Shtuka cohomology and special values of Goss L-functions

Mornev, M.

Author: Mornev, M.
Title: Shtuka cohomology and special values of Goss L-functions
Issue Date: 2018-02-13

CHAPTER 8

The motive and the Hom shtuka

Let A be a coefficient ring, F the local field of A at infinity and ω the module of Kähler differentials of A over $\mathbb{F}_q$. Let E be a Drinfeld A-module over a reduced $\mathbb{F}_q$-algebra B and let $M = \operatorname{Hom}(E, \mathbb{G}_a)$ be its motive. Rather than working with M directly we will work with the Hom shtuka

$$\mathcal{H}\mathrm{om}_{A\otimes B}(M,\, \omega \otimes B).$$

This chapter has three main results. The first is purely algebraic and deals with a Drinfeld A-module E over an arbitrary reduced $\mathbb{F}_q$-algebra B. We show that

$$\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}_{A\otimes B}(M,\, \omega \otimes B)) \cong E(B)[-1],$$
$$\mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}_{A\otimes B}(M,\, \omega \otimes B)) \cong \mathrm{Lie}_E(B)[-1].$$

This result is related to the formulas of Barsotti-Weil type obtained by Papanikolas-Ramachandran [18] and Taelman [22]. In some sense it is analogous to the classical Barsotti-Weil isomorphism

$$\mathrm{Ext}^1(h(E), \mathbb{1}) \cong E(k)$$

for the motive $h(E)$ of an abelian variety E over a field k.

The second main result deals with a Drinfeld module E over a finite product K of local fields containing $\mathbb{F}_q$. We assume that the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F. Under this condition there is a natural map $\exp\colon \mathrm{Lie}_E(K) \to E(K)$. We show that

$$\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}_{A\otimes K}(M,\, b(F/A, K))) \cong \big[\, \mathrm{Lie}_E(K) \xrightarrow{\exp} E(K) \big],$$

where $b(F/A, K)$ denotes the space of bounded functions as introduced in Chapter 2. This result is inspired by the work of Anderson [1, Section 2].

Hochschild cohomology of function spaces plays an essential role in the proofs. The subspace $a(F/A, K) \subset b(F/A, K)$ of locally constant functions is naturally isomorphic to $\omega \otimes K$, and this provides a link between the two results.

The third group of results relates Tate modules of a Drinfeld module over a field k to the Hom shtuka $\mathcal{H}\mathrm{om}_{A\otimes k}(M,\, \omega \otimes k)$. In essence we translate certain results of David Goss [10, Section 5.6] concerning the motive M of E to the language of Hom shtukas.

Our theorems generalize to Anderson modules [1] in place of Drinfeld modules. Their proofs extend without change. We limit our exposition to Drinfeld modules since other important parts of our theory still depend on their special properties.

1. Hochschild cohomology in general

Let A be an associative unital $\mathbb{F}_q$-algebra and let M be a complex of (A,A)-bimodules. The Hochschild cohomology of A with coefficients in M is by definition the complex

$$\mathrm{RHom}_{A\otimes A^\circ}(A, M).$$

where A° is the opposite of A and the (A,A)-bimodule structure on A is the diagonal one. We denote this complex $\mathrm{RH}(A,M)$. Its cohomology groups will be denoted $\mathrm{H}^n(A,M)$.

Even though Hochschild cohomology will figure prominently in our computations, no nontrivial properties of it will be used.

2. Hochschild cohomology of coefficient rings

Let A be a coefficient ring in the sense of Definition 7.2.1. Given an (A,A)-bimodule M we denote $D(A,M)$ the module of M-valued derivations over $\mathbb{F}_q$, i.e. $\mathbb{F}_q$-linear maps $d\colon A \to M$ which satisfy the Leibniz identity

$$d(a_0a_1) = a_0d(a_1) + d(a_1)a_0.$$

We denote $\partial\colon M \to D(A,M)$ the map which sends an element $m \in M$ to the derivation $a \mapsto am - ma$.

Proposition 2.1. *For every (A,A)-bimodule M there is a natural quasi-isomorphism*

$$\mathrm{RH}(A,M) \cong \Big[M \xrightarrow{\partial} D(A,M)\Big]$$

Proof. The (A,A)-bimodule A has a canonical resolution $0 \to I \to A\otimes A \to A \to 0$. The $A\otimes A$-modules I and $A\otimes A$ are invertible, so that

$$\mathrm{RHom}_{A\otimes A}(A,M) = \Big[M \to \mathrm{Hom}_{A\otimes A}(I,M)\Big].$$

