Shtuka cohomology and special values of Goss L-functions

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CHAPTER 10

Change of coefficients

This chapter is of a technical nature. In it we verify that the constructions of Chapter 9 are compatible with restriction of the coefficient ring A. This result will be used in Chapter 11 where we show that the regulator of a local model is the inverse of the exponential map. We will do it by reduction to an explicit computation in the case $A = \mathbb{F}_q[t]$.

1. Duality for Hom shtukas

In this section we describe a general duality construction for Hom shtukas. It will be used several times in the rest of the chapter. We begin with an auxillary result.

Lemma 1.1. *Let R be a τ-ring. If $M = [M_0 \underset{j_M}{\overset{i_M}{\rightrightarrows}} M_1]$ is an R-module shtuka and N a left $R\{\tau\}$-module then*

$$\mathcal{H}\mathrm{om}_R(M, N) = \Big[\mathrm{Hom}_R(M_1, N) \underset{j}{\overset{i}{\rightrightarrows}} \mathrm{Hom}_R(\tau^* M_0, N) \Big]$$

where

$$i(f)\colon r \otimes m \mapsto r f j_M(m),$$
$$j(f)\colon r \otimes m \mapsto r\tau \cdot f i_M(m).$$

Proof. Follows directly from the definition of $\mathcal{H}\mathrm{om}$ (Definition 1.12.1). □

If $\varphi\colon R \to S$ is a ring homomorphism, M an S-module and N an R-module then the R-modules $\mathrm{Hom}_R(M, N)$ and $\mathrm{Hom}_R(S, N)$ carry natural S-module structures: via M in the first case and via S in the second. Furthermore the natural duality map

$$\mathrm{Hom}_S(M, \mathrm{Hom}_R(S, N)) \to \mathrm{Hom}_R(M, N), \quad f \mapsto [m \mapsto f(m)(1)]$$

is an S-module isomorphism. We would like to establish an analog of this duality for a τ-ring homomorphism $\varphi\colon (R, \tau) \to (S, \sigma)$ and $\mathcal{H}\mathrm{om}$ in place of Hom. In the following it will be important to distinguish the τ-endomorphisms of R and S. We thus denote them by different letters.

Let $\varphi\colon (R, \tau) \to (S, \sigma)$ be a homomorphism of τ-rings. For every S-module M there is a natural base change map

$$\mu_M\colon \tau^* M \to \sigma^* M, \quad r \otimes m \mapsto \varphi(r) \otimes m.$$

In particular we have a base change map $\mu_S\colon \tau^*S \to \sigma^*S = S$.

Consider the commutative diagram

$$(*)\qquad \begin{tikzcd} S \arrow[r, "\sigma"] & S \\ R \arrow[u, "\varphi"] \arrow[r, "\tau"] & R \arrow[u, "\varphi"'] \end{tikzcd}$$

For the duality statements below to work it will be necessary to assume that $(*)$ is cocartesian in the category of rings. It is cocartesian if and only if the base change map $\mu_S\colon \tau^*S \to S$ is an isomorphism.

Proposition 1.2. *Let $\varphi\colon (R,\tau) \to (S,\sigma)$ be a homomorphism of τ-rings and let N be a left $R\{\tau\}$-module. Consider the shtuka*

$$\mathcal{H}\mathrm{om}_R(S,N) = \Big[\operatorname{Hom}_R(S,N) \underset{j}{\overset{i}{\rightrightarrows}} \operatorname{Hom}_R(\tau^*S,N)\Big].$$

If the commutative diagram of rings $(*)$ is cocartesian then the following holds:

(1) *i is an isomorphism.*
(2) *The endomorphism $i^{-1}j$ of $\operatorname{Hom}_R(S,N)$ makes it into a left $S\{\sigma\}$-module.*
(3) *If $f \in \operatorname{Hom}_R(S,N)$ then $g = i^{-1}j(f)$ is the unique R-linear map such that*
$$g\big[\varphi(r)\sigma(s)\big] = r\tau \cdot f(s)$$
for every $r \in R$ and $s \in S$.

Proof. (1) Let $\tau^a\colon \tau^*S \to S$ be the adjoint of the τ-multiplication map of S. By definition
$$\tau^a(r \otimes s) = \varphi(r)\sigma(s).$$
At the same time
$$\mu_S(r \otimes s) = \varphi(r) \otimes s = \varphi(r)\sigma(s).$$
Thus $\tau^a = \mu_S$ is an isomorphism and we conclude that i is an isomorphism from Lemma 1.1.

(2) We will deduce from (3) that the endomorphism $i^{-1}j$ is σ-linear. Let us temporarily denote $e = i^{-1}j$. Let $f \in \operatorname{Hom}_R(S,N)$. If $s_1 \in S$ then the maps $e(f \cdot s_1)$ and $e(f) \cdot \sigma(s_1)$ satisfy
$$e(f \cdot s_1)\big[\varphi(r)\sigma(s)\big] = r\tau \cdot f(s_1 s)$$
$$e(f) \cdot \sigma(s_1)\big[\varphi(r)\sigma(s)\big] = e(f)\big[\sigma(s_1)\varphi(r)\sigma(s)\big] = r\tau \cdot f(s_1 s)$$
for every $r \in R$, $s \in S$. Thus e is σ-linear.

