Shtuka cohomology and special values of Goss L-functions

Mornev, M.

Author: Mornev, M.
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CHAPTER 9

Local models

Fix a coefficient ring A as in Definition 7.2.1 and let F be the local field of A at infinity. We denote $\mathcal{O}_F \subset F$ the ring of integers and $\mathfrak{m}_F \subset \mathcal{O}_F$ the maximal ideal. Let K be a finite product of local fields containing $\mathbb{F}_q$. As in Section 3.5 $\mathcal{O}_K \subset K$ stands for the ring of integers and $\mathfrak{m}_K \subset \mathcal{O}_K$ denotes the Jacobson radical. The τ-ring and τ-module structures used in this chapter are as described in Section 3.11.

We study Drinfeld modules in a local situation. Namely we work with a Drinfeld A-module E over K by the means of the $F \check{\otimes} K$-module shtuka $\mathcal{H}\mathrm{om}(M, a(F,K))$. Here M is the motive of E and $a(F,K)$ is the space of locally constant bounded $\mathbb{F}_q$-linear maps from F to K as in Definition 2.10.1. We introduce the notion of a local model which is an $\mathcal{O}_F \check{\otimes} \mathcal{O}_K$-module subshtuka $\mathcal{M} \subset \mathcal{H}\mathrm{om}(M, a(F,K))$ with certain properties. Informally speaking, $\mathcal{M}$ compactifies $\mathcal{H}\mathrm{om}(M, a(F,K))$ in the direction of the coefficients $\mathcal{O}_F \subset F$ and the base $\mathcal{O}_K \subset K$. One important result of this chapter is Theorem 7.5 which implies that local models exist. Another important result is Theorem 9.6 which states that a local model is an elliptic shtuka in the sense of Chapter 5.

The constructions in this chapter are algebraic in nature and the topology on the various tensor product rings plays no essential role. Even though the (abstract) rings $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ and $\mathcal{O}_F \check{\otimes} \mathcal{O}_K$ are the same we use the latter notation to stress that it is a subring of $F \check{\otimes} K$.

1. Lattices

Throughout the rest of the text we will often use the notion of a lattice in a module or a shtuka.

Definition 1.1. Let $R_0 \to R$ be a ring homomorphism, M an R-module and $M_0 \subset M$ an R_0-submodule. We say that M_0 is an R_0-*lattice* in M if the natural map $R \otimes_{R_0} M_0 \to M$ is an isomorphism.

Let $R_0 \to R$ be a homomorphism of τ-rings, M an R-module shtuka and $M_0 \subset M$ an R_0-module subshtuka. We say that M_0 is an R_0-lattice in M if the underlying modules of M_0 are R_0-lattices in the underlying modules of M.

2. Reflexive sheaves

The aim of this section is to review some properties of reflexive sheaves on a scheme such as $Y = \operatorname{Spec} k[[z, \zeta]]$ with k a field and the open subscheme $U \subset Y$ which is the complement of the closed point. The main result states that every locally free sheaf on U extends uniquely to a locally free sheaf on Y. While the contents of the section is widely known, it does not seem to appear in the literature in the form which we need.

Let F be a local field containing $\mathbb{F}_q$ and let K be a finite product of local fields containing $\mathbb{F}_q$. We denote $\mathcal{O}_F \subset F$ and $\mathcal{O}_K \subset K$ the corresponding rings of integers. For the rest of this section let us fix uniformizers $z \in \mathcal{O}_F$ and $\zeta \in \mathcal{O}_K$. Let $Y = \operatorname{Spec}(\mathcal{O}_F \mathbin{\check{\otimes}} \mathcal{O}_K)$ and let $U = D(z) \cup D(\zeta) \subset Y$.

Lemma 2.1. *The ring $\mathcal{O}_F \mathbin{\check{\otimes}} \mathcal{O}_K$ has the following properties:*

(1) *$\mathcal{O}_F \mathbin{\check{\otimes}} \mathcal{O}_K$ is a finite product of complete regular 2-dimensional local rings.*
(2) *The maximal ideals of $\mathcal{O}_F \mathbin{\check{\otimes}} \mathcal{O}_K$ are precisely the prime ideals containing z and ζ.*

Proof. By Proposition 3.4.13 the natural map $\mathcal{O}_F \mathbin{\check{\otimes}} \mathcal{O}_K \to \mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ is an isomorphism. So the result follows at once from Proposition 3.6.7. $\square$

Recall that a coherent sheaf $\mathcal{F}$ is called *reflexive* if the natural map from $\mathcal{F}$ to its double dual $\mathcal{F}^{**}$ is an isomorphism. A locally free sheaf is automatically reflexive. We use [13] as the reference for the theory of reflexive sheaves.

Lemma 2.2. *If $\mathcal{F}$ is a reflexive sheaf on Y then the natural map $\Gamma(Y, \mathcal{F}) \to \Gamma(U, \mathcal{F})$ is an isomorphism.*

Proof. Lemma 2.1 implies the following two statements. First, Y is a finite disjoint union of regular schemes. Second, all the prime ideals in the complement of U are of height 2. Therefore the result follows from [13], Proposition 1.6 (i) $\Rightarrow$ (iii). $\square$

Let $\iota\colon U \to Y$ be the open embedding.

Lemma 2.3. *Every coherent sheaf on U is globally generated.*

Proof. Let $\mathcal{F}$ be a coherent sheaf on U. The morphism ι is quasi-compact quasi-separated so $\iota_*\mathcal{F}$ is quasi-coherent. Let $f \in \mathcal{F}(D(z))$. As $\iota_*\mathcal{F}$ is quasi-coherent and Y is affine there exists an $n \gg 0$ such that fz^n lifts to a global section of $\iota_*\mathcal{F}$ or equivalently, to a global section of $\mathcal{F}$. The same argument applies to $\mathcal{F}(D(\zeta))$. We can therefore lift all the generators of $\mathcal{F}(D(z))$ and $\mathcal{F}(D(\zeta))$ to global sections of $\mathcal{F}$. $\square$

Lemma 2.4. *If $\mathcal{F}$ is a reflexive sheaf on U then $\iota_*\mathcal{F}$ is reflexive.*

Proof. First we prove that $\iota_*\mathcal{F}$ is coherent. The sheaf $\mathcal{F}^*$ is coherent. By Lemma 2.3 there is a surjection $\mathcal{O}_U^n \to \mathcal{F}^*$ for $n \gg 0$. Dualizing it and taking the composition with the natural isomorphism $\mathcal{F} \cong \mathcal{F}^{**}$ we obtain an embedding $\mathcal{F} \hookrightarrow \mathcal{O}_U^n$. The sheaf $\mathcal{O}_Y$ is reflexive so Lemma 2.2 implies that $\iota_* \mathcal{O}_U^n = \mathcal{O}_Y^n$. Therefore $\iota_* \mathcal{F}$ embeds into $\mathcal{O}_Y^n$.