The natural map $\mathrm{Hom}_{A\otimes A}(I,M) \to D(A,M)$, $f \mapsto (a \mapsto f(a\otimes 1 - 1\otimes a))$, is an isomorphism of (A,A)-bimodules. It identifies the complex above with the complex in the statement of the lemma. ☐

We denote ω the module of Kähler differentials of A over $\mathbb{F}_q$. Let M be an A-module. Using the universal property of ω one easily proves the following lemma:

Lemma 2.2. *The map $M \to D(A, \omega\otimes_A M)$, $m \mapsto (a \mapsto da\otimes_A m)$ is an isomorphism.* ☐

Lemma 2.3. *The sequence*

$$0 \to \omega \otimes M \xrightarrow{\partial} D(A, \omega \otimes M) \xrightarrow{-D(\mu)} D(A, \omega \otimes_A M) \to 0 \tag{2.1}$$

is exact. Here $D(\mu)$ is the map induced by the contraction map $\mu\colon \omega \otimes M \to \omega \otimes_A M$.

Observe that in (2.1) we use the morphism $-D(\mu)$ instead of the more natural $D(\mu)$. With this choice of sign the statements of several results become more straightforward. One examples is Proposition 3.6.

Proof of Lemma 2.3. Let $0 \to I \to A \otimes A \to A \to 0$ be the canonical resolution of the diagonal (A, A)-bimodule A. The composition with the multiplication map $A \otimes A \to A$ induces a homomorphism $D(A, A \otimes A) \to D(A, A)$. One easily shows that the natural sequence

$$0 \to A \otimes A \to D(A, A \otimes A) \to D(A, A) \to 0 \tag{2.2}$$

is exact. The tensor product of (2.2) with $\omega \otimes M$ over $A \otimes A$ coincides with the sequence (2.1). Moreover

$$D(A, A) = \mathrm{Hom}_{A \otimes A}(I, A) = \mathrm{Hom}_A(I/I^2, A) = \mathrm{Hom}_A(\omega, A).$$

So to prove the lemma it is enough to show that

$$\mathrm{Tor}_1^{A \otimes A}(\mathrm{Hom}_A(\omega, A), \omega \otimes M) = 0$$

with the diagonal (A, A)-bimodule structure on $\mathrm{Hom}_A(\omega, A)$. However

$$\mathrm{Hom}_A(\omega, A) \otimes^{\mathbf{L}}_{A \otimes A} (\omega \otimes M) = \mathrm{Hom}_A(\omega, A) \otimes^{\mathbf{L}}_A \omega \otimes^{\mathbf{L}}_A M = M[0].$$

So we are done. □

Proposition 2.4. *For every A-module M there is a natural A-linear quasi-isomorphism*

$$\mathrm{RH}(A, \omega \otimes M) \cong M[-1].$$

It is the composition

$$\mathrm{RH}(A, \omega \otimes M) \cong \Big[\omega \otimes M \xrightarrow{\partial} D(A, \omega \otimes M)\Big] \cong D(A, \omega \otimes_A M)[-1] \cong M[-1].$$

of the quasi-isomorphisms given by Proposition 2.1, Lemma 2.3 and Lemma 2.2 respectively. □

3. Function spaces

As before A stands for a coefficient ring in the sense of Definition 7.2.1. We denote F the local field of A at ∞. Let M be a locally compact A-module. In this section we will study Hochschild cohomology of the following objects:

- $a(F/A, M)$, the space of bounded locally constant $\mathbb{F}_q$-linear maps from F/A to M (Definition 2.10.1).
- $b(F/A, M)$, the space of bounded $\mathbb{F}_q$-linear maps from F/A to M (Definition 2.9.1).

- $g(F/A, M)$, the space of germs of continuous $\mathbb{F}_q$-linear maps from F/A to M (Definition 2.11.1).

Each of them is an (A, A)-bimodule in a natural way: A acts via M on the left and via F/A on the right. To improve the legibility we will generally write $\mathrm{RH}(-)$ instead of $\mathrm{RH}(A, -)$.

Let ω be the module of Kähler differentials of A over $\mathbb{F}_q$. In the following we denote $\mathrm{res}\colon \omega \otimes_A F \to \mathbb{F}_q$ the composition of the residue map at ∞ with the trace map to $\mathbb{F}_q$.

Lemma 3.1. *For every locally compact A-module M the natural map*

$$\omega \otimes M \to a(F/A, M), \quad \eta \otimes m \mapsto (x \mapsto \mathrm{res}(\eta x) \cdot m)$$

is an isomorphism of (A, A)-bimodules.