(3) Let $\tau_N^a\colon \tau^*N \to N$ be the adjoint of the τ-multiplication map. If $f \in \operatorname{Hom}_R(S,N)$ then according to Lemma 1.1
$$j(f)\colon r \otimes s \mapsto r\tau \cdot f(s)$$

for every $r \in R$ and $s \in S$. As we saw in (1) the map i is given by the composition with the base change map $\mu_S\colon r \otimes s \mapsto \varphi(r)\sigma(s)$. The result is now clear. $\square$

Proposition 1.3. *Let $\varphi\colon (R,\tau) \to (S,\sigma)$ be a homomorphism of τ-rings. Let*

$$M = \Big[M_0 \overset{i_M}{\underset{j_M}{\rightrightarrows}} M_1\Big]$$

be an S-module shtuka and N a left $R\{\tau\}$-module. If the commutative diagram of rings $(*)$ is cocartesian then the maps

$$\mathrm{Hom}_S(M_1, \mathrm{Hom}_R(S,N)) \xrightarrow{\quad\text{duality}\quad} \mathrm{Hom}_R(M_1, N),$$
$$\mathrm{Hom}_S(\sigma^* M_0, \mathrm{Hom}_R(S,N)) \xrightarrow{(\mu_{M_0})^* \circ\, \text{duality}} \mathrm{Hom}_R(\tau^* M_0, N)$$

define an isomorphism

$$\mathcal{H}\mathrm{om}_S(M, \mathrm{Hom}_R(S,N)) \cong \mathcal{H}\mathrm{om}_R(M,N)$$

of R-module shtukas. The left $S\{\sigma\}$-module structure on $\mathrm{Hom}_R(S,N)$ *is as constructed in Proposition 1.2.*

Proof. Denote the duality maps

$$\begin{aligned}\eta_0\colon\ &\mathrm{Hom}_S(M_1, \mathrm{Hom}_R(S,N)) \to \mathrm{Hom}_R(M,N),\\ \eta_1\colon\ &\mathrm{Hom}_S(\sigma^* M_0, \mathrm{Hom}_R(S,N)) \to \mathrm{Hom}_R(\tau^* M_0, N).\end{aligned}$$

If $(*)$ is cocartesian then the base change map μ_{M_0} is an isomorphism. The inverse of μ_{M_0} is given by the formula

$$\varphi(r)\sigma(s) \otimes m \mapsto r \otimes sm.$$

Thus η_0 and η_1 define an isomorphism of R-module shtukas provided they form a morphism of shtukas. Let us show that it is indeed the case.

Let i_S, j_S be the arrows of $\mathcal{H}\mathrm{om}_S(M, \mathrm{Hom}_R(S,N))$ and let i_R, j_R be the arrows of $\mathcal{H}\mathrm{om}_R(M,N)$. If $f \in \mathrm{Hom}_S(M_1, \mathrm{Hom}_R(S,N))$ then

$$\eta_0(f)\colon m \mapsto f(m)(1).$$

So Lemma 1.1 implies that

$$j_R(\eta_0(f))\colon r \otimes m \mapsto r\tau \cdot [f i_M(m)](1).$$

By the same Lemma

$$j_S(f)\colon s \otimes m \mapsto s\sigma \cdot f i_M(m).$$

If $g \in \mathrm{Hom}_S(\sigma^* M_0, \mathrm{Hom}_R(S,N))$ then

$$\eta_1(g)\colon r \otimes m \mapsto g(\varphi(r) \otimes m)(1).$$

Therefore

$$\begin{aligned}\eta_1(j_S(f))\colon r\otimes m \mapsto{} & [j_S(f)(\varphi(r)\otimes m)](1)\\ ={} & [\varphi(r)\sigma\cdot fi_M(m)](1)\\ ={} & [\sigma\cdot fi_M(m)](\varphi(r)).\end{aligned}$$

According to Proposition 1.2

$$\sigma\cdot fi_M(m)\colon \varphi(r)\mapsto r\tau\cdot[fi_M(m)](1).$$

Hence

$$\eta_1(j_S(f))\colon r\otimes m\mapsto r\tau\cdot[fi_M(m)](1).$$

We conclude that $\eta_1(j_S(f)) = j_R(\eta_0(f))$. It is easy to see that $\eta_1(i_S(f)) = i_R(\eta_0(f))$. ☐

2. Tensor products

Lemma 2.1. *Let T be a locally compact $\mathbb{F}_q$-algebra and let $S'\subset S$ be an extension of locally compact $\mathbb{F}_q$-algebras. If S is finitely generated free as a topological S'-module then the natural map $S\otimes_{S'}(S'\mathbin{\check{\otimes}} T)\to S\mathbin{\check{\otimes}} T$ is an isomorphism.*

Proof. We rewrite the natural map in question as

$$(S\otimes_{\mathrm{ic}} T)\otimes_{(S'\otimes_{\mathrm{ic}} T)}(S'\mathbin{\check{\otimes}} T)\to S\mathbin{\check{\otimes}} T.$$

