# Shtuka cohomology and special values of Goss L-functions

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CHAPTER 11

# Regulators of local models

Let $E$ be a Drinfeld $A$-module over $K$, a finite product of local fields containing $\mathbb{F}_q$. As in Chapter 9 we assume that the action of $A$ on $\mathrm{Lie}_E(K)$ extends to a continuous action of $F$. In that chapter we introduced the notion of a local model $\mathcal{M}$, its regulator $\rho\colon \mathrm{H}^1(\mathcal{M}) \xrightarrow{\sim} \mathrm{H}^1(\nabla\mathcal{M})$ and its exponential map $\exp\colon F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\nabla\mathcal{M}) \xrightarrow{\sim} F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M})$. Our goal is to show that the square

$$\begin{tikzcd} \mathrm{H}^1(\nabla\mathcal{M}) \arrow[r] & F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "\exp", "\wr"'] \\ \mathrm{H}^1(\mathcal{M}) \arrow[u, "\rho", "\wr"'] \arrow[r] & F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M}) \end{tikzcd}$$

commutes. Unfortunately the only proof we have proceeds by reduction to the case $A = \mathbb{F}_q[t]$ (using the results of Chapter 10) and a brute force computation. On the positive side, this chapter makes the abstract machinery of Chapter 9 more explicit.

## 1. The setting

From now on we consider the special case $A = \mathbb{F}_q[t]$. As usual $F = \mathbb{F}_q(\!(t^{-1})\!)$ stands for the local field of $A$ at infinity and $\mathcal{O}_F = \mathbb{F}_q[[t^{-1}]]$ denotes the ring of integers of $F$. Let $K$ be a local field containing $\mathbb{F}_q$. As before its ring of integers is denoted $\mathcal{O}_K$ and $\mathfrak{m}_K \subset \mathcal{O}_K$ stands for the maximal ideal. We also fix a norm on $K$ such that $|\zeta| = q^{-1}$ for a uniformizer $\zeta \in K$.

Fix a Drinfeld $A$-module $E$ of rank $r$ over $K$. Let $\varphi\colon A \to K\{\tau\}$ be the ring homomorphism determined by the action of $A$ on $E$. We define the elements $\theta, \alpha_1, \ldots, \alpha_r \in K$ via the equation

$$\varphi(t) = \theta + \alpha_1\tau + \ldots + \alpha_r\tau^r.$$

As in Chapter 9 we assume that the action of $A$ on $\mathrm{Lie}_E(K)$ extends to a continuous action of $F$.

**Lemma 1.1.** *The following are equivalent:*

1. *The action of $A$ on $\mathrm{Lie}_E(K)$ extends to a continuous action of $F$.*
2. *$|\theta| > 1$.* □

According to Definition 9.6.4 the conductor $\mathfrak{f} \subset \mathcal{O}_K$ is the ideal generated by $\mathcal{O}_K$ under the homomorphism $F \to K$ induced by the action of $A$ on $\mathrm{Lie}_E(K)$.

**Lemma 1.2.** $\mathfrak{f} = \theta^{-1}\mathcal{O}_K$. □

Throughout this chapter $M = \mathrm{Hom}(E, \mathbb{G}_a)$ stands for the motive of $E$. Without loss of generality we assume that the underlying group scheme of $E$ is $\mathbb{G}_a$ so that we can identify $M$ with $K\{\tau\}$.

**Lemma 1.3.** *The motive $M = K\{\tau\}$ has the following properties.*

(1) *$M$ is a free $A \otimes K$-module of rank $r$ with a basis $1, \ldots, \tau^{r-1}$.*
(2) *$M^{\geqslant 1}$ is a free $A \otimes K$-module of rank $r$ with a basis $\tau, \ldots, \tau^r$.*
(3) *We have a relation*

$$\tau^r = \alpha_r^{-1}\big((t-\theta)\cdot 1 - \alpha_1\tau - \alpha_2\tau^2 - \ldots - \alpha_{r-1}\tau^{r-1}\big).$$

*in the $A \otimes K$-module $M$.*

*Proof.* (1) follows from Lemma 7.3.3 (2). In view of (1) the result (2) is a corollary of Proposition 7.1.10. (3) is a consequence of (1). □

## 2. Coefficient compactification

**Lemma 2.1.** *Let $k \geqslant 1$ be an integer.*

(1) *Let $f \in \mathrm{Hom}(M, a(F,K))$. The following are equivalent:*
 (a) *$f(\tau^n)$ vanishes on $\mathcal{O}_F$ for all $n \leqslant kr$.*
 (b) *$f(1), \ldots, f(\tau^{r-1})$ vanish on $t^{k-1}\mathcal{O}_F$ and furthermore $f(1)$ vanishes on $t^k\mathcal{O}_F$.*
(2) *Let $g \in \mathrm{Hom}(M^{\geqslant 1}, a(F,K))$. The following are equivalent:*
 (a) *$g(\tau^n)$ vanishes on $\mathcal{O}_F$ for all $n \leqslant kr$.*
 (b) *$g(\tau), \ldots, g(\tau^r)$ vanish on $t^{k-1}\mathcal{O}_F$.*

*Proof.* (2) From the relation

$$\tau^n t = \theta^{q^n}\tau^n + \alpha_1^{q^n}\tau^{n+1} + \ldots + \alpha_r^{q^n}\tau^{n+r}.$$

one concludes that if $g(\tau^n), \ldots, g(\tau^{n+r-1})$ vanish on an open neighbourhood $U \subset F$ of 0 then the following are equivalent:

(i) $g(\tau^{n+r})$ vanishes on $U$.
(ii) $g(\tau^n)$ vanishes on $tU$.

With (i) one gets (a) ⇒ (b) and (ii) implies (b) ⇒ (a). The argument for (1) is the same as for (2). □

**Proposition 2.2.** *The $\mathcal{O}_F \check{\otimes} K$-modules $\mathcal{M}_0^c$, $\mathcal{M}_1^c$ in the coefficient compactification*

$$\Big[\mathcal{M}_0^c \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1^c\Big] \subset \mathcal{H}\mathrm{om}(M, a(F,K)).$$

*admit the following description:*

$$\begin{aligned}\mathcal{M}_0^c \;&= \big\{ f\colon M \to a(F,K) \;\mid\; f(\tau),\dots,f(\tau^{r-1}) \text{ vanish on } t^{-1}\mathcal{O}_F \\ &\qquad\qquad\qquad\qquad\qquad \text{and } f(1) \text{ vanishes on } \mathcal{O}_F \big\},\\ \mathcal{M}_1^c \;&= \big\{ g\colon M \to a(F,K) \;\mid\; g(\tau),\dots,g(\tau^{r}) \text{ vanish on } t^{-1}\mathcal{O}_F \big\}.\end{aligned}$$

*Proof.* Let $C$ be the compactification of $\operatorname{Spec} A$ and let $\iota\colon \operatorname{Spec}(A\otimes K) \to C\times K$ be the open embedding. We denote

$$\mathcal{E} = \Big[\mathcal{E}_{-1} \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{E}_0\Big] \subset \Big[M \overset{1}{\underset{\tau}{\rightrightarrows}} M\Big]$$

the shtuka produced by Theorem 7.5.5. By definition $\mathcal{M}_0^c$ consists of those $f\colon M \to a(F,K)$ which send the submodule

$$\mathcal{E}_0(\mathcal{O}_F \mathbin{\check{\otimes}} K) \subset M \otimes_{A\otimes K} (F \mathbin{\check{\otimes}} K)$$

to $a(F/\mathcal{O}_F, K) \subset a(F,K)$.