Next we have a natural commutative diagram

$$\begin{tikzcd} \Gamma(Y, \iota_* \mathcal{F}) \arrow[r] \arrow[d] & \Gamma(Y, (\iota_* \mathcal{F})^{**}) \arrow[d] \\ \Gamma(U, \mathcal{F}) \arrow[r] & \Gamma(U, \mathcal{F}^{**}). \end{tikzcd} \tag{2.1}$$

Here the horizontal arrows are induced by the natural maps $\mathcal{F} \to \mathcal{F}^{**}$ and $\iota_* \mathcal{F} \to (\iota_* \mathcal{F})^{**}$, the left vertical arrow is the restriction while the right vertical arrow is the composition

$$\Gamma(Y, (\iota_* \mathcal{F})^{**}) \to \Gamma(U, (\iota_* \mathcal{F})^{**}) \to \Gamma(U, \mathcal{F}^{**})$$

of the restriction map and the map which results from applying $\Gamma(U, -^{**})$ to the adjunction morphism $\iota^* \iota_* \mathcal{F} \to \mathcal{F}$.

Let us prove that the right vertical arrow in (2.1) is an isomorphism. By [13], Corollary 1.2 the sheaf $(\iota_* \mathcal{F})^{**}$ is reflexive. Hence the restriction map $\Gamma(Y, (\iota_* \mathcal{F})^{**}) \to \Gamma(U, (\iota_* \mathcal{F})^{**})$ is an isomorphism by Lemma 2.2. The adjunction map $\iota^* \iota_* \mathcal{F} \to \mathcal{F}$ is an isomorphism since ι is an open embedding. So our claim follows.

The right vertical arrow in (2.1) is an isomorphism by construction while the bottom horizontal arrow is an isomorphism since $\mathcal{F}$ is reflexive. It follows that the top horizontal arrow is an isomorphism whence $\iota_* \mathcal{F}$ is reflexive. $\square$

Lemma 2.5. *Every reflexive sheaf on* $Y = \operatorname{Spec} \mathcal{O}_F \check{\otimes} \mathcal{O}_K$ *is locally free.*

Proof. According to Lemma 2.1 the ring $\mathcal{O}_F \check{\otimes} \mathcal{O}_K$ is a finite product of regular local 2-dimensional rings. So the result follows from [13], Corollary 1.4. $\square$

3. Reflexive sheaves and lattices

Using the results of the previous section we prove several technical lemmas on modules over the ring $\mathcal{O}_F \check{\otimes} \mathcal{O}_K$.

Lemma 3.1. *Let* $z \in \mathcal{O}_F$ *and* $\zeta \in \mathcal{O}_K$ *be uniformizers.*

(1) *The natural map* $\operatorname{Spec}(F \check{\otimes} \mathcal{O}_K) \to \operatorname{Spec}(\mathcal{O}_F \check{\otimes} \mathcal{O}_K)$ *is an open embedding with image* $D(z)$.
(2) *The natural map* $\operatorname{Spec}(\mathcal{O}_F \check{\otimes} K) \to \operatorname{Spec}(\mathcal{O}_F \check{\otimes} \mathcal{O}_K)$ *is an open embedding with image* $D(\zeta)$.
(3) *The natural map* $\operatorname{Spec}(F \check{\otimes} K) \to \operatorname{Spec}(\mathcal{O}_F \check{\otimes} \mathcal{O}_K)$ *is an open embedding with image* $D(z) \cap D(\zeta)$.

Proof. Proposition 3.6.2 shows that

$$(\mathcal{O}_F \check{\otimes} \mathcal{O}_K)[z^{-1}] = F \check{\otimes} \mathcal{O}_K,$$
$$(\mathcal{O}_F \check{\otimes} \mathcal{O}_K)[\zeta^{-1}] = \mathcal{O}_F \check{\otimes} K$$

where $z \in \mathcal{O}_F$ and $\zeta \in \mathcal{O}_K$ are uniformizers. Furthermore Proposition 3.6.3 tells us that

$$(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)[(z\zeta)^{-1}] = F \breve{\otimes} K.$$

So the result follows. □

Lemma 3.2. *Let N be a locally free $\mathcal{O}_F \breve{\otimes} \mathcal{O}_K$-module. Consider the modules*

$$N^c = N \otimes_{\mathcal{O}_F \breve{\otimes} \mathcal{O}_K} (\mathcal{O}_F \breve{\otimes} K), \quad N^b = N \otimes_{\mathcal{O}_F \breve{\otimes} \mathcal{O}_K} (F \breve{\otimes} \mathcal{O}_K)$$

We have $N = N^c \cap N^b$ as submodules of $N \otimes_{\mathcal{O}_F \breve{\otimes} \mathcal{O}_K} (F \breve{\otimes} K)$. □

Lemma 3.3. *Let N be a locally free $F \breve{\otimes} K$-module of finite rank. If $N^c \subset N$ is a locally free $\mathcal{O}_F \breve{\otimes} K$-lattice and $N^b \subset N$ a locally free $F \breve{\otimes} \mathcal{O}_K$-lattice then $N^c \cap N^b$ is a locally free $\mathcal{O}_F \breve{\otimes} \mathcal{O}_K$-lattice in N^c and N^b.*

Proof. Let $z \in \mathcal{O}_F$ and $\zeta \in \mathcal{O}_K$ be uniformizers. By Lemma 3.1 $\operatorname{Spec}(F \breve{\otimes} \mathcal{O}_K) = D(z)$ and $\operatorname{Spec}(\mathcal{O}_F \breve{\otimes} K) = D(\zeta)$ are open subschemes in $\operatorname{Spec}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ whose intersection is $\operatorname{Spec}(F \breve{\otimes} K)$. By assumption N^c and N^b restrict to the same module N on the intersection $D(z) \cap D(\zeta) = \operatorname{Spec}(F \breve{\otimes} K)$. Hence they define a locally free sheaf $\mathcal{N}$ on $D(z) \cup D(\zeta)$. Let $\iota\colon D(z) \cup D(\zeta) \to \operatorname{Spec}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$ be the embedding map. Lemmas 2.4 and 2.5 imply that $\iota_* \mathcal{N}$ is a locally free sheaf on $\operatorname{Spec}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K)$. By construction

$$\Gamma(\operatorname{Spec}(\mathcal{O}_F \breve{\otimes} \mathcal{O}_K), \iota_* \mathcal{N}) = N^c \cap N^b$$

so the result follows. □

4. Hom shtukas

Fix a Drinfeld A-module E over K. The motive $M = \operatorname{Hom}(E, \mathbb{G}_a)$ of E carries a natural structure of a left $A \otimes K\{\tau\}$-module with A acting on the right via E and $K\{\tau\}$ acting on the left via $\mathbb{G}_a$. Let F be the local field of A at infinity. In this section we list some properties of the shtuka $\mathcal{H}\mathrm{om}_{A\otimes K}(M, N)$ where N is one of the function spaces

$$a(F, K), \quad b(F, K), \quad a(F/A, K), \quad b(F/A, K)$$

or the germ space $g(F, K)$. These are immediate consequences of results in Chapters 2 and 3. The τ-module structures on these spaces are as described in Section 3.11.