By assumption S is a finitely generated free topological S'-module. Therefore $S\otimes_{\mathrm{ic}} T$ is a finitely generated free topological $S'\otimes_{\mathrm{ic}} T$-module. The result now follows from Lemma 3.2.4. ☐

Lemma 2.2. *Let T be a locally compact $\mathbb{F}_q$-algebra and let $S'\subset S$ be an extension of locally compact $\mathbb{F}_q$-algebras. If S is locally free of finite rank as an S'-module without topology then the natural map $S\otimes_{S'}((S')^{\#}\mathbin{\widehat{\otimes}} T)\to S^{\#}\mathbin{\widehat{\otimes}} T$ is an isomorphism.*

Proof. We rewrite the natural map in question as

$$(S^{\#}\otimes_{\mathrm{c}} T)\otimes_{((S')^{\#}\otimes_{\mathrm{c}} T)}((S')^{\#}\mathbin{\widehat{\otimes}} T)\to S^{\#}\mathbin{\widehat{\otimes}} T.$$

By assumption S is locally free of finite rank as an S'-module without topology. Therefore $S^{\#}\otimes_{\mathrm{c}} T$ is a topological direct summand of a finitely generated free $(S')^{\#}\otimes_{\mathrm{c}} T$-module. The result now follows from Lemma 3.2.4. ☐

3. The setting

We now start with the main part of this chapter. The setting is as follows. Fix a coefficient ring A as in Definition 7.2.1. As usual we denote F the local field of A at infinity, $\mathcal{O}_F\subset F$ the ring of integers and $\mathfrak{m}_F\subset\mathcal{O}_F$ the maximal ideal. Let K be a finite product of local fields containing $\mathbb{F}_q$. Fix a Drinfeld A-module E over K and let $M=\operatorname{Hom}(E,\mathbb{G}_a)$ be its motive. Throughout the chapter we assume that the action of A on $\operatorname{Lie}_E(K)$ extends to a continuous action of F. This is the context in which we defined and studied local models in Chapter 9.

Fix an $\mathbb{F}_q$-subalgebra $A' \subset A$ such that A is finite flat over A'. Note that A' is itself a coefficient ring. We denote F' the local field of A' at infinity, $\mathcal{O}_{F'} \subset F'$ the ring of integers and $\mathfrak{m}_{F'} \subset \mathcal{O}_{F'}$ the maximal ideal.

Lemma 3.1. *The motive of E viewed as a Drinfeld A'-module is M with its natural left $A' \otimes K\{\tau\}$-module structure.* $\square$

4. Hom shtukas

Lemma 4.1. *The commutative square of rings*

$$\begin{tikzcd} A \otimes K \arrow[r] & F \check{\otimes} K \\ A' \otimes K \arrow[r] \arrow[u] & F' \check{\otimes} K \arrow[u] \end{tikzcd}$$

is cocartesian.

Proof. Indeed $A \otimes_{A'} F' = F$ so

$$A \otimes_{A'} (F' \check{\otimes} K) = (A \otimes_{A'} F') \otimes_{F'} (F' \check{\otimes} K) = F \otimes_{F'} (F' \check{\otimes} K).$$

Now Lemma 2.1 shows that $F \otimes_{F'} (F' \check{\otimes} K) = F \check{\otimes} K$ and the result follows. $\square$

Corollary 4.2. *The natural map $M \otimes_{A' \otimes K} (F' \check{\otimes} K) \to M \otimes_{A \otimes K} (F \check{\otimes} K)$ is an isomorphism of left $F' \check{\otimes} K\{\tau\}$-modules.* $\square$

Lemma 4.3. *The commutative square of rings*

$$\begin{tikzcd} F \check{\otimes} K \arrow[r, "\tau"] & F \check{\otimes} K \\ F' \check{\otimes} K \arrow[r, "\tau"] \arrow[u] & F' \check{\otimes} K \arrow[u] \end{tikzcd}$$

is cocartesian.

Proof. Immediate from Lemma 2.1. $\square$

We are thus in position to apply the duality machinery of Section 1 to the τ-ring homomorphism $F' \check{\otimes} K \to F \check{\otimes} K$. Proposition 1.2 equips the $F \check{\otimes} K$-module

$$\mathrm{Hom}_{F' \check{\otimes} K}(F \check{\otimes} K, a(F', K))$$

with the structure of a left $F \check{\otimes} K\{\tau\}$-module.

Lemma 4.4. *The natural map*

$$\mathrm{Hom}_{F' \check{\otimes} K}(F \check{\otimes} K, a(F', K)) \to \mathrm{Hom}_{F'}(F, a(F', K))$$

is an isomorphism of $F \check{\otimes} K$-modules.

Proof. Follows since $F \otimes_{F'} (F' \check{\otimes} K) = F \check{\otimes} K$ by Lemma 2.1. $\square$

So we get a left $F \check{\otimes} K\{\tau\}$-module structure on $\mathrm{Hom}_{F'}(F, a(F', K))$.

Lemma 4.5. *If $g \in \mathrm{Hom}_{F'}(F, a(F', K))$ then $\tau \cdot g$ maps $x \in F$ to $\tau(g(x))$.* $\square$

Lemma 4.6. *The map*

$$a(F, K) \to \mathrm{Hom}_{F'}(F, a(F', K)), \quad f \mapsto [x \mapsto (y \mapsto f(yx))]$$

is an isomorphism of left $F \check{\otimes} K\{\tau\}$-modules.

