circuit-coprime.

Proof. We start with part i). Suppose Γ is circuit-coprime. Let L be a line bundle on $\mathcal{X}_K$. By theorem 9.5, there exists an integer $n \geq 0$ such that L extends to a line bundle $\widetilde{\mathcal{L}}$ on $\mathcal{X}_n$. Let (Γ_n, l_n) be the labelled graph of $\mathcal{X}_n$, which is the n-th blow-up graph of Γ. Denote by $\alpha \in \mathbb{Z}^{V_n}$ the vertex-labelling assigning to each vertex v the degree of the restriction of $\widetilde{\mathcal{L}}$ to the component of $(\mathcal{X}_n)_k$ corresponding to v. By proposition 11.21, the map $H \to H_n$ is an isomorphism; hence there exists a Cartier vertex labelling φ on (Γ_n, l_n) such that $\delta(\varphi) + \alpha$ is in the image of the map $\mathbb{Z}^V \to \mathbb{Z}^{V_n}$. Equivalently (by lemma 12.2) there exists a Cartier divisor $D \in \mathcal{C}(\mathcal{X}_n)$, such that $\delta_{\mathcal{X}_n}(D) + \alpha$ is in the image of $\mathbb{Z}^V \to \mathbb{Z}^{V_n}$, i.e., $\delta_{\mathcal{X}}(D) + \alpha$ has value zero on all new vertices of Γ_n. This means precisely that $\mathcal{O}_{\mathcal{X}_n}(D) \otimes \widetilde{\mathcal{L}}$ has degree zero on every component of the exceptional locus of $\pi_n \colon \mathcal{X}_n \to \mathcal{X}$. By proposition 10.2, $\mathcal{L} := (\pi_n)_*(\widetilde{\mathcal{L}} \otimes \mathcal{O}(D))$ is a line bundle on $\mathcal{X}$, which restricts to L on the generic fibre.

Let's turn to part ii). Suppose that Γ is not circuit-coprime. Then there exists $n \geq 0$ such that the map $H \to H_n$ is not surjective. Let α be a basis element of $\mathbb{Z}^{V_n}$ such that the image of α in $H_n = \mathbb{Z}^{V_n}/\delta_n(\mathcal{C}_n)$ is not in the image of $H \to H_n$. Then α takes value 1 on some vertex v of Γ_n and value zero on all other vertices. The vertex v corresponds to an exceptional component $C \cong \mathbb{P}^1_k$ of $\pi_n \colon \mathcal{X}_n \to \mathcal{X}$. Let p be a k-rational point of $(\mathcal{X}_n)^{sm}_k$ lying on C, which exists as k is separably closed. Since the base is henselian, p can be extended to a section $s \colon S \to \mathcal{X}_n$. The image $D \subset \mathcal{X}_n$ of s defines a Cartier divisor. Let $L := \mathcal{O}(D)_{|K}$ be its restriction to the generic fibre. Assume by contradiction that L can be extended to a line bundle $\mathcal{L}$ on $\mathcal{X}$. Then $\mathcal{F} := \mathcal{O}(D) \otimes \pi_n^* \mathcal{L}^{-1}$ is generically trivial. Let D' be a Cartier divisor supported on the special fibre of $\mathcal{X}_n$ such that $\mathcal{O}(D') \cong \mathcal{F}$. Then D' corresponds to a Cartier-vertex labelling φ of Γ_n, and $\alpha - \delta_n(\varphi)$ is the vertex-labelling associated to the multidegree of $\pi_n^* \mathcal{L}$. As $\pi_n^* \mathcal{L}$ has degree zero on every component of the exceptional fibre of $\pi_n \colon \mathcal{X}_n \to \mathcal{X}$, $\alpha - \delta_n(\varphi)$ has value zero on every new vertex of Γ_n. In particular, $\alpha \delta_n(\varphi)$ is in the image of $H \to H_n$, and so is α, yielding a contradiction.