Proof. According to Theorem 3.10.1 the residue map induces a topological isomorphism $\omega \xrightarrow{\sim} (F/A)^*$ where ω is understood to carry the discrete topology. As $(F/A)^*$ is discrete we have $(F/A)^* \otimes M = (F/A)^* \check{\otimes} M$. The natural map $(F/A)^* \check{\otimes} M \to a(F/A, M)$ is an isomorphism of (A, A)-bimodules by Lemma 3.8.7. Whence the result. □

Corollary 3.2. *For every locally compact A-module M there is a natural quasi-isomorphism*

$$\mathrm{RH}(A, a(F/A, M)) \xrightarrow{\sim} M[-1].$$

It is given by the composition $\mathrm{RH}(a(F/A, M)) \xrightarrow{\sim} \mathrm{RH}(\omega \otimes M) \xrightarrow{\sim} M[-1]$ of the quasi-isomorphism induced by Lemma 3.1 and the quasi-isomorphism of Proposition 2.4. □

We will use the following lemma to prove that certain maps between function spaces induce quasi-isomorphisms on Hochschild cohomology.

Lemma 3.3. *If M is an $A \widehat{\otimes} F$-module then $\mathrm{RH}(A, M) = 0$.*

Proof. Indeed $A \widehat{\otimes} F$ is flat over $A \otimes A$ and the pullback of the diagonal $A \otimes A$-module A to $A \widehat{\otimes} F$ is zero. So $\mathrm{RHom}_{A \otimes A}(A, M) = \mathrm{RHom}_{A \widehat{\otimes} F}(0, M) = 0$ by extension of scalars. □

Let M be a locally compact A-module. According to Proposition 2.11.2 the natural sequence of (A, A)-bimodules $0 \to a(F/A, M) \to b(F/A, M) \to g(F/A, M) \to 0$ is exact. It gives rise to a canonical distinguished triangle

$$\mathrm{RH}(a(F/A, M)) \to \mathrm{RH}(b(F/A, M)) \to \mathrm{RH}(g(F/A, M)) \xrightarrow{\delta} [1]. \tag{3.1}$$

Lemma 3.4. *If V is a locally compact F-vector space then $\mathrm{RH}(b(F/A, V)) = 0$.*

Proof. Proposition 3.8.4 shows that the (A, A)-bimodule structure on the space $b(F/A, V)$ extends to an $A \widehat{\otimes} F$-module structure. The result now follows from Lemma 3.3. □

Corollary 3.5. *If V is a locally compact F-vector space then the natural map*

$$\mathrm{RH}(g(F/A,V)) \xrightarrow{\delta} \mathrm{RH}(a(F/A,V))[1]$$

is a quasi-isomorphism. □

By Proposition 2.1 we have

$$\mathrm{RH}(g(F/A,V)) = \left[g(F/A,V) \xrightarrow{\partial} D(A, g(F/A,V)) \right]$$

Proposition 3.6. *If V is a locally compact F-vector space then the map*

$$V \to g(F/A,V), \quad \alpha \mapsto (x \mapsto x\alpha)$$

induces a quasi-isomorphism $V[0] \xrightarrow{\sim} \mathrm{RH}(g(F/A,V))$. This quasi-isomorphism fits into a commutative square

$$\begin{tikzcd}
V[0] \arrow[rr, equal] \arrow[d, "\wr"'] & & V[0] \\
\mathrm{RH}(g(F/A,V)) \arrow[rr, "\delta", "\sim"'] & & \mathrm{RH}(a(F/A,V))[1] \arrow[u, "\wr"']
\end{tikzcd}$$

where the right arrow is the quasi-isomorphism of Corollary 3.2 and δ is the boundary morphism in (3.1).

Although the statement appears natural and innocent, we do not know an easy proof. The hardest part is to show that the square is commutative. We split the proof into several auxillary lemmas. Our first task is to understand the Hochschild cohomology of the (A,A)-bimodule $a(F,F)$.

The natural sequence $0 \to a(F,F) \to b(F,F) \to g(F,F) \to 0$ is exact by Proposition 2.11.2. It induces a canonical distinguished triangle

$$\mathrm{RH}(a(F,F)) \to \mathrm{RH}(b(F,F)) \to \mathrm{RH}(g(F,F)) \xrightarrow{\delta} [1]. \tag{3.2}$$

Lemma 3.7. *The map $\delta\colon \mathrm{RH}(g(F,F)) \to \mathrm{RH}(a(F,F))[1]$ is a quasi-isomorphism.*

Proof. Indeed Proposition 3.8.4 shows that the (A,A)-bimodule structure on $b(F,F)$ extends to an $A \mathbin{\widehat{\otimes}} F$-module structure and Lemma 3.3 implies that $\mathrm{RH}(b(F,F)) = 0$. □

Lemma 3.8. *The natural map $\mathrm{RH}(a(F/A,F)) \to \mathrm{RH}(a(F,F))$ is a quasi-isomorphism.*