Proof. For a finite-dimensional F'-vector space V let α_V be the map

$$\alpha_V : a(V, K) \to \mathrm{Hom}_{F'}(V, a(F', K)), \quad f \mapsto [v \mapsto (y \mapsto f(yv))].$$

It is clearly an $F' \check{\otimes} K$-linear isomorphism if V is of dimension 1. The map α_V is also natural in V. Hence α_V is an $F' \check{\otimes} K$-linear isomorphism for any finite-dimensional F'-vector space V and in particular for $V = F$.

The map α_F is also F-linear. By Lemma 2.1 we have $F \otimes_{F'} (F' \check{\otimes} K) = F \check{\otimes} K$. Hence α_F is $F \check{\otimes} K$-linear. A simple computation shows that it commutes witht the action of τ. So we get the result. $\square$

Proposition 4.7. *The restriction map $a(F, K) \to a(F', K)$ induces an isomorphism*

$$\mathcal{H}\mathrm{om}_{A \otimes K}(M, a(F, K)) \cong \mathcal{H}\mathrm{om}_{A' \otimes K}(M, a(F', K))$$

of $F' \check{\otimes} K$-module shtukas.

Proof. We have

$$\mathcal{H}\mathrm{om}_{A \otimes K}(M, a(F, K)) = \mathcal{H}\mathrm{om}_{F \check{\otimes} K}(M \otimes_{A \otimes K} (F \check{\otimes} K), a(F, K)).$$

Lemma 4.6 gives us a natural isomorphism

$$\begin{aligned} &\mathcal{H}\mathrm{om}_{F \check{\otimes} K}(M \otimes_{A \otimes K} (F \check{\otimes} K), a(F, K)) \\ \cong\, &\mathcal{H}\mathrm{om}_{F \check{\otimes} K}(M \otimes_{A \otimes K} (F \check{\otimes} K), \mathrm{Hom}_{F'}(F, a(F', K)))). \end{aligned}$$

In view of Lemma 4.3 we can apply Proposition 1.3 to get an isomorphism

$$\begin{aligned} &\mathcal{H}\mathrm{om}_{F \check{\otimes} K}(M \otimes_{A \otimes K} (F \check{\otimes} K), \mathrm{Hom}_{F'}(F, a(F', K))) \\ \cong\, &\mathcal{H}\mathrm{om}_{F' \check{\otimes} K}(M \otimes_{A \otimes K} (F \check{\otimes} K), a(F', K)) \end{aligned}$$

of $F' \check{\otimes} K$-module shtukas. Corollary 4.2 identifies $M \otimes_{A \otimes K} (F \check{\otimes} K)$ with $M \otimes_{A' \otimes K} (F' \check{\otimes} K)$. So we get an isomorphism

$$\mathcal{H}\mathrm{om}_{A \otimes K}(M, a(F, K)) \cong \mathcal{H}\mathrm{om}_{A' \otimes K}(M, a(F', K))$$

of $F' \check{\otimes} K$-module shtukas. A straightforward computation shows that this isomorphism is induced by the restriction map $a(F, K) \to a(F', K)$. $\square$

5. Coefficient compactifications

We denote C the projective compactification of $\operatorname{Spec} A$ and C' the projective compactification of A'. Let $\rho\colon C \times K \to C' \times K$ be the map induced by the inclusion $A' \subset A$.

Lemma 5.1. *The commutative square of schemes*

$$\begin{tikzcd}
\operatorname{Spec} \mathcal{O}_F \mathbin{\check{\otimes}} K \arrow[r] \arrow[d] & C \times X \arrow[d, "\rho"] \\
\operatorname{Spec} \mathcal{O}_{F'} \mathbin{\check{\otimes}} K \arrow[r] & C' \times X
\end{tikzcd}$$

is cartesian.

Proof. Lemma 2.1 implies that the square

$$\begin{tikzcd}
\mathcal{O}_F \otimes K \arrow[r] & \mathcal{O}_F \mathbin{\check{\otimes}} K \\
\mathcal{O}_{F'} \otimes K \arrow[r] \arrow[u] & \mathcal{O}_{F'} \mathbin{\check{\otimes}} K \arrow[u]
\end{tikzcd}$$

is cocartesian. Whence the result. □

Corollary 5.2. *For every quasi-coherent sheaf $\mathcal{E}$ on $C \times X$ the base change map*

$$(\rho_*\mathcal{E})(\mathcal{O}_{F'} \mathbin{\check{\otimes}} K) \to \mathcal{E}(\mathcal{O}_F \mathbin{\check{\otimes}} K)$$

is an isomorphism of $\mathcal{O}_{F'} \mathbin{\check{\otimes}} K$-modules. □

Lemma 5.3. *If $\mathcal{E}$ is the shtuka on $C \times K$ which corresponds to the left $A \otimes K\{\tau\}$-module M by Theorem 7.5.5 then $\rho_*\mathcal{E}$ is the shtuka on $C' \times K$ which corresponds to M viewed as a left $A' \otimes K\{\tau\}$-module.*