Lemma 4.1. *$\mathcal{H}\mathrm{om}(M, a(F, K))$ is a locally free $F \breve{\otimes} K$-module shtuka and $\mathcal{H}\mathrm{om}(M, b(F, K))$ is a locally free $F^{\#} \widehat{\otimes} K$-module shtuka.*

Proof. The $A \otimes K$-module M is locally free of finite rank by definition. Corollary 3.10.3 tells us that the function space $a(F, K)$ is a free $F \breve{\otimes} K$-module of rank 1 while $b(F, K)$ is a free $F^{\#} \widehat{\otimes} K$-module of rank 1. The result now follows from the definition of $\mathcal{H}\mathrm{om}$. □

Lemma 4.2. *$\mathcal{H}\mathrm{om}(M, a(F, K))$ is an $F \breve{\otimes} K$-lattice in $\mathcal{H}\mathrm{om}(M, b(F, K))$.*

Proof. According to Corollary 3.10.3 the function space $a(F,K)$ is an $F \check{\otimes} K$-lattice in $b(F,K)$. The result now follows from the definition of $\mathcal{H}\mathrm{om}$ since M is a locally free $A \otimes K$-module of finite rank. □

Lemma 4.3. *The shtuka $\mathcal{H}\mathrm{om}(M, a(F/A,K))$ has the following properties:*

(1) *It is a locally free $A \otimes K$-module shtuka.*
(2) *It is an $A \otimes K$-lattice in the $F \check{\otimes} K$-module shtuka $\mathcal{H}\mathrm{om}(M, a(F,K))$.*

Proof. By definition M is a locally free $A \otimes K$-module. By Corollaries 3.10.2 and 3.10.5 the space $a(F/A,K)$ is a locally free $A \otimes K$-lattice in the $F \check{\otimes} K$-module $a(F,K)$. So (1) and (2) follow from the definition of $\mathcal{H}\mathrm{om}$. □

Lemma 4.4. *The shtuka $\mathcal{H}\mathrm{om}(M, b(F/A,K))$ has the following properties:*

(1) *It is a locally free $A \widehat{\otimes} K$-module shtuka.*
(2) *It is an $A \widehat{\otimes} K$-lattice in the $F^{\#} \widehat{\otimes} K$-module shtuka $\mathcal{H}\mathrm{om}(M, b(F,K))$.*

Proof. By definition M is a locally free $A \otimes K$-module. By Corollaries 3.10.2 and 3.10.5 the space $b(F/A,K)$ is a locally free $A \widehat{\otimes} K$-lattice in the $F^{\#} \widehat{\otimes} K$-module $b(F,K)$. So (1) and (2) follow from the definition of $\mathcal{H}\mathrm{om}$. □

Lemma 4.5. *The shtuka $\mathcal{H}\mathrm{om}(M, a(F/A,K))$ is a locally free $A \otimes K$-lattice in the $A \widehat{\otimes} K$-module shtuka $\mathcal{H}\mathrm{om}(M, b(F/A,K))$.*

Proof. Follows from Corollary 3.10.2 since M is a locally free $A \otimes K$-module of finite rank. □

5. Hom shtukas and nilpotence

Let (R,τ) be a τ-ring. According to Definition 7.5.4 an R-module shtuka

$$M = \left[M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1 \right]$$

is called co-nilpotent if the adjoint $j^a \colon \tau^* M_0 \to M_1$ of the map $j \colon M_0 \to \tau_* M_1$ is an isomorphism and the composition

$$\tau^{*n}(u) \circ \ldots \circ u, \quad u = (j^a)^{-1} \circ i$$

is zero for $n \gg 0$.

Lemma 5.1. *Let N be a left $R\{\tau\}$-module and M an S-module shtuka given by a diagram*

$$M = \left[M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1 \right].$$

If M is co-nilpotent then $\mathcal{H}\mathrm{om}_R(M,N)$ is nilpotent.

Proof. Follows directly from the definition of $\mathcal{H}\mathrm{om}$ (Definition 1.12.1). In verifying this it is convenient to identify M_1 with $\tau^* M_0$ via j^a. □

6. The notion of a local model

As before we work with a fixed Drinfeld module E over K. We denote $M = \mathrm{Hom}(E, \mathbb{G}_a)$ the motive of E as in Definition 7.1.4. Throughout the rest of the chapter we assume that the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F, the local field of A at infinity.

The shtuka $\mathcal{H}\mathrm{om}(M, a(F, K))$ is a locally free $F \mathbin{\check{\otimes}} K$-module shtuka. In this chapter we will study models of this shtuka over various subrings of $F \mathbin{\check{\otimes}} K$. In particular we will introduce the notions of

(1) the *coefficient compactification* (a model over $\mathcal{O}_F \mathbin{\check{\otimes}} K$),
(2) a *base compactification* (a model over $F \mathbin{\check{\otimes}} \mathcal{O}_K$),
(3) a *local model* (over $\mathcal{O}_F \mathbin{\check{\otimes}} \mathcal{O}_K$).

Let C be projective compactification of $\mathrm{Spec}\, A$ and let $\iota\colon \mathrm{Spec}(A \otimes K) \to C \times K$ be the natural open immersion. We denote

$$\mathcal{E} \subset \iota_*\Big[M \overset{1}{\underset{\tau}{\rightrightarrows}} M\Big]$$

the shtuka constructed in Theorem 7.5.5. By Proposition 3.8.8 the function space $a(F/\mathcal{O}_F, K)$ carries a natural structure of an $\mathcal{O}_F \mathbin{\check{\otimes}} K$-module.