Take $k \gg 0$ such that $\mathcal{E}_0(k)$ is globally generated. Theorem 7.5.5 implies that the $\mathcal{O}_F \mathbin{\check{\otimes}} K$-submodule

$$\mathcal{E}_0(k)(\mathcal{O}_F \mathbin{\check{\otimes}} K) \subset M \otimes_{A\otimes K} (F \mathbin{\check{\otimes}} K)$$

is generated by $\mathrm{H}^0(C\times K, \mathcal{E}_0(k)) = M^{kr}$. By Lemma 2.1 (1) a morphism $f\colon M \to a(F,K)$ sends this submodule to $a(F/\mathcal{O}_K, K)$ if and only if $f(1)$ vanishes on $t^k\mathcal{O}_F$ and $f(\tau),\dots,f(\tau^{r-1})$ vanish on $t^{k-1}\mathcal{O}_F$. However

$$\mathcal{E}_0(\mathcal{O}_F \mathbin{\check{\otimes}} K) = t^{-k}\mathcal{E}_0(k)(\mathcal{O}_F \mathbin{\check{\otimes}} K)$$

so we get the result for $\mathcal{M}_0^c$. The case of $\mathcal{M}_1^c$ is handled in a similar manner. □

## 3. Explicit models

We equip the space $b(F,K)$ and its subspace $a(F,K)$ with the sup-norm induced by the norm on $K$.

**Lemma 3.1.** *Let $\mu\colon F \to \mathbb{F}_q$ be a nonzero continuous $\mathbb{F}_q$-linear map.*

(1) *$\mu$ generates $a(F,K)$ as an $F \mathbin{\check{\otimes}} K$-module.*
(2) *For every $\beta \in K^\times$ the subspace $\{f : |f| \leqslant |\beta|\} \subset a(F,K)$ is a free $F \mathbin{\check{\otimes}} \mathcal{O}_K$-submodule generated by $\beta\cdot\mu$.*

*Proof.* (1) Let $F^*$ be the continuous $\mathbb{F}_q$-linear dual of $F$. According to Theorem 3.10.1 the topological $F$-module $F^*$ is free of rank one. So a nonzero function $\mu\colon F \to \mathbb{F}_q$ generates $F^*$ as an $F$-module. Now Lemma 3.8.10 implies that $F^*$ is an $F$-lattice in the $F \mathbin{\check{\otimes}} K$-module $a(F,K)$ so we get the result.

(2) Without loss of generality we assume that $\beta = 1$. In this case the subspace of $a(F,K)$ in question is $a(F,\mathcal{O}_K)$. By Lemma 3.8.10 the space $F^*$ is an $F$-lattice in the $F \mathbin{\check{\otimes}} \mathcal{O}_K$-module $a(F,\mathcal{O}_K)$. Whence the result. □

**Definition 3.2.** Let $b, b_1, \dots, b_{r-1} \in q^{\mathbb{Z}}$ be real numbers. We introduce conditions on $A \otimes K$-linear maps $f\colon M \to b(F, K)$ and $g\colon M^{\geqslant 1} \to b(F, K)$:

(EM$_0$) $\quad |f(1)| \leqslant |\alpha_r \theta^{-1}| b, \quad |f(\tau)| \leqslant b_1, \quad \dots \quad |f(\tau^{r-1})| \leqslant b_{r-1},$

(EM$_1$) $\quad |g(\tau)| \leqslant b_1, \quad \dots \quad |g(\tau^{r-1})| \leqslant b_{r-1}, \quad |g(\tau^r)| \leqslant b.$

**Proposition 3.3.** *Let $b, b_1 \dots b_{r-1} \in q^{\mathbb{Z}}$ be real numbers. Consider the $F \mathbin{\check{\otimes}} \mathcal{O}_K$-submodules*

$$\mathcal{M}_0 = \big\{ \ f \in \operatorname{Hom}(M, a(F, K)) \mid f \text{ satisfies } (\mathrm{EM}_0) \ \big\},$$
$$\mathcal{M}_1 = \big\{ \ g \in \operatorname{Hom}(M^{\geqslant 1}, a(F, K)) \mid g \text{ satisfies } (\mathrm{EM}_1) \ \big\}.$$

*If $b, b_1 \dots b_{r-1}$ satisfy the inequalities*

(EP$_1$) $\quad \Big|\dfrac{\alpha_n}{\alpha_r}\Big| \dfrac{b_n}{b} \leqslant 1, \quad n \in \{1, \dots, r-1\}$

(EP$_2$) $\quad \Big|\dfrac{\alpha_r^q}{\theta^q}\Big| \dfrac{b^q}{b_1} \leqslant |\theta^{-1}|; \quad \dfrac{b_{r-1}^q}{b} \leqslant |\theta^{-1}|; \quad \dfrac{b_n^q}{b_{n+1}} \leqslant |\theta^{-1}|, \quad n \in \{1, \dots, r-2\}$

*then $\mathcal{M}_0$, $\mathcal{M}_1$ define an $F \mathbin{\check{\otimes}} \mathcal{O}_K$-subshtuka*

$$\mathcal{M} = [\mathcal{M}_0 \rightrightarrows \mathcal{M}_1] \subset \mathcal{H}\mathrm{om}(M, a(F, K))$$

*which is a base compactification in the sense of Definition 9.6.6.*

*Proof.* It will be convenient for us to work with bases of various modules. Our first goal is to construct bases for $\operatorname{Hom}(M, a(F, K))$ and $\operatorname{Hom}(M^{\geqslant 1}, a(F, K))$.