Proof. Suppose that $\mathcal{E}$ is given by the diagram

$$\Big[\mathcal{E}_{-1} \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{E}_0\Big] \subset \Big[M \overset{1}{\underset{\tau}{\rightrightarrows}} M\Big].$$

Let f be the degree of the residue field of F over $\mathbb{F}_q$ and let r be the rank of M as an $A \otimes K$-module. By Theorem 7.5.5 the sheaves $\mathcal{E}_{-1}$, $\mathcal{E}_0$ are locally free of rank r and have the following property:

$$\begin{aligned}
\mathrm{H}^0(C \times K, \mathcal{E}_0(n)) &= M^{frn}, \\
\mathrm{H}^0(C \times K, \mathcal{E}_{-1}(n)) &= M^{frn-1}.
\end{aligned}$$

Let $d = [F : F']$. Observe that $f = df'$ where f' is the degree of the residue field of F' over $\mathbb{F}_q$. The morphism $\rho\colon C \times K \to C' \times K$ is finite flat of degree

d. Hence $\rho_*\mathcal{E}_{-1}$, $\rho_*\mathcal{E}_0$ are locally free of rank dr and

$$\mathrm{H}^0(C \times K, \rho_*\mathcal{E}_0(n)) = M^{f'drn},$$
$$\mathrm{H}^0(C \times K, \rho_*\mathcal{E}_{-1}(n)) = M^{f'drn-1}.$$

The unicity part of Theorem 7.5.5 now implies the result. □

We next study the function space $a(F/\mathcal{O}_F, K)$.

Lemma 5.4. *The commutative square of rings*

$$\begin{array}{ccc} \mathcal{O}_F \,\check{\otimes}\, K & \xrightarrow{\tau} & \mathcal{O}_F \,\check{\otimes}\, K \\ \uparrow & & \uparrow \\ \mathcal{O}_{F'} \,\check{\otimes}\, K & \xrightarrow{\tau} & \mathcal{O}_{F'} \,\check{\otimes}\, K \end{array}$$

is cocartesian.

Proof. Immediate from Lemma 2.1. □

So we can apply the duality constructions of Section 1 to the τ-ring homomorphism $\mathcal{O}_{F'} \check{\otimes} K \to \mathcal{O}_F \check{\otimes} K$. Proposition 1.2 equips the $\mathcal{O}_F \check{\otimes} K$-module

$$\mathrm{Hom}_{\mathcal{O}_{F'}\check{\otimes}K}(\mathcal{O}_F \,\check{\otimes}\, K, a(F'/\mathcal{O}_{F'}, K))$$

with the structure of a left $\mathcal{O}_F \,\check{\otimes}\, K\{\tau\}$-module.

Lemma 5.5. *The natural map*

$$\mathrm{Hom}_{\mathcal{O}_{F'}\check{\otimes}K}(\mathcal{O}_F \,\check{\otimes}\, K, a(F'/\mathcal{O}_{F'}, K)) \to \mathrm{Hom}_{\mathcal{O}_{F'}}(\mathcal{O}_F, a(F'/\mathcal{O}_{F'}, K))$$

is an isomorphism of $\mathcal{O}_F \,\check{\otimes}\, K$-modules. □

So we get a left $\mathcal{O}_F \check{\otimes} K\{\tau\}$-module structure on $\mathrm{Hom}_{\mathcal{O}_{F'}}(\mathcal{O}_F, a(F'/\mathcal{O}_{F'}, K))$.

Lemma 5.6. *If $g \in \mathrm{Hom}_{\mathcal{O}_{F'}}(\mathcal{O}_F, a(F'/\mathcal{O}_{F'}, K))$ then $\tau \cdot g$ maps $x \in \mathcal{O}_F$ to $\tau(g(x))$.* □

Lemma 5.7. *The map*

$$a(F/\mathcal{O}_F, K) \to \mathrm{Hom}_{\mathcal{O}_{F'}}(\mathcal{O}_F, a(F'/\mathcal{O}_{F'}, K)), \quad f \mapsto [x \mapsto (y \mapsto f(yx))]$$

is an isomorphism of left $\mathcal{O}_F \,\check{\otimes}\, K\{\tau\}$-modules.

Proof. We view $a(F/\mathcal{O}_F, K)$ as a subspace of $a(F, K)$ consisting of functions which vanish on $\mathcal{O}_F$, and similarly for $a(F'/\mathcal{O}_{F'}, K)$. Note that

$$\mathrm{Hom}_{F'}(F, a(F, K)) = \mathrm{Hom}_{\mathcal{O}_{F'}}(\mathcal{O}_F, a(F', K)).$$

We can thus identify

$$\mathrm{Hom}_{\mathcal{O}_{F'}}(\mathcal{O}_F, a(F'/\mathcal{O}_{F'}, K))$$

with a submodule of $\mathrm{Hom}_{F'}(F, a(F, K))$. In view of this remark the result follows from Lemma 4.6. □

Proposition 5.8. *The restriction isomorphism of Proposition 4.7 identifies the coefficient compactification of* $\mathcal{H}\mathrm{om}_{A\otimes K}(M, a(F,K))$ *with the coefficient compactification of* $\mathcal{H}\mathrm{om}_{A'\otimes K}(M, a(F',K))$.