Definition 6.1. The *coefficient compactification* $\mathcal{M}^c \subset \mathcal{H}\mathrm{om}(M, a(F, K))$ is the $\mathcal{O}_F \mathbin{\check{\otimes}} K$-subshtuka

$$\mathcal{H}\mathrm{om}_{\mathcal{O}_F \check{\otimes} K}(\mathcal{E}(\mathcal{O}_F \mathbin{\check{\otimes}} K), a(F/\mathcal{O}_F, K)).$$

The superscript "c" stands for "coefficients".

Lemma 6.2. *$\mathcal{M}^c \subset \mathcal{H}\mathrm{om}(M, a(F, K))$ is a locally free $\mathcal{O}_F \mathbin{\check{\otimes}} K$-lattice.*

Proof. By construction $\iota^*\mathcal{E}$ coincides with the shtuka given by the left $A \otimes K\{\tau\}$-module M. Hence $\mathcal{E}(\mathcal{O}_F \mathbin{\check{\otimes}} K)$ is a locally free $\mathcal{O}_F \mathbin{\check{\otimes}} K$-lattice in the pullback of M to $F \mathbin{\check{\otimes}} K$. According to Corollaries 3.10.4 and 3.10.5 the function space $a(F/\mathcal{O}_F, K)$ is a locally free $\mathcal{O}_F \mathbin{\check{\otimes}} K$-lattice in $a(F, K)$. The result now follows from the definition of $\mathcal{H}\mathrm{om}$. □

Lemma 6.3. *$\mathcal{M}^c(\mathcal{O}_F/\mathfrak{m}_F \otimes K)$ is nilpotent.*

Proof. By Theorem 7.5.5 the shtuka $\mathcal{E}(\mathcal{O}_F/\mathfrak{m}_F \otimes K)$ is co-nilpotent so the result follows by Lemma 5.1. □

By our assumption the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F so that we have a continuous homomorphism $F \to K$. Such a homomorphism necessarily maps $\mathcal{O}_F$ to $\mathcal{O}_K$.

Definition 6.4. We define the *conductor* $\mathfrak{f} \subset \mathcal{O}_K$ to be the ideal generated by $\mathfrak{m}_F$ in $\mathcal{O}_K$.

Lemma 6.5. *The ideal $\mathfrak{f}$ is open and is contained in the Jacobson radical $\mathfrak{m}_K$.* □

Definition 6.6. A *base compactification* of $\mathcal{H}\mathrm{om}(M, a(F,K))$ is a locally free $F\check{\otimes}\mathcal{O}_K$-lattice $\mathcal{M}^b$ such that $\mathcal{M}^b(F\otimes\mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathcal{M}^b(F\otimes\mathcal{O}_K/\mathfrak{f})$ is linear. The superscript "b" stands for "base".

Definition 6.7. A *local model* of $\mathcal{H}\mathrm{om}(M, a(F,K))$ is a locally free $\mathcal{O}_F \check{\otimes} \mathcal{O}_K$-lattice $\mathcal{M}$ such that:

(1) The subshtuka $\mathcal{M}(\mathcal{O}_F \check{\otimes} K) \subset \mathcal{H}\mathrm{om}(M, a(F,K))$ coincides with $\mathcal{M}^c$.
(2) $\mathcal{M}(F \check{\otimes} \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathcal{M}(F \check{\otimes} \mathcal{O}_K/\mathfrak{f})$ is linear.

Proposition 6.8. *Let $\mathcal{M} \subset \mathcal{A}$ be an $\mathcal{O}_F \check{\otimes} \mathcal{O}_K$-subshtuka. The following are equivalent:*

(1) *$\mathcal{M}$ is a local model.*
(2) *The subshtuka $\mathcal{M}(F \check{\otimes} \mathcal{O}_K) \subset \mathcal{H}\mathrm{om}(M, a(F,K))$ is a base compactification and $\mathcal{M} = \mathcal{M}^c \cap \mathcal{M}(F \check{\otimes} \mathcal{O}_K)$*

Proof. (1) $\Rightarrow$ (2). It follows directly from the definition of a local model that $\mathcal{M}(F \check{\otimes} \mathcal{O}_K)$ is a base compactification and $\mathcal{M}^c = \mathcal{M}(\mathcal{O}_F \check{\otimes} K)$. Lemma 3.2 implies that $\mathcal{M} = \mathcal{M}^c \cap \mathcal{M}(F \check{\otimes} \mathcal{O}_K)$. (2) $\Rightarrow$ (1). Lemma 3.3 shows that $\mathcal{M}$ is a locally free $\mathcal{O}_F \check{\otimes} \mathcal{O}_K$-lattice in $\mathcal{M}^c$ so (1) follows. $\square$

Lemma 6.9. *If $\mathcal{M} \subset \mathcal{H}\mathrm{om}(M, a(F,K))$ is a local model then the natural map $\mathcal{M}(F^\# \widehat{\otimes} \mathcal{O}_K) \to \mathcal{H}\mathrm{om}(M, b(F,K))$ is an inclusion of an $F^\# \widehat{\otimes} \mathcal{O}_K$-lattice.*

Proof. $\mathcal{M}(F \check{\otimes} K) = \mathcal{H}\mathrm{om}(M, a(F,K))$ by definition and $\mathcal{H}\mathrm{om}(M, a(F,K))$ is an $F\check{\otimes}K$-lattice in $\mathcal{H}\mathrm{om}(M, b(F,K))$ by Lemma 4.2. Therfore the natural map $\mathcal{M}(F^\#\widehat{\otimes}K) \to \mathcal{H}\mathrm{om}(M, b(F,K))$ is an isomorphism. According to Proposition 3.6.1 the ring $F^\# \widehat{\otimes} K$ is a localization of $F^\# \widehat{\otimes} \mathcal{O}_K$ at a uniformizer of $\mathcal{O}_K$. As $\mathcal{M}$ is locally free it follows that $\mathcal{M}(F^\# \widehat{\otimes} \mathcal{O}_K) \to \mathcal{M}(F^\# \widehat{\otimes} K)$ is a lattice inclusion. $\square$

The natural map $\mathcal{O}_F \check{\otimes} \mathcal{O}_K \to \mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$ is an isomorphism by Proposition 3.4.13. So we can view an $\mathcal{O}_F \check{\otimes} \mathcal{O}_K$-module shtuka $\mathcal{M}$ as an $\mathcal{O}_F \widehat{\otimes} \mathcal{O}_K$-module shtuka. In particular the constructions of Chapter 5 apply to shtukas on $\mathcal{O}_F \check{\otimes} \mathcal{O}_K$.

