Shtuka cohomology and special values of Goss L-functions

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CHAPTER 12

Global models and the class number formula

In this chapter we introduce global shtuka models of a Drinfeld module E over a Dedekind domain, and use them to deduce the class number formula for Drinfeld modules.

1. The setting

Fix a coefficient ring A. As before we denote F the local field of A at infinity, $\mathcal{O}_F \subset F$ the ring of integers and $\mathfrak{m}_F \subset \mathcal{O}_F$ the maximal ideal. We denote C the compactification of $\operatorname{Spec} A$.

Fix a finite flat A-algebra R. Throughout the chapter R is assumed to be a domain. We equip R with the discrete topology. We denote X the smooth projective connected curve over $\mathbb{F}_q$ which compactifies $\operatorname{Spec} R$.

Fix a Drinfeld A-module E over R. By definition such E has good reduction at all primes of R. Throughout this chapter $M = \operatorname{Hom}(E, \mathbb{G}_a)$ denotes the motive of E. Let $\mu\colon A \to R$ be the natural map. We assume that elements $a \in A$ act on Lie_E by multiplication by $\mu(a)$.

Set $K = R \otimes_A F$. By construction K is a finite product of local fields containing $\mathbb{F}_q$. As usual we denote $\mathcal{O}_K \subset K$ the ring of integers and $\mathfrak{m}_K \subset \mathcal{O}_K$ the Jacobson radical. Since $K = R \otimes_A F$ the action of A on $\operatorname{Lie}_E(K)$ extends to a continuous action of F. We denote $\mathfrak{f}$ the ideal generated by $\mathfrak{m}_F$ in $\mathcal{O}_K$ under the natural map $F \to K$. We call $\mathfrak{f}$ the conductor.

The notation of this chapter differs from the notation used in the introduction. Namely we write F in place of F_∞ and K in place of K_∞.

2. Hom shtukas

We begin with some elementary properties of $\mathcal{H}\mathrm{om}(M, a(F/A, R))$.

Lemma 2.1. *$\mathcal{H}\mathrm{om}(M, a(F/A, R))$ is a locally free $A \otimes R$-module shtuka. $\mathcal{H}\mathrm{om}(M, a(F, R))$ is a locally free $F \otimes R$-module shtuka.*

Proof. By Corollary 3.10.2 the $A \otimes R$-module $a(F/A, R)$ is locally free of rank 1. Corollary 3.10.3 shows that the $F \otimes R$-module $a(F, R)$ is locally free of rank 1. As M is a locally free $A \otimes R$-module the result follows from the definition of $\mathcal{H}\mathrm{om}$. ☐

Lemma 2.2. *The shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, R))$ is an $A \otimes R$-lattice in the following shtukas:*

(1) *the* $F \otimes R$*-module shtuka* $\mathcal{H}\mathrm{om}(M, a(F, R))$,
(2) *the* $A \otimes K$*-module shtuka* $\mathcal{H}\mathrm{om}(M, a(F/A, K))$,
(3) *the* $F \check{\otimes} K$*-module shtuka* $\mathcal{H}\mathrm{om}(M, a(F, K))$.
(4) *the* $A \widehat{\otimes} K$*-module shtuka* $\mathcal{H}\mathrm{om}(M, b(F/A, K))$.

Proof. (1) By Corollary 3.10.5 the $A \otimes R$-module $a(F/A, R)$ is a lattice in the $F \otimes R$-module $a(F, R)$ so the result follows. (2) Let ω be the module of Kähler differentials of A over $\mathbb{F}_q$. Lemma 8.3.1 implies that $a(F/A, R) = \omega \otimes R$ and $a(F/A, K) = \omega \otimes K$. Whence the result. (3) By Lemma 9.4.3 the shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, K))$ is an $A \otimes K$-module lattice in the $F \check{\otimes} K$-module shtuka $\mathcal{H}\mathrm{om}(M, a(F, K))$. So the result follows from (2). (4) The shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, K))$ is an $A \otimes K$-module lattice in the $A \widehat{\otimes} K$-module shtuka $\mathcal{H}\mathrm{om}(M, b(F/A, K))$ by Lemma 8.4.5. So (2) implies what we need. $\square$

3. The notion of a global model

To the Drinfeld module E with motive M we associate the $A \otimes R$-shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, R))$. By Corollary 3.10.2 it is locally free. To prove the class number formula for E we will extend this shtuka to a locally free shtuka on $C \times X$. In this section we introduce the appropriate notion of such an extension which we call a global model.

Let $\iota^c\colon \operatorname{Spec} A \otimes R \to C \times R$ be the open immersion. We denote

$$\mathcal{E} \subset \iota^c_* \Big[M \underset{\tau}{\overset{1}{\rightrightarrows}} M \Big]$$

the shtuka constructed in Theorem 7.5.5.

Definition 3.1. The *coefficient compactification* $\mathcal{M}^c \subset \mathcal{H}\mathrm{om}(M, a(F, R))$ is the $\mathcal{O}_F \otimes R$-subshtuka

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_F \otimes R}(\mathcal{E}(\mathcal{O}_F \otimes R), a(F/\mathcal{O}_F, R)).$$

The superscript "c" stands for "coefficients".

Note that $\mathcal{M}^c$ is a locally free $\mathcal{O}_F \otimes R$-module shtuka which is a lattice in the $F \otimes R$-module shtuka $\mathcal{H}\mathrm{om}(M, a(F, R))$.

Lemma 3.2. $\mathcal{M}^c(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ *is nilpotent.*

Proof. By Theorem 7.5.5 the shtuka $\mathcal{E}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is co-nilpotent. So the result follows from Lemma 9.5.1. $\square$

Let $\iota\colon \operatorname{Spec}(A \otimes R) \to C \times X$ be the natural open immersion.

Definition 3.3. A *global model* of $\mathcal{H}\mathrm{om}(M, a(F/A, R))$ is a subshtuka

$$\mathcal{M} \subset \iota_* \mathcal{H}\mathrm{om}(M, a(F/A, R))$$

which has the following properties:

(1) $\mathcal{M}$ is a locally free shtuka on $C \times X$ which coincides with the shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, R))$ on $\operatorname{Spec}(A \otimes R)$.

(2) $\mathcal{M}(\mathcal{O}_F \otimes R)$ coincides with $\mathcal{M}^c$ as a subshtuka of

$$\mathcal{H}\mathrm{om}(M, a(F, R)) = \mathcal{H}\mathrm{om}(M, a(F/A, R)) \otimes_{A\otimes R} (F \otimes R).$$

The equality holds by Lemma 2.2 (1).

(3) $\mathcal{M}(A \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathcal{M}(A \otimes \mathcal{O}_K/\mathfrak{f})$ is linear.

This definition has been designed to make the following two results true.

Proposition 3.4. *If $\mathcal{M}$ is a global model then $\mathcal{M}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ is a local model in the sense of Definition 9.6.7.*

Proof. The shtuka $\mathcal{M}(F \breve{\otimes} K)$ can be identified with $\mathcal{H}\mathrm{om}(M, a(F, K))$ as follows. By Lemma 2.2 (3) the $A \otimes R$-module shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, R))$ is a lattice in the $F \breve{\otimes} K$-module shtuka $\mathcal{H}\mathrm{om}(M, a(F, K))$. As the restriction of $\mathcal{M}$ to $\operatorname{Spec} A \otimes R$ coincides with $\mathcal{H}\mathrm{om}(M, a(F/A, R))$ we conclude that $\mathcal{M}(F \breve{\otimes} K) = \mathcal{H}\mathrm{om}(M, a(F, K))$.