According to Lemma 1.3 an $A \otimes K$-basis for $M$ is given by elements $1, \tau, \dots, \tau^{r-1}$. Similarly the elements $\tau, \dots, \tau^r$ form a basis of $M^{\geqslant 1}$. Fix a nonzero continuous $\mathbb{F}_q$-linear map $\mu\colon F \to \mathbb{F}_q$. Lemma 3.1 (1) shows that $\mu$ generates $a(F, K)$ as an $F \mathbin{\check{\otimes}} K$-module. We thus get $F \mathbin{\check{\otimes}} K$-module bases of $\operatorname{Hom}(M, a(F, K))$, $\operatorname{Hom}(M^{\geqslant 1}, a(F, K))$ which are dual to the aforementioned bases of $M$.

Let $\beta, \beta_1 \dots \beta_{r-1} \in K^\times$ be such that $|\beta| = b, |\beta_1| = b_1 \dots |\beta_{r-1}| = b_{r-1}$. Lemma 3.1 (2) implies that the modules $\mathcal{M}_0$, $\mathcal{M}_1$ have $F \mathbin{\check{\otimes}} \mathcal{O}_K$-bases given by matrices

$$\begin{pmatrix} \alpha_r \theta^{-1} \beta & 0 & & 0 \\ 0 & \beta_1 & & 0 \\ & & \ddots & \\ 0 & 0 & & \beta_{r-1} \end{pmatrix}, \quad \begin{pmatrix} \beta_1 & & 0 & 0 \\ & \ddots & & \\ 0 & & \beta_{r-1} & 0 \\ 0 & & 0 & \beta \end{pmatrix}.$$

with respect to the fixed bases of $\operatorname{Hom}(M, a(F, K))$ and $\operatorname{Hom}(M^{\geqslant 1}, a(F, K))$. We conclude that $\mathcal{M}_0$, $\mathcal{M}_1$ are $F \mathbin{\check{\otimes}} \mathcal{O}_K$-lattices in these modules.

Let $i$ and $j$ be the arrows of the shtuka

$$\mathcal{H}\mathrm{om}(M, a(F, K)) = \Big[ \operatorname{Hom}(M, a(F, K)) \underset{j}{\overset{i}{\rightrightarrows}} \operatorname{Hom}(M^{\geqslant 1}, a(F, K)) \Big].$$

According to Lemma 8.7.1 the map $i$ is the restriction to $M^{\geqslant 1}$. Hence the matrix of the map $i$ in the fixed bases of $\mathcal{H}\mathrm{om}(M, a(F,K))$ is

$$\begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \\ \frac{t-\theta}{\alpha_r} & -\frac{\alpha_1}{\alpha_r} & & -\frac{\alpha_{r-1}}{\alpha_r} \end{pmatrix}$$

Rewriting it in the bases of $\mathcal{M}_0$, $\mathcal{M}_1$ gives

$$\begin{pmatrix} 0 & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \\ t\theta^{-1} - 1 & -\frac{\alpha_1\beta_1}{\alpha_r\beta} & & -\frac{\alpha_{r-1}\beta_{r-1}}{\alpha_r\beta} \end{pmatrix}$$

The assumptions $|\theta| > 1$ and (EP$_1$) imply that the bottom row is in $F\check{\otimes}\mathcal{O}_K$ and therefore $i(\mathcal{M}_0) \subset \mathcal{M}_1$. The determinant of the matrix is $(-1)^r(1-\theta^{-1}t)$. As $|\theta| > 1$ it reduces to $(-1)^r$ modulo $F \check{\otimes} \mathfrak{m}_K$ and in particular $i$ is an isomorphism modulo $F \check{\otimes} \mathfrak{m}_K$.

According to Lemma 8.7.1 the map $j$ sends $f \in \mathrm{Hom}(M, a(F,K))$ to the map

$$\tau m \mapsto \tau f(m).$$

Hence the matrix of $j$ in the bases of $\mathcal{M}_0$, $\mathcal{M}_1$ is

$$\begin{pmatrix} \frac{\alpha_r^q\beta^q}{\theta^q\beta_1} & 0 & & 0 & 0 \\ 0 & \frac{\beta_1^q}{\beta_2} & & 0 & 0 \\ & & \ddots & & \\ 0 & 0 & & \frac{\beta_{r-2}^q}{\beta_{r-1}} & 0 \\ 0 & 0 & & 0 & \frac{\beta_{r-1}^q}{\beta} \end{pmatrix}$$

In our case the conductor $\mathfrak{f}$ is the ideal $\theta^{-1}\mathcal{O}_K$. Hence the assumptions (EP$_2$) imply that the matrix above lies in $\mathfrak{f}$ whence $j(\mathcal{M}_0) \subset \mathcal{M}_1$ and $j$ reduces to zero modulo $\mathfrak{f}$. Above we demonstrated that $i$ reduces to an isomorphism modulo $\mathfrak{m}_K$. Hence $\mathcal{M}$ is a base compactification as claimed. □

**Proposition 3.4.** *There exist $b, b_1, \ldots, b_{r-1} \in q^{\mathbb{Z}}$ satisfying* (EP$_1$) *and* (EP$_2$).

*Proof.* A direct verification shows that for all $\varepsilon \in q^{\mathbb{Z}}$ small enough the real numbers

$$b = \varepsilon^2,\ b_1 = \varepsilon^3, \ldots, b_{r-1} = \varepsilon^3$$

satisfy (EP$_1$) and (EP$_2$). □

**Definition 3.5.** Let $b, b_1, \ldots, b_{r-1} \in q^{\mathbb{Z}}$ be real numbers satisfying ($\mathrm{EP}_1$), ($\mathrm{EP}_2$). The *explicit model* $\mathcal{M}$ of parameters $b, b_1 \ldots b_{r-1}$ is the subshtuka in $\mathcal{H}\mathrm{om}(M, a(F,K))$ defined as the intersection

$$\mathcal{M} = \mathcal{M}^c \cap \mathcal{M}^b$$

where $\mathcal{M}^c$ is the coefficient compactification of Definition 9.6.1 and $\mathcal{M}^b \subset \mathcal{H}\mathrm{om}(M, a(F,K))$ is the subshtuka desribed in Proposition 3.3. The number $b$ will be called the *leading parameter*.

It follows from Proposition 9.6.8 that an explicit model is a local model in the sense of Definition 9.6.7. In particular we have the twists $\mathfrak{f}^n\mathcal{M}$ of a local model as in Proposition 9.6.10.

**Proposition 3.6.** *If $\mathcal{M}$ is an explicit model of parameters $b, b_1, \ldots, b_{r-1}$ then $\mathfrak{f}\mathcal{M}$ is an explicit model of parameters $b|\theta^{-1}|, b_1|\theta^{-1}|, \ldots, b_{r-1}|\theta^{-1}|$.*

*Proof.* Follows since $\mathfrak{f} = \theta^{-1}\mathcal{O}_K$. $\square$

Next we prove a technical statement on explicit models as subshtukas of $\mathcal{H}\mathrm{om}(M, b(F,K))$.