Proof. Lemma 5.7 gives us a natural isomorphism

$$\begin{aligned}&\mathcal{H}\mathrm{om}_{\mathcal{O}_F\check{\otimes}K}(\mathcal{E}(\mathcal{O}_F \check{\otimes} K), a(F/\mathcal{O}_F, K))\\ \cong\,&\mathcal{H}\mathrm{om}_{\mathcal{O}_F\check{\otimes}K}(\mathcal{E}(\mathcal{O}_F \check{\otimes} K), \mathrm{Hom}_{\mathcal{O}_{F'}}(\mathcal{O}_F, a(F'/\mathcal{O}_{F'}, K))).\end{aligned}$$

In view of Lemma 5.4 we can apply Proposition 1.3 to get an isomorphism

$$\begin{aligned}&\mathcal{H}\mathrm{om}_{\mathcal{O}_F\check{\otimes}K}(\mathcal{E}(\mathcal{O}_F \check{\otimes} K), \mathrm{Hom}_{\mathcal{O}_{F'}}(\mathcal{O}_F, a(F'/\mathcal{O}_{F'}, K)))\\ \cong\,&\mathcal{H}\mathrm{om}_{\mathcal{O}_{F'}\check{\otimes}K}(\mathcal{E}(\mathcal{O}_F \check{\otimes} K), a(F'/\mathcal{O}_{F'}, K))\end{aligned}$$

of $\mathcal{O}_F \check{\otimes} K$-module shtukas. It is easy to see that the resulting isomorphism

$$\begin{aligned}&\mathcal{H}\mathrm{om}_{\mathcal{O}_F\check{\otimes}K}(\mathcal{E}(\mathcal{O}_F \check{\otimes} K), a(F/\mathcal{O}_F, K))\\ \cong\,&\mathcal{H}\mathrm{om}_{\mathcal{O}_{F'}\check{\otimes}K}(\mathcal{E}(\mathcal{O}_F \check{\otimes} K), a(F'/\mathcal{O}_{F'}, K))\end{aligned}$$

is induced by the restriction map $a(F/\mathcal{O}_F, K) \to a(F'/\mathcal{O}_{F'}, K)$. Now Corollary 5.2 implies that the natural map $\rho_*\mathcal{E}(\mathcal{O}_{F'} \check{\otimes} K) \to \mathcal{E}(\mathcal{O}_F \check{\otimes} K)$ is an isomorphism. Lemma 5.3 implies that the shtuka

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_{F'}\check{\otimes}K}(\rho_*\mathcal{E}(\mathcal{O}_{F'} \check{\otimes} K), a(F'/\mathcal{O}_{F'}, K))$$

is the coefficient compactification of $\mathcal{H}\mathrm{om}_{A'\otimes K}(M, a(F', K))$. Whence the result. □

6. Base compactifications

Lemma 6.1. *The commutative square of rings*

$$\begin{array}{ccc} F \check{\otimes} \mathcal{O}_K & \longrightarrow & F \check{\otimes} K \\ \uparrow & & \uparrow \\ F' \check{\otimes} \mathcal{O}_K & \longrightarrow & F' \check{\otimes} K \end{array}$$

is cocartesian.

Proof. Follows from Lemma 2.1. □

Lemma 6.2. *For every open ideal* $I \subset \mathcal{O}_K$ *the commutative square of rings*

$$\begin{array}{ccc} F \check{\otimes} \mathcal{O}_K & \longrightarrow & F \otimes \mathcal{O}_K/I \\ \uparrow & & \uparrow \\ F' \check{\otimes} \mathcal{O}_K & \longrightarrow & F' \otimes \mathcal{O}_K/I \end{array}$$

is cocartesian.

Proof. Indeed Lemma 2.1 implies that the square

$$\begin{array}{ccc} F \otimes \mathcal{O}_K & \longrightarrow & F \check{\otimes} \mathcal{O}_K \\ \uparrow & & \uparrow \\ F' \otimes \mathcal{O}_K & \longrightarrow & F' \check{\otimes} \mathcal{O}_K \end{array}$$

is cocartesian. Whence the result. □

According to our assumptions the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F so that we have a continuous homomorphism $F \to K$. Recall that the *conductor* $\mathfrak{f}$ is the ideal generated by $\mathfrak{m}_F$ in $\mathcal{O}_K$ (Definition 9.6.4). In the same manner we get a conductor $\mathfrak{f}' \subset \mathcal{O}_K$ for the coefficient subring $A' \subset A$.