The shtuka $\mathcal{M}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ is a locally free $\mathcal{O}_F \breve{\otimes} \mathcal{O}_K$-lattice in $\mathcal{M}(F \breve{\otimes} K) = \mathcal{H}\mathrm{om}(M, a(F, K))$. To prove that $\mathcal{M}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ is a local model of $\mathcal{H}\mathrm{om}(M, a(F, K))$ we need to show the following:

(1) $\mathcal{M}(\mathcal{O}_F \breve{\otimes} K)$ is the coefficient compactification in the sense of Definition 9.6.1.

(2) $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{f})$ is linear.

It follows from Lemma 3.8.10 that $a(F/\mathcal{O}_F, R)$ is an $A \otimes R$-lattice in the $F \otimes R$-module $a(F/\mathcal{O}_F, K)$. Hence the pullback of $\mathcal{M}^c$ to $\mathcal{O}_F \breve{\otimes} K$ is

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_F \breve{\otimes} K}(\mathcal{E}(\mathcal{O}_F \breve{\otimes} K), a(F/\mathcal{O}_F, K)).$$

Since the construction of Theorem 7.5.5 is compatible with base change we conclude that the shtuka above is the coefficient compactification of the shtuka $\mathcal{H}\mathrm{om}(M, a(F, K))$ in the sense of Definition 9.6.1. We thus get (1) and (2) follows directly from the definition of a global model. □

Proposition 3.5. *If $\mathcal{M}$ is a global model then the restriction of $\mathcal{M}$ to $\mathcal{O}_F \times X$ is an elliptic shtuka of conductor $\mathfrak{f}$ in the sense of Definition 6.8.2.*

Proof. We need to prove the following:

(1) $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes R)$ is nilpotent.

(2) $\mathcal{M}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ is an elliptic shtuka of conductor $\mathfrak{f}$.

Now $\mathcal{M}(\mathcal{O}_F \otimes R) = \mathcal{M}^c$ so (1) follows by Lemma 3.2. According to Proposition 3.4 the shtuka $\mathcal{M}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ is a local model so (2) follows by Theorem 9.9.6. □

4. Existence of global models

In this section ω denotes the invertible sheaf of Kähler differentials on the coefficient curve $C/\mathbb{F}_q$. We write $\omega(A)$ and $\omega(\mathcal{O}_F)$ for its sections on $\operatorname{Spec} A$ and $\operatorname{Spec} \mathcal{O}_F$ respectively. Let $\iota^c \colon \operatorname{Spec} A \otimes R \to C \times R$ be the natural map.

Lemma 4.1. *There exists a subshtuka $\mathcal{M} \subset \iota^c_* \mathcal{H}\mathrm{om}(M, a(F/A, R))$ with the following properties:*

(1) *$\mathcal{M}$ is a locally free shtuka on $C \times R$.*
(2) *$\mathcal{M}(A \otimes R) = \mathcal{H}\mathrm{om}(M, a(F/A, R))$.*
(3) *$\mathcal{M}(\mathcal{O}_F \otimes R)$ coincides with the coefficient compactification $\mathcal{M}^c \subset \mathcal{H}\mathrm{om}(M, a(F, R))$ of Definition 3.1.*

Proof. Theorem 7.5.5 provides us with a shtuka

$$\mathcal{E} = \Big[\mathcal{E}_{-1} \underset{j_{\mathcal{E}}}{\overset{i_{\mathcal{E}}}{\rightrightarrows}} \mathcal{E}_0\Big] \subset \iota^c_*\Big[M \underset{\tau}{\overset{1}{\rightrightarrows}} M\Big].$$

Consider the shtuka $\mathcal{M}$ on $C \times R$ given by the diagram

$$\mathcal{H}\mathrm{om}_{C\times R}(\mathcal{E}_0, \omega \otimes R) \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{H}\mathrm{om}_{C\times R}(\mathcal{E}_{-1}, \omega \otimes R).$$

Here $\mathcal{H}\mathrm{om}_{C\times R}$ denotes the sheaf Hom on $C \times R$ and $\omega \otimes R$ is the pullback of ω to $C \times R$. The maps i and j are given by the formulas

$$\begin{aligned} i(f) &= f \circ j^a_{\mathcal{E}}, \\ j(f) &= \tau^a \circ \tau^*(f) \circ \tau^*(i_{\mathcal{E}}) \end{aligned}$$

where $j^a_{\mathcal{E}} \colon \tau^*\mathcal{E}_{-1} \to \mathcal{E}_0$ is the adjoint of $j_{\mathcal{E}}$ and $\tau^a \colon \tau^*(\omega \otimes R) \to \omega \otimes R$ is the adjoint of the map $1 \otimes \tau_R$.

By construction $\mathcal{M}$ restricts to $\mathcal{H}\mathrm{om}(M, \omega(A) \otimes R)$ on $\operatorname{Spec} A \otimes R$ and to $\mathcal{H}\mathrm{om}(\mathcal{E}(\mathcal{O}_F \otimes R), \omega(\mathcal{O}_F) \otimes R)$ on $\operatorname{Spec} \mathcal{O}_F \otimes R$. Now Lemma 8.3.1 identifies $\omega(A) \otimes R$ with $a(F/A, R)$. Combining Lemma 3.8.10 with Theorem 3.10.1 we get an identification $a(F/\mathcal{O}_F, R) = \omega(\mathcal{O}_F) \otimes R$. Whence the result. □

Let $\iota^b \colon \operatorname{Spec} A \otimes R \to A \times X$ be the natural map.

Lemma 4.2. *There exists a subshtuka $\mathcal{M} \subset \iota^b_* \mathcal{H}\mathrm{om}(M, a(F/A, R))$ with the following properties:*

(1) *$\mathcal{M}$ is a locally free shtuka on $A \times X$.*
(2) *$\mathcal{M}(A \otimes R) = \mathcal{H}\mathrm{om}(M, a(F/A, R))$.*
(3) *$\mathcal{M}(A \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathcal{M}(A \otimes \mathcal{O}_K/\mathfrak{f})$ is linear.*

Proof. By Theorem 9.7.5 the shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, K))$ admits an $A \otimes \mathcal{O}_K$-lattice $\mathcal{M}^b$ lattice with the following properties:

(1) $\mathcal{M}^b$ is locally free.
(2) $\mathcal{M}^b(A \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathcal{M}^b(A \otimes \mathcal{O}_K/\mathfrak{f})$ is linear.

Using the Beauville-Laszlo Theorem [0BP2] we glue the $A \otimes R$-module shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, R))$ to the $A \mathbin{\widehat{\otimes}} \mathcal{O}_K$-module shtuka $\mathcal{M}^b(A \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ over $A \mathbin{\widehat{\otimes}} K$ and obtain the desired shtuka $\mathcal{M}$ on $A \times X$. □

Let U denote the union of open subschemes $C \times R$ and $A \times X$ in $C \times X$ and let $\iota^\infty \colon U \to C \times X$ be the open immersion. Note that the complement of U in $C \times X$ consists of finitely many points.