**Lemma 3.7.** *For every $\beta \in K^\times$ the subspace $\{f \in a(F,K) : |f| \leqslant |\beta|\}$ is an $F \mathbin{\breve{\otimes}} \mathcal{O}_K$-lattice in the $F^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K$-module $\{f \in b(F,K) : |f| \leqslant |\beta|\}$.*

*Proof.* Without loss of generality we assume that $\beta = 1$. In this case the spaces in question are $a(F, \mathcal{O}_K)$ and $b(F, \mathcal{O}_K)$ so the result follows from Corollary 3.10.3. $\square$

If $\mathcal{M} \subset \mathcal{H}\mathrm{om}(M, a(F,K))$ is a local model then Lemma 9.6.9 shows that the natural map

$$\mathcal{M}(F^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K) \to \mathcal{H}\mathrm{om}(M, b(F,K))$$

is an inclusion of an $F^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K$-lattice. In particular we can view $\mathcal{M}(F^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ as a subshtuka of $\mathcal{H}\mathrm{om}(M, b(F,K))$.

**Lemma 3.8.** *If $\mathcal{M} = [\mathcal{M}_0 \rightrightarrows \mathcal{M}_1]$ is an explicit model of parameters $b, b_1, \ldots, b_{r-1}$ then*

$$\mathcal{M}_0(F^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K) = \{\ f \in \mathrm{Hom}(M, b(F,K)) \mid f \textit{ satisfies } (\mathrm{EM}_0)\ \},$$
$$\mathcal{M}_1(F^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K) = \{\ g \in \mathrm{Hom}(M^{\geqslant 1}, b(F,K)) \mid g \textit{ satisfies } (\mathrm{EM}_1)\ \}.$$

*Proof.* Follows immediately from Lemma 3.7. $\square$

## 4. Exponential maps of Drinfeld modules

As we identified the underlying group scheme of the Drinfeld module $E$ with $\mathbb{G}_a$ we can view the exponential map of $E$ as a function $K \to K$. The map $\exp\colon K \to K$ is a local isomorphism of topological $\mathbb{F}_q$-vector spaces. In the following we will need a slightly more precise version of this result.

**Lemma 4.1.** *There exists a constant* $B_{\exp} \in \mathbb{R}$ *such that*

(1) $0 < B_{\exp} \leqslant \frac{1}{q}$.
(2) *If* $|z| \leqslant B_{\exp}$ *then* $|\exp z - z| \leqslant |z|^q$. *In particular* $|\exp z| = |z|$.

*Proof.* Indeed $\exp z$ is given by an everywhere convergent power series

$$\exp z = z + a_1 z^q + a_2 z^{q^2} + \dots$$

Therefore

$$\exp z - z = z^q(a_1 + a_2 z^q + \dots).$$

The power series in brackets also converges everywhere and defines a continuous function from $K$ to $K$. Hence there exists a nonzero constant $B_{\exp}$ such that as soon as $|z| \leqslant B_{\exp}$ the values of this function have norm less than or equal to 1. Without loss of generality we may assume that $B_{\exp} \leqslant \frac{1}{q}$. We thus get (1) and (2). □

Let $B_{\exp}$ be a constant as in Lemma 4.1 and let $U \subset K$ be the ball of radius $B_{\exp}$ around 0. Lemma 4.1 (2) implies that $\exp(U) = U$ and the induced map $\exp\colon U \to U$ is an isomorphism of topological $\mathbb{F}_q$-vector spaces.

**Definition 4.2.** In the following we denote $\log\colon U \to U$ the inverse of exp on $U$. We call it the *logarithmic map* of the Drinfeld module $E$.

Denote $\varphi_t\colon K \to K$ the map given by the action of $t$ on $K = E(K)$. In other words

$$\varphi_t(z) = \theta z + \alpha_1 z^q + \dots + \alpha_r z^{q^r}.$$

**Lemma 4.3.** *Let* $z \in K$. *If* $|\varphi_t(z)| \leqslant B_{\exp}$ *and* $|z| \leqslant B_{\exp}$ *then* $\log(\varphi_t(z)) = \theta \log(z)$.

*Proof.* There exists an $y \in K$ such that $\exp(y) = z$. Furthermore $\varphi_t(\exp(y)) = \exp(\theta y)$ since exp is an $A$-linear map from $\mathrm{Lie}_E(K)$ to $E(K)$. The result follows since log is the inverse of exp. □

## 5. Exponential maps of explicit models

Let $\mathcal{M}$ be an explicit model. In Section 9.10 we introduced the exponential map of $\mathcal{M}$:

$$\exp\colon F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\nabla\mathcal{M}) \to F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M}).$$

In this section we will describe this map on explicit representatives of cohomology classes.

**Definition 5.1.** Let $h \in b(F, K)$. We define $A \otimes K$-linear maps

$$\nabla g(h)\colon M^{\geqslant 1} \to b(F, K),$$
$$g(h)\colon M^{\geqslant 1} \to b(F, K)$$

by prescribing them on the basis $\tau, \ldots, \tau^r$ of $M^{\geqslant 1}$ as follows:

$$\begin{aligned}\nabla g(h)\colon \tau^1, \ldots, \tau^{r-1} &\mapsto 0,\\ \tau^r &\mapsto \alpha_r^{-1}(ht - \theta h),\\ g(h)\colon \tau^1, \ldots, \tau^{r-1} &\mapsto 0,\\ \tau^r &\mapsto \alpha_r^{-1}\exp\circ(ht - \theta h).\end{aligned}$$

where $\exp\colon K \to K$ is the exponential map of the Drinfeld module $E$. Note that $g(h)(\tau^r)$ maps $x \in F$ to $\alpha_r^{-1}\exp(h(tx) - \theta h(x))$ in $K$.

We define an $A \otimes K$-linear map $\nabla f(h)\colon M \to b(F, K)$ by prescribing it on the basis $1, \ldots, \tau^{r-1}$ of $M$ as follows:

$$\nabla f(h)\colon 1 \mapsto h, \quad \tau, \ldots, \tau^{r-1} \mapsto 0.$$

We will use $g(h)$ and $\nabla g(h)$ as representatives for classes in the cohomology groups $\mathrm{H}^1(\mathcal{M})$ and $\mathrm{H}^1(\nabla\mathcal{M})$ respectively.