Proof. We have a commutative square

$$\begin{tikzcd}
\mathrm{RH}(a(F,F)) \arrow[rr, "\delta", "\sim"'] & & \mathrm{RH}(g(F,F))[1] \\
\mathrm{RH}(a(F/A,F)) \arrow[u] \arrow[rr, "\delta", "\sim"'] & & \mathrm{RH}(g(F/A,F))[1] \arrow[u, "\wr"']
\end{tikzcd}$$

The quotient map $F \to F/A$ is a local isomorphism so that the induced map $g(F/A, F) \to g(F, F)$ is an isomorphism of (A, A)-bimodules. As a consequence the right arrow in the square above is a quasi-isomorphism. The top arrow is a quasi-isomorphism by Lemma 3.7 while the bottom arrow is a quasi-isomorphism by Corollary 3.5. $\square$

So Corollary 3.2 provides us with a natural quasi-isomorphism

$$\mathrm{RH}(a(F, F)) \cong F[-1].$$

We would like to describe this quasi-isomorphism explicitly. In the following we denote F^* the continuous $\mathbb{F}_q$-linear dual of F (see Definition 2.4.3).

Lemma 3.9. *There exists a unique continuous map $\mu\colon a(F, F) \to F^*$ with the following property. If $f\colon F \to \mathbb{F}_q$ is a continuous $\mathbb{F}_q$-linear map and $\alpha \in F$ then the image of the map $x \mapsto f(x)\alpha$ under μ is the map $x \mapsto f(x\alpha)$.*

Proof. Consider the lth $\mathbb{F}_q$-vector space $F^* \otimes_{\mathrm{ic}} F$ (see Definition 2.7.1). Let $\mu\colon F^* \otimes_{\mathrm{ic}} F \to F^*$ be the map which sends a tensor $f \otimes \alpha$ to the function $x \mapsto f(\alpha x)$. This map is easily shown to be continuous in the ind-tensor product topology. It induces a unique continuous map $F^* \breve{\otimes} F \to F^*$ by completion. Lemma 3.8.7 identifies $F^* \breve{\otimes} F$ with $a(F, F)$ so we get the result. $\square$

We denote $\rho\colon \omega \otimes_A F \xrightarrow{\sim} F^*$ the isomorphism induced by the residue pairing (Theorem 3.10.1).

Lemma 3.10. *The sequence*

$$0 \to a(F, F) \xrightarrow{\partial} D(A, a(F, F)) \xrightarrow{-D(\mu)} D(A, F^*) \to 0$$

is exact. Here $\mu\colon a(F, F) \to F^*$ is the map of Lemma 3.9.

Proof. Consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & a(F,F) \arrow[r] & D(A, a(F,F)) \arrow[r, "-D(\mu)"] & D(A, F^*) \arrow[r] & 0 \\
0 \arrow[r] & \omega \otimes F \arrow[r] \arrow[u] & D(A, \omega \otimes F) \arrow[r, "-D(\mu)"] \arrow[u] & D(A, \omega \otimes_A F) \arrow[r] \arrow[u, "\rho"', "\wr"] & 0
\end{tikzcd} \tag{3.3}$$

where the bottom row is the short exact sequence of Lemma 2.3 and the unmarked vertical arrows are given by the map $\omega \otimes F \to a(F, F)$, $\eta \otimes \alpha \mapsto (x \mapsto \mathrm{res}(x\alpha\eta))$. The left square is clearly commutative. Let us show that the right square is also commutative.

Let $\eta \in \omega$ and $\alpha \in F$. The element $\eta \otimes \alpha$ maps to the function $f\colon x \mapsto \mathrm{res}(x\eta) \cdot \alpha$. Hence $\mu(f)$ is the function $x \mapsto \mathrm{res}(x\alpha\eta)$. It is the image of the element $\eta \otimes_A \alpha \in \omega \otimes_A F$ under ρ. So the right square in (3.3) is commutative. Thus the top row is exact. $\square$

Lemma 3.11. *There exists a quasi-isomorphism* $\mathrm{RH}(a(F,F)) \cong F[-1]$. *It is the composition*

$$\mathrm{RH}(a(F,F)) \cong \Big[a(F,F) \xrightarrow{\partial} D(A, a(F,F))\Big] \cong$$
$$\cong D(A, F^*)[-1] \cong D(A, \omega \otimes_A F) \cong F[-1]$$

of the quasi-isomorphisms given by Proposition 2.1, Lemma 3.10, the map ρ *and Lemma 2.2 respectively. This quasi-isomorphism fits to a commutative square*

$$\begin{tikzcd}
\mathrm{RH}(a(F/A,F)) \arrow[r, "\sim"] \arrow[d, "\wr"'] & \mathrm{RH}(a(F,F)) \arrow[d, "\wr"] \\
F[-1] \arrow[r, equal] & F[-1]
\end{tikzcd}$$

where the left arrow is the quasi-isomorphism of Corollary 3.2.