□

Remark 12.4. The assumption that $\Gamma(S, \mathcal{O}_S)$ is strictly-henselian can be replaced by the weaker assumption: for each irreducible component Y of $\mathcal{X}_k$, there exists a line bundle $\mathcal{L}_Y$ on $\mathcal{X}$ whose restriction to $\mathcal{X}_k$ has degree 1 on Y and degree 0 on all other components.

Corollary 12.5. *Hypotheses as in theorem 12.3. Let* $\pi\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ *be the blowing-up of* $\mathcal{X}$ *at the finite union of closed points* $\mathcal{X}^{nreg} \cap \mathcal{X}_k$. *The restriction map*

$$\mathrm{Pic}(\widetilde{\mathcal{X}}) \to \mathrm{Pic}(\mathcal{X}_K)$$

is surjective.

Proof. Let (Γ, l) be the labelled graph of $\mathcal{X} \to S$. The labelled graph $(\widetilde{\Gamma}, \widetilde{l})$ of $\widetilde{\mathcal{X}} \to S$ is the first-blow-up graph of Γ (definition 11.17). Every edge of $\widetilde{\Gamma}$ with a label different from 1 is adjacent to exactly two edges, both with label 1. Hence $\widetilde{\Gamma}$ is circuit-coprime, and we conclude by theorem 12.3.

□

Corollary 12.6. *Hypotheses as in theorem 12.3. Suppose that the special fibre* $\mathcal{X}_k$ *is of compact-type (i.e. its dual graph* Γ *is a tree). Then the restriction map*

$$\mathrm{Pic}(\mathcal{X}) \to \mathrm{Pic}(\mathcal{X}_K)$$

is surjective.

Proof. The dual graph Γ of the special fibre has no circuits, hence the labelled graph (Γ, l) is circuit-coprime. □

In general, semi-factoriality of nodal curves over traits does not descend along étale base change, and we cannot drop the assumption in theorem 12.3 that the special fibre of the curve has split singularities. Here is an example.

Example 12.7. Let $R = \mathbb{Q}[[t]]$, $K = \mathrm{Frac}\, R$, $S = \mathrm{Spec}\, R$, and

$$\mathcal{X} = \mathrm{Proj}\, \frac{R[x,y,z]}{x^2 + y^2 - t^2 z^2}.$$

The curve $\mathcal{X} \to S$ has smooth generic fibre $\mathcal{X}_K/K$, and a node $P = (t = 0, x = 0, y = 0, z = 1)$ on the special fibre. The section $s\colon S \to \mathcal{X}$ given by $x = t, y = 0, z = 1$ goes through the node P . The Cartier divisor on $\mathcal{X}_K$ given by the image of $s_K\colon \mathrm{Spec}\, K \to \mathcal{X}_K$ does not extend to a Cartier divisor on $\mathcal{X}$. Indeed, if by contradiction it extended to a Cartier divisor D on $\mathcal{X}$, the difference $D - s$ as Weil divisors would be a Weil divisor supported on the special fibre; hence a Weil divisor linearly equivalent to zero, since the special fibre is irreducible. Then s would be Cartier, which it is not, and we have the contradiction.

On the other hand, the base change of $\mathcal{X}/R$ by the étale map $R \to R' := \mathbb{Q}(i)[[t]]$ is semi-factorial, since its special fibre has split singularities and its graph is a tree. We see that, denoting by X_1 and X_2 the two components of the special fibre, the Weil divisors $s_{R'} - X_1$ and $s_{R'} - X_2$ are both Cartier, and both extend the Cartier divisor on $\mathcal{X}_{K'}$ given by $s_{K'}$.