**Lemma 5.2.** *Let $i$, $j$ be the arrows of the shtuka*

$$\mathcal{H}\mathrm{om}(M, b(F,K)) = \Big[\mathrm{Hom}(M, b(F,K)) \underset{j}{\overset{i}{\rightrightarrows}} \mathrm{Hom}(M^{\geqslant 1}, b(F,K))\Big].$$

*If $h \in b(F,K)$ then $i(\nabla f(h)) = \nabla g(h)$.*

*Proof.* According to Lemma 8.7.1 the map $i(\nabla f(h))$ is the restriction of $\nabla f(h)$ to $M^{\geqslant 1}$. Applying $\nabla f(h)$ to the relation

$$\tau^r = \alpha_r^{-1}\big((t-\theta)\cdot 1 - \alpha_1\tau - \alpha_2\tau^2 - \ldots - \alpha_{r-1}\tau^{r-1}\big)$$

we conclude that $\nabla f(h)(\tau^r) = \alpha_r^{-1}(ht - \theta h)$. □

**Lemma 5.3.** *Let $\mathcal{M}$ be an explicit model given by a diagram*

$$\mathcal{M}_0 \underset{j}{\overset{i}{\rightrightarrows}} \mathcal{M}_1$$

*and let $c \in \mathrm{H}^1(\nabla\mathcal{M})$ be a cohomology class. There exists a function $h \in b(F,K)$ such that the following holds:*

(1) *$\nabla g(h)$ belongs to $\mathcal{M}_1$ and represents $c$ in $\mathrm{H}^1(\nabla\mathcal{M}) = \mathrm{coker}(i)$.*
(2) *$\nabla f(h) \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$.*

*Proof.* Let $g \in \mathcal{M}_1 \subset \mathrm{Hom}(M^{\geqslant 1}, a(F,K))$ be a representative of the cohomology class $c$. According to Proposition 9.10.2 there exists a unique $f \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K) \subset \mathrm{Hom}(M, b(F,K))$ such that $i(f) = g$. Set $h = f(1)$.

Since $f$ is an element of $\mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ Lemma 3.8 shows that the function $h$ satisfies $|h| \leqslant |\alpha_r\theta^{-1}|b$. The same lemma then implies that $\nabla f(h) \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ which is the claim (2). In order to deduce (1) it is enough to show that the difference

$$\delta = f - \nabla f(h)$$

is an element of $\mathcal{M}_0$. Indeed $i(\nabla f(h)) = \nabla g(h)$ by Lemma 5.2 so $\nabla g(h) = g - i(\delta)$ is an element of $\mathcal{M}_1$ which represents the same cohomology class as $g$.

According to Lemma 8.7.1 the map $i(f) = g$ is the restriction of $f$ to $M^{\geqslant 1}$. Therefore the difference $\delta$ acts on the generators of $M$ as follows:

$$\delta\colon 1 \mapsto 0,\ \tau^n \mapsto g(\tau^n),\ n \in \{1, \dots, r-1\}.$$

Let $\mathcal{M}^c = [\mathcal{M}_0^c \rightrightarrows \mathcal{M}_1^c]$ be the coefficient compactification. By definition of an explicit model

$$\mathcal{M}_0 = \mathcal{M}_0^c \cap \mathcal{M}_0(F \check{\otimes} \mathcal{O}_K).$$

We first prove that $\delta \in \mathcal{M}_0^c$. As $g \in \mathcal{M}_1^c$ Proposition 2.2 shows that the functions $g(\tau), \dots, g(\tau^r)$ vanish on $t^{-1}\mathcal{O}_F$. Since $\delta$ maps 1 to 0 the same Proposition implies that $\delta \in \mathcal{M}_0^c$.

Now we prove that $\delta \in \mathcal{M}_0(F \check{\otimes} \mathcal{O}_K)$. By definition $g$ is an element of $\mathcal{M}_1$ so ($\mathrm{EM}_1$) implies that

$$|\delta(\tau^n)| = |g(\tau^n)| \leqslant b_n, \quad n \in \{1, \dots, r-1\}$$

where $b_n$ are the parameters of $\mathcal{M}$. Moreover $\delta(1) = 0$ so $\delta \in \mathcal{M}_0(F \check{\otimes} \mathcal{O}_K)$ by ($\mathrm{EM}_0$). □

In the following let us fix a constant $B_{\exp}$ satisfying the assumptions of Lemma 4.1.

**Lemma 5.4.** *Let $\mathcal{M}$ be an explicit model given by a diagram*

$$\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1.$$

*Let $h \in b(F, K)$ be a function such that*

$$\nabla g(h), g(h) \in \mathcal{M}_1, \quad \nabla f(h) \in \mathcal{M}_0(F^{\#} \widehat{\otimes} \mathcal{O}_K).$$

*Assume that the leading parameter $b$ of $\mathcal{M}$ satisfies $|\alpha_r| b \leqslant B_{\exp}$. If an element $f \in \mathcal{M}_0(F^{\#} \widehat{\otimes} \mathcal{O}_K)$ satisfies $(i-j)(f) = g(h)$ then $f(1) = \exp h$.*

*Proof.* Set $h_1 = f(1)$. As before we denote $\varphi_t\colon K \to K$ the action of $t$ on $K = E(K)$. We have

$$\varphi_t(z) = \theta z + \alpha_1 z^q + \ldots + \alpha_r z^{q^r}.$$

We split the proof in several steps.

**Step 1.** $h_1 \circ t - \varphi_t \circ h_1 = \exp \circ (ht - \theta h)$ *in* $b(F, K)$.

According to Lemma 8.7.1 the map $(i-j)(f)$ satisfies

$$(i-j)(f)\colon \tau^{n+1} \mapsto f(\tau^{n+1}) - \tau f(\tau^n)$$

for all $n \geqslant 0$. Comparing with the definition of $g(h)$ we deduce that

$$f(\tau^n) = \tau^n f(1) = \tau^n h_1 \tag{5.1}$$

for all $n \in \{0, \dots, r-1\}$ and that

$$f(\tau^r) = \tau^r h_1 + \alpha_r^{-1} \exp(ht - \theta h). \tag{5.2}$$

Appling the $A \otimes K$-linear map $f$ to the relation

$$1 \cdot t - \theta \cdot 1 = \alpha_1 \tau + \ldots + \alpha_r \tau^r$$

in $M$, we obtain what we need.

**Step 2.** *The function* $\varphi_t \circ h_1$ *in* $b(F, K)$ *satisfies* $|\varphi_t \circ h_1| \leqslant |\alpha_r| b$.

