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Title:  Shtuka cohomology and special values of Goss L-functions
Issue Date: 2018-02-13
CHAPTER 10

Change of coefficients

This chapter is of a technical nature. In it we verify that the constructions of Chapter 9 are compatible with restriction of the coefficient ring $A$. This result will be used in Chapter 11 where we show that the regulator of a local model is the inverse of the exponential map. We will do it by reduction to an explicit computation in the case $A = \mathbb{F}_q[t]$.

1. Duality for Hom shtukas

In this section we describe a general duality construction for Hom shtukas. It will be used several times in the rest of the chapter. We begin with an auxiliary result.

**Lemma 1.1.** Let $R$ be a $\tau$-ring. If $M = \left[M_0 \xrightarrow{i_M} M_1 \right]$ is an $R$-module shtuka and $N$ a left $R\{\tau\}$-module then

$$\mathcal{H}om_R(M, N) = \left[\text{Hom}_R(M_1, N) \xrightarrow{i} \text{Hom}_R(\tau^* M_0, N)\right]$$

where

\[i(f): r \otimes m \mapsto rfj_M(m),\]

\[j(f): r \otimes m \mapsto r\tau \cdot f i_M(m).\]

**Proof.** Follows directly from the definition of $\mathcal{H}om$ (Definition 1.12.1). □

If $\varphi: R \to S$ is a ring homomorphism, $M$ an $S$-module and $N$ an $R$-module then the $R$-modules $\text{Hom}_R(M, N)$ and $\text{Hom}_R(S, N)$ carry natural $S$-module structures: via $M$ in the first case and via $S$ in the second. Furthermore the natural duality map

$$\text{Hom}_S(M, \text{Hom}_R(S, N)) \to \text{Hom}_R(M, N), \quad f \mapsto [m \mapsto f(m)(1)]$$

is an $S$-module isomorphism. We would like to establish an analog of this duality for a $\tau$-ring homomorphism $\varphi: (R, \tau) \to (S, \sigma)$ and $\mathcal{H}om$ in place of $\text{Hom}$. In the following it will be important to distinguish the $\tau$-endomorphisms of $R$ and $S$. We thus denote them by different letters.

Let $\varphi: (R, \tau) \to (S, \sigma)$ be a homomorphism of $\tau$-rings. For every $S$-module $M$ there is a natural base change map

$$\mu_M: \tau^* M \to \sigma^* M, \quad r \otimes m \mapsto \varphi(r) \otimes m.$$
In particular we have a base change map $\mu_S: \tau^* S \to \sigma^* S = S$.

Consider the commutative diagram

\[
\begin{array}{ccc}
S & \xrightarrow{\sigma} & S \\
\varphi \uparrow & & \varphi \uparrow \\
R & \xrightarrow{\tau} & R
\end{array}
\]

For the duality statements below to work it will be necessary to assume that \((\ast)\) is cocartesian in the category of rings. It is cocartesian if and only if the base change map $\mu_S: \tau^* S \to S$ is an isomorphism.

**Proposition 1.2.** Let $\varphi: (R, \tau) \to (S, \sigma)$ be a homomorphism of $\tau$-rings and let $N$ be a left $R\{\tau\}$-module. Consider the shtuka

\[
\mathbb{Hom}_R(S, N) = \left[ \mathbb{Hom}_R(S, N) \xrightarrow{i} \mathbb{Hom}_R(\tau^* S, N) \right].
\]

If the commutative diagram of rings \((\ast)\) is cocartesian then the following holds:

1. $i$ is an isomorphism.
2. The endomorphism $i^{-1} j$ of $\mathbb{Hom}_R(S, N)$ makes it into a left $S\{\sigma\}$-module.
3. If $f \in \mathbb{Hom}_R(S, N)$ then $g = i^{-1} j(f)$ is the unique $R$-linear map such that

\[
ge\left[ \varphi(r)\sigma(s) \right] = r\tau \cdot f(s)
\]

for every $r \in R$ and $s \in S$.

**Proof.** (1) Let $\tau^o: \tau^* S \to S$ be the adjoint of the $\tau$-multiplication map of $S$. By definition

\[
\tau^o(r \otimes s) = \varphi(r)\sigma(s).
\]

At the same time

\[
\mu_S(r \otimes s) = \varphi(r) \otimes s = \varphi(r)\sigma(s).
\]

Thus $\tau^o = \mu_S$ is an isomorphism and we conclude that $i$ is an isomorphism from Lemma 1.1

(2) We will deduce from (3) that the endomorphism $i^{-1} j$ is $\sigma$-linear. Let us temporarily denote $e = i^{-1} j$. Let $f \in \mathbb{Hom}_R(S, N)$. If $s_1 \in S$ then the maps $e(f \cdot s_1)$ and $e(f) \cdot \sigma(s_1)$ satisfy

\[
e(f \cdot s_1)\left[ \varphi(r)\sigma(s) \right] = r\tau \cdot f(s_1 s)
\]

\[
e(f) \cdot \sigma(s_1)\left[ \varphi(r)\sigma(s) \right] = e(f)\left[ \sigma(s_1)\varphi(r)\sigma(s) \right] = r\tau \cdot f(s_1 s)
\]

for every $r \in R$, $s \in S$. Thus $e$ is $\sigma$-linear.