Proof. Indeed the diagram (3.3) is commutative so the square in the statement of the lemma commutes. $\square$

Lemma 3.12. *If* V *is a finite-dimensional discrete* $\mathbb{F}_q$*-vector space,* $f\colon F \to V$ *and* $g\colon V \to F$ *continuous* $\mathbb{F}_q$*-linear maps then* $\mu(g \circ f)\colon F \to \mathbb{F}_q$ *is the map* $x \mapsto \mathrm{tr}_V(f \circ x \circ g)$.

Proof. If $V = \mathbb{F}_q$ then the claim follows from the definition of μ. Next assume that $V = V_1 \oplus V_2$. Let f_1, f_2 be the compositions of f with the projections to V_1, V_2 and let g_1, g_2 be the restrictions of g to V_1, V_2. We then have

$$\mu(g \circ f) = \mu(g_1 \circ f_1) + \mu(g_2 \circ f_2).$$

At the same time

$$\mathrm{tr}_V(f \circ x \circ g) = \mathrm{tr}_{V_1}(f_1 \circ x \circ g_1) + \mathrm{tr}_{V_2}(f_2 \circ x \circ g_2).$$

So the result follows by induction on the dimension of V. $\square$

Lemma 3.13. *The map* $F \to g(F,F)$, $\alpha \mapsto (x \mapsto x\alpha)$ *induces a quasi-isomorphism*

$$F[0] \xrightarrow{\sim} \mathrm{RH}(g(F,F)).$$

This quasi-isomorphism fits into a commutative square

$$\begin{tikzcd}
F[0] \arrow[r, equal] \arrow[d, "\wr"'] & F[0] \\
\mathrm{RH}(g(F,F)) \arrow[r, "\delta", "\sim"'] & \mathrm{RH}(a(F,F))[1] \arrow[u, "\wr"']
\end{tikzcd}$$

where the right arrow is the quasi-isomorphism of Lemma 3.11 and δ *is the boundary morphism in* (3.1).

Proof. To prove the lemma it is enough to show that the square commutes for then the map $F[0] \to \mathrm{RH}(g(F,F))$ will automatically be a quasi-isomorphism.

All the maps in the square above are F-linear. Indeed the right map is F-linear by construction while the map δ is F-linear since $a(F,F)$, $b(F,F)$ and $g(F,F)$ are $A \otimes F$-modules so that

$$\begin{aligned}\mathrm{RH}(a(F,F)) &= \mathrm{RHom}_{A\otimes F}(F, a(F,F)),\\ \mathrm{RH}(b(F,F)) &= \mathrm{RHom}_{A\otimes F}(F, b(F,F)),\\ \mathrm{RH}(g(F,F)) &= \mathrm{RHom}_{A\otimes F}(F, g(F,F))\end{aligned}$$

with the diagonal $A \otimes F$-module structure on F. So to prove that the square is commutative it is enough to consider the images of the element $1 \in F$.

Consider the square

$$\begin{tikzcd} F[0] \arrow[r, equal] \arrow[d, "\wr"'] & F[0] \\ \mathrm{H}^0(A, g(F,F)) \arrow[r, "\delta", "\sim"'] & \mathrm{H}^1(A, a(F,F)) \arrow[u, "\wr"'] \end{tikzcd}$$

The image of $1 \in F$ under the composition of the left, bottom and right arrows can be described in the following way. Pick a bounded function $f\colon F \to F$ such that $f(x) = x$ for all x in an open neighbourhood of 0 in F. Let $D\colon A \to a(F,F)$ be the derivation given by the formula $D(a) = af - fa$. We denote $\rho\colon F^* \xrightarrow{\sim} \omega \otimes_A F$ the isomorphism induced by the residue pairing. Let $a \in A$ be an arbitrary nonconstant element. There exists a unique $\alpha \in F$ such that

$$\rho\mu(D(a)) = da \otimes_A \alpha \in \omega \otimes_A F.$$

The image of $1 \in F$ is the element $-\alpha \in F$. It does not depend on the choice of the function $f\colon F \to F$ and the nonconstant element $a \in A$. Our goal is to prove that $\alpha = -1$.

In the following we denote F_0 the fraction field of A and Ω the module of Kähler differentials of F_0 over $\mathbb{F}_q$. By construction we have

$$\omega \otimes_A F = \Omega \otimes_{F_0} F.$$

Pick elements $a \in A$, $b \in A$ such that $z = ab^{-1}$ is a uniformizer of F. Let k be the residue field of F so that $F = k(\!(z)\!)$.