Lemma 4.3. *If $\mathcal{F}$ is a locally free sheaf on U then $\iota^\infty_*\mathcal{F}$ is a locally free sheaf on $C \times X$.*

Proof. Consider the cartesian square of schemes

$$\begin{tikzcd} U' \arrow[r, "\iota"] \arrow[d] & \operatorname{Spec} \mathcal{O}_F \check{\otimes} \mathcal{O}_K \arrow[d] \\ U \arrow[r, "\iota^\infty"] & C \times X \end{tikzcd}$$

where U' is the complement of the closed points of $\operatorname{Spec} \mathcal{O}_F \check{\otimes} \mathcal{O}_K$. Observe that the pullback of $\iota^\infty_*\mathcal{F}$ to $\mathcal{O}_F \check{\otimes} \mathcal{O}_K$ coincides with $\iota_*\mathcal{F}'$ where $\mathcal{F}'$ is the pullback of $\mathcal{F}$ to U'. Now Lemmas 9.2.4 and 9.2.5 imply that $\iota_*\mathcal{F}'$ is a locally free sheaf on $\operatorname{Spec} \mathcal{O}_F \check{\otimes} \mathcal{O}_K$. Whence $\iota^\infty_*\mathcal{F}$ is locally free. $\square$

Proposition 4.4. *The shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, R))$ admits a global model.*

Proof. Lemmas 4.1 and 4.2 give us a locally free shtuka $\mathcal{M}$ on the union U of $C \times R$ and $A \times X$. By Lemma 4.3 we extend it to a locally free shtuka $\iota^\infty_*\mathcal{M}$ on $C \times X$. By construction this shtuka is a global model. $\square$

5. Cohomology of the Hom shtukas

Definition 5.1. The *complex of units* of the Drinfeld module E is the A-module complex

$$U_E = \Big[\operatorname{Lie}_E(K) \xrightarrow{\exp} \frac{E(K)}{E(R)}\Big]$$

where $\exp\colon \operatorname{Lie}_E(K) \to E(K)$ is the exponential map of E.

Our goal is to construct quasi-isomorphisms

$$\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{H}\mathrm{om}(M, a(F/A, R))) \xrightarrow{\sim} U_E[-1],$$
$$\mathrm{R}\Gamma_{\mathrm{c}}(\nabla\, \mathcal{H}\mathrm{om}(M, a(F/A, R))) \xrightarrow{\sim} \operatorname{Lie}_E(R)[-1]$$

where $\mathrm{R}\Gamma_{\mathrm{c}}$ is the compactly supported cohomology of shtukas on $A \otimes R$ (Definition 4.5.1). In the next section we will use these quasi-isomorphisms to study the cohomology of global models.

We first study the germ cohomology

$$\mathrm{R}\Gamma_g(A \otimes K,\, \mathcal{H}\mathrm{om}(M, a(F/A, K))),$$
$$\mathrm{R}\Gamma_g(A \otimes K,\, \nabla\, \mathcal{H}\mathrm{om}(M, a(F/A, K)))$$

(Definition 4.1.1). To improve legibility we will write $\mathrm{R}\Gamma_g(-)$ in place of $\mathrm{R}\Gamma_g(A\otimes K,-)$. By Lemma 9.4.5 the shtuka $\mathcal{H}\mathrm{om}(M,a(F/A,K))$ is an $A\otimes K$-lattice in the shtuka $\mathcal{H}\mathrm{om}(M,b(F/A,K))$. So we have natural quasi-isomorphisms

$$\begin{aligned}&\mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M,a(F/A,K)))\cong\\&\qquad\cong\Big[\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,a(F/A,K)))\to\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,b(F/A,K)))\Big]\end{aligned}$$

and

$$\begin{aligned}&\mathrm{R}\Gamma_g(\nabla\,\mathcal{H}\mathrm{om}(M,a(F/A,K)))\cong\\&\qquad\cong\Big[\mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M,a(F/A,K)))\to\mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M,b(F/A,K)))\Big]\end{aligned}$$

by definition of germ cohomology. According to Lemma 8.10.4 the natural sequence

$$\begin{aligned}&0\to\mathcal{H}\mathrm{om}(M,a(F/A,K))\to\mathcal{H}\mathrm{om}(M,b(F/A,K))\to\\&\qquad\qquad\to\mathcal{H}\mathrm{om}(M,g(F/A,K))\to 0\end{aligned}$$

is exact. We thus obtain natural quasi-isomorphisms

$$\mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M,a(F/A,K)))\cong\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,g(F/A,K)))[-1],\tag{5.1}$$

$$\mathrm{R}\Gamma_g(\nabla\,\mathcal{H}\mathrm{om}(M,a(F/A,K)))\cong\mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M,g(F/A,K)))[-1].\tag{5.2}$$

Definition 5.2. We define natural quasi-isomorphisms

$$\begin{aligned}\mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M,a(F/A,K)))&\cong\mathrm{Lie}_E(K)[-1],\\\mathrm{R}\Gamma_g(\nabla\,\mathcal{H}\mathrm{om}(M,a(F/A,K)))&\cong\mathrm{Lie}_E(K)[-1]\end{aligned}$$

as the compositions

$$\begin{aligned}&\mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M,a(F/A,K)))\xrightarrow{(5.1)}\\&\qquad\longrightarrow\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,g(F/A,K)))[-1]\xrightarrow{\sim}\mathrm{Lie}_E(K)[-1]\end{aligned}$$

and

$$\begin{aligned}&\mathrm{R}\Gamma_g(\nabla\,\mathcal{H}\mathrm{om}(M,a(F/A,K)))\xrightarrow{(5.2)}\\&\qquad\longrightarrow\mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M,g(F/A,K)))[-1]\xrightarrow{\sim}\mathrm{Lie}_E(K)[-1]\end{aligned}$$

where the unlabelled arrows are the quasi-isomorphisms of Propositions 8.10.6 and 8.10.5 respectively.

In Section 4.5 we constructed a natural morphism $\mathrm{R}\Gamma_c(\mathcal{M})\to\mathrm{R}\Gamma_g(A\otimes K,\mathcal{M})$ for every quasi-coherent shtuka $\mathcal{M}$ on $A\otimes R$. By Lemma 2.2 (2) the shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, R))$ is an $A \otimes R$-lattice in $\mathcal{H}\mathrm{om}(M, a(F/A, K))$. We thus get natural morphisms

$$\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{H}\mathrm{om}(M, a(F/A, R))) \to \mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M, a(F/A, K))) \tag{5.3}$$

$$\mathrm{R}\Gamma_{\mathrm{c}}(\nabla\, \mathcal{H}\mathrm{om}(M, a(F/A, R))) \to \mathrm{R}\Gamma_g(\nabla\, \mathcal{H}\mathrm{om}(M, a(F/A, K))). \tag{5.4}$$

Proposition 5.3. *There exists a quasi-isomorphism*

$$\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{H}\mathrm{om}(M, a(F/A, R))) \xrightarrow{\sim} U_E[-1]$$

such that the square

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{H}\mathrm{om}(M, a(F/A, R))) \arrow[r, "(5.3)"] \arrow[d, "\wr"'] & \mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M, a(F/A, K))) \arrow[d, "\wr"] \\
U_E[-1] \arrow[r] & \mathrm{Lie}_E(K)[-1]
\end{tikzcd}$$

is commutative. Here the right arrow is the quasi-isomorphism of Definition 5.2 and the bottom arrow is given by the identity map in degree 1*.*

Proof. We split the construction into several steps.