13 Application to Néron lft-models of jacobians of nodal curves

13.1 Representability of the relative Picard functor

Let S be a scheme and $\mathcal{X} \to S$ a curve. We denote by $\mathrm{Pic}_{\mathcal{X}/S}$ the relative Picard functor, that is, the fppf-sheafification of the functor

$$\begin{array}{rcl} (\mathbf{Sch}/S)^{opp} & \to & \mathbf{Sets} \\ T & \mapsto & \{\text{invertible sheaves on } \mathcal{X}_T\}/\cong \end{array}$$

We start with a result on representability of the Picard functor:

Theorem 13.1 ([BLR90] 9.4/1)**.** *Let $f\colon \mathcal{X} \to S$ be a nodal curve. Then the relative Picard functor $\mathrm{Pic}_{\mathcal{X}/S}$ is representable by an algebraic space*[2]*, smooth over S.*

Lemma 13.2. *Let $f\colon \mathcal{X} \to S$ be a nodal curve admitting a section $s\colon S \to \mathcal{X}$. Then for any S-scheme T the natural map*

$$\mathrm{Pic}(\mathcal{X} \times_S T)/\mathrm{Pic}(T) \to \mathrm{Pic}_{\mathcal{X}/S}(T)$$

is an isomorphism.

Proof. See the discussion about rigidified line bundles on [BLR90] 8.1. □

13.2 Néron lft-models

Let S be a Dedekind scheme, that is, a noetherian normal scheme of dimension ≤ 1. Then S is a disjoint union of integral Dedekind schemes S_i. The *ring of rational functions* of S is the direct sum $K := \bigoplus_i k(\eta_i)$, where the points $\{\eta_i\}$ are the generic points of the S_i.

[2] Defined as in [BLR90] 8.3/4

Definition 13.3 ([BLR90], 10.1/1)**.** Let S be a Dedekind scheme, with ring of rational functions K. Let A be a K-scheme. A *Néron lft-model* over S for A is the datum of a smooth separated scheme $\mathcal{A} \to S$ and a K-isomorphism $\varphi\colon \mathcal{A} \times_S K \to A$ satisfying the following universal property: for any smooth map of schemes $T \to S$ and K-morphism $f\colon T_K \to A$, there exists a unique S-morphism $F\colon T \to \mathcal{A}$ with $F_K = f$.

A Néron lft-model differs from a Néron model in that the former is not required to be quasi-compact.

Proposition 13.4 ([BLR90], 10.1/2)**.** *Let S be a trait and G a smooth separated S-group scheme. The following are equivalent:*

i) G is a Néron lft-model of its generic fibre;

ii) for every essentially smooth local extension of traits $S' \to S$, with $K' = \operatorname{Frac}\Gamma(S, \mathcal{O}_S)$, the map $G(S') \to G(K')$ is surjective.

Lemma 13.5. *Let $\mathcal{X} \to S$ be a nodal curve over a trait. Let $\operatorname{cl}(e_K) \subset \operatorname{Pic}_{\mathcal{X}/S}$ be the schematic closure of the unit section $e_K\colon \operatorname{Spec} K \to \operatorname{Pic}_{\mathcal{X}_K/K}$. Then the fppf-quotient sheaf $\mathcal{N} = \operatorname{Pic}_{\mathcal{X}/S} / \operatorname{cl}(e_K)$ is representable by a smooth separated S-group scheme. Moreover, the quotient morphism $\operatorname{Pic}_{\mathcal{X}/S} \to \mathcal{N}$ is étale.*

Proof. As $\operatorname{cl}(e_K)$ is flat over S, the fppf-quotient of sheaves $\mathcal{N} = \operatorname{Pic}_{\mathcal{X}/S} / \operatorname{cl}(e_K)$ is a group algebraic space, smooth over S because $\operatorname{Pic}_{\mathcal{X}/S}$ is; as $\operatorname{cl}(e_K)$ is closed in $\operatorname{Pic}_{\mathcal{X}/S}$, $\mathcal{N}$ is separated over S. In particular, $\mathcal{N}$ is a separated group algebraic space locally of finite type over S, so it is a group scheme by [Ana73], Chapter IV, Theorem 4.B. Finally, to show that $\operatorname{Pic}_{\mathcal{X}/S} \to \mathcal{N}$ is étale we prove that $\operatorname{cl}(e_K)$ is étale over S. As the property is étale local on S, we may assume that $\mathcal{X} \to S$ has special fibre with split singularities. The multidegree map $E(\mathcal{X}) \to \mathbb{Z}^V$ (lemma 12.2, ii)) is injective, hence the intersection of $\operatorname{cl}(e_K)$ with the identity component $\operatorname{Pic}^0_{\mathcal{X}/S} \subset \operatorname{Pic}_{\mathcal{X}/S}$ is trivial and it follows that $\operatorname{cl}(e_K)$ is étale over S. □