(3) Let $\tau^o_N: \tau^* N \to N$ be the adjoint of the $\tau$-multiplication map. If $f \in \mathbb{Hom}_R(S, N)$ then according to Lemma 1.1

\[
j(f): r \otimes s \mapsto r\tau \cdot f(s)
\]
for every \( r \in R \) and \( s \in S \). As we saw in (1) the map \( i \) is given by the composition with the base change map \( \mu_S : r \otimes s \mapsto \varphi(r)\sigma(s) \). The result is now clear. \( \square \)

**Proposition 1.3.** Let \( \varphi : (R, \tau) \to (S, \sigma) \) be a homomorphism of \( \tau \)-rings. Let

\[
M = \left[ M_0 \xrightarrow{i_M} M_1 \right]
\]

be an \( S \)-module shtuka and \( N \) a left \( R\{\tau\} \)-module. If the commutative diagram of rings \( \star \) is cocartesian then the maps

\[
\begin{align*}
\text{duality} & \quad \text{Hom}_S(M_1, \text{Hom}_R(S, N)) \quad \text{Hom}_R(M_1, N), \\
(\mu_{M_0})^* \circ \text{duality} & \quad \text{Hom}_S(\sigma^* M_0, \text{Hom}_R(S, N)) \quad \text{Hom}_R(\tau^* M_0, N)
\end{align*}
\]

define an isomorphism

\[ \mathcal{H}\text{om}_S(M, \text{Hom}_R(S, N)) \cong \mathcal{H}\text{om}_R(M, N) \]

of \( R \)-module shtukas. The left \( S\{\sigma\} \)-module structure on \(\text{Hom}_R(S, N)\) is as constructed in Proposition 1.2.

**Proof.** Denote the duality maps

\[
\begin{align*}
\eta_0 : \text{Hom}_S(M_1, \text{Hom}_R(S, N)) & \to \text{Hom}_R(M_1, N), \\
\eta_1 : \text{Hom}_S(\sigma^* M_0, \text{Hom}_R(S, N)) & \to \text{Hom}_R(\tau^* M_0, N).
\end{align*}
\]

If \( \star \) is cocartesian then the base change map \( \mu_{M_0} \) is an isomorphism. The inverse of \( \mu_{M_0} \) is given by the formula

\[ \varphi(r)\sigma(s) \otimes m \mapsto r \otimes sm. \]

Thus \( \eta_0 \) and \( \eta_1 \) define an isomorphism of \( R \)-module shtukas provided they form a morphism of shtukas. Let us show that it is indeed the case.

Let \( i_S, j_S \) be the arrows of \( \mathcal{H}\text{om}_S(M, \text{Hom}_R(S, N)) \) and let \( i_R, j_R \) be the arrows of \( \mathcal{H}\text{om}_R(M, N) \). If \( f \in \text{Hom}_S(M_1, \text{Hom}_R(S, N)) \) then

\[ \eta_0(f) : m \mapsto f(m)(1). \]

So Lemma 1.1 implies that

\[ j_R(\eta_0(f)) : r \otimes m \mapsto r\tau \cdot [fi_M(m)](1). \]

By the same Lemma

\[ j_S(f) : s \otimes m \mapsto s\sigma \cdot fi_M(m). \]

If \( g \in \text{Hom}_S(\sigma^* M_0, \text{Hom}_R(S, N)) \) then

\[ \eta_1(g) : r \otimes m \mapsto g(\varphi(r) \otimes m)(1). \]
Therefore
\[ \eta_1(j_S(f)) : r \otimes m \mapsto [j_S(f)(\varphi(r) \otimes m)](1) \]
\[ = [\varphi(r) \cdot f_iM(m)](1) \]
\[ = [\sigma \cdot f_iM(m)](\varphi(r)). \]

According to Proposition 1.2
\[ \sigma \cdot f_iM(m) : \varphi(r) \mapsto r^\tau \cdot [f_iM(m)](1). \]

Hence
\[ \eta_1(j_S(f)) : r \otimes m \mapsto r^\tau \cdot [f_iM(m)](1). \]
We conclude that
\[ \eta_1(i_S(f)) = i_R(\eta_0(f)). \] It is easy to see that \[ \eta_1(i_S(f)) = i_R(\eta_0(f)). \]

2. Tensor products

**Lemma 2.1.** Let \( T \) be a locally compact \( \mathbb{F}_q \)-algebra and let \( S' \subset S \) be an extension of locally compact \( \mathbb{F}_q \)-algebras. If \( S \) is finitely generated free as a topological \( S' \)-module then the natural map \( S \otimes_{S'} (S' \hat{\otimes} T) \to S \hat{\otimes} T \) is an isomorphism.