Step 1. Lemma 2.2 (4) tells us that $\mathcal{H}\mathrm{om}(M, a(F/A, R))$ is an $A \otimes R$-lattice in the $A \widehat{\otimes} K$-module shtuka $\mathcal{H}\mathrm{om}(M, b(F/A, K))$. So we have a natural quasi-isomorphism

$$\begin{aligned}
&\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{H}\mathrm{om}(M, a(F/A, R))) \cong \\
&\qquad \cong \Big[\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, a(F/A, R))) \to \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, b(F/A, K)))\Big]
\end{aligned}$$

by definition of compactly supported cohomology (Definition 4.5.1). As explained above we also have a natural quasi-isomorphism

$$\begin{aligned}
&\mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M, a(F/A, K))) \cong \\
&\qquad \cong \Big[\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, a(F/A, K))) \to \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, b(F/A, K)))\Big].
\end{aligned}$$

Under these identifications the canonical map

$$\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{H}\mathrm{om}(M, a(F/A, R))) \to \mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M, a(F/A, K))) \tag{5.5}$$

is given by the natural map of complexes on the right hand side.

Step 2. Let ω be the module of Kähler differentials of A over $\mathbb{F}_q$. Lemma 8.3.1 identifies $\omega \otimes R$ with $a(F/A, R)$. We thus get quasi-isomorphisms

$$\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, a(F/A, R))) \cong E(R)[-1], \tag{5.6}$$

$$\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, a(F/A, K))) \cong E(K)[-1] \tag{5.7}$$

by Theorem 8.9.4. Moreover Theorem 8.10.7 provides us with a quasi-isomorphism

$$\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, b(F/A, K))) \cong C_{\exp} \tag{5.8}$$

where

$$C_{\exp} = \left[\mathrm{Lie}_E(K) \xrightarrow{\exp} E(K)\right].$$

According to Theorem 8.10.7 the square

$$\begin{tikzcd}
\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, a(F/A, K))) \arrow[r] \arrow[d, "(5.7)"', "\wr"] & \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, b(F/A, K))) \arrow[d, "\wr"', "(5.8)"] \\
E(K)[-1] \arrow[r] & C_{\exp}
\end{tikzcd}$$

is commutative. Here the bottom arrow is given by the identity in degree 1. As the quasi-isomorphisms (5.6) and (5.7) of Theorem 8.9.4 are natural we also get a commutative square

$$\begin{tikzcd}
\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, a(F/A, R))) \arrow[r] \arrow[d, "(5.6)"', "\wr"] & \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, b(F/A, K))) \arrow[d, "\wr"', "(5.8)"] \\
E(R)[-1] \arrow[r] & C_{\exp}
\end{tikzcd}$$

where the bottom arrow is given by the natural map $E(R) \to E(K)$ in degree 1.

Step 3. Combining the two squares of Step 2 with Step 1 we get a commutative square

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{H}\mathrm{om}(M, a(F/A, R))) \arrow[r] \arrow[d, "\wr"'] & \mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M, a(F/A, K))) \arrow[d, "\wr"] \\
\left[E(R)[-1] \to C_{\exp}\right] \arrow[r] & \left[E(K)[-1] \to C_{\exp}\right].
\end{tikzcd}$$

Together with the canonical quasi-isomorphisms $[E(R)[-1] \to C_{\exp}] \cong U_E[-1]$ and $[E(K)[-1] \to C_{\exp}] \cong \mathrm{Lie}_E(K)[-1]$ it gives us a commutative square of the form

$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathrm{c}}(\mathcal{H}\mathrm{om}(M, a(F/A, R))) \arrow[r] \arrow[d, "\wr"'] & \mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M, a(F/A, K))) \arrow[d, "\wr"] \\
U_E[-1] \arrow[r] & \mathrm{Lie}_E(K)[-1].
\end{tikzcd}$$

Step 4. It remains to verify that the right arrow in the square above is as in the statement of the proposition. Theorem 8.10.7 shows that the square

$$\begin{tikzcd}
\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, b(F/A, K))) \arrow[r] \arrow[d, "(5.8)"', "\wr"] & \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, g(F/A, K))) \\
C_{\exp} \arrow[r] & \mathrm{Lie}_E(K)[0] \arrow[u, "\wr"']
\end{tikzcd}$$

is commutative. Here the bottom arrow is given by the identity in degree 1 and the right arrow is the quasi-isomorphism of Proposition 8.10.6. As a consequence the quasi-isomorphism

$$\mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M, a(F/A, K))) \cong \mathrm{Lie}_E(K)[-1]$$

of Step 3 coincides with the quasi-isomorphism of Definition 5.2. Whence the result. ☐

Proposition 5.4. *There exists a quasi-isomorphism*

$$\mathrm{R}\Gamma_c(\nabla\,\mathcal{H}\mathrm{om}(M, a(F/A, R))) \xrightarrow{\sim} \mathrm{Lie}_E(R)[-1]$$

such that the square

$$\begin{tikzcd} \mathrm{R}\Gamma_c(\nabla\,\mathcal{H}\mathrm{om}(M, a(F/A, R))) \arrow[r, "(5.4)"] \arrow[d, "\wr"'] & \mathrm{R}\Gamma_g(\nabla\,\mathcal{H}\mathrm{om}(M, a(F/A, K))) \arrow[d, "\wr"] \\ \mathrm{Lie}_E(R)[-1] \arrow[r] & \mathrm{Lie}_E(K)[-1] \end{tikzcd}$$

is commutative. Here the right arrow is the quasi-isomorphism of Definition 5.2 and the bottom arrow is given by the natural map $\mathrm{Lie}_E(R) \to \mathrm{Lie}_E(K)$ *in degree* 1*.*

Proof. Same as the proof of Proposition 5.3. ☐

6. Cohomology of global models

Let $\mathcal{M}$ be a global model. In this section we construct quasi-isomorphisms

$$\mathrm{R}\Gamma(A \times X, \mathcal{M}) \xrightarrow{\sim} U_E[-1],$$
$$\mathrm{R}\Gamma(A \times X, \nabla\mathcal{M}) \xrightarrow{\sim} \mathrm{Lie}_E(R)[-1].$$

According to Proposition 3.4 the shtuka $\mathcal{M}(\mathcal{O}_F \check{\otimes} \mathcal{O}_K)$ is a local model. So Lemma 9.9.2 shows that the maps of Definition 9.9.1 induce F-linear quasi-isomorphisms

$$\gamma\colon\ \mathrm{R}\Gamma(F \check{\otimes} \mathcal{O}_K, \mathcal{M}) \xrightarrow{\sim} \mathrm{Lie}_E(K)[-1],$$
$$\nabla\gamma\colon\ \mathrm{R}\Gamma(F \check{\otimes} \mathcal{O}_K, \nabla\mathcal{M}) \xrightarrow{\sim} \mathrm{Lie}_E(K)[-1]$$

Theorem 6.1. *Let* $\mathcal{M}$ *be a global model. There exists a quasi-isomorphism*

$$\mathrm{R}\Gamma(A \times X, \mathcal{M}) \xrightarrow{\sim} U_E[-1]$$

such that the square

$$\begin{tikzcd} \mathrm{R}\Gamma(A \times X, \mathcal{M}) \arrow[r] \arrow[d, "\wr"'] & \mathrm{R}\Gamma(F \check{\otimes} \mathcal{O}_K, \mathcal{M}) \arrow[d, "\wr\ \gamma"] \\ U_E[-1] \arrow[r] & \mathrm{Lie}_E(K)[-1] \end{tikzcd}$$

is commutative. Here the top arrow is the pullback morphism and the bottom arrow is given by the identity in degree 1*.*