Given a nodal curve $\mathcal{X} \to S$ over a trait, we can associate to it the labelled graph (Γ, l) of the base change $\mathcal{X} \times_S S' \to S'$, where S' is the spectrum of the strict henselization of $\Gamma(S, \mathcal{O}_S)$ with respect to some algebraic closure of the residue field k. The graph (Γ, l) does not depend on the choice of an algebraic closure of k.

Theorem 13.6. *Let $\mathcal{X} \to S$ be a nodal curve over a trait with perfect fraction field K. The S-group scheme $\mathcal{N} = \operatorname{Pic}_{\mathcal{X}/S} / \operatorname{cl}(e_K)$ is a Néron lft-model for $\operatorname{Pic}_{\mathcal{X}_K/K}$ over S if and only if the labelled graph (Γ, l) of $\mathcal{X} \to S$ is circuit-coprime.*

Proof. Let $S^{sh} \to S$ be a strict henselization of S with respect to some algebraic closure of the residue field, and denote by K^{sh} its fraction field. If (Γ, l) is not circuit-coprime, the map

$$\operatorname{Pic}(\mathcal{X}_{S^{sh}}) \to \operatorname{Pic}(\mathcal{X}_{K^{sh}})$$

is not surjective, by theorem 12.3. Now, as the special fibre of $\mathcal{X}_{S^{sh}}/S^{sh}$ is generically smooth, $\mathcal{X}_{S^{sh}} \to S^{sh}$ admits a section; hence, we can apply lemma 13.2 and find that

$$\operatorname{Pic}_{\mathcal{X}/S}(S^{sh}) \to \operatorname{Pic}_{\mathcal{X}_K/K}(K^{sh})$$

is not surjective. As the quotient $\operatorname{Pic}_{\mathcal{X}/S} \to \mathcal{N}$ is an étale surjective morphism of S^{sh}-algebraic spaces (lemma 13.5), the map $\operatorname{Pic}_{\mathcal{X}/S}(S^{sh}) \to \mathcal{N}(S^{sh})$ is surjective. We deduce that $\mathcal{N}(S^{sh}) \to \operatorname{Pic}_{\mathcal{X}_K/K}(K^{sh})$ is not surjective. Then for some étale extension of discrete valuation rings $S' \to S$, $\mathcal{N}(S') \to \operatorname{Pic}_{\mathcal{X}_K/K}(K')$ is not surjective, hence $\mathcal{N}$ is not a Néron model of $\operatorname{Pic}_{\mathcal{X}_K/K}$.

Now assume that (Γ, l) is circuit coprime. Assume first that S is strictly henselian. By proposition 13.4 it is enough to prove that for all essentially smooth local extensions $R \to R'$ of discrete valuation rings, the map

$$\mathcal{N}(R') \to \operatorname{Pic}_{\mathcal{X}_K/K}(K')$$

is surjective. As $\mathcal{X} \to S$ admits a section, we may apply lemma 13.2 and just show that $\operatorname{Pic}(\mathcal{X}_{R'}) \to \operatorname{Pic}(\mathcal{X}_{K'})$ is surjective. The map $R \to R'$ has ramification index 1, i.e. it sends a uniformizer to a uniformizer. Therefore the labelled graph (Γ', l') associated to $\mathcal{X}_{R'}$ is again circuit-coprime, and in fact $(\Gamma', l') = (\Gamma, l)$. Now we conclude by theorem 12.3.