Let us recall the twisting construction of Section 5.7. Let $I \subset \mathcal{O}_F \check{\otimes} \mathcal{O}_K$ be a τ-invariant ideal. Given an $\mathcal{O}_F \check{\otimes} \mathcal{O}_K$-module shtuka

$$\mathcal{M} = \Big[M_0 \overset{i}{\underset{j}{\rightrightarrows}} M_1\Big]$$

we define

$$I\mathcal{M} = \Big[IM_0 \overset{i}{\underset{j}{\rightrightarrows}} IM_1\Big].$$

The fact that I is an invariant ideal guarantees that the diagram on the right indeed defines a shtuka. We will use twists by the invariant ideal $\mathcal{O}_F \check{\otimes} \mathfrak{f}$. To improve legibility we will write $\mathfrak{f}\mathcal{M}$ in place of $(\mathcal{O}_F \check{\otimes} \mathfrak{f})\mathcal{M}$.

Proposition 6.10. *If $\mathcal{M}$ is a local model then $\mathfrak{f}\mathcal{M}$ is a local model.*

Proof. Indeed $(\mathfrak{f}\mathcal{M})(\mathcal{O}_F \check{\otimes} K) = \mathcal{M}(\mathcal{O}_F \check{\otimes} K)$ so $(\mathfrak{f}\mathcal{M})(\mathcal{O}_F \check{\otimes} K)$ coincides with the coefficient compactification of $\mathcal{H}\mathrm{om}(M, a(F, K))$. Lemma 5.7.5 shows that $(\mathfrak{f}\mathcal{M})(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent while Proposition 5.7.6 implies that $(\mathfrak{f}\mathcal{M})(F \otimes \mathcal{O}_K/\mathfrak{f})$ is linear. Since $\mathfrak{f}\mathcal{M}$ is a locally free $\mathcal{O}_F \check{\otimes} \mathcal{O}_K$-lattice in $\mathcal{H}\mathrm{om}(M, a(F, K))$ by construction we conclude that $\mathfrak{f}\mathcal{M}$ is a local model. $\square$

Proposition 6.11. *If $\mathcal{M}$, $\mathcal{N}$ are local models then there exists $n \geqslant 0$ such that $\mathfrak{f}^n \mathcal{M} \subset \mathcal{N}$.*

Proof. By Proposition 6.8 we have

$$\mathcal{M} = \mathcal{M}^c \cap \mathcal{M}(F \check{\otimes} \mathcal{O}_K), \quad \mathcal{N} = \mathcal{M}^c \cap \mathcal{N}(F \check{\otimes} \mathcal{O}_K)$$

where $\mathcal{M}^c \subset \mathcal{H}\mathrm{om}(M, a(F, K))$ is the coefficient compactification. So to prove the proposition it is enough to find an integer $n \geqslant 0$ such that $(\mathfrak{f}^n \mathcal{M})(F \check{\otimes} \mathcal{O}_K) \subset \mathcal{N}(F \check{\otimes} \mathcal{O}_K)$. Observe that the conductor ideal $\mathfrak{f}$ is contained in the Jacobson radical $\mathfrak{m}_K$ by construction. In particular it contains a power of a uniformizer of $\mathcal{O}_K$. The result follows since $\mathcal{M}(F \check{\otimes} \mathcal{O}_K)$ and $\mathcal{N}(F \check{\otimes} \mathcal{O}_K)$ are $F \check{\otimes} \mathcal{O}_K$-lattices in $\mathcal{H}\mathrm{om}(M, a(F, K))$. $\square$

7. Existence of base compactifications

We keep the assumptions and the notation of the previous section. In this section we prove that the $F \check{\otimes} K$-module shtuka $\mathcal{H}\mathrm{om}(M, a(F, K))$ admits a base compactification over $F \check{\otimes} \mathcal{O}_K$ in the sense of the preceding section. In fact we go one step further and show that the $A \otimes K$-module shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, K))$ admits a base compactification over $A \otimes \mathcal{O}_K$. This result was inspired by the construction of extension by zero from the theory of Böckle-Pink [3, Section 4.5].

Lemma 7.1. *If N is a finitely generated reflexive module over the ring $A \widehat{\otimes} \mathcal{O}_K$ then N is locally free.*

Proof. The ring $A \otimes \mathcal{O}_K$ is noetherian, regular and of Krull dimension 2. Hence so is its completion $A \widehat{\otimes} \mathcal{O}_K$ at the Jacobson radical $\mathfrak{m}_K \subset \mathcal{O}_K$. So the result follows from [13, Corollary 1.4]. $\square$

Lemma 7.2. *Every locally free $A \widehat{\otimes} K$-module admits a locally free $A \widehat{\otimes} \mathcal{O}_K$-lattice.*

Proof. Let N be a locally free $A \widehat{\otimes} K$-module. Clearly there exists a finitely generated $A \widehat{\otimes} \mathcal{O}_K$-lattice $N_0 \subset N$. A priori N_0 need not be locally free. But the double dual N_0^{**} is still a lattice in N and is a reflexive $A \widehat{\otimes} \mathcal{O}_K$-module by [13, Corollary 1.2]. Now Lemma 7.1 shows that N_0^{**} is a locally free $A \widehat{\otimes} \mathcal{O}_K$-module. $\square$

Proposition 7.3. *Let N be a left $A \widehat{\otimes} K\{\tau\}$-module which is locally free of finite rank as an $A \widehat{\otimes} K$-module.*

(1) *There exists an $A \widehat{\otimes} \mathcal{O}_K\{\tau\}$-submodule $N_0 \subset N$ such that N_0 is a locally free $A \widehat{\otimes} \mathcal{O}_K$-lattice in the $A \widehat{\otimes} K$-module N.*

(2) *Given an open ideal $I \subset \mathcal{O}_K$ one can choose N_0 in such a way as to ensure that τ acts by zero on $N \otimes_{A \widehat{\otimes} \mathcal{O}_K} (A \otimes \mathcal{O}_K/I)$.*

Proof. (1) By Lemma 7.2 the $A \widehat{\otimes} K$-module N admits a locally free $A \widehat{\otimes} \mathcal{O}_K$-lattice $N_1 \subset N$. Let $\zeta \in \mathcal{O}_K$ be a uniformizer. According to Proposition 3.6.1 the ring $A \widehat{\otimes} K$ is the localization of $A \widehat{\otimes} \mathcal{O}_K$ at ζ. Hence $N_1[\zeta^{-1}] = N$. As a consequence there exists an $n \geqslant 0$ such that $\tau N_1 \subset \zeta^{-n} N_1$. The locally free $A \widehat{\otimes} \mathcal{O}_K$-lattice $N_0 = \zeta^n N_1$ has the property that $\tau N_0 \subset \zeta^{(q-1)n} N_1$. As $q > 1$ it follows that N_0 is a left $A \widehat{\otimes} \mathcal{O}_K\{\tau\}$-submodule.