First let us construct a suitable bounded function $f\colon F \to F$. Consider the function $f\colon F \to F$ defined by the formula

$$f\Big(\sum_n \alpha_n z^n\Big) = \sum_{n\geqslant 0} \alpha_n z^n$$

with $\alpha_n \in k$. This function is clearly bounded and satisfies $f(x) = x$ for all $x \in \mathcal{O}_F$. Moreover for every $\alpha \in k$ and $n \in \mathbb{Z}$ we have

$$zf(\alpha z^n) - f(\alpha z^{n+1}) = \begin{cases} -\alpha, & n = -1,\\ 0, & n \neq -1.\end{cases}$$

Applying Lemma 3.12 to $zf - fz$ with $V = k$ and $g\colon k \hookrightarrow F$ the natural inclusion we conclude that $\mu(zf - fz)$ is the function $x \mapsto -\operatorname{res}(x\,dz)$. Therefore

$$\rho\mu(zf - fz) = -dz \otimes_{F_0} 1$$

in $\Omega \otimes_{F_0} F = \omega \otimes_A F$.

Next we extend the derivation $D\colon A \to a(F, F)$ to a derivation $D\colon F \to a(F, F)$ using the formula $D(x) = xf - fx$. We then have

$$D(a) = D(zb) = zD(b) + D(z)b.$$

By assumption $\rho\mu(D(a)) = da \otimes_A \alpha$ and $\rho\mu(D(b)) = db \otimes_A \alpha$ so that

$$da \otimes_A \alpha = db \otimes_A z\alpha + b\rho\mu(D(z)).$$

At the same time there is an identity $da = d(zb) = z\,db + b\,dz$ in Ω. Comparing with the identity above we conclude that

$$\rho\mu(D(z)) = dz \otimes_{F_0} \alpha$$

as an element of $\Omega \otimes_{F_0} F = \omega \otimes_A F$. However we observed above that

$$\rho\mu(D(z)) = \rho\mu(zf - fz) = -dz \otimes_{F_0} 1.$$

Hence $\alpha = -1$. □

Proof of Proposition 3.6. The locally compact F-vector space V is finite-dimensional by Weil's theorem [26]. Thus by naturality we can assume that $V = F$. In this case Lemma 3.11 implies that the natural maps $g(F/A, F) \to g(F, F)$ and $a(F/A, F) \to a(F, F)$ identify the square of Proposition 3.6 with the square of Lemma 3.13. The latter square is commutative. □

4. The exponential map

Let A be a coefficient ring in the sense of Definition 7.2.1 and let K be a finite product of local fields containing $\mathbb{F}_q$. Fix a Drinfeld A-module E over K. The A-module $\mathrm{Lie}_E(K)$ is naturally a free K-module of rank 1. Throughout this section we assume that the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F.

Definition 4.1. The *exponential map* of the Drinfeld module E is a map

$$\exp\colon \mathrm{Lie}_E(K) \to E(K)$$

satisfying the following conditions:

(1) exp is a homomorphism of A-modules,
(2) exp is an analytic function with derivative 1 at zero in the following sense. Fix an $\mathbb{F}_q$-linear isomorphism of group schemes $E \cong \mathbb{G}_a$. It identifies $E(K)$ with K while its differential identifies $\mathrm{Lie}_E(K)$ with K. We demand that the resulting map $\exp\colon K \to K$ is given by an everywhere convergent power series of the form

$$\exp(z) = z + a_1 z^q + a_2 z^{q^2} + \ldots$$

with coefficients in K.

Proposition 4.2. *The exponential map exists, is unique and has the following properties:*

(1) $\ker\exp \subset \mathrm{Lie}_E(K)$ *is a locally free A-module of finite rank.*
(2) $\exp$ *is a local isomorphism in the sense of Definition 2.11.3.*

Proof. For the existence and unicity see [6, Theorem 2.1]. (1) Since exp is a nonzero $\mathbb{F}_q$-linear analytic function its kernel is discrete. As exp is also A-linear it follows that the kernel is a discrete A-submodule of a finite-dimensional F-vector space $\mathrm{Lie}_E(K)$. As a consequence $\ker\exp$ is a projective A-module of finite type. (2) follows since exp has nonzero derivative at zero. □

In the following we denote $C_{\exp}$ the A-module complex

$$\Big[\mathrm{Lie}_E(K) \xrightarrow{\exp} E(K)\Big].$$

The complex $C_{\exp}$ sits in a canonical distinguished triangle

$$E(K)[-1] \to C_{\exp} \to \mathrm{Lie}_E(K)[0] \xrightarrow{\exp} E(K)[0] \tag{4.1}$$

where the first arrow is given by the identity map in degree 1 and the second arrow is given by the identity map in degree 0.