Now let $\mathcal{X} \to S$ be any nodal curve with circuit-coprime labelled graph. Let $p\colon S' \to S$ be a strict henselization of S. Consider the smooth separated S-group scheme $\mathcal{N} = \operatorname{Pic}_{\mathcal{X}/S} / \operatorname{cl}(e_K)$. As taking the schematic closure commutes with flat base change, $p^*\mathcal{N}$ is canonically isomorphic to $\operatorname{Pic}_{\mathcal{X}'/S'} / \operatorname{cl}(e_{K'})$, hence is a Néron lft-model for $\operatorname{Pic}_{\mathcal{X}_{K'}/K'}$ over S'. We show that $\mathcal{N}$ is a Néron lft-model of its generic fibre. Let $T \to S$ be a smooth S-scheme, $f\colon T_K \to \operatorname{Pic}_{\mathcal{X}_K/K}$ a K-morphism. The base change $p^*f\colon T_{K'} \to \operatorname{Pic}_{\mathcal{X}_{K'}/K'}$ extends uniquely to an S'-morphism $g\colon p^*T \to \mathcal{N}'$. Let $S'' := S' \times_S S'$, $p_1, p_2\colon S'' \to S'$ the two projections, and $q\colon S'' \to S$ the composition. The two maps $p_1^*g, p_2^*g\colon q^*T \to q^*\mathcal{N}$ both coincide with q^*f when restricted to q^*T_K. As $q^*T \to S$ is flat, q^*T_K is schematically dense in q^*T. Since moreover $q^*\mathcal{N}$ is separated, we have that $p_1^*g = p_2^*g$. Hence g descends to a morphism $T \to \mathcal{N}$ extending f. Again, the extension is unique because $\mathcal{N} \to S$ is separated and T_K is schematically dense in T.

□

Corollary 13.7. *Let $\mathcal{X} \to S$ be a nodal curve over a trait. Let $\pi\colon \widetilde{\mathcal{X}} \to \mathcal{X}$ be the blowing-up of $\mathcal{X}$ at the finite union of closed points $\mathcal{X}^{nreg} \cap \mathcal{X}_k$. Then $\mathcal{N} = \operatorname{Pic}_{\widetilde{\mathcal{X}}/S} / \operatorname{cl}(e_K)$ is a Néron lft-model for $\operatorname{Pic}_{\mathcal{X}_K/K}$ over S.*

Proof. It is enough to check that the labelled graph $(\widetilde{\Gamma}, \widetilde{l})$ of $\widetilde{\mathcal{X}} \to S$ is circuit-coprime, by the previous Theorem. As labelled graphs are preserved under étale extensions of the base trait, we may assume that $\mathcal{X} \to S$ has special fibre with split singularities. Then the same argument as in the proof of corollary 12.5 shows that $(\widetilde{\Gamma}, \widetilde{l})$ is circuit-coprime. □

Corollary 13.8. *Let $\mathcal{X} \to S$ be a nodal curve over a trait with perfect fraction field K. Let $\overline{k}$ be a separable closure of the residue field of S and suppose that the graph of $\mathcal{X}_{\overline{k}}$ is a tree. Then $\mathcal{N} = \operatorname{Pic}_{\mathcal{X}/S} / \operatorname{cl}(e_K)$ is a Néron lft-model for $\operatorname{Pic}_{\mathcal{X}_K/K}$ over S.*

We have shown how to construct Néron lft-models for the group scheme $\operatorname{Pic}_{\mathcal{X}_K/K}$, without ever imposing bounds on the degree of line bundles; the following lemma allows us to retrieve lft-Néron models for subgroup schemes of $\operatorname{Pic}_{\mathcal{X}_K/K}$, and applies in particular to subgroup schemes that are open and closed, such as the connected component of the identity $\operatorname{Pic}^{[0]}_{\mathcal{X}_K/K}$.

Lemma 13.9. *Let $\mathcal{X}/S$ be a nodal curve over a trait, and $H \subset \operatorname{Pic}_{\mathcal{X}_K/K}$ a K-smooth closed subgroup scheme of $\operatorname{Pic}_{\mathcal{X}_K/K}$. Let $\mathcal{N} \to S$ be the Néron model of $\operatorname{Pic}_{\mathcal{X}_K/K}$. Then H admits a Néron lft-model $\mathcal{H}$ over S, which is obtained as a group smoothening of the schematic closure of H inside $\mathcal{N}$.*

Proof. This is a special case of [BLR90], 10.1/4.