**Proof.** We rewrite the natural map in question as
\[ (S \hat{\otimes}_{ic} T) \otimes_{(S' \hat{\otimes} ic)T} (S' \hat{\otimes} T) \to S \hat{\otimes} T. \]
By assumption \( S \) is a finitely generated free topological \( S' \)-module. Therefore \( S \hat{\otimes}_{ic} T \) is a finitely generated free topological \( S' \hat{\otimes} ic T \)-module. The result now follows from Lemma 3.2.4.

**Lemma 2.2.** Let \( T \) be a locally compact \( \mathbb{F}_q \)-algebra and let \( S' \subset S \) be an extension of locally compact \( \mathbb{F}_q \)-algebras. If \( S \) is locally free of finite rank as an \( S' \)-module without topology then the natural map \( S \otimes_{S'} ((S')^\# \hat{\otimes} T) \to S^\# \hat{\otimes} T \) is an isomorphism.

**Proof.** We rewrite the natural map in question as
\[ (S^\# \otimes_c T) \otimes_{((S')^\# \otimes_c T)} ((S')^\# \hat{\otimes} T) \to S^\# \hat{\otimes} T. \]
By assumption \( S \) is locally free of finite rank as an \( S' \)-module without topology. Therefore \( S^\# \otimes_c T \) is a topological direct summand of a finitely generated free \( (S')^\# \otimes_c T \)-module. The result now follows from Lemma 3.2.4.

3. The setting

We now start with the main part of this chapter. The setting is as follows. Fix a coefficient ring \( A \) as in Definition 7.2.1. As usual we denote \( F \) the local field of \( A \) at infinity, \( \mathcal{O}_F \subset F \) the ring of integers and \( m_F \subset \mathcal{O}_F \) the maximal ideal. Let \( K \) be a finite product of local fields containing \( \mathbb{F}_q \). Fix a Drinfeld \( A \)-module \( E \) over \( K \) and let \( M = \text{Hom}(E, G_a) \) be its motive. Throughout the chapter we assume that the action of \( A \) on \( \text{Lie}_E(K) \) extends to a continuous
action of $F$. This is the context in which we defined and studied local models in Chapter 9.

Fix an $\mathbb{F}_q$-subalgebra $A' \subset A$ such that $A$ is finite flat over $A'$. Note that $A'$ is itself a coefficient ring. We denote $F'$ the local field of $A'$ at infinity, $\mathcal{O}_{F'} \subset F'$ the ring of integers and $\mathfrak{m}_{F'} \subset \mathcal{O}_{F'}$ the maximal ideal.

**Lemma 3.1.** The motive of $E$ viewed as a Drinfeld $A'$-module is $M$ with its natural left $A' \otimes K\{\tau\}$-module structure. □

4. Hom shtukas

**Lemma 4.1.** The commutative square of rings

$$
\begin{array}{ccc}
A \otimes K & \longrightarrow & F \otimes K \\
\uparrow & & \uparrow \\
A' \otimes K & \longrightarrow & F' \otimes K
\end{array}
$$

is cocartesian.

**Proof.** Indeed $A \otimes_{A'} F' = F$ so

$$A \otimes_{A'} (F' \otimes K) = (A \otimes_{A'} F') \otimes_{F'} (F' \otimes K) = F \otimes_{F'} (F' \otimes K).$$

Now Lemma 2.1 shows that $F \otimes_{F'} (F' \otimes K) = F \otimes K$ and the result follows. □

**Corollary 4.2.** The natural map $M \otimes_{A' \otimes K} (F' \otimes K) \rightarrow M \otimes_{A \otimes K} (F \otimes K)$ is an isomorphism of left $F' \otimes K\{\tau\}$-modules. □

**Lemma 4.3.** The commutative square of rings

$$
\begin{array}{ccc}
F \otimes K & \longrightarrow & F \otimes K \\
\uparrow & & \uparrow \\
F' \otimes K & \longrightarrow & F' \otimes K
\end{array}
$$

is cocartesian.

**Proof.** Immediate from Lemma 2.1. □

We are thus in position to apply the duality machinery of Section 1 to the $\tau$-ring homomorphism $F' \otimes K \rightarrow F \otimes K$. Proposition 1.2 equips the $F \otimes K$-module

$$\text{Hom}_{F' \otimes K}(F \otimes K, a(F', K))$$

with the structure of a left $F \otimes K\{\tau\}$-module.