Lemma 4.3. *If $f\colon C_{\exp} \to C_{\exp}$ is a morphism in the derived category of A-modules such that the square*

$$\begin{tikzcd} E(K)[-1] \arrow[r] \arrow[d, equal] & C_{\exp} \arrow[d, "f"] \\ E(K)[-1] \arrow[r] & C_{\exp} \end{tikzcd}$$

is commutative then $f = 1$.

Proof. Applying $\mathrm{Hom}(-, C_{\exp})$ to the distinguished triangle (4.1) we get an exact sequence

$$\mathrm{Hom}(\mathrm{Lie}_E(K)[0], C_{\exp}) \to \mathrm{Hom}(C_{\exp}, C_{\exp}) \to \mathrm{Hom}(E(K)[-1], C_{\exp}).$$

To prove the claim we need to show that $\mathrm{Hom}(\mathrm{Lie}_E(K)[0], C_{\exp}) = 0$. As A is of global dimension 1 there exists a non-canonical quasi-isomorphism

$$C_{\exp} \cong \mathrm{H}^0(C_{\exp})[0] \oplus \mathrm{H}^1(C_{\exp})[-1].$$

Hence $\mathrm{Hom}(\mathrm{Lie}_E(K)[0], C_{\exp}) = \mathrm{Hom}_A(\mathrm{Lie}_E(K), \mathrm{H}^0(C_{\exp}))$. The latter Hom module is zero. Indeed $\mathrm{Lie}_E(K)$ is a uniquely divisible A-module while the A-module $\mathrm{H}^0(C_{\exp}) = \ker\exp$ is locally free of finite rank. □

5. Function spaces and Drinfeld modules

As before A is a coefficient ring, K is a finite product of local fields containing $\mathbb{F}_q$ and E is a Drinfeld A-module over K. Let F be the local field of A at infinity. As in the previous section we assume that the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F. The natural topology on $\mathrm{Lie}_E(K)$ is locally compact so it becomes a finite-dimensional F-vector space.

The goal of this section is to compute Hochschild cohomology of the function spaces $a(F/A, E(K))$, $b(F/A, E(K))$ and $g(F/A, E(K))$. We begin with the easier cohomology of the function spaces with the codomain $\mathrm{Lie}_E(K)$. By Proposition 2.11.2 the natural sequence

$$0 \to a(F/A, \mathrm{Lie}_E(K)) \to b(F/A, \mathrm{Lie}_E(K)) \to g(F/A, \mathrm{Lie}_E(K)) \to 0$$

is exact. Applying the Hochschild cohomology functor $\mathrm{RH}(-) = \mathrm{RH}(A, -)$ we obtain a canonical distinguished triangle

$$\begin{aligned}\mathrm{RH}(a(F/A, \mathrm{Lie}_E(K))) \to \mathrm{RH}(b(F/A, \mathrm{Lie}_E(K))) \to &\\ \to \mathrm{RH}(g(F/A, \mathrm{Lie}_E(K))) \xrightarrow{\delta} [1].&\end{aligned}$$

Lemma 5.1. *The map* $\mathrm{RH}(g(F/A, \mathrm{Lie}_E(K))) \xrightarrow{\delta} \mathrm{RH}(a(F/A, \mathrm{Lie}_E(K)))[1]$ *is a quasi-isomorphism.*

Proof. Follows from Corollary 3.5 since $\mathrm{Lie}_E(K)$ is a locally compact F-vector space. □

Proposition 5.2. *The map* $\mathrm{Lie}_E(K) \to g(F/A, \mathrm{Lie}_E(K))$*,* $\alpha \mapsto (x \mapsto x\alpha)$ *induces a quasi-isomorphism*

$$\mathrm{Lie}_E(K)[0] \xrightarrow{\sim} \mathrm{RH}(g(F/A, \mathrm{Lie}_E(K))).$$

This quasi-isomorphism fits into a commutative square

$$\begin{tikzcd} \mathrm{Lie}_E(K)[0] \arrow[r, equal] \arrow[d, "\wr"'] & \mathrm{Lie}_E(K)[0] \\ \mathrm{RH}(g(F/A, \mathrm{Lie}_E(K))) \arrow[r, "\delta", "\sim"'] & \mathrm{RH}(a(F/A, \mathrm{Lie}_E(K)))[1] \arrow[u, "\wr"'] \end{tikzcd}$$

where the right arrow is the quasi-isomorphism of Corollary 3.2.