□

We remark that, if the generic fibre $\mathcal{X}_K/K$ is not smooth, $\operatorname{Pic}^{[0]}_{\mathcal{X}_K/K}$ is an extension of an abelian variety by a torus; if the torus contains a copy of $\mathbb{G}_{m,K}$, the Néron lft-model of $\operatorname{Pic}^{[0]}_{\mathcal{X}_K/K}$ is not quasi-compact.

Bibliography

[AK80] Allen B. Altman and Steven L. Kleiman. Compactifying the Picard scheme. *Adv. in Math.*, 35(1):50–112, 1980.

[Ana73] Sivaramakrishna Anantharaman. Schémas en groupes, espaces homogènes et espaces algébriques sur une base de dimension 1. pages 5–79. Bull. Soc. Math. France, Mém. 33, 1973.

[BLR90] Siegfried Bosch, Werner Lütkebohmert, and Michel Raynaud. *Néron Models*, volume 21 of *Ergebnisse der Mathematik und ihrer Grenzgebiete*. Springer-Verlag, 1990.

[Cap08] Lucia Caporaso. Néron models and compactified Picard schemes over the moduli stack of stable curves. *Amer. J. Math.*, 130(1):1–47, 2008.

[Del85] Pierre Deligne. Le lemme de Gabber. *Astérisque*, 127:131–150, 1985. Seminar on arithmetic bundles: the Mordell conjecture (Paris, 1983/84).

[Die05] Reinhard Diestel. *Graph theory*, volume 173 of *Graduate Texts in Mathematics*. Springer-Verlag, Berlin, third edition, 2005.

[dJ96] A. J. de Jong. Smoothness, semi-stability and alterations. *Inst. Hautes Études Sci. Publ. Math.*, (83):51–93, 1996.

[Edi92] Bas Edixhoven. Néron models and tame ramification. *Compositio Mathematica*, 81(3):291–306, 1992.

[GD67] Alexander Grothendieck and Jean Dieudonné. *Éléments de géométrie algébrique IV*, volume 20, 24, 28, 32 of *Publications Mathématiques*. Institute des Hautes Études Scientifiques., 1964-1967.

[GR71] L. Gruson and M. Raynaud. Critères de platitude et de projectivité. Techniques de "platification" d'un module. *Inventiones mathematicae*, 13:1–89, 1971.

[Gro71] Alexander Grothendieck. *Revêtements étales et groupe fondamental (SGA 1)*, volume 224 of *Lecture notes in mathematics*. Springer-Verlag, 1971.

[GRR72] Alexander Grothendieck, Michel Raynaud, and Dock Sang Rim. *Groupes de monodromie en géométrie algébrique. I.* Lecture Notes in Mathematics, Vol. 288. Springer-Verlag, 1972. Séminaire de Géométrie Algébrique du Bois-Marie 1967–1969 (SGA 7 I).

[Hol17a] David Holmes. Extending the double ramification cycle by resolving the Abel-Jacobi map. 2017.

[Hol17b] David Holmes. Néron models of jacobians over base schemes of dimension greater than 1. *To appear in Journal für die reine und angewandte Mathematik*, 2017.

[Liu02] Qing Liu. *Algebraic geometry and arithmetic curves*, volume 6 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2002. Translated from the French by Reinie Erné, Oxford Science Publications.

[MB85] Laurent Moret-Bailly. Pinceaux de variétés abéliennes. *Astérisque*, (129):266, 1985.

[Nér64] André Néron. Modèles minimaux des variétés abéliennes sur les corps locaux et globaux. *Inst. Hautes Études Sci. Publ.Math. No.*, 21:128, 1964.

[Ore17] Giulio Orecchia. Semi-factorial nodal curves and Néron models of jacobians. *Manuscripta Mathematica*, 154(3):309–341, Nov 2017.

[Pép13] Cédric Pépin. Modèles semi-factoriels et modèles de Néron. *Math. Ann.*, 355(1):147–185, 2013.