**Lemma 4.4.** The natural map

$$\text{Hom}_{F' \otimes K}(F \otimes K, a(F', K)) \rightarrow \text{Hom}_{F'}(F, a(F', K))$$

is an isomorphism of $F \otimes K$-modules.
Proof. Follows since $F \otimes_{F'} (F' \otimes K) = F \otimes K$ by Lemma 2.1.

So we get a left $F \otimes K\{\tau\}$-module structure on $\text{Hom}_{F'}(F, a(F', K))$.

Lemma 4.5. If $g \in \text{Hom}_{F'}(F, a(F', K))$ then $\tau \cdot g$ maps $x \in F$ to $\tau(g(x))$.

Proof. For a finite-dimensional $F'$-vector space $V$ let $\alpha_V$ be the map

$$\alpha_V : a(V, K) \rightarrow \text{Hom}_{F'}(V, a(F', K)), \quad f \mapsto [v \mapsto (y \mapsto f(yv))]$$

It is clearly an $F' \otimes K$-linear isomorphism if $V$ is of dimension 1. The map $\alpha_V$ is also natural in $V$. Hence $\alpha_V$ is an $F' \otimes K$-linear isomorphism for any finite-dimensional $F'$-vector space $V$ and in particular for $V = F$.

The map $\alpha_F$ is also $F$-linear. By Lemma 2.1 we have $F \otimes_{F'} (F' \otimes K) = F \otimes K$. Hence $\alpha_F$ is $F \otimes K$-linear. A simple computation shows that it commutes with the action of $\tau$. So we get the result.

Proposition 4.7. The restriction map $a(F, K) \rightarrow a(F', K)$ induces an isomorphism

$$\mathcal{H}\text{om}_{A \otimes K}(M, a(F, K)) \cong \mathcal{H}\text{om}_{A' \otimes K}(M, a(F', K))$$

of $F' \otimes K$-module shtukas.

Proof. We have

$$\mathcal{H}\text{om}_{A \otimes K}(M, a(F, K)) = \mathcal{H}\text{om}_{F \otimes K}(M \otimes_{A \otimes K} (F \otimes K), a(F, K))$$

Lemma 4.6 gives us a natural isomorphism

$$\mathcal{H}\text{om}_{F \otimes K}(M \otimes_{A \otimes K} (F \otimes K), a(F, K)) \cong \mathcal{H}\text{om}_{F \otimes K}(M \otimes_{A \otimes K} (F \otimes K), \text{Hom}_{F'}(F, a(F', K))).$$

In view of Lemma 4.3 we can apply Proposition 1.3 to get an isomorphism

$$\mathcal{H}\text{om}_{F \otimes K}(M \otimes_{A \otimes K} (F \otimes K), \text{Hom}_{F'}(F, a(F', K))) \cong \mathcal{H}\text{om}_{F' \otimes K}(M \otimes_{A \otimes K} (F' \otimes K), a(F', K))$$

of $F' \otimes K$-module shtukas. Corollary 4.2 identifies $M \otimes_{A \otimes K} (F \otimes K)$ with $M \otimes_{A' \otimes K} (F' \otimes K)$. So we get an isomorphism

$$\mathcal{H}\text{om}_{A \otimes K}(M, a(F, K)) \cong \mathcal{H}\text{om}_{A' \otimes K}(M, a(F', K))$$

of $F' \otimes K$-module shtukas. A straightforward computation shows that this isomorphism is induced by the restriction map $a(F, K) \rightarrow a(F', K)$. □
5. Coefficient compactifications

We denote $C$ the projective compactification of Spec $A$ and $C'$ the projective compactification of $A'$. Let $\rho: C \times K \to C' \times K$ be the map induced by the inclusion $A' \subset A$.

**Lemma 5.1.** The commutative square of schemes

$$
\begin{array}{ccc}
\text{Spec } \mathcal{O}_F \otimes K & \longrightarrow & C \times X \\
\downarrow & & \downarrow \rho \\
\text{Spec } \mathcal{O}_{F'} \otimes K & \longrightarrow & C' \times X
\end{array}
$$

is cartesian.