We prove it estimating the expression

$$\varphi_t \circ h_1 = \theta h_1 + \alpha_1 \tau h_1 + \ldots + \alpha_r \tau^r h_1$$

term by term. As $f \in \mathcal{M}_0(F^\# \widehat{\otimes} \mathcal{O}_K)$ Lemma 3.8 implies that $|h_1| \leqslant |\alpha_r \theta^{-1} b|$. Hence

$$|\theta h_1| \leqslant |\alpha_r| b. \tag{5.3}$$

Next let $b_1, \ldots, b_{r-1}$ be the parameters of $\mathcal{M}$ and let $n \in \{1, \ldots, r-1\}$. According to (5.1) we have $f(\tau^n) = \tau^n h_1$. The map $f$ belongs to $\mathcal{M}_0(F^\# \widehat{\otimes} \mathcal{O}_K)$. Hence Lemma 3.8 shows that

$$|f(\tau^n)| \leqslant b_n.$$

However the conditions ($\mathrm{EP}_1$) imply that

$$|\alpha_n| b_n \leqslant |\alpha_r| b.$$

One thus obtains the inequality

$$|\alpha_n \tau^n h_1| \leqslant |\alpha_n| b_n \leqslant |\alpha_r| b. \tag{5.4}$$

It remains to estimate $\alpha_r \tau^r h_1$. The equation (5.2) implies that

$$\tau^r h_1 = f(\tau^r) - g(h)(\tau^r).$$

The element $f$ belongs to $\mathcal{M}_0(F^\# \widehat{\otimes} \mathcal{O}_K)$ by assumption. Hence $i(f) \in \mathcal{M}_1(F^\# \widehat{\otimes} \mathcal{O}_K)$. According to Lemma 8.7.1 the map $i(f)$ is the restriction of $f$ to $M^{\geqslant 1}$. Therefore Lemma 3.8 implies that $|f(\tau^r)| \leqslant b$. As $g(h) \in \mathcal{M}_1$ we deduce that $|g(h)(\tau^r)| \leqslant b$. Hence

$$|\tau^r h_1| \leqslant b. \tag{5.5}$$

Combining (5.3), (5.4), (5.5) we get the inequality $|\varphi_t \circ h_1| \leqslant |\alpha_r| b$.

**Step 3.** $\log h_1 \in b(F, K)$ *is well-defined.*

The logarithmic map is defined on the ball of radius $B_{\exp}$ around 0 (see Definition 4.2). The function $h_1$ satisfies $|h_1| \leqslant |\alpha_r \theta^{-1}| b$ because $f \in \mathcal{M}_0(F^\# \widehat{\otimes} \mathcal{O}_K)$. Since $|\theta| > 1$ and $|\alpha_r| b \leqslant B_{\exp}$ by assumption we conclude that $\log h_1$ is well-defined.

**Step 4.** $\nabla g(\log h_1) = \nabla g(h)$.

According to Step 1 we have

$$h_1 \circ t - \varphi_t \circ h_1 = \exp \circ (ht - \theta h). \tag{5.6}$$

We would like to apply log to this equation. Step 2 shows that

$$|\varphi_t \circ h_1| \leqslant |\alpha_r| b \leqslant B_{\exp}$$

so $\log \varphi_t h_1$ is well-defined. Moreover $\log h_1$ is well-defined by Step 3. Now $f(h) \in \mathcal{M}_0(F^\# \widehat{\otimes} \mathcal{O}_K)$ by assumption so Lemma 3.8 shows that

$$|h| \leqslant |\alpha_r \theta^{-1}| b \leqslant |\theta^{-1}| B_{\exp}.$$

According to Lemma 4.1 we have $|\exp(z)| = |z|$ provided $|z| \leqslant B_{\exp}$. Therefore

$$|\exp\circ(ht - \theta h)| \leqslant |ht - \theta h| \leqslant B_{\exp}$$

and we conclude that $\log(\exp\circ(ht-\theta h))$ is well-defined. Applying log to (5.6) we get

$$\log h_1 t - \log \varphi_t h_1 = ht - \theta h.$$

Lemma 4.3 shows that $\log(\varphi_t(z)) = \theta \log z$ provided $\varphi_t(z)$ and $z$ are in the domain of definition of log. As a consequence $\log \varphi_t h_1 = \theta \log h_1$ and

$$\log h_1 t - \theta \log h_1 = ht - \theta h.$$

It follows that $\nabla g(\log h_1) = \nabla g(h)$ by definition of $\nabla g$ and $g$.

**Step 5.** $\nabla f(\log h_1) \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$.

From Lemma 4.1 it follows that $|\log z| = |z|$ for all $z \in K$ satisfying $|z| \leqslant B_{\exp}$. Thus $|\log h_1| = |h_1| \leqslant |\alpha_r \theta^{-1}|b$. Applying Lemma 3.8 we conclude that $\nabla f(\log h_1) \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$.

**Step 6.** $h_1 = \exp h$. According to Lemma 5.2

$$\begin{aligned} i(\nabla f(h)) &= \nabla g(h), \\ i(\nabla f(\log h_1)) &= \nabla g(\log h_1). \end{aligned}$$

Furthermore $\nabla g(\log h_1) = g(h)$ by Step 4. Now $\nabla f(h)$ belongs to $\mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ by assumption while $\nabla f(\log h_1) \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ by Step 5. Hence the unicity part of Proposition 9.10.1 implies that $\nabla f(h) = \nabla f(\log h_1)$. By definition of $\nabla f$ it means that $h = \log h_1$. Therefore $h_1 = \exp h$. ☐

**Lemma 5.5.** *Let $\mathcal{M} = [\mathcal{M}_0 \rightrightarrows \mathcal{M}_1]$ be an explicit model with leading parameter $b$ satisfying $|\alpha_r|b \leqslant B_{\exp}$. If $h \in b(F,K)$ is a function such that $\nabla g(h) \in \mathcal{M}_1(F \mathbin{\check{\otimes}} \mathcal{O}_K)$ then*

$$\nabla g(h) - g(h) \in I\mathcal{M}_1(F \mathbin{\check{\otimes}} \mathcal{O}_K)$$

*where $I = \{x \in K : |x| \leqslant (|\alpha_r|b)^{q-1}\} \subset \mathcal{O}_K$.*

*Proof.* The fact that $\nabla g(h)$ is an element of $\mathcal{M}_1(F \mathbin{\check{\otimes}} \mathcal{O}_K)$ implies a bound

$$|ht - \theta h| \leqslant |\alpha_r|b.$$

According to Lemma 4.1 $|\exp z - z| \leqslant |z|^q$ as soon as $|z| \leqslant B_{\exp}$. As $|\alpha_r|b \leqslant B_{\exp}$ we conclude that

$$|\exp(ht - \theta h) - (ht - \theta h)| \leqslant |ht - \theta h|^q \leqslant (|\alpha_r|b)^q.$$