Proposition 6.3. *Let $\mathcal{M} \subset \mathcal{H}\mathrm{om}_{A\otimes K}(M, a(F, K))$ be a base compactification. If $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{f}')$ is linear then the image of $\mathcal{M}$ under the restriction isomorphism of Proposition 4.7 is a base compactification of $\mathcal{H}\mathrm{om}_{A'\otimes K}(M, a(F', K))$.*

Proof. Let $\mathcal{M}'$ be the image of $\mathcal{M}$ in $\mathcal{H}\mathrm{om}_{A'\otimes K}(M, a(F', K))$. The shtuka $\mathcal{M}$ is an $F \check{\otimes} \mathcal{O}_K$-lattice in $\mathcal{H}\mathrm{om}_{A\otimes K}(M, a(F, K))$ so Lemma 6.1 implies that $\mathcal{M}'$ is an $F' \check{\otimes} \mathcal{O}_K$-lattice in $\mathcal{H}\mathrm{om}_{A'\otimes K}(M, a(F', K))$. According to Lemma 6.2 we have natural isomorphisms

$$\mathcal{M}'(F' \otimes \mathcal{O}_K/\mathfrak{m}) \cong \mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}),$$
$$\mathcal{M}'(F' \otimes \mathcal{O}_K/\mathfrak{f}') \cong \mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{f}).$$

By definition $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m})$ is nilpotent while $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{f}')$ is linear by assumption. It follows that $\mathcal{M}'$ is a base compactification. □

7. Local models

Proposition 7.1. *Let $\mathcal{M} \subset \mathcal{H}\mathrm{om}_{A\otimes K}(M, a(F, K))$ be a local model. If $\mathcal{M}(F\otimes\mathcal{O}_K/\mathfrak{f}')$ is linear then the image of $\mathcal{M}$ under the restriction isomorphism of Proposition 4.7 is a local model of the shtuka $\mathcal{H}\mathrm{om}_{A'\otimes K}(M, a(F', K))$.*

Proof. By Proposition 9.6.8 the local model $\mathcal{M}$ is the intersection of the coefficient compactification $\mathcal{M}^c$ and a base compactification $\mathcal{M}^b$. Proposition 5.8 claims that $\mathcal{M}^c$ is mapped isomorphically onto the coefficient compactification of $\mathcal{H}\mathrm{om}_{A'\otimes K}(M, a(F', K))$. The image of $\mathcal{M}^b$ in $\mathcal{H}\mathrm{om}_{A'\otimes K}(M, a(F', K))$ is a base compactification by Propostion 6.3. According to Proposition 9.6.8 their intersection is a local model. □

Proposition 7.2. *Let $\mathcal{M} \subset \mathcal{H}\mathrm{om}_{A\otimes K}(M, a(F, K))$ be a local model such that $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{f}')$ is linear. Let $\mathcal{M}' \subset \mathcal{H}\mathrm{om}_{A'\otimes K}(M, a(F', K))$ be the image of*

$\mathcal{M}$ under the restriction isomorphism of Proposition 4.7. The diagram

$$\begin{tikzcd} \mathrm{H}^1(\mathcal{M}) \arrow[r, "\rho"] \arrow[d, "res."', "\wr"] & \mathrm{H}^1(\nabla\mathcal{M}) \arrow[d, "\wr"', "res."] \\ \mathrm{H}^1(\mathcal{M}') \arrow[r, "\rho'"] & \mathrm{H}^1(\nabla\mathcal{M}') \end{tikzcd}$$

is commutative. Here ρ is the regulator of $\mathcal{M}$ and ρ' is the regulator of $\mathcal{M}'$.

Proof. Recall that the natural maps

$$\mathcal{O}_F \mathbin{\check{\otimes}} \mathcal{O}_K \to \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K,$$
$$\mathcal{O}_{F'} \mathbin{\check{\otimes}} \mathcal{O}_K \to \mathcal{O}_{F'} \mathbin{\widehat{\otimes}} \mathcal{O}_K$$

are isomorphisms by Proposition 3.7.8. We need to prove that the regulator of $\mathcal{M}$ viewed as an elliptic $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$-shtuka of conductor $\mathfrak{f}$ coincides with the regulator of $\mathcal{M}$ viewed as an elliptic $\mathcal{O}_{F'} \mathbin{\widehat{\otimes}} \mathcal{O}_K$-shtuka of conductor $\mathfrak{f}'$. The field extension F/F' is totally ramified of degree d. As a consequence $\mathfrak{f}' = \mathfrak{f}^d$. The result now follows from Theorem 5.14.5. □

Next we prove that the exponential maps of local models are stable under restriction of coefficients.

Lemma 7.3. *The restriction map $b(F, K) \to b(F', K)$ induces an isomorphism*

$$\mathcal{H}\mathrm{om}_{A\otimes K}(M, b(F, K)) \cong \mathcal{H}\mathrm{om}_{A'\otimes K}(M, b(F', K))$$

of $F^\# \mathbin{\widehat{\otimes}} K$-module shtukas.

Proof. The argument is the same as in Proposition 4.7 save for the fact that one needs to use Lemma 2.2 in place of Lemma 2.1. □

Lemma 7.4. *The commutative square of rings*

$$\begin{tikzcd} \mathcal{O}_F \mathbin{\check{\otimes}} \mathcal{O}_K \arrow[r] & F^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K \\ \mathcal{O}_{F'} \mathbin{\check{\otimes}} \mathcal{O}_K \arrow[r] \arrow[u] & (F')^\# \mathbin{\widehat{\otimes}} \mathcal{O}_K \arrow[u] \end{tikzcd}$$

is cocartesian.