**Proof.** Lemma 2.1 implies that the square

$$
\begin{array}{ccc}
\mathcal{O}_F \otimes K & \longrightarrow & \mathcal{O}_F \otimes K \\
\uparrow & & \uparrow \\
\mathcal{O}_{F'} \otimes K & \longrightarrow & \mathcal{O}_{F'} \otimes K
\end{array}
$$

is cocartesian. Whence the result. \qed

**Corollary 5.2.** For every quasi-coherent sheaf $\mathcal{E}$ on $C \times X$ the base change map

$$
(\rho_* \mathcal{E})(\mathcal{O}_{F'} \otimes K) \to \mathcal{E}(\mathcal{O}_F \otimes K)
$$

is an isomorphism of $\mathcal{O}_{F'} \otimes K$-modules. \qed

**Lemma 5.3.** If $\mathcal{E}$ is the shtuka on $C \times K$ which corresponds to the left $A \otimes K\{\tau\}$-module $M$ by Theorem 7.5.5 then $\rho_* \mathcal{E}$ is the shtuka on $C' \times K$ which corresponds to $M$ viewed as a left $A' \otimes K\{\tau\}$-module.

**Proof.** Suppose that $\mathcal{E}$ is given by the diagram

$$
\begin{bmatrix}
\mathcal{E}_{-1} & \xrightarrow{i} & \mathcal{E}_0 \\
\xrightarrow{j} & & \mathcal{E} \\
\end{bmatrix} \subset \begin{bmatrix}
M & \xrightarrow{1} & M \\
\tau & & \tau
\end{bmatrix}.
$$

Let $f$ be the degree of the residue field of $F$ over $\mathbb{F}_q$ and let $r$ be the rank of $M$ as an $A \otimes K$-module. By Theorem 7.5.5 the sheaves $\mathcal{E}_{-1}, \mathcal{E}_0$ are locally free of rank $r$ and have the following property:

$$
H^0(C \times K, \mathcal{E}_0(n)) = M^{fr}n,
$$

$$
H^0(C \times K, \mathcal{E}_{-1}(n)) = M^{fr}n-1.
$$

Let $d = [F : F']$. Observe that $f = df'$ where $f'$ is the degree of the residue field of $F'$ over $\mathbb{F}_q$. The morphism $\rho: C \times K \to C' \times K$ is finite flat of degree
d. Hence $\rho_0 E_0$ are locally free of rank $dr$ and
\[ H^0(C \times K, \rho_0 E_0(n)) = M^{drn}, \]
\[ H^0(C \times K, \rho_0 E_0(n)) = M^{drn-1}. \]
The unicity part of Theorem 7.5.5 now implies the result.

We next study the function space $a(F/O_F, K)$.

**Lemma 5.4.** The commutative square of rings
\[
\begin{array}{ccc}
O_F \otimes K & \overset{\tau}{\longrightarrow} & O_F \otimes K \\
\uparrow & & \uparrow \\
O_{F'} \otimes K & \overset{\tau}{\longrightarrow} & O_{F'} \otimes K
\end{array}
\]
is cocartesian.

**Proof.** Immediate from Lemma 2.1. □

So we can apply the duality constructions of Section 1 to the $\tau$-ring homomorphism $O_F \otimes K \rightarrow O_F \otimes K$. Proposition 1.2 equips the $O_F \otimes K$-module $\text{Hom}_{O_F \otimes K}(O_F \otimes K, a(F'/O_{F'}, K))$ with the structure of a left $O_F \otimes K\{\tau\}$-module.

**Lemma 5.5.** The natural map
\[
\text{Hom}_{O_{F'} \otimes K}(O_F \otimes K, a(F'/O_{F'}, K)) \rightarrow \text{Hom}_{O_{F'}(O_F, a(F'/O_{F'}, K))}
\]
is an isomorphism of $O_F \otimes K$-modules. □

So we get a left $O_F \otimes K\{\tau\}$-module structure on $\text{Hom}_{O_{F'}}(O_F, a(F'/O_{F'}, K))$.

**Lemma 5.6.** If $g \in \text{Hom}_{O_{F'}}(O_F, a(F'/O_{F'}, K))$ then $\tau \cdot g$ maps $x \in O_F$ to $\tau(g(x))$. □

**Lemma 5.7.** The map
\[
a(F/O_F, K) \rightarrow \text{Hom}_{O_{F'}}(O_F, a(F'/O_{F'}, K)), \quad f \mapsto [x \mapsto (y \mapsto f(yx))]
\]
is an isomorphism of left $O_F \otimes K\{\tau\}$-modules.

**Proof.** We view $a(F/O_F, K)$ as a subspace of $a(F, K)$ consisting of functions which vanish on $O_F$, and similarly for $a(F'/O_{F'}, K)$. Note that
\[
\text{Hom}_{F'}(F, a(F, K)) = \text{Hom}_{O_{F'}}(O_F, a(F', K)).
\]
We can thus identify
\[
\text{Hom}_{O_{F'}}(O_F, a(F'/O_{F'}, K))
\]
with a submodule of $\text{Hom}_{F'}(F, a(F, K))$. In view of this remark the result follows from Lemma 4.6. □
Proposition 5.8. The restriction isomorphism of Proposition 4.7 identifies the coefficient compactification of $\text{Hom}_{A \otimes K}(M, a(F, K))$ with the coefficient compactification of $\text{Hom}_{A' \otimes K}(M, a(F', K))$.