[Ray70a] M. Raynaud. Spécialisation du foncteur de Picard. *Inst. Hautes Études Sci. Publ. Math.*, (38):27–76, 1970.

[Ray70b] Michel Raynaud. *Faisceaux amples sur les schémas en groupes et les espaces homogènes.* Lecture Notes in Mathematics, Vol. 119. Springer-Verlag, Berlin-New York, 1970.

[Sch94] Claus Scheiderer. *Real and étale cohomology*, volume 1588 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1994.

[Sha13] Igor R. Shafarevich. *Basic algebraic geometry. 1.* Springer, Heidelberg, third edition, 2013. Varieties in projective space.

[Sta16] The Stacks Project Authors. *Stacks Project.* `http://stacks.math.columbia.edu`*, 2016.*

[TS92] K. Thulasiraman and M. N. S. Swamy. Graphs: theory and algorithms. *A Wiley-Interscience Publication. John Wiley & Sons, Inc., New York, 1992.*

Acknowledgements

This thesis owes a great deal to the guidance I have received from my two supervisors, David Holmes and Qing Liu.

As I spent roughly the 80% of my PhD in Leiden, David was my primary supervisor, and he introduced me to the topics treated in the thesis. Throughout my PhD, his commitment, advice, and patience were invaluable. His unceasing support during highs and lows gave me the confidence which I needed – especially during the lows.

I would like to thank Qing Liu for kindly accepting to take over as co-supervisor (when my original co-supervisor Jilong Tong moved to China), for useful discussions, and for helping me in my search for future jobs.

I would like to thank the reading committee and the anonymous referee for proofreading my thesis and providing many valuable comments.

Another person who deserves a special thank is Bas Edixhoven. In my first year in Leiden he supervised my master thesis project; he shaped my taste for algebraic geometry and inspired me to become a better mathematician. During my PhD he was always available for math discussions in front of his digital blackboard.

I would like to thank Michel Raynaud for a very useful exchange of emails in November 2016, and Raymond van Bommel for being kind enough to provide a proof of lemma 4.8 and for several math discussions over four years in Leiden.

I would like to thanks the Scuola Galileiana di Studi Superiori in Padova for offering me the opportunity to study for three years in an exceptionally stimulating environment, during my bachelor years.

I thank all my friends and relatives around the world for their support.

Finally, I would like to thank my family, who first nurtured my interest in mathematics and constantly encourages and supports me.

Abstract

This thesis is subdivided in two parts.

In the first part, we introduce a new condition, called toric-additivity, on a family of abelian varieties degenerating to a semi-abelian scheme over a normal crossing divisor. The condition depends only on the Tate module $T_l A(K^{sep})$ of the generic fibre, for a prime l invertible on the base. We show that toric-additivity is a sufficient condition for the existence of a Néron model if the base is a $\mathbb{Q}$-scheme. In the case of the jacobian of a smooth curve with semi-stable reduction, we obtain the same result without assumptions on the base characteristic; and we show that toric-additivity is also necessary for the existence of a Néron model, when the base is a $\mathbb{Q}$-scheme.

In the second part, we consider the case of a family of nodal curves over a discrete valuation ring, having split singularities. We say that such a family is semi-factorial if every line bundle on the generic fibre extends to a line bundle on the total space. We give a necessary and sufficient condition for semi-factoriality, in terms of combinatorics of the dual graph of the special fibre. In particular, we show that performing one blow-up with center the non-regular closed points yields a semi-factorial model of the generic fibre.

As an application, we extend the result of Raynaud relating Néron models of smooth curves and Picard functors of their regular models to the case of nodal curves having a semi-factorial model.

Samenvatting

Dit proefschrift bestaat uit twee delen.