Proof. Lemma 2.1 implies that the square

$$\begin{tikzcd} \mathcal{O}_F \otimes \mathcal{O}_K \arrow[r] & \mathcal{O}_F \mathbin{\check{\otimes}} \mathcal{O}_K \\ \mathcal{O}_{F'} \otimes \mathcal{O}_K \arrow[r] \arrow[u] & \mathcal{O}_{F'} \mathbin{\check{\otimes}} \mathcal{O}_K \arrow[u] \end{tikzcd}$$

is cocartesian. At the same time the square

$$\begin{CD} F \otimes \mathcal{O}_K @>>> F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K \\ @AAA @AAA \\ F' \otimes \mathcal{O}_K @>>> (F')^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K \end{CD}$$

is cocartesian by Lemma 2.2. Whence the result. $\square$

Proposition 7.5. *Let $\mathcal{M} \subset \mathcal{H}\mathrm{om}_{A\otimes K}(M, a(F,K))$ be a local model such that $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{f}')$ is linear. Let $\mathcal{M}' \subset \mathcal{H}\mathrm{om}_{A'\otimes K}(M, a(F',K))$ be the image of $\mathcal{M}$ under the restriction isomorphism of Proposition 4.7. The diagram*

$$\begin{CD} F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\nabla\mathcal{M}) @>{\exp}>> F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M}) \\ @V{res.}V{\wr}V @V{\wr}V{res.}V \\ F' \otimes_{\mathcal{O}_{F'}} \mathrm{H}^1(\nabla\mathcal{M}') @>{\exp'}>> F' \otimes_{\mathcal{O}'_F} \mathrm{H}^1(\mathcal{M}') \end{CD}$$

is commutative. Here $\exp$ *is the exponential map of $\mathcal{M}$ and* $\exp'$ *is the exponential map of $\mathcal{M}'$.*

Proof. Suppose that $\mathcal{M}$ is given by a diagram

$$\Big[\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1\Big].$$

By definition the exponential map of $\mathcal{M}$ is the composition $\mathrm{H}^1(\gamma) \circ \mathrm{H}^1(\nabla\gamma)^{-1}$ of the maps

$$\mathrm{H}^1(\gamma)\colon F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M}) \to \mathrm{Lie}_E(K),$$
$$\mathrm{H}^1(\nabla\gamma)\colon F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\nabla\mathcal{M}) \to \mathrm{Lie}_E(K)$$

of Definition 9.9.1. Similarly the exponential map of $\mathcal{M}'$ is the composition of the maps

$$\mathrm{H}^1(\gamma')\colon F' \otimes_{\mathcal{O}_{F'}} \mathrm{H}^1(\mathcal{M}) \to \mathrm{Lie}_E(K),$$
$$\mathrm{H}^1(\nabla\gamma')\colon F' \otimes_{\mathcal{O}_{F'}} \mathrm{H}^1(\nabla\mathcal{M}) \to \mathrm{Lie}_E(K).$$

To prove the proposition it is enough to show that $\mathrm{H}^1(\gamma)$ and $\mathrm{H}^1(\nabla\gamma)$ are compatible with the corresponding maps of $\mathcal{M}'$.

Let $c \in \mathrm{H}^1(\mathcal{M})$ be a cohomology class and let α be the image of c under $\mathrm{H}^1(\gamma)$. Let $g \in \mathcal{M}_1$ be an element representing c. According to Proposition 9.10.1 there exists a unique element

$$f \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K) \subset \mathcal{H}\mathrm{om}_{A\otimes K}(M, b(F,K))$$

such that $(i-j)(f) = g$. The element $\alpha \in \mathrm{Lie}_E(K)$ is characterized by the fact that for all x in an open neighbourhood of 0 in F and for all $m \in M^0$ we have $f(m)(x) = m\exp(x\alpha)$.

Lemma 7.3 identifies $\mathcal{H}\mathrm{om}_{A\otimes K}(M, b(F,K))$ with $\mathcal{H}\mathrm{om}_{A'\otimes K}(M, b(F',K))$ while Lemma 7.4 implies that the natural map $\mathcal{M}'((F')^{\#} \widehat{\otimes} \mathcal{O}_K) \to \mathcal{M}(F^{\#} \widehat{\otimes} \mathcal{O}_K)$ is an isomorphism. Using Proposition 9.10.1 we conclude that the square

$$\begin{tikzcd} \mathrm{H}^1(\mathcal{M}) \arrow[r, "\mathrm{H}^1(\gamma)"] \arrow[d, "\mathrm{res.}"', "\wr"] & \mathrm{Lie}_E(K) \arrow[d, equal] \\ \mathrm{H}^1(\mathcal{M}') \arrow[r, "\mathrm{H}^1(\gamma')"] & \mathrm{Lie}_E(K) \end{tikzcd}$$

is commutative. The same argument shows that the square

$$\begin{tikzcd} \mathrm{H}^1(\nabla\mathcal{M}) \arrow[r, "\mathrm{H}^1(\nabla\gamma)"] \arrow[d, "\mathrm{res.}"', "\wr"] & \mathrm{Lie}_E(K) \arrow[d, equal] \\ \mathrm{H}^1(\nabla\mathcal{M}') \arrow[r, "\mathrm{H}^1(\nabla\gamma')"] & \mathrm{Lie}_E(K) \end{tikzcd}$$

is commutative. So we get the result. $\square$