Proof. As the restriction of $\mathcal{M}$ to $\operatorname{Spec} A \otimes R$ is $\mathcal{H}\mathrm{om}(M, a(F/A, R))$ we get a quasi-isomorphism

$$\mathrm{R}\Gamma_{\mathrm{c}}(A \otimes R, \mathcal{M}) \cong U_E[-1].$$

by Proposition 5.3. The shtuka $\mathcal{M}(A \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent so the natural map

$$\mathrm{R}\Gamma_{\mathrm{c}}(A \otimes R, \mathcal{M}) \to \mathrm{R}\Gamma(A \times X, \mathcal{M})$$

is a quasi-isomorphism by Proposition 4.5.2. We claim that the resulting quasi-isomorphism

$$\mathrm{R}\Gamma(A \times X, \mathcal{M}) \cong U_E[-1] \tag{6.1}$$

makes the square above commutative. We verify it in several steps.

Step 1. Consider the square

$$\begin{tikzcd}
U_E[-1] \arrow[r] & \mathrm{Lie}_E(K)[-1] \\
\mathrm{R}\Gamma(A \times X, \mathcal{M}) \arrow[u, "(6.1)", "\wr"'] \arrow[r, "\text{global}"] & \mathrm{R}\Gamma_g(A \otimes K, \mathcal{M}) \arrow[u, "\wr"]
\end{tikzcd}$$

where the top arrow is given by the identity map in degree 1, the right arrow is the quasi-isomorphism of Definition 5.2 and the arrow labelled "global" is the global germ map

$$\mathrm{R}\Gamma(A \times X, \mathcal{M}) \xleftarrow{\sim} \mathrm{R}\Gamma_{\mathrm{c}}(A \otimes R, \mathcal{M}) \to \mathrm{R}\Gamma_g(A \otimes K, \mathcal{M})$$

of Definition 4.5.3. Proposition 5.3 implies that this square is commutative.

Step 2. Consider the diagram

$$\begin{tikzcd}
\mathrm{R}\Gamma(A \times X, \mathcal{M}) \arrow[r, "\text{global}"] \arrow[d] & \mathrm{R}\Gamma_g(A \otimes K, \mathcal{M}) \arrow[d] \\
\mathrm{R}\Gamma(F \times X, \mathcal{M}) \arrow[r, "\text{global}"] \arrow[d] & \mathrm{R}\Gamma_g(F \check{\otimes} K, \mathcal{M}) \arrow[d, equal] \\
\mathrm{R}\Gamma(F \check{\otimes} \mathcal{O}_K, \mathcal{M}) \arrow[r, "\text{local}"] & \mathrm{R}\Gamma_g(F \check{\otimes} K, \mathcal{M})
\end{tikzcd}$$

where the arrow labelled "local" is the local germ map of Definition 4.2.4 and the unlabelled arrows are the pullback morphisms. The top square of this diagram commutes by naturality of the global germ map. The commutativity of the bottom square follows from Theorem 4.6.1.

Step 3. Combining Step 1 and Step 2 we get a commutative diagram

$$\begin{tikzcd}
U_E[-1] \arrow[r] & \mathrm{Lie}_E(K)[-1] \\
\mathrm{R}\Gamma(A \times X, \mathcal{M}) \arrow[u, "(6.1)", "\wr"'] \arrow[r, "\text{global}"] \arrow[d] & \mathrm{R}\Gamma_g(A \otimes K, \mathcal{M}) \arrow[u, "\wr", "\text{Definition 5.2}"'] \arrow[d] \\
\mathrm{R}\Gamma(F \breve{\otimes} \mathcal{O}_K, \mathcal{M}) \arrow[r, "\text{local}"] & \mathrm{R}\Gamma_g(F \breve{\otimes} K, \mathcal{M}).
\end{tikzcd}$$

Step 4. We claim that the arrow $\mathrm{R}\Gamma_g(A \otimes K, \mathcal{M}) \to \mathrm{R}\Gamma_g(F \breve{\otimes} K, \mathcal{M})$ is a quasi-isomorphism and that the composition

$$\mathrm{R}\Gamma(F \breve{\otimes} \mathcal{O}_K, \mathcal{M}) \xrightarrow{\text{local}} \mathrm{R}\Gamma_g(F \breve{\otimes} K, \mathcal{M}) \xleftarrow{\sim} \\ \xleftarrow{\sim} \mathrm{R}\Gamma_g(A \otimes K, \mathcal{M}) \xrightarrow{\text{Def. 5.2}} \mathrm{Lie}_E(K)[-1]$$

coincides with γ. Together with Step 3 this claim immediately implies the theorem.

To prove this claim we first observe that the square

$$\begin{tikzcd}
\mathrm{R}\Gamma_g(A \otimes K, \mathcal{M}) \arrow[r, "\text{Def. 5.2}", "\sim"'] \arrow[d] & \mathrm{Lie}_E(K)[-1] \arrow[d, equal] \\
\mathrm{R}\Gamma_g(F \breve{\otimes} K, \mathcal{M}) \arrow[r, "\text{Def. 9.8.3}", "\sim"'] & \mathrm{Lie}_E(K)[-1]
\end{tikzcd}$$

is commutative by construction. By definition the quasi-isomorphism γ is the composition

$$\mathrm{R}\Gamma(F \breve{\otimes} \mathcal{O}_K, \mathcal{M}) \xrightarrow{\text{local}} \mathrm{R}\Gamma_g(F \breve{\otimes} K, \mathcal{M}) \xrightarrow{\text{Def. 9.8.3}} \mathrm{Lie}_E(K)[-1].$$

Hence the claim and the theorem follow. ☐

Theorem 6.2. *Let $\mathcal{M}$ be a global model. There exists a quasi-isomorphism*

$$\mathrm{R}\Gamma(A \times X, \nabla\mathcal{M}) \xrightarrow{\sim} \mathrm{Lie}_E(R)[-1]$$

such that the square

$$\begin{tikzcd}
\mathrm{R}\Gamma(A \times X, \mathcal{M}) \arrow[r] \arrow[d, "\wr"] & \mathrm{R}\Gamma(F \breve{\otimes} \mathcal{O}_K, \mathcal{M}) \arrow[d, "\wr"', "\nabla\gamma"] \\
\mathrm{Lie}_E(R)[-1] \arrow[r] & \mathrm{Lie}_E(K)[-1]
\end{tikzcd}$$

is commutative. Here the top arrow is the pullback morphism and the bottom arrow is given by the natural map $\mathrm{Lie}_E(R) \to \mathrm{Lie}_E(K)$ *in degree* 1.

Proof. Same as the proof of Theorem 6.1. ☐

The constructions of quasi-isomorphisms in Propositions 5.3 and 5.4 involve no choices. So the quasi-isomorphisms of Theorems 6.1 and 6.2 are in fact canonical. Unfortunately we know no easy way to characterize them as morphisms in the derived category of A-modules. Still one can prove that these quasi-isomorphisms are natural with respect to the global model $\mathcal{M}$. In a suitable sense they are natural with respect to the Drinfeld module E too.