Proof. Lemma 5.7 gives us a natural isomorphism

$$\text{Hom}_{O_F \hat{\otimes} K}(E(O_F \hat{\otimes} K), a(F/O_F, K)) \cong \text{Hom}_{O_{F'} \hat{\otimes} K}(E(O_F \hat{\otimes} K), a(F'/O_{F'}, K)).$$

In view of Lemma 5.4 we can apply Proposition 1.3 to get an isomorphism

$$\text{Hom}_{O_F \hat{\otimes} K}(E(O_F \hat{\otimes} K), a(F/O_F, K)) \cong \text{Hom}_{O_{F'} \hat{\otimes} K}(E(O_F \hat{\otimes} K), a(F'/O_{F'}, K))$$

of $O_F \hat{\otimes} K$-module shtukas. It is easy to see that the resulting isomorphism

$$\text{Hom}_{O_F \hat{\otimes} K}(E(O_F \hat{\otimes} K), a(F/O_F, K)) \cong \text{Hom}_{O_{F'} \hat{\otimes} K}(E(O_F \hat{\otimes} K), a(F'/O_{F'}, K))$$

is induced by the restriction map $a(F/O_F, K) \to a(F'/O_{F'}, K)$. Now Corollary 5.2 implies that the natural map $\rho_* E(O_{F'} \hat{\otimes} K) \to E(O_F \hat{\otimes} K)$ is an isomorphism. Lemma 5.3 implies that the shtuka

$$\text{Hom}_{O_{F'} \hat{\otimes} K}(\rho_* E(O_{F'} \hat{\otimes} K), a(F'/O_{F'}, K))$$

is the coefficient compactification of $\text{Hom}_{A' \otimes K}(M, a(F', K))$. Whence the result. □

6. Base compactifications

Lemma 6.1. The commutative square of rings

$$\begin{array}{ccc}
F \hat{\otimes} O_K & \longrightarrow & F \hat{\otimes} K \\
\uparrow & & \uparrow \\
F' \hat{\otimes} O_K & \longrightarrow & F' \hat{\otimes} K
\end{array}$$

is cocartesian.

Proof. Follows from Lemma 2.1 □

Lemma 6.2. For every open ideal $I \subset O_K$ the commutative square of rings

$$\begin{array}{ccc}
F \hat{\otimes} O_K & \longrightarrow & F \otimes O_K/I \\
\uparrow & & \uparrow \\
F' \hat{\otimes} O_K & \longrightarrow & F' \otimes O_K/I
\end{array}$$

is cocartesian.
Proof. Indeed Lemma 2.1 implies that the square

\[
\begin{array}{ccc}
F \otimes \mathcal{O}_K & \longrightarrow & F \otimes \mathcal{O}_K \\
\uparrow & & \uparrow \\
F' \otimes \mathcal{O}_K & \longrightarrow & F' \otimes \mathcal{O}_K
\end{array}
\]

is cocartesian. Whence the result. \qed

According to our assumptions the action of \( A \) on \( \text{Lie}_E(K) \) extends to a continuous action of \( F \) so that we have a continuous homomorphism \( F \rightarrow K \). Recall that the conductor \( f \) is the ideal generated by \( m_F \) in \( \mathcal{O}_K \) (Definition 9.6.4). In the same manner we get a conductor \( f' \subset \mathcal{O}_K \) for the coefficient subring \( A' \subset A \).

**Proposition 6.3.** Let \( M \subset \mathcal{H}\mathcal{O}\mathcal{M}_{A \otimes K}(M, a(F, K)) \) be a base compactification. If \( M(F \otimes \mathcal{O}_K/f') \) is linear then the image of \( M \) under the restriction isomorphism of Proposition 4.7 is a base compactification of \( \mathcal{H}\mathcal{O}\mathcal{M}_{A' \otimes K}(M, a(F', K)) \).

