Shtuka cohomology and special values of Goss L-functions

Mornev, M.

Author: Mornev, M.
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Stellingen

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Shtuka cohomology and special values of Goss L-functions
van Maxim Mornev

Fix a finite field $\mathbb{F}_q$. In the following all tensor and fiber products without subscripts will be over $\mathbb{F}_q$. Fix an $\mathbb{F}_q$-algebra A which is a coefficient ring in the sense of Definition 7.2.1. Let ω be the module of Kähler differentials of A over $\mathbb{F}_q$. Given a Drinfeld A-module E over an $\mathbb{F}_q$-algebra R we denote $M(E) = \mathrm{Hom}(E, \mathbb{G}_a)$ the motive of E.

1. Let R be an $\mathbb{F}_q$-algebra. For every Drinfeld A-module E over R there exists a natural quasi-isomorphism
$$\mathrm{RHom}_{A\otimes R\{\tau\}}(M(E),\, \omega \otimes R) \cong E(R)[-1].$$

2. Let K be a local field containing $\mathbb{F}_q$ and let E be a Drinfeld A-module over K. If the natural map $A \to K$ is injective with discrete image then there exists a natural quasi-isomorphism
$$\mathrm{RHom}_{A\otimes K\{\tau\}}(M(E),\, \omega \mathbin{\widehat{\otimes}} K) \cong \Big[\mathrm{Lie}_E(K) \xrightarrow{\exp} E(K)\Big].$$

Let F and K be local fields containing $\mathbb{F}_q$, $\mathcal{O}_F \subset F$ and $\mathcal{O}_K \subset K$ their rings of integers and $\mathfrak{m} \subset \mathcal{O}_F$ the maximal ideal.

3. Let $\mathcal{M}$ be a locally free $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$-module shtuka. If $\mathcal{M}(\mathcal{O}_F/\mathfrak{m} \otimes K)$ is nilpotent then $\mathrm{H}^0(\mathcal{M}) = 0$ and $\mathrm{H}^1(\mathcal{M})$ is a finitely generated free $\mathcal{O}_F$-module.

4. Let $\mathfrak{f} \subset \mathcal{O}_K$ be a nonzero ideal. There exists a unique $\mathcal{O}_F$-linear natural isomorphism $\rho\colon \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M})$ of functors on the category of elliptic $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$-module shtukas of conductor $\mathfrak{f}$ such that for every elliptic shtuka $\mathcal{M}$ the diagram
$$\begin{tikzcd} \mathrm{H}^1(\mathcal{M}) \arrow[r, "\rho"] \arrow[d] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d] \\ \mathrm{H}^1(\mathcal{M}/\mathfrak{f}^n) \arrow[r, "1"] & \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{f}^n) \end{tikzcd}$$
is commutative provided $\mathcal{M}/\mathfrak{f}^{2d}$ is linear.

Let X be a smooth connected projective curve over $\mathbb{F}_q$. Let $\operatorname{Spec} R \subset X$ be an affine open subscheme. We denote $\mathcal{O}_K$ the product of complete local rings of the points in the complement of $\operatorname{Spec} R$. Fix an ideal $\mathfrak{f} \subset \mathcal{O}_K$ such that the quotient $\mathcal{O}_K/\mathfrak{f}$ is finite.

5. Let $\mathcal{M}$ be an elliptic shtuka on $\operatorname{Spec}\mathcal{O}_F \times X$ of conductor $\mathfrak{f}$. Let $\mathcal{M}_0$, $\mathcal{M}_1$ be its underlying sheaves, i and j the arrows. Assume that $\mathcal{M}_0$, $\mathcal{M}_1$ have cohomology concentrated in degree 1. If the conductor $\mathfrak{f}$ is contained in the Jacobson radical of $\mathcal{O}_K$ then
$$\zeta_{\mathcal{M}} = L(\mathcal{M}) \cdot \det_{\mathcal{O}_F}(\rho_{\mathcal{M}}).$$
Here $\zeta_{\mathcal{M}}$ is the ζ-isomorphism, $\rho\colon \mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(\nabla\mathcal{M})$ is the regulator of $\mathcal{M}$ and
$$L(\mathcal{M}) = \prod_{\mathfrak{m}} \det_{\mathcal{O}_F}\Big(1 - i^{-1}j \,\Big|\, \mathcal{M}_0(\mathcal{O}_F \otimes R/\mathfrak{m})\Big)^{-1}.$$
where $\mathfrak{m}$ runs over all the maximal ideals of R.

6. Let F be a non-archimedean local field, $\mathcal{O}_F \subset F$ its ring of integers and $\mathfrak{m} \subset \mathcal{O}_F$ the maximal ideal. Let M be a compact Hausdorff $\mathcal{O}_F$-module. The following are equivalent:
   (a) $M/\mathfrak{m}$ is finite.
   (b) M is finitely generated as an $\mathcal{O}_F$-module without topology.

7. Let k be a commutative ring, A and B associative unital k-algebras. Assume that A is a projective k-module and B is a flat k-module. For all left $A \otimes_k B$-modules M and N there exists a natural quasi-isomorphism
$$\mathrm{RHom}_{A\otimes_k B}(M,N) \cong \mathrm{RHom}_{A\otimes_k A^\circ}(A, \mathrm{RHom}_B(M,N))$$
where A° is the opposite of A and the (A,A)-bimodule A on the right hand side is the diagonal one.

8. Let V and W be locally compact $\mathbb{F}_q$-vector spaces. The natural map
$$V \mathbin{\check{\otimes}} W \to (V^\# \mathbin{\widehat{\otimes}} W) \times_{V \widehat{\otimes} W} (V \mathbin{\widehat{\otimes}} W^\#).$$
is an isomorphism in the category of topological $\mathbb{F}_q$-vector spaces.

9. Let A be a coefficient ring, F its local field at infinity, ω the module of Kähler differentials of A over $\mathbb{F}_q$ and K a local field containing $\mathbb{F}_q$. There exists an $A \otimes K$-linear isomorphism
$$\frac{\omega \mathbin{\widehat{\otimes}} K}{\omega \otimes K} \cong g(F,K)$$
where $g(F,K)$ is the space of germs of continuous $\mathbb{F}_q$-linear maps from F to K.

10. Alexander Grothendieck once mistook the number 57 for a prime, which made this number famous. Among its many properties is the fact that 57 is indeed a prime, in base 8.