(2) Without loss of generality we may assume that $I = \zeta^n \mathcal{O}_K$ for some $n \geqslant 0$. Let $N_0 \subset N$ be a left $A \widehat{\otimes} \mathcal{O}_K\{\tau\}$-submodule as in (1) and let $N_1 = \zeta^n N_0$. We have $\tau(N_1) \subset \zeta^{(q-1)n} N_1$. The result follows since $q > 1$. ☐

Definition 7.4. A *base compactification* of $\mathcal{H}\mathrm{om}(M, a(F/A, K))$ is a locally free $A \otimes \mathcal{O}_K$-lattice $\mathcal{M}^b$ such that $\mathcal{M}^b(A \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathcal{M}^b(A \otimes \mathcal{O}_K/\mathfrak{f})$ is linear. Here $\mathfrak{f} \subset \mathcal{O}_K$ is the conductor ideal of Definition 6.4.

Any base compactification of $\mathcal{H}\mathrm{om}(M, a(F/A, K))$ induces a base compactification of $\mathcal{H}\mathrm{om}(M, a(F, K))$ in the sense of Definition 6.6.

Theorem 7.5. *The shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, K))$ admits a base compactification.*

Proof. The $A \otimes K$-module shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, K))$ is locally free by Lemma 4.3 while the $A \widehat{\otimes} K$-module shtuka $\mathcal{H}\mathrm{om}(M, b(F/A, K))$ is locally free by Lemma 4.4. According to Lemma 4.5 the shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, K))$ is an $A \otimes K$-lattice in $\mathcal{H}\mathrm{om}(M, b(F/A, K))$. Hence Beauville-Laszlo glueing theorem [0BP2] implies that to prove the existence of a base compactification it is enough to construct a locally free $A \widehat{\otimes} \mathcal{O}_K$-lattice $\mathcal{M} \subset \mathcal{H}\mathrm{om}(M, b(F/A, K))$ such that $\mathcal{M}(A \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent and $\mathcal{M}(A \otimes \mathcal{O}_K/\mathfrak{f})$ is linear.

Now consider the shtuka

$$\mathcal{H}\mathrm{om}(M, b(F/A, K)) = \Big[\mathrm{Hom}(M, b(F/A, K)) \overset{i}{\underset{j}{\rightrightarrows}} \mathrm{Hom}(\tau^* M, b(F/A, K)) \Big].$$

Proposition 8.10.3 shows that $\mathrm{R}\Gamma(\nabla \mathcal{H}\mathrm{om}(M, b(F/A, K))) = 0$. Hence the arrow i in the diagram above is an isomorphism. If we let τ act on the $A \widehat{\otimes} K$-module $\mathrm{Hom}(M, b(F/A, K))$ via the endomorphism $i^{-1} \circ j$ then it becomes a left $A \widehat{\otimes} K\{\tau\}$-module. By construction $\mathcal{H}\mathrm{om}(M, b(F/A, K))$ is isomorphic to the shtuka defined by $\mathrm{Hom}(M, b(F/A, K))$. Therefore applying Proposition 7.3 to $N = \mathrm{Hom}(M, b(F/A, K))$ with $I = \mathfrak{f}$ we get the result. ☐

8. Cohomology of the Hom shtukas

We keep the notation and the assumptions of the previous section. Our goal for the moment is to construct natural quasi-isomorphisms

$$\mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M, a(F, K))) \cong \mathrm{Lie}_E(K)[-1]$$
$$\mathrm{R}\Gamma_g(\nabla \mathcal{H}\mathrm{om}(M, a(F, K))) \cong \mathrm{Lie}_E(K)[-1].$$

where $\mathrm{R}\Gamma_g$ is the germ cohomology functor of Definition 4.1.1.

According to Lemma 4.2 the shtuka $\mathcal{H}\mathrm{om}(M, a(F,K))$ is a locally free $F \mathbin{\check{\otimes}} K$-lattice in the locally free $F^{\#} \mathbin{\widehat{\otimes}} K$-module shtuka $\mathcal{H}\mathrm{om}(M, b(F,K))$. So we have natural quasi-isomorphisms

$$\mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M, a(F,K))) \cong \\ \cong \Big[\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, a(F,K))) \to \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, b(F,K)))\Big]$$

and

$$\mathrm{R}\Gamma_g(\nabla\,\mathcal{H}\mathrm{om}(M, a(F,K))) \cong \\ \cong \Big[\mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M, a(F,K))) \to \mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M, b(F,K)))\Big]$$

by definition of germ cohomology.

Lemma 8.1. *The natural sequence*

$$0 \to \mathcal{H}\mathrm{om}(M, a(F,K)) \to \mathcal{H}\mathrm{om}(M, b(F,K)) \to \mathcal{H}\mathrm{om}(M, g(F,K)) \to 0$$

is exact.

Proof. Indeed the natural sequence $0 \to a(F,K) \to b(F,K) \to g(F,K) \to 0$ is exact by Proposition 2.11.2. The functor $\mathcal{H}\mathrm{om}(M, -)$ transforms it to the sequence in question. As M is a locally free $A \otimes K$-module by definition the functor $\mathcal{H}\mathrm{om}(M, -)$ preserves exactness. □

We thus obtain natural quasi-isomorphisms

$$\mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M, a(F,K))) \cong \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, g(F,K)))[-1], \tag{8.1}$$

$$\mathrm{R}\Gamma_g(\nabla\,\mathcal{H}\mathrm{om}(M, a(F,K))) \cong \mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M, g(F,K)))[-1]. \tag{8.2}$$

Let $\exp\colon \mathrm{Lie}_E(K) \to E(K)$ be the exponential map. As before, given a morphism $m\colon E \to \mathbb{G}_a$ we denote $dm\colon \mathrm{Lie}_E \to \mathrm{Lie}_{\mathbb{G}_a}$ the induced map.