Therefore

$$|\nabla g(h)(\tau^r) - g(h)(\tau^r)| \leqslant (|\alpha_r|b)^{q-1} \cdot b$$

which implies that $\nabla g(h) - g(h) \in I\mathcal{M}_1(F \mathbin{\check{\otimes}} \mathcal{O}_K)$. ☐

**Proposition 5.6.** *Let $\mathcal{M} = [\mathcal{M}_0 \rightrightarrows \mathcal{M}_1]$ be an explicit model with leading parameter $b$. If $|\alpha_r|b \leqslant B_{\exp}$ then for every cohomology class $c \in \mathrm{H}^1(\nabla\mathcal{M})$ there exists $h \in b(F,K)$ such that the following holds.*

(1) *$\nabla g(h)$ belongs to $\mathcal{M}_1$ and represents $c$.*
(2) *$g(h)$ belongs to $\mathcal{M}_1$.*
(3) *$\exp[\nabla g(h)] = [g(h)]$ in $F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M})$.*
(4) *$\nabla g(h) - g(h) \in I\mathcal{M}_1$ where $I = \{x \in K : |x| \leqslant (|\alpha_r|b)^{q-1}\} \subset \mathcal{O}_K$.*

*Here the brackets $[\,]$ denote cohomology classes and* exp *is the exponential map of $\mathcal{M}$ as in Definition 9.10.3.*

*Proof.* According to Lemma 5.3 there exists an $h \in b(F,K)$ such that $\nabla g(h) \in \mathcal{M}_1$ represents the class $c$ and $\nabla f(h) \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$.

We claim that $\nabla g(h) - g(h) \in I\mathcal{M}_1$. To prove it we consider the coefficient compactification $\mathcal{M}^c = [\mathcal{M}_0^c \rightrightarrows \mathcal{M}_1^c]$. As $\nabla g(h) \in \mathcal{M}_1^c$ Proposition 2.2 implies that $g(h) \in \mathcal{M}_1^c$. Lemma 5.5 shows that

$$\nabla g(h) - g(h) \in I\mathcal{M}_1(F \mathbin{\check{\otimes}} \mathcal{O}_K)$$

By definition of an explicit model

$$\mathcal{M}_1^c \cap I\mathcal{M}_1(F \mathbin{\check{\otimes}} \mathcal{O}_K) = I\mathcal{M}_1.$$

Hence $\nabla g(h) - g(h) \in I\mathcal{M}_1$ and $g(h) \in \mathcal{M}_1$.

It remains to prove that $\exp[\nabla g(h)] = [g(h)]$. Consider the isomorphisms

$$\gamma\colon F \otimes_{\mathcal{O}_F} \mathrm{R\Gamma}(\mathcal{M}) \xrightarrow{\sim} \mathrm{Lie}_E(K)[-1],$$
$$\nabla\gamma\colon F \otimes_{\mathcal{O}_F} \mathrm{R\Gamma}(\nabla\mathcal{M}) \xrightarrow{\sim} \mathrm{Lie}_E(K)[-1]$$

of Definition 9.9.1. By Definition 9.10.3 the exponential map of $\mathcal{M}$ is the composition

$$\mathrm{H}^1(\gamma) \circ \mathrm{H}^1(\nabla\gamma)^{-1}.$$

Let $\alpha \in \mathrm{Lie}_E(K)$ be such that $\mathrm{H}^1(\nabla\gamma)(c) = \alpha$. Proposition 9.10.2 shows that the element $\alpha$ is characterized by the following property:

$$\nabla f(h)(1) = h \sim (x \mapsto x\alpha).$$

Let $i$ and $j$ be the arrows of $\mathcal{M}$. According to Proposition 9.10.1 there exists a unique $f \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ such that $(i-j)f = g(h)$. Lemma 5.4 tells us that $f(1) = \exp h$. Hence

$$f(1) \sim (x \mapsto \exp(x\alpha)).$$

Thus $\mathrm{H}^1(\gamma)([g(h)]) = \alpha$ by Proposition 9.10.1. We conclude that $\exp[\nabla g(h)] = [g(h)]$. ∎

## 6. Regulators of explicit models

Let $\mathcal{M}$ be a local model. By Theorem 9.9.5 the cohomology modules $\mathrm{H}^1(\mathcal{M})$, $\mathrm{H}^1(\nabla\mathcal{M})$ are free $\mathcal{O}_F$-modules of finite rank.

**Lemma 6.1.** *Let $\mathcal{M}$ be an explicit model with leading parameter $b$. If $|\alpha_r|b \leqslant B_{\mathrm{exp}}$ then the exponential map $\exp\colon F\otimes_{\mathcal{O}_F}\mathrm{H}^1(\nabla\mathcal{M}) \to F\otimes_{\mathcal{O}_F}\mathrm{H}^1(\mathcal{M})$ sends the $\mathcal{O}_F$-submodule $\mathrm{H}^1(\nabla\mathcal{M})$ to $\mathrm{H}^1(\mathcal{M})$.*

*Proof.* Immediate from Proposition 5.6. □

As in Section 5.7 we denote $\mathcal{M}/\mathfrak{f}^n$ the quotient $\mathcal{M}/(\mathfrak{f}^n\mathcal{M})$.

**Lemma 6.2.** *Let $\mathcal{M}$ be an explicit model with leading parameter $b$ satisfying $|\alpha_r|b \leqslant B_{\mathrm{exp}}$. Let*

$$I = \{x \in K : |x| \leqslant (|\alpha_r|b)^{q-1}\} \subset \mathcal{O}_K.$$

*Let $n \geqslant 0$. If $I \subset \mathfrak{f}^n$ and $\mathcal{M}/\mathfrak{f}^n$ is linear then the following square is commutative:*

$$\begin{tikzcd} \mathrm{H}^1(\nabla\mathcal{M}) \arrow[r] \arrow[d, "\exp"] & \mathrm{H}^1(\nabla\mathcal{M}/\mathfrak{f}^n) \arrow[d, "1"] \\ \mathrm{H}^1(\mathcal{M}) \arrow[r] & \mathrm{H}^1(\mathcal{M}/\mathfrak{f}^n). \end{tikzcd}$$

*Here the horizontal arrows are induced by reduction modulo $J$ and the right vertical arrow comes from the identity of the complexes $\mathrm{R}\Gamma(\nabla\mathcal{M}/\mathfrak{f}^n)$ and $\mathrm{R}\Gamma(\mathcal{M}/\mathfrak{f}^n)$.*

