Shtuka cohomology and special values of Goss L-functions

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The references of the form $[wxyz]$ point to tags in the Stacks project [27] as explained in the chapter "Notation and conventions".

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Summary

Let C be a connected smooth projective curve over a finite field $\mathbb{F}_q$. Fix a closed point $\infty \in C$ and set $A = \Gamma(C - \{\infty\}, \mathcal{O}_C)$. We call A the *coefficient ring*. Let F be the fraction field of A. Fix a finite field extension K of F.

Drinfeld A-modules over K behave in a way similar to abelian varieties over number fields. To such a Drinfeld module E and a prime $\mathfrak{p} \subset A$ one can associate the $\mathfrak{p}$-adic Tate module $T_{\mathfrak{p}}E$. It is a finitely generated free module over the completion of A at $\mathfrak{p}$. It carries a natural continuous action of the Galois group of K which is unramified at almost all primes. Given such a prime $\mathfrak{m}$ it makes sense to consider the inverse characteristic polynomial $P_{\mathfrak{m}}(T)$ of the geometric Frobenius element at $\mathfrak{m}$ acting on $T_{\mathfrak{p}}E$. This polynomial has coefficients in F and is independent of the choice of $\mathfrak{p}$.

Assume that the Drinfeld module E has good reduction everywhere. In this case we have a characteristic polynomial $P_{\mathfrak{m}}(T)$ for every prime $\mathfrak{m}$ of K not diving ∞. One can show that the formal product

$$L(E^*, 0) = \prod_{\mathfrak{m}} \frac{1}{P_{\mathfrak{m}}(1)}$$

converges in the local field F_∞ of the curve C at ∞. The construction of $L(E^*, 0)$ resembles the classical construction of an L-function of an abelian variety. Indeed $L(E^*, 0)$ is the value of a certain L-function at $s = 0$, the Goss L-function of the strictly compatible family of Galois representations given by the Tate modules $T_{\mathfrak{p}}E$.

In the case of abelian varieties one expects that the values of L-functions at integral points reflect subtle arithmetic invariants of the varieties in question. The precise relation is given by the celebrated conjecture of Birch and Swinnerton-Dyer and more generally by the equivariant Tamagawa number conjecture. These conjectures are still very far from being solved.

It was a wonderful discovery of Taelman [25] that an analog of the BSD conjecture holds for Goss L-functions of Drinfeld modules with the coefficient ring $A = \mathbb{F}_q[t]$. Building on the work of Taelman, Böckle and Pink [3], Fang [9] and V. Lafforgue [17] we extended the result of Taelman to Drinfeld modules over arbitrary coefficient rings A. Our approach differs substantially from that of Taelman. It is based on a theory of shtukas and their cohomology which we developed for this purpose.

Samenvatting

Laat C een samenhangende gladde projectieve kromme zijn over een eindig lichaam $\mathbb{F}_q$. Laat $\infty \in C$ een gesloten punt zijn, en laat $A = \Gamma(C - \{\infty\}, \mathcal{O}_C)$. We noemen A de *ring van coëfficiënten*. Laat F het breukenlichaam van A zijn, en K een eindige uitbreiding van F.

Drinfeld A-modulen over K gedragen zich vergelijkbaar met abelse variëteiten over getallenlichamen. Aan zo'n Drinfeld A-moduul en een priem $\mathfrak{p} \subset A$ kan men het $\mathfrak{p}$-adisch Tate moduul $T_\mathfrak{p}E$ toekennen. Dit is een eindig voortgebract vrij moduul over de completering van A bij $\mathfrak{p}$. Het heeft een natuurlijke continue actie van de Galoisgroep van K die bij bijna alle priemen onvertakt is. Gegeven zo'n priem $\mathfrak{m}$ hebben we het inverse karakteristieke polynoom $P_\mathfrak{m}(T)$ van het meetkundige Frobenius-element bij $\mathfrak{m}$ werkend op $T_\mathfrak{p}E$. Dit polynoom heeft coëfficiënten in F en is onafhankelijk van de keuze van $\mathfrak{p}$.

Neem aan dat het Drinfeld-moduul E overal goede reductie heeft. Dan is er voor elke eindige priem $\mathfrak{m}$ van K een karakteristiek polynoom $P_\mathfrak{m}(T)$. Men kan bewijzen dat het formele product $L(E^*, 0) = \prod_\mathfrak{m} P_\mathfrak{m}(1)^{-1}$ convergeert in het locale lichaam F_∞ van de kromme C bij ∞. De constructie van $L(E^*, 0)$ lijkt op de klassieke constructie van een L-functie van een abelse variëteit. Inderdaad is $L(E^*, 0)$ de waarde van een zekere L-functie in $s = 0$, de Goss L-functie van het strikte compatibele systeem van Galois-representaties gegeven door de Tate-modulen $T_\mathfrak{p}E$.

In het geval van abelse variëteiten verwacht men dat de waarden van L-functies in gehele punten subtiele aritmetische invarianten van de betreffende variëteiten weerspiegelen. De preciese relatie is gegeven door het gevierde vermoeden van Birch en Swinnerton-Dyer en algemener door het equivariante vermoeden over Tamagawa-getallen. Deze vermoedens zijn nog verre van een oplossing.

Het was een wonderbaarlijke ontdekking van Taelman [25] dat een analogon van het BSD vermoeden waar is voor Goss L-functies van Drinfeld-modulen met ring van coëfficiënten $A = \mathbb{F}_q[t]$. Voortbouwend op het werk van Taelman, Böckle en Pink [3], Fang [9] en V. Lafforgue [17] hebben wij het resultaat van Taelman gegeneraliseerd naar Drinfeld-modulen over willekeurige ringen van coëfficiënten A. Onze benadering verschilt substantieel van die van Taelman. Die van ons is gebaseerd op een theorie van stukken (dingen) en hun cohomologie die we voor dit doel hebben ontwikkeld.

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In the Fall semester of 2016 I had a pleasure to assist Peter Bruin and Arno Kret in teaching a course on Galois representations and automorphic forms at the University of Amsterdam. Even though the subject of this course is only tangentially related to my thesis, it greatly inspired my work. I learned a lot from it, and I hope that so did my students.

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Curriculum Vitae

Maxim Mornev was born on October 16, 1986 in a small mining town of Kushva in the Ural mountains. There he attended State School No. 6, graduating in 2003. He went on to obtain a bachelor (2009) and a master (2011) degree in computer science from what was then the Ural State University in Yekaterinburg.

In 2011 Maxim won an ALGANT master scholarship. He spent his first year at Universiteit Leiden and the second year at Université Paris-Sud. His master thesis, "Zero cycles on surfaces", was written under supervision of François Charles. Maxim defended it in 2013 and obtained a joint master degree in mathematics from both universities.

The same year he was awarded a joint doctoral scholarship from the ALGANT consortium and Universiteit Leiden. He began to work on his doctoral thesis at Universiteit Leiden under supervision of Lenny Taelman. He was co-supervised by Fabrizio Andreatta at Università degli Studi di Milano. A large part of his PhD project was completed at Universiteit van Amsterdam where his adviser Lenny Taelman moved in 2014.