Lemma 8.2. *The map* $\mathrm{Lie}_E(K) \to \mathrm{Hom}(M, g(F,K))$, $\alpha \mapsto (m \mapsto (x \mapsto m\exp(x\alpha)))$ *induces a quasi-isomorphism*

$$\mathrm{Lie}_E(K)[0] \xrightarrow{\sim} \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, g(F,K))).$$

The map $\mathrm{Lie}_E(K) \to \mathrm{Hom}(M, g(F,K))$, $\alpha \mapsto (m \mapsto (x \mapsto dm(x\alpha)))$ *induces a quasi-isomorphism*

$$\mathrm{Lie}_E(K)[0] \xrightarrow{\sim} \mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M, g(F,K))).$$

Proof. As the quotient map $F \to F/A$ is a local isomorphism in the sense of Definition 2.11.3 it follows that the induced map $g(F/A, K) \to g(F,K)$ is an isomorphism of left $A \otimes K\{\tau\}$-modules. Hence the result follows from Propositions 8.10.5 and 8.10.6. □

Definition 8.3. We define natural quasi-isomorphisms

$$\mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M, a(F,K))) \cong \mathrm{Lie}_E(K)[-1],$$
$$\mathrm{R}\Gamma_g(\nabla\,\mathcal{H}\mathrm{om}(M, a(F,K))) \cong \mathrm{Lie}_E(K)[-1]$$

as the compositions

$$\mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M, a(F,K))) \xrightarrow{(8.1)} \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, g(F,K)))[-1] \cong \mathrm{Lie}_E(K)[-1],$$

$$\mathrm{R}\Gamma_g(\nabla\,\mathcal{H}\mathrm{om}(M, a(F,K))) \xrightarrow{(8.2)} \mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M, g(F,K)))[-1] \cong \mathrm{Lie}_E(K)[-1]$$

where the unlabelled arrows are the quasi-isomorphisms of Lemma 8.2.

9. Local models as elliptic shtukas

We keep the notation and the assumptions of the previous section.

Definition 9.1. Let $\mathcal{M} \subset \mathcal{H}\mathrm{om}(M, a(F,K))$ be a local model. We define a map

$$\gamma\colon\ \mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{Lie}_E(K)[-1]$$

as the composition of the following maps:

- The natural map $\mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(F \check{\otimes} \mathcal{O}_K, \mathcal{M})$.
- The local germ map $\mathrm{R}\Gamma(F\check{\otimes}\mathcal{O}_K, \mathcal{M}) \xrightarrow{\sim} \mathrm{R}\Gamma_g(F\check{\otimes}K, \mathcal{M})$ of Definition 4.2.4.
- The isomorphism $\mathrm{R}\Gamma_g(F \check{\otimes} K, \mathcal{M}) = \mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M, a(F,K)))$ which results from the equality $\mathcal{M}(F \check{\otimes} K) = \mathcal{H}\mathrm{om}(M, a(F,K))$.
- The quasi-isomorphism $\mathrm{R}\Gamma_g(\mathcal{H}\mathrm{om}(M, a(F,K))) \cong \mathrm{Lie}_E(K)[-1]$ of Definition 8.3.

We define a map

$$\nabla\gamma\colon\ \mathrm{R}\Gamma(\nabla\mathcal{M}) \to \mathrm{Lie}_E(K)[-1]$$

in the same way.

Observe that the map γ is $\mathcal{O}_F$-linear by construction while $\nabla\gamma$ is both $\mathcal{O}_F$-linear and $\mathcal{O}_K$-linear.

Lemma 9.2. *If $\mathcal{M}$ is a locally free $\mathcal{O}_F \check{\otimes} \mathcal{O}_K$-module shtuka then the natural map $F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{R}\Gamma(F \check{\otimes} \mathcal{O}_K, \mathcal{M})$ is a quasi-isomorphism.*

Proof. According to Proposition 3.6.2 the natural map $F \otimes_{\mathcal{O}_F} (\mathcal{O}_F \check{\otimes} \mathcal{O}_K) \to F\check{\otimes}\mathcal{O}_K$ is an isomorphism. The differentials in the complex computing $\mathrm{R}\Gamma(\mathcal{M})$ are $\mathcal{O}_F$-linear so the result follows. $\square$

Corollary 9.3. *The F-linear extensions*

$$\gamma\colon F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M}) \to \mathrm{Lie}_E(K)[-1],$$
$$\nabla\gamma\colon F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{M}) \to \mathrm{Lie}_E(K)[-1]$$

of the maps γ and $\nabla\gamma$ are quasi-isomorphisms. $\square$

Proposition 9.4. *If $\mathcal{M} \hookrightarrow \mathcal{N}$ is an inclusion of local models then the induced maps*

$$F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{M}) \to F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\mathcal{N}),$$
$$F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{M}) \to F \otimes_{\mathcal{O}_F} \mathrm{R}\Gamma(\nabla\mathcal{N})$$

are quasi-isomorphisms.

Proof. Since γ and $\nabla\gamma$ are natural the result follows from Corollary 9.3. □

Theorem 9.5. *Let $\mathcal{M}$ be a local model.*

(1) $\mathrm{H}^0(\mathcal{M}) = 0$ *and* $\mathrm{H}^1(\mathcal{M})$ *is a finitely generated free $\mathcal{O}_F$-module.*
(2) $\mathrm{H}^0(\nabla\mathcal{M}) = 0$ *and* $\mathrm{H}^1(\nabla\mathcal{M})$ *is a finitely generated free $\mathcal{O}_F$-module.*

Proof. (1) By definition $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes K) = \mathcal{M}^c(\mathcal{O}_F/\mathfrak{m}_F \otimes K)$ where $\mathcal{M}^c \subset \mathcal{H}\mathrm{om}(M, a(F, K))$ is the coefficient compactification. Lemma 6.3 claims that $\mathcal{M}^c(\mathcal{O}_F/\mathfrak{m}_F \otimes K)$ is nilpotent. Hence the result follows Theorem 5.4.2. (2) Nilpotence is preserved under linearization. So the result follows from Theorem 5.4.2 as well. □

The main result of this section is the following theorem:

Theorem 9.6. *A local model $\mathcal{M}$ is an elliptic shtuka of conductor $\mathfrak{f}$ where $\mathfrak{f} \subset \mathcal{O}_K$ is the ideal of Definition 6.4.*

Proof. We verify the conditions of Definition 5.6.2 for $\mathcal{M}$.