*Proof.* Let $\mathcal{M} = [\mathcal{M}_0 \rightrightarrows \mathcal{M}_1]$ and let $c \in \mathrm{H}^1(\nabla\mathcal{M})$ be a cohomology class. According to Proposition 5.6 there exists a function $h \in b(F, K)$ such that

$$\begin{gathered} \nabla g(h), g(h) \in \mathcal{M}_1, \\ \nabla g(h) - g(h) \in I\mathcal{M}_1, \\ [\nabla g(h)] = c, \\ \exp[\nabla g(h)] = [g(h)]. \end{gathered}$$

Here the brackets [ ] denote cohomology classes. We get the result since the images of $\nabla g(h)$ and $g(h)$ in $\mathcal{M}_1/\mathfrak{f}^n$ are the same. □

**Theorem 6.3.** *Let $\mathcal{M}$ be an explicit model with leading parameter $b$. If $|\alpha_r|b \leqslant B_{\mathrm{exp}}$ then the exponential map $\exp\colon \mathrm{H}^1(\nabla\mathcal{M}) \to \mathrm{H}^1(\mathcal{M})$ is the inverse of the regulator $\rho\colon \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M})$.*

*Proof.* Let $n \geqslant 0$. By Proposition 3.6 the twist $\mathfrak{f}^n\mathcal{M}$ is an explicit model with leading parameter $b|\theta^{-n}|$. Consider the ideal

$$I_n = \{x \in K : |x| \leqslant (|\alpha_r\theta^{-n}|b)^{q-1}\} \subset \mathcal{O}_K.$$

Since $|\alpha_r|b \leqslant B_{\exp} < 1$ and $\mathfrak{f} = \theta^{-1}\mathcal{O}_K$ we conclude that $I_n \subset \mathfrak{f}^n$. The shtuka $(\mathfrak{f}^n\mathcal{M})/\mathfrak{f}^n$ is linear by Proposition 5.7.6. So Lemma 6.2 implies that

$$\begin{tikzcd} \mathrm{H}^1(\nabla\mathfrak{f}^n\mathcal{M}) \arrow[r] \arrow[d, "\exp"] & \mathrm{H}^1(\nabla(\mathfrak{f}^n\mathcal{M})/\mathfrak{f}^n) \arrow[d, "1"] \\ \mathrm{H}^1(\mathfrak{f}^n\mathcal{M}) \arrow[r] & \mathrm{H}^1((\mathfrak{f}^n\mathcal{M})/\mathfrak{f}^n). \end{tikzcd}$$

Now Theorem 5.14.5 shows that exp is the inverse of the regulator map. □

## 7. Regulators in general

In this section we let $A$ be an arbitrary coefficient ring and $K$ a finite product of local fields containing $\mathbb{F}_q$. As before we fix a Drinfeld $A$-module $E$ over $K$ with motive $M$. We assume that the action of $A$ on $\mathrm{Lie}_E(K)$ extends to a continuous action of $F$.

We are finally ready to prove the following result:

**Theorem 7.1.** *If $\mathcal{M} \subset \mathcal{H}\mathrm{om}(M, a(F,K))$ is a local model then the diagram*

$$\begin{tikzcd} \mathrm{H}^1(\nabla\mathcal{M}) \arrow[r] & F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "\exp", "\wr"'] \\ \mathrm{H}^1(\mathcal{M}) \arrow[u, "\rho", "\wr"'] \arrow[r] & F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M}) \end{tikzcd}$$

*is commutative.*

*Proof.* We split the proof into several steps.

**Step 1.** *Let $\mathcal{N} \subset \mathcal{M}$ be an inclusion of local models. The theorem holds for $\mathcal{N}$ if and only if it holds for $\mathcal{M}$.* Indeed Proposition 9.9.4 shows that the maps

$$F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{N}) \to F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M}),$$
$$F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{N}) \to F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{M})$$

are quasi-isomorphisms. The result follows since both $\rho$ and exp are natural transformations of functors on the category of local models.

**Step 2.** *The theorem holds in the case $A = \mathbb{F}_q[t]$.*

Without loss of generality we assume that $K$ is a single local field. Let $\mathcal{M}$ be a local model. According to Proposition 3.4 there exists an explicit model $\mathcal{N}$ with some parameters $b$, $b_1, \ldots, b_{r-1}$. Proposition 9.6.11 shows that the local model $\mathfrak{f}^n\mathcal{N}$ is a subshtuka of $\mathcal{M}$ for all $n \gg 0$. By Proposition 3.6 the twist $\mathfrak{f}^n\mathcal{N}$ is an explicit model with leading parameter $b|\theta^{-n}|$. So taking $n \gg 0$ we can ensure that $|\alpha_r|b \leqslant B_{\exp}$. In this situation Theorem 6.3 shows that the result holds for $\mathfrak{f}^n\mathcal{N}$. Step 1 implies that it holds for $\mathcal{M}$ as well.

Next let us fix a nonconstant element $a \in A$. We denote $A' = \mathbb{F}_q[a]$ and $F'$ the local field of $A'$ at infinity and $\mathfrak{f}' \subset \mathcal{O}_K$ the ideal generated by $\mathfrak{m}_{F'}$ under the map $F' \to K$ given by the action of $A'$ on $\mathrm{Lie}_E(K)$.

**Step 3.** *If $\mathcal{M}$ is a local model such that $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{f}')$ is linear then the theorem holds for $\mathcal{M}$.*

Let $\mathcal{M}'$ be $\mathcal{M}$ viewed as an $\mathcal{O}_{F'} \mathbin{\check{\otimes}} \mathcal{O}_K$-module shtuka. By Proposition 10.7.1 the shtuka $\mathcal{M}'$ is a local model of the Drinfeld $A'$-module $E$. Proposition 10.7.2 shows that the regulator of $\mathcal{M}'$ is compatible with the regulator of $\mathcal{M}$ while Proposition 10.7.5 does the same for the exponential map. Hence the theorem holds for $\mathcal{M}$ with the coefficient ring $A$ if and only if it holds for $\mathcal{M}'$ with the coefficient ring $A'$. Applying Step 2 to $\mathcal{M}'$ we get the result.

**Step 4.** *The theorem holds for an arbitrary local model.*

Let $\mathcal{M}$ be a local model. The conductor $\mathfrak{f}'$ is a power of $\mathfrak{f}$ by construction. Hence Proposition 9.6.10 demonstrates that $\mathfrak{f}'\mathcal{M}$ is a local model. Proposition 5.7.6 shows that $(\mathfrak{f}'\mathcal{M})(F \otimes \mathcal{O}_K/\mathfrak{f}')$ is linear. Therefore the theorem holds for $\mathfrak{f}'\mathcal{M}$ by Step 3. As $\mathfrak{f}'\mathcal{M} \subset \mathcal{M}$ Step 1 implies that the theorem holds for $\mathcal{M}$. $\square$