7. Regulators

Definition 7.1. The *arithmetic regulator*

$$\rho_E\colon F \otimes_A U_E \to \mathrm{Lie}_E(K)[0]$$

of the Drinfeld module E is the F-linear extension of the morphism $U_E \to \mathrm{Lie}_E(K)[0]$ given by the identity in degree zero.

Taelman [23] demonstrated that ρ_E is a quasi-isomorphism. This will also follow from the results below.

Let $\mathcal{M}$ be a global model. By Proposition 3.5 the restriction of $\mathcal{M}$ to $\mathcal{O}_F \times X$ is an elliptic shtuka of conductor $\mathfrak{f}$. We can thus make the following definition.

Definition 7.2. The *regulator*

$$\rho\colon\ \mathrm{R}\Gamma(\mathcal{O}_F \times X, \mathcal{M}) \xrightarrow{\sim} \mathrm{R}\Gamma(\mathcal{O}_F \times X, \nabla\mathcal{M})$$

of a global model $\mathcal{M}$ is the regulator of the elliptic shtuka $\mathcal{M}|_{\mathcal{O}_F\times X}$ (Definition 6.8.5). We will also denote ρ its F-linear extension

$$\rho\colon\ \mathrm{R}\Gamma(F \times X, \mathcal{M}) \xrightarrow{\sim} \mathrm{R}\Gamma(F \times X, \nabla\mathcal{M}).$$

Theorem 7.3. *If $\mathcal{M}$ is a global model then the square*

$$\begin{tikzcd}
\mathrm{R}\Gamma(F \times X, \mathcal{M}) \arrow[r, "\rho"] \arrow[d, "\wr"'] & \mathrm{R}\Gamma(F \times X, \nabla\mathcal{M}) \arrow[d, "\wr"] \\
F \otimes_A U_E[-1] \arrow[r, "\rho_E[-1]"] & \mathrm{Lie}_E(K)[-1]
\end{tikzcd}$$

is commutative. Here the vertical arrows are the F-linear extensions of the quasi-isomorphisms of Theorems 6.1, 6.2.

Proof. According to Proposition 3.4 the shtuka $\mathcal{M}(\mathcal{O}_F \check{\otimes} \mathcal{O}_K)$ is a local model of $\mathcal{H}\mathrm{om}(M, a(F,K))$. So it is an elliptic shtuka of conduction $\mathfrak{f}$ by Theorem 9.9.6. As such it has a regulator

$$\mathrm{R}\Gamma(\mathcal{O}_F \check{\otimes} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(\mathcal{O}_F \check{\otimes} \mathcal{O}_K, \nabla\mathcal{M}).$$

By Lemma 9.9.2 its F-linear extension can be identified with a quasi-isomorphism

$$\check{\rho}\colon\ \mathrm{R}\Gamma(F \check{\otimes} \mathcal{O}_K, \mathcal{M}) \to \mathrm{R}\Gamma(F \check{\otimes} \mathcal{O}_K, \nabla\mathcal{M}).$$

By definition of the regulator (Definition 6.8.5) the natural square

$$\begin{tikzcd}
\mathrm{R}\Gamma(F \times X, \mathcal{M}) \arrow[r, "\rho"] \arrow[d, "\wr"'] & \mathrm{R}\Gamma(F \times X, \nabla\mathcal{M}) \arrow[d, "\wr"] \\
\mathrm{R}\Gamma(F \,\breve{\otimes}\, \mathcal{O}_K, \mathcal{M}) \arrow[r, "\breve{\rho}"] & \mathrm{R}\Gamma(F \,\breve{\otimes}\, \mathcal{O}_K, \nabla\mathcal{M})
\end{tikzcd}$$

is commutative. Moreover Lemma 6.8.4 implies that the vertical arrows in this square are quasi-isomorphisms. Now consider the diagram

$$\begin{tikzcd}
F \otimes_A \mathrm{R}\Gamma(A \times X, \mathcal{M}) \arrow[r, "\sim"] \arrow[d] & F \otimes_A U_E[-1] \arrow[d, "\wr"', "\rho_E"] \\
\mathrm{R}\Gamma(F \,\breve{\otimes}\, \mathcal{O}_K, \mathcal{M}) \arrow[r, "\gamma", "\sim"'] \arrow[d, "\breve{\rho}"'] & \mathrm{Lie}_E(K)[-1] \arrow[d, equal] \\
\mathrm{R}\Gamma(F \,\breve{\otimes}\, \mathcal{O}_K, \nabla\mathcal{M}) \arrow[r, "\nabla\gamma", "\sim"'] & \mathrm{Lie}_E(K)[-1] \\
F \otimes_A \mathrm{R}\Gamma(A \times X, \nabla\mathcal{M}) \arrow[r, "\sim"] \arrow[u] & F \otimes_A \mathrm{Lie}_E(R)[-1] \arrow[u, "\wr"']
\end{tikzcd}$$

The top and bottom squares are induced by the commutative squares of Theorems 6.1 and 6.2 respectively. The middle square commutes by Theorem 11.7.1.

Observe that the composition

$$F \otimes_A U_E[-1] \xrightarrow{\sim} F \otimes_A \mathrm{R}\Gamma(A \times X, \mathcal{M}) \to \mathrm{R}\Gamma(F \,\breve{\otimes}\, \mathcal{O}_K, \mathcal{M}) \xleftarrow{\sim} \mathrm{R}\Gamma(F \times X, \mathcal{M})$$

is the inverse of the arrow $F \otimes_A U_E[-1] \xrightarrow{\sim} \mathrm{R}\Gamma(F \times X, \mathcal{M})$ in the statement of the theorem. An analogous observation applies to the right arrow. We thus get the result. □

8. Euler products

Given a maximal ideal $\mathfrak{p}$ of A we denote $A_\mathfrak{p}$ the completion of A at $\mathfrak{p}$ and $F_\mathfrak{p}$ the field of fractions of $A_\mathfrak{p}$. Let F_0 be the fraction field of A and Ω the module of Kähler differentials of F_0 over $\mathbb{F}_q$. Let K_0 be the fraction field of R and K_0^{sep} a separable closure of K_0. We denote $G = \mathrm{Aut}(K_0^{\mathrm{sep}}/K_0)$ the corresponding Galois group.

With the choice of K_0^{sep} fixed we have for every prime $\mathfrak{p}$ of A the $\mathfrak{p}$-adic Tate module

$$T_\mathfrak{p} E = \mathrm{Hom}_A(F_\mathfrak{p}/A_\mathfrak{p},\, E(K_0^{\mathrm{sep}})).$$

Let $\mu\colon A \to R$ be the natural homomorphism. If $\mathfrak{m} \subset R$ is a maximal ideal such that $\mathfrak{p} \neq \mu^{-1}(\mathfrak{m})$ then $T_\mathfrak{p} E$ is unramified at $\mathfrak{m}$. We denote

$$P_\mathfrak{m}(T) = \det{}_{A_\mathfrak{p}}\big(1 - T\sigma_\mathfrak{m}^{-1} \,\big|\, T_\mathfrak{p} E\big) \in A_\mathfrak{p}[T]$$

where $\sigma_{\mathfrak{m}}^{-1} \in G$ is a geometric Frobenius element at $\mathfrak{m}$.