Proof. Let \( M' \) be the image of \( M \) in \( \mathcal{H}\mathcal{O}\mathcal{M}_{A' \otimes K}(M, a(F', K)) \). The shtuka \( M \) is an \( F \otimes \mathcal{O}_K \)-lattice in \( \mathcal{H}\mathcal{O}\mathcal{M}_{A \otimes K}(M, a(F, K)) \) so Lemma 6.1 implies that \( M' \) is an \( F' \otimes \mathcal{O}_K \)-lattice in \( \mathcal{H}\mathcal{O}\mathcal{M}_{A' \otimes K}(M, a(F', K)) \). According to Lemma 6.2 we have natural isomorphisms

\[
M'(F' \otimes \mathcal{O}_K/m) \cong M(F \otimes \mathcal{O}_K/m),
\]

\[
M'(F' \otimes \mathcal{O}_K/f') \cong M(F \otimes \mathcal{O}_K/f).
\]

By definition \( M(F \otimes \mathcal{O}_K/m) \) is nilpotent while \( M(F \otimes \mathcal{O}_K/f') \) is linear by assumption. It follows that \( M' \) is a base compactification. \qed

7. Local models

**Proposition 7.1.** Let \( M \subset \mathcal{H}\mathcal{O}\mathcal{M}_{A \otimes K}(M, a(F, K)) \) be a local model. If \( M(F \otimes \mathcal{O}_K/f') \) is linear then the image of \( M \) under the restriction isomorphism of Proposition 4.7 is a local model of the shtuka \( \mathcal{H}\mathcal{O}\mathcal{M}_{A' \otimes K}(M, a(F', K)) \).

Proof. By Proposition 9.6.8 the local model \( M \) is the intersection of the coefficient compactification \( M^c \) and a base compactification \( M^b \). Proposition 5.8 claims that \( M^c \) is mapped isomorphically onto the coefficient compactification of \( \mathcal{H}\mathcal{O}\mathcal{M}_{A' \otimes K}(M, a(F', K)) \). The image of \( M^b \) in \( \mathcal{H}\mathcal{O}\mathcal{M}_{A' \otimes K}(M, a(F', K)) \) is a base compactification by Proposition 6.3. According to Proposition 9.6.8 their intersection is a local model. \qed

**Proposition 7.2.** Let \( M \subset \mathcal{H}\mathcal{O}\mathcal{M}_{A \otimes K}(M, a(F, K)) \) be a local model such that \( M(F \otimes \mathcal{O}_K/f') \) is linear. Let \( M' \subset \mathcal{H}\mathcal{O}\mathcal{M}_{A' \otimes K}(M, a(F', K)) \) be the image of
\(M\) under the restriction isomorphism of Proposition 4.7. The diagram

\[
\begin{align*}
\xymatrix{H^1(M) \ar[r]^{\rho} \ar[d]^\text{res.} & H^1(\nabla M) \ar[d]^\text{res.} \\
H^1(M') \ar[r]^{\rho'} & H^1(\nabla M')}
\end{align*}
\]

is commutative. Here \(\rho\) is the regulator of \(M\) and \(\rho'\) is the regulator of \(M'\).

**Proof.** Recall that the natural maps

\[
\begin{align*}
\mathcal{O}_F \otimes \mathcal{O}_K & \to \mathcal{O}_F \hat{\otimes} \mathcal{O}_K, \\
\mathcal{O}_{F'} \otimes \mathcal{O}_K & \to \mathcal{O}_{F'} \hat{\otimes} \mathcal{O}_K
\end{align*}
\]

are isomorphisms by Proposition 3.7.8. We need to prove that the regulator of \(M\) viewed as an elliptic \(\mathcal{O}_F \hat{\otimes} \mathcal{O}_K\)-shtuka of conductor \(\mathfrak{f}\) coincides with the regulator of \(M\) viewed as an elliptic \(\mathcal{O}_{F'} \hat{\otimes} \mathcal{O}_K\)-shtuka of conductor \(\mathfrak{f}'\). The field extension \(F/F'\) is totally ramified of degree \(d\). As a consequence \(\mathfrak{f}' = \mathfrak{f}^d\).

The result now follows from Theorem 5.14.5. \(\square\)

Next we prove that the exponential maps of local models are stable under restriction of coefficients.

**Lemma 7.3.** The restriction map \(b(F, K) \to b(F', K)\) induces an isomorphism

\[
\text{Hom}_{A \otimes K}(M, b(F, K)) \cong \text{Hom}_{A' \otimes K}(M, b(F', K))
\]

of \(F^\# \hat{\otimes} K\)-module shtukas.

**Proof.** The argument is the same as in Proposition 4.7 save for the fact that one needs to use Lemma 2.2 in place of Lemma 2.1. \(\square\)

**Lemma 7.4.** The commutative square of rings

\[
\begin{align*}
\xymatrix{\mathcal{O}_F \hat{\otimes} \mathcal{O}_K & F^\# \hat{\otimes} \mathcal{O}_K \\
\mathcal{O}_{F'} \hat{\otimes} \mathcal{O}_K & (F')^\# \hat{\otimes} \mathcal{O}_K}
\end{align*}
\]

is cocartesian.