In het eerste deel introduceren we een nieuwe voorwaarde, torische-additiviteit genaamd, voor een familie van abelse variëteiten die tot een semi-abelse schema degenereren boven een divisor met normale kruisingen. De voorwaarde hangt alleen af van het Tate-moduul $T_l A(K^{sep})$ van de generieke vezel, voor een priemgetal l dat inverteerbaar is op de basis. We laten zien dat torische-additiviteit een voldoende voorwaarde is voor het bestaan van een Néron model, als de basis een $\mathbb{Q}$-schema is. In het geval van de jacobian van een gladde kromme met semi-stabiele reductie, verkrijgen we hetzelfde resultaat zonder veronderstellingen over de karakteristiek van de basis; bovendien, laten we zien dat torische-additiviteit ook nodig is voor het bestaan van een Néron-model, wanneer de basis een $\mathbb{Q}$-schema is.

In het tweede deel beschouwen we het geval van een familie van semi-stabiele krommen over een discrete valuatie ring, met gespleten singulariteiten. We zeggen dat zo'n familie semi-factorieel is als elke lijnbundel op de generieke vezel de restrictie is van een lijnbundel op de totale ruimte. We geven een noodzakelijke en voldoende voorwaarde voor semi-factorialiteit, in termen van de combinatoriek van de duale graaf van de speciale vezel. In het bijzonder laten we zien dat het uitvoeren van één blow-up met centrum de niet-reguliere gesloten punten een semi-factorieel model oplevert van de generieke vezel.

Als toepassing, breiden we het resultaat van Raynaud met betrekking tot Néron-modellen van gladde krommen en Picard-functoren van hun reguliere modellen uit naar het geval van (mogelijk singuliere) krommen met een semi-factorieel model.

Résumé

Cette thèse est divisée en deux parties. Dans la première partie, nous introduisons une nouvelle condition, appellée additivité torique, sur une famille de variétés abéliennes qui dégénèrent en un schéma semi-abelien au-dessus d'un diviseur à croisements normaux. La condition ne dépend que du module de Tate $T_l A(K^{sep})$ de la fibre générique. Nous montrons que l'additivité torique est une condition suffisante pour l'existence d'un modèle de Néron, si la base est un schéma de caractéristique nulle. Dans le cas de la jacobienne d'une courbe lisse à réduction semi-stable, on obtient le même résultat sans aucune hypothèse sur la caractéristique de base; et nous montrons que l'additivité torique est aussi nécessaire pour l'existence d'un modèle de Néron, si la base est un schéma de caractéristique nulle.

Dans la deuxième partie, on considère le cas d'une famille de courbes nodales sur un anneau de valuation discrète. On donne une condition combinatoire sur le graphe dual de la fibre spéciale, appellée semi-factorialité, qui équivaut au fait que tous les faisceaux inversibles sur la fibre générique s'étendent en des faisceaux inversibles sur l'espace total de la courbe. Il est démontré par la suite que cette condition est automatiquement satisfaite après un éclatement centré aux points fermés non-réguliers de la famille de courbes.

On applique le résultat ci-dessus pour généraliser un théorème de Raynaud sur le modèle de Néron des jacobiennes de courbes lisses, au cas des courbes nodales.

Curriculum Vitae

Giulio Orecchia was born on January 11, 1990, in Rapallo, Italy. He lived until 2009 in Genova, where he attended high school at Liceo Scientifico Gian Domenico Cassini, which offered him the opportunity to participate in a number of mathematical competitions for high school students.

After finishing high school, he moved to Padova for his undergraduate studies in mathematics. During those years, he was also a student at the Scuola Galileiana di Studi Superiori, from which he obtained a diploma in 2015.

In 2012, Giulio was awarded an Erasmus Mundus Master scholarship to pursue his master degree within the ALGANT program at Concordia University and Universiteit Leiden. The program had a strong focus on Algebra, Geometry and Number theory courses. He graduated in July 2014 with a thesis in algebraic geometry titled "Torsion-free rank one sheaves on a semi-stable curve", written under the supervision of Prof. Bas Edixhoven.

In September of the same year, he began his Ph.D. in mathematics, again within the ALGANT doctorate program, under the joint supervision of Dr. David Holmes (Universiteit Leiden) and Prof. Qing Liu (Université de Bordeaux). He plans to defend his thesis in February 2018.