Proposition 8.1. *The characteristic polynomial $P_{\mathfrak{m}}$ has coefficients in F_0 and is independent of the choice of $\mathfrak{p}$.*

Proof. Theorem 8.11.9 implies that

$$P_{\mathfrak{m}}(T^d) = \det_{F_0}\big(1 - T(i^{-1}j) \,\big|\, \mathrm{Hom}(M,\, \Omega \otimes R/\mathfrak{m})\big)$$

where i and j are the arrows of the shtuka $\mathcal{H}\mathrm{om}(M,\, \Omega \otimes R/\mathfrak{m})$. □

Definition 8.2. We define the formal product $L(E^*, 0) \in F$ as follows:

$$L(E^*, 0) = \prod_{\mathfrak{m}} \frac{1}{P_{\mathfrak{m}}(1)}$$

where $\mathfrak{m} \subset R$ ranges over the maximal ideals.

In a moment we will see that this product indeed converges.

Proposition 8.3. *Let $\mathcal{M}$ be a global model. For every maximal ideal $\mathfrak{m}$ the invariant $L(\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m})) \in \mathcal{O}_F$ of Definition 6.9.2 coincides with $P_{\mathfrak{m}}(1) \in F_0$ as an element of F.*

Proof. Suppose that $\mathcal{M}$ is given by the diagram

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1.$$

According to Proposition 3.5 the restriction of $\mathcal{M}$ to $\mathcal{O}_F \times X$ is an elliptic shtuka. In particular the shtuka $\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m})$ is nilpotent so that its i-arrow is invertible. By definition

$$L(\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m})) = \det_{\mathcal{O}_F}\big(1 - i^{-1}j \,\big|\, \mathcal{M}_0(\mathcal{O}_F \otimes R/\mathfrak{m})\big).$$

As $\mathcal{M}$ is a global model the shtuka $\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m})$ is a locally free $\mathcal{O}_F \otimes R/\mathfrak{m}$-lattice in the $F \otimes R/\mathfrak{m}$-module shtuka $\mathcal{H}\mathrm{om}(M,\, a(F, R/\mathfrak{m}))$. Hence

$$L(\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m})) = \det_F\big(1 - i^{-1}j \,\big|\, \mathrm{Hom}(M,\, a(F, R/\mathfrak{m}))\big).$$

Next $\Omega \otimes R/\mathfrak{m}$ is an $F_0 \otimes R/\mathfrak{m}$-lattice in $a(F, R/\mathfrak{m})$ so that

$$\det_F\big(1 - i^{-1}j \,\big|\, \mathrm{Hom}(M,\, a(F, R/\mathfrak{m}))\big) = \det_{F_0}\big(1 - i^{-1}j \,\big|\, \mathrm{Hom}(M, \Omega \otimes R/\mathfrak{m}))\big).$$

Theorem 8.11.9 now shows that the determinant on the right coincides with $P_{\mathfrak{m}}(1)$. □

Let $\mathcal{M}$ be a global model. By Proposition 3.5 the restriction of $\mathcal{M}$ to $\mathcal{O}_F \times X$ is an elliptic shtuka. So we can make the following definition.

Definition 8.4. We set $L(\mathcal{M}) = L(\mathcal{M}|_{\mathcal{O}_F \times X})$ where $L(\mathcal{M}|_{\mathcal{O}_F \times X}) \in \mathcal{O}_F$ is the invariant of Definition 6.9.4.

Proposition 8.5. *The invariant $L(E^*, 0)$ has the following properties.*

(1) *$L(E^*, 0)$ converges to an element of $\mathcal{O}_F$.*
(2) *$L(E^*, 0) \equiv 1 \pmod{\mathfrak{m}_F}$.*
(3) *For every global model $\mathcal{M}$ we have $L(\mathcal{M}) = L(E^*, 0)$.*

Proof. Let $\mathcal{M}$ be a global model. By definition

$$L(\mathcal{M}) = L(\mathcal{M}|_{\mathcal{O}_F \times X}) = \prod_{\mathfrak{m}} \frac{1}{L(\mathcal{M}(\mathcal{O}_F \otimes R/\mathfrak{m}))}$$

where $\mathfrak{m}$ runs over all the maximal ideals. The product defining $L(\mathcal{M})$ converges by Lemma 6.9.3. Proposition 8.3 now implies that $L(\mathcal{M}) = L(E^*, 0)$ and we get (3). Proposition 6.9.6 shows that $L(\mathcal{M}) \equiv 1 \pmod{\mathfrak{m}_F}$. Since global models exist by Proposition 4.4 we get (1) and (2). □

9. Trace formula

Definition 9.1. Let $I \subset \mathcal{O}_K$ be an open ideal and let $\mathcal{M}$ be a locally free shtuka on $C \times X$ given by a diagram

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

We define the *twist* $I\mathcal{M}$ to be the locally free subshtuka

$$\mathcal{I}\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{I}\mathcal{M}_1$$

of $\mathcal{M}$. Here $\mathcal{I} \subset \mathcal{O}_{C\times X}$ is the pullback of the unique ideal sheaf $\mathcal{I}_0 \subset \mathcal{O}_X$ which satisfies $\mathcal{I}_0(\operatorname{Spec} R) = R$ and $\mathcal{I}_0(\operatorname{Spec} \mathcal{O}_K) = I$.

Proposition 9.2. *If $\mathcal{M}$ is a global model then $\mathfrak{f}\mathcal{M}$ is a global model.*

Proof. Observe that the restrictions of $\mathfrak{f}\mathcal{M}$ and $\mathcal{M}$ to $C \times \operatorname{Spec} R$ coincide. So in order to prove that $\mathfrak{f}\mathcal{M}$ is a global model we only need to show that $(\mathfrak{f}\mathcal{M})(A \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $(\mathfrak{f}\mathcal{M})(A \otimes \mathcal{O}_K/\mathfrak{f})$ is linear.

Suppose that the $A \mathbin{\widehat{\otimes}} \mathcal{O}_K$-module shtuka $\mathcal{N} = \mathcal{M}(A \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ is given by the diagram

$$M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1.$$

If $I \subset \mathcal{O}_K$ is an open ideal then

$$(I\mathcal{M})(A \mathbin{\widehat{\otimes}} \mathcal{O}_K) = (A \mathbin{\widehat{\otimes}} I) \cdot M_0 \overset{i}{\underset{j}{\rightrightarrows}} (A \mathbin{\widehat{\otimes}} I) \cdot M_1.$$

Let us denote this shtuka $I\mathcal{N}$.

We have a short exact sequence

$$0 \to \mathfrak{f}\mathcal{N}/\mathfrak{m}_K\mathfrak{f}\mathcal{N} \to \mathcal{N}/\mathfrak{m}_K\mathfrak{f}\mathcal{N} \to \mathcal{N}/\mathfrak{m}_K\mathcal{N} \to 0$$

of $A \mathbin{\widehat{\otimes}} \mathcal{O}_K$-module shtukas. The shtuka $\mathcal{N}/\mathfrak{m}_K\mathcal{N} = \mathcal{M}(A \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent by assumption. Proposition 1.9.4 implies that $\mathcal{N}/\mathfrak{m}_K\mathfrak{f}\mathcal{N}$ is nilpotent. As a consequence $\mathfrak{f}\mathcal{N}/\mathfrak{m}_K\mathfrak{f}\mathcal{N}$ is nilpotent. However the shtuka $\mathfrak{f}\mathcal{N}/\mathfrak{m}_K\mathfrak{f}\mathcal{N}$ is canonically isomorphic to $(\mathfrak{f}\mathcal{M})(A \otimes \mathcal{O}_K/\mathfrak{m}_K)$. We conclude that the latter shtuka is nilpotent.