**Proof.** Lemma 2.1 implies that the square

\[
\begin{align*}
\xymatrix{\mathcal{O}_F \otimes \mathcal{O}_K & \mathcal{O}_F \hat{\otimes} \mathcal{O}_K \\
\mathcal{O}_{F'} \otimes \mathcal{O}_K & \mathcal{O}_{F'} \hat{\otimes} \mathcal{O}_K}
\end{align*}
\]

is cocartesian.
is cocartesian. At the same time the square
\[
\begin{array}{ccc}
F \otimes O_K & \xrightarrow{\exp} & F^\# \otimes O_K \\
\downarrow & & \downarrow \\
F' \otimes O_K & \xrightarrow{\exp'} & (F')^\# \otimes O_K
\end{array}
\]
is cocartesian by Lemma 2.2. Whence the result. \qed

**Proposition 7.5.** Let \( \mathcal{M} \subset \mathcal{H}om_{A \otimes K}(M, a(F, K)) \) be a local model such that \( \mathcal{M}(F \otimes O_K/f') \) is linear. Let \( \mathcal{M}' \subset \mathcal{H}om_{A' \otimes K}(M, a(F', K)) \) be the image of \( \mathcal{M} \) under the restriction isomorphism of Proposition 4.7. The diagram
\[
\begin{array}{ccc}
F \otimes O_F H^1(\nabla M) & \xrightarrow{\exp} & F \otimes O_F H^1(M) \\
\downarrow \text{res.} & & \downarrow \text{res.} \\
F' \otimes O_{F'} H^1(\nabla M') & \xrightarrow{\exp'} & F' \otimes O_{F'} H^1(M')
\end{array}
\]
is commutative. Here \( \exp \) is the exponential map of \( \mathcal{M} \) and \( \exp' \) is the exponential map of \( \mathcal{M}' \).

**Proof.** Suppose that \( \mathcal{M} \) is given by a diagram
\[
\begin{bmatrix} M_0 & \\ i & \Rightarrow & j & M_1 \end{bmatrix}.
\]

By definition the exponential map of \( \mathcal{M} \) is the composition \( H^1(\gamma) \circ H^1(\nabla \gamma)^{-1} \) of the maps
\[
H^1(\gamma): F \otimes O_F H^1(M) \to \text{Lie}_E(K),
\]
\[
H^1(\nabla \gamma): F \otimes O_F H^1(\nabla M) \to \text{Lie}_E(K)
\]
of Definition 9.9.1. Similarly the exponential map of \( \mathcal{M}' \) is the composition of the maps
\[
H^1(\gamma'): F' \otimes O_{F'} H^1(M) \to \text{Lie}_E(K),
\]
\[
H^1(\nabla \gamma'): F' \otimes O_{F'} H^1(\nabla M) \to \text{Lie}_E(K).
\]

To prove the proposition it is enough to show that \( H^1(\gamma) \) and \( H^1(\nabla \gamma) \) are compatible with the corresponding maps of \( \mathcal{M}' \).

Let \( c \in H^1(M) \) be a cohomology class and let \( \alpha \) be the image of \( c \) under \( H^1(\gamma) \). Let \( g \in M_1 \) be an element representing \( c \). According to Proposition 9.10.1 there exists a unique element
\[
f \in M_0(F^\# \otimes O_K) \subset \mathcal{H}om_{A \otimes K}(M, b(F, K))
\]
such that \( (i - j)(f) = g \). The element \( \alpha \in \text{Lie}_E(K) \) is characterized by the fact that for all \( x \) in an open neighbourhood of 0 in \( F \) and for all \( m \in M^0 \) we have \( f(m)(x) = m \exp(x\alpha) \).
Lemma 7.3 identifies $\mathcal{H}om_{A \otimes K}(M, b(F, K))$ with $\mathcal{H}om_{A' \otimes K}(M, b(F', K))$ while Lemma 7.4 implies that the natural map $\mathcal{M}'((F')^\# \otimes \mathcal{O}_K) \to \mathcal{M}(F^\# \otimes \mathcal{O}_K)$ is an isomorphism. Using Proposition 9.10.1 we conclude that the square

\[
\begin{array}{ccc}
H^1(\mathcal{M}) & \xrightarrow{H^1(\gamma)} & \text{Lie}_E(K) \\
\downarrow \text{res} & & \downarrow \\
H^1(\mathcal{M}') & \xrightarrow{H^1(\gamma')} & \text{Lie}_E(K)
\end{array}
\]

is commutative. The same argument shows that the square

\[
\begin{array}{ccc}
H^1(\nabla \mathcal{M}) & \xrightarrow{H^1(\nabla \gamma)} & \text{Lie}_E(K) \\
\downarrow \text{res} & & \downarrow \\
H^1(\nabla \mathcal{M}') & \xrightarrow{H^1(\nabla \gamma')} & \text{Lie}_E(K)
\end{array}
\]

is commutative. So we get the result. \qed