(E0) Indeed $\mathcal{M}$ is a locally free $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$-module shtuka by definition.
(E1) By construction $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes K) = \mathcal{M}^c(\mathcal{O}_F/\mathfrak{m}_F \otimes K)$ where $\mathcal{M}^c \subset \mathcal{H}\mathrm{om}(M, a(F, K))$ is the coefficient compactification. Thus the shtuka $\mathcal{M}(\mathcal{O}_F/\mathfrak{m}_F \otimes K)$ is nilpotent by Lemma 6.3.
(E2) $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{m}_K)$ is nilpotent by definition.
(E3) Consider the map $\mathrm{H}^1(\nabla\gamma)\colon \mathrm{H}^1(\nabla\mathcal{M}) \to \mathrm{Lie}_E(K)$. This map is $\mathcal{O}_F$-linear and $\mathcal{O}_K$-linear by construction. Corollary 9.3 in combination with Theorem 9.5 implies that the map is injective. Therefore

$$\mathfrak{m}_F \cdot \mathrm{H}^1(\nabla\mathcal{M}) = \mathfrak{f} \cdot \mathrm{H}^1(\nabla\mathcal{M})$$

by definition of the conductor ideal $\mathfrak{f}$.
(E4) The shtuka $\mathcal{M}(F \otimes \mathcal{O}_K/\mathfrak{f})$ is linear by definition. As $\mathcal{M}$ is locally free it follows that $\mathcal{M}(\mathcal{O}_F \otimes \mathcal{O}_K/\mathfrak{f})$ is linear. □

Theorem 9.6 allows us to make the following important definition:

Definition 9.7. The *regulator*

$$\rho\colon \mathrm{H}^1(\mathcal{M}) \xrightarrow{\sim} \mathrm{H}^1(\nabla\mathcal{M})$$

of a local model $\mathcal{M}$ is the regulator of $\mathcal{M}$ viewed as an elliptic shtuka over $\mathcal{O}_F \mathbin{\widehat{\otimes}} \mathcal{O}_K$ (see Definition 5.14.1).

10. The exponential map

We keep the assumptions and the notation of the previous section. In this section we describe in detail how the maps γ and $\nabla\gamma$ act on cohomology classes and introduce the exponential map of a local model. Before we begin let us make a remark. Let $h_1, h_2 \in b(F, K)$. Recall that the expression

$$h_1 \sim h_2$$

signifies that the image of $h_1 - h_2$ in $g(F, K)$ is zero. In other words there exists an open neighbourhood $U \subset F$ of 0 such that $h_1|_U = h_2|_U$.

Proposition 10.1. *Let $\mathcal{M} = [\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1]$ be a local model and let $g \in \mathcal{M}_1$.*

(1) *There exists a unique $f \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ such that $(i - j)(f) = g$.*
(2) *Let $[g]$ be the cohomology class of g in $\mathrm{H}^1(\mathcal{M})$ and let $\alpha = \gamma[g]$. The element $\alpha \in \mathrm{Lie}_E(K)$ is uniquely characterized by the property that for every $m \in M^0$ one has*

$$f(m) \sim (x \mapsto m\exp(x\alpha)).$$

Here we view f as an element of $\mathrm{Hom}(M, b(F, K))$ via the natural inclusion $\mathcal{M}(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K) \subset \mathcal{H}\mathrm{om}(M, b(F, K))$ of Lemma 6.9.

Proof. (1) is a direct consequence of Proposition 4.2.6 (1). The part (2) of this proposition implies that the image of f in $\mathrm{Hom}(M, g(F, K))$ represents the image of $[g] \in \mathrm{H}^1(\mathcal{M})$ under the local germ map

$$\mathrm{H}^1(\mathcal{M}) \xrightarrow{\sim} \mathrm{H}^0_{\mathrm{g}}(F \mathbin{\breve{\otimes}} K, \mathcal{M}) = \mathrm{H}^0(\mathcal{H}\mathrm{om}(M, g(F, K))).$$

Now Lemma 8.2 implies that for every $m \in M$ we have $f(m) \sim (x \mapsto m\exp(x\alpha))$. It remains to show that if $\beta \in \mathrm{Lie}_E(K)$ is an element such that $f(m) \sim (x \mapsto m\exp(x\beta))$ for all $m \in M^0 \subset M$ then $\beta = \alpha$.

According to Proposition 7.1.6 M^0 generates M as a left $K\{\tau\}$-module. As the image of f in $\mathrm{Hom}(M, g(F, K))$ represents a class in $\mathrm{H}^0(\mathcal{H}\mathrm{om}(M, g(F, K)))$ Proposition 1.12.3 shows that the map $M \to g(F, K)$ induced by f is in fact a homomorphism of left $A \otimes K\{\tau\}$-modules. As a consequence $f(m) \sim (x \mapsto m\exp(x\beta))$ for all $m \in M$. Whence the result. ☐

Proposition 10.2. *Let $\mathcal{M} = [\mathcal{M}_0 \overset{i}{\underset{j}{\rightrightarrows}} \mathcal{M}_1]$ be a local model and let $g \in \mathcal{M}_1$.*

(1) *There exists a unique $f \in \mathcal{M}_0(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K)$ such that $i(f) = g$.*
(2) *Let $[g]$ be the cohomology class of g in $\mathrm{H}^1(\nabla\mathcal{M})$ and let $\alpha = \nabla\gamma[g]$. The element $\alpha \in \mathrm{Lie}_E(K)$ is uniquely characterized by the property that for every $m \in M^0$ one has*

$$f(m) \sim (x \mapsto dm(x\alpha)).$$

Here we view f as an element of $\mathrm{Hom}(M, b(F, K))$ via the natural inclusion $\mathcal{M}(F^{\#} \mathbin{\widehat{\otimes}} \mathcal{O}_K) \subset \mathcal{H}\mathrm{om}(M, b(F, K))$ of Lemma 6.9. Given

$m \in M = \mathrm{Hom}(E, \mathbb{G}_a)$ we denote $dm\colon \mathrm{Lie}_E \to \mathrm{Lie}_{\mathbb{G}_a}$ the induced map of Lie algebras.

Proof. Same as the proof of Proposition 10.1. □

In the following it would be convenient for us to assemble the maps γ and $\nabla\gamma$ into a map acting on the cohomology of a local model.

Definition 10.3. Let $\mathcal{M}$ be a local model. We define the *exponential map*

$$\exp\colon F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\nabla\mathcal{M}) \to F \otimes_{\mathcal{O}_F} \mathrm{H}^1(\mathcal{M})$$

as the composition $\mathrm{H}^1(\gamma) \circ \mathrm{H}^1(\nabla\gamma)^{-1}$.

Even though this exponential map is induced by the identity map on the Lie algebra, one can justify its name by looking at what it does to the auxillary elements $f \in \mathrm{Hom}(M, b(F, K))$ as described in Propositions 10.2 and 10.1.

We will show in the subsequent chapters that the exponential map is nothing but the inverse of the regulator map $\rho\colon \mathrm{H}^1(\mathcal{M}) \to \mathrm{H}^1(\nabla\mathcal{M})$ which we introduced in Definition 9.7. This important result appears to be neither easy nor evident. For one thing, the regulator is a purely shtuka-theoretic construct while the exponential map uses the arithmetic data of the Drinfeld module in an essential way. The only proof we have at the moment is rather technical and is based on explicit computations.