By construction $(\mathfrak{f}\mathcal{M})(A \otimes \mathcal{O}_K/\mathfrak{f}) = \mathfrak{f}\mathcal{N}/\mathfrak{f}^2\mathcal{N}$. So to show that $(\mathfrak{f}\mathcal{M})(A \otimes \mathcal{O}_K/\mathfrak{f})$ is linear it is enough to prove that $j((A \mathbin{\widehat{\otimes}} \mathfrak{f}) \cdot M_0) \subset (A \mathbin{\widehat{\otimes}} \mathfrak{f}^2) \cdot M_1$. This is immediate since $\tau(A \mathbin{\widehat{\otimes}} \mathfrak{f}) \subset A \mathbin{\widehat{\otimes}} \mathfrak{f}^q$. □

By Proposition 4.9.1 every global model $\mathcal{M}$ has a ζ-isomorphism

$$\zeta_{\mathcal{M}} \colon \det_A \mathrm{R}\Gamma(A \times X, \mathcal{M}) \to \det_A \mathrm{R}\Gamma(A \times X, \nabla\mathcal{M}).$$

Theorem 9.3. *If $\mathcal{M}$ is a global model then for all $n \gg 0$ we have*

$$\zeta_{\mathfrak{f}^n\mathcal{M}} = L(\mathfrak{f}^n\mathcal{M}) \cdot \det_F(\rho_{\mathfrak{f}^n\mathcal{M}}).$$

The theorem should hold for $n = 0$ as well. See the remark below Theorem 6.10.4.

Proof of Theorem 9.3. Let $\mathcal{N}$ be the restriction of $\mathcal{M}$ to $\mathcal{O}_F \times X$. It is an elliptic shtuka of conductor $\mathfrak{f}$ by Proposition 3.5. Let $\mathcal{N}_0$ and $\mathcal{N}_1$ be the underlying sheaves of $\mathcal{N}$. Lemma 6.1.1 shows that for all $n \gg 0$ and for all $* \in \{0, 1\}$ the module $\mathrm{H}^0(\mathcal{O}_F \times X, \mathfrak{f}^n\mathcal{N}_*)$ is zero and $\mathrm{H}^1(\mathcal{O}_F \times X, \mathfrak{f}^n\mathcal{N}_*)$ is a free $\mathcal{O}_F$-module of finite rank. Here the twists $\mathfrak{f}^n\mathcal{N}_*$ are in the sense of Definition 6.8.12. We are thus in position to apply Theorem 6.10.4. It implies that for all $n \gg 0$ we have

$$\zeta_{\mathfrak{f}^n\mathcal{N}} = L(\mathfrak{f}^n\mathcal{N}) \cdot \det_{\mathcal{O}_F}(\rho_{\mathfrak{f}^n\mathcal{N}}).$$

Here $\mathfrak{f}^n\mathcal{N}$ is the twist of $\mathcal{N}$ in the sense of Definition 6.8.14.

Observe that the pullback of $\mathfrak{f}^n\mathcal{M}$ to $\mathcal{O}_F \times X$ coincides with the twist $\mathfrak{f}^n\mathcal{N}$ by construction. Hence the regulator $\rho_{\mathfrak{f}^n\mathcal{M}}$ coincides with the F-linear extension of the regulator of $\mathfrak{f}^n\mathcal{N}$ and the invariants $L(\mathfrak{f}^n\mathcal{M})$ and $L(\mathfrak{f}^n\mathcal{N})$ agree. Moreover Proposition 4.9.2 implies that

$$F \otimes_{\mathcal{O}_F} \zeta_{\mathfrak{f}^n\mathcal{N}} = F \otimes_A \zeta_{\mathfrak{f}^n\mathcal{M}}.$$

We thus get the result. □

10. The class number formula

Recall that the complex of units of the Drinfeld module E is the A-module complex

$$U_E = \Big[\, \mathrm{Lie}_E(K) \xrightarrow{\ \exp\ } \frac{E(K)}{E(R)} \,\Big]$$

where $\exp\colon \mathrm{Lie}_E(K) \to E(K)$ is the exponential map of E. The arithmetic regulator

$$\rho_E \colon F \otimes_A U_E \to \mathrm{Lie}_E(K)[0]$$

is the F-linear extension of the morphism $U_E \to \mathrm{Lie}_E(K)[0]$ given by the identity in degree zero. Taelman [23] observed that U_E is perfect and ρ_E is a quasi-isomorphism.

Theorem 10.1. *There exists a unique A-module isomorphism*

$$\zeta\colon \det_A U_E \xrightarrow{\sim} \det_A \mathrm{Lie}_E(R)$$

such that

$$\det_F(\rho_E) = L(E^*, 0) \cdot \zeta$$

as maps from $F \otimes_A \det_A U_E$ to $\det_F \mathrm{Lie}_E(K) = F \otimes_A \det_A \mathrm{Lie}_E(R)$.

Proof. Proposition 4.4 associates global shtuka models $\mathcal{M}$ with the Drinfeld module E. Theorem 9.3 shows that after replacing such a model $\mathcal{M}$ with a twist $\mathfrak{f}^n\mathcal{M}$ by a high enough power of $\mathfrak{f}^n$ we have

$$\zeta_{\mathcal{M}} = L(\mathcal{M}) \cdot \det_F(\rho_{\mathcal{M}}) \tag{10.1}$$

where

$$\zeta_{\mathcal{M}}\colon \det_A \mathrm{R\Gamma}(A \times X, \mathcal{M}) \xrightarrow{\sim} \det_A \mathrm{R\Gamma}(A \times X, \nabla\mathcal{M}),$$
$$\rho_{\mathcal{M}}\colon \mathrm{R\Gamma}(F \times X, \mathcal{M}) \xrightarrow{\sim} \mathrm{R\Gamma}(F \times X, \nabla\mathcal{M})$$

are the ζ-isomorphism and the regulator of $\mathcal{M}$ respectively (see sections 9 and 7 above).

Theorem 6.1 provides us with quasi-isomorphisms

$$\mathrm{R\Gamma}(A \times X, \mathcal{M}) \cong U_E[-1],$$
$$\mathrm{R\Gamma}(A \times X, \nabla\mathcal{M}) \cong \mathrm{Lie}_E(R)[-1].$$

Hence the ζ-isomorphism $\zeta_{\mathcal{M}}$ induces an A-module isomorphism

$$\det_A U_E[-1] \xrightarrow{\sim} \det_A \mathrm{Lie}_E(R)[-1].$$

Taking the inverse of its dual we obtain an isomorphism

$$\zeta\colon \det_A U_E \xrightarrow{\sim} \det_A \mathrm{Lie}_E(R).$$

Thanks to Theorem 7.3 we know that under the quasi-isomorphisms above the regulator $\rho_{\mathcal{M}}$ matches with the shifted arithmetic regulator

$$\rho_E[-1]\colon F \otimes_A U_E[-1] \to \mathrm{Lie}_E(K)[-1].$$

Moreover $L(\mathcal{M}) = L(E^*, 0)$ by Proposition 8.5. As $\det_F(\rho_E[-1])$ is the inverse of the dual of $\det_F(\rho_E)$ we conclude that (10.1) implies the theorem. ∎