Proof. Since $\mathrm{Lie}_E(K)$ is a locally compact F-vector space the result follows instantly from Proposition 3.6. □

Now we move from $\mathrm{Lie}_E(K)$ to $E(K)$. As before we have a canonical distinguished triangle

$$\mathrm{RH}(a(F/A, E(K))) \to \mathrm{RH}(b(F/A, E(K))) \to \mathrm{RH}(g(F/A, E(K))) \xrightarrow{\delta} [1].$$

Let $\exp\colon \mathrm{Lie}_E(K) \to E(K)$ be the exponential map of E in the sense of Definition 4.1.

Proposition 5.3. *The map*

$$\mathrm{Lie}_E(K) \to g(F/A, E(K)), \quad \alpha \mapsto (x \mapsto \exp(x\alpha))$$

induces a quasi-isomorphism $\mathrm{Lie}_E(K)[0] \xrightarrow{\sim} \mathrm{RH}(g(F/A, E(K)))$*. This quasi-isomorphism fits into a commutative square*

$$\begin{tikzcd}
\mathrm{Lie}_E(K)[0] \arrow[r, "\exp"] \arrow[d, "\wr"'] & E(K)[0] \\
\mathrm{RH}(g(F/A, E(K))) \arrow[r, "\delta"] & \mathrm{RH}(a(F/A, E(K)))[1] \arrow[u, "\wr"']
\end{tikzcd}$$

where the right arrow is the quasi-isomorphism of Corollary 3.2.

Proof. The exponential map $\exp\colon \mathrm{Lie}_E(K) \to E(K)$ induces a morphism of short exact sequences

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
a(F/A, \mathrm{Lie}_E(K)) \arrow[r, "\exp"] \arrow[d] & a(F/A, E(K)) \arrow[d] \\
b(F/A, \mathrm{Lie}_E(K)) \arrow[r, "\exp"] \arrow[d] & b(F/A, E(K)) \arrow[d] \\
g(F/A, \mathrm{Lie}_E(K)) \arrow[r, "\exp"] \arrow[d] & g(F/A, E(K)) \arrow[d] \\
0 & 0
\end{tikzcd}$$

The exponential map is an A-linear local isomorphism. Hence the induced map $g(F/A, \mathrm{Lie}_E(K)) \to g(F/A, E(K))$ is an isomorphism of (A, A)-bimodules. Taking the cohomology and applying Proposition 5.2 we get the result. $\square$

As in the previous section we denote $C_{\exp}$ the A-module complex

$$\Big[\mathrm{Lie}_E(K) \xrightarrow{\exp} E(K)\Big].$$

Theorem 5.4. *There exists a quasi-isomorphism*

$$\mathrm{RH}(b(F/A, E(K))) \xrightarrow{\sim} C_{\exp}$$

with the following properties.

(1) *It is the unique map in the derived category of A-modules such that the square*

$$\begin{tikzcd}
\mathrm{RH}(a(F/A, E(K))) \arrow[r] \arrow[d, "\wr"] & \mathrm{RH}(b(F/A, E(K))) \arrow[d] \\
E(K)[-1] \arrow[r] & C_{\exp}
\end{tikzcd}$$

is commutative. Here the left arrow is the quasi-isomorphism of Corollary 3.2 and the bottom arrow is given by the identity in degree 1*.*

(2) *The square*

$$\begin{tikzcd}
\mathrm{RH}(b(F/A, E(K))) \arrow[r] \arrow[d, "\wr"'] & \mathrm{RH}(g(F/A, E(K))) \\
C_{\exp} \arrow[r] & \mathrm{Lie}_E(K)[0] \arrow[u, "\wr"']
\end{tikzcd}$$

is commutative. Here the right arrow is the quasi-isomorphism of Proposition 5.3 and the bottom arrow is given by the identity in degree 0*.*

Rather remarkably Theorem 5.4 identifies the cohomology of a purely algebraic object $b(F/A, E(K))$ with the complex $C_{\exp}$ which is defined in terms of the analytic function exp.

Proof of Theorem 5.4. Consider the diagram

$$\begin{tikzcd}
\mathrm{RH}(a(F/A, E(K))) \arrow[r, "\sim"] \arrow[d] & E(K)[-1] \arrow[d] \\
\mathrm{RH}(b(F/A, E(K))) \arrow[d] & C_{\exp} \arrow[d] \\
\mathrm{RH}(g(F/A, E(K))) \arrow[r, "\sim"] \arrow[d] & \mathrm{Lie}_E(K)[0] \arrow[d, "\exp"] \\
{[1]} & E(K)[0]
\end{tikzcd}$$

where the first horizontal arrow is the quasi-isomorphism of Corollary 3.2, the second horizontal arrow is the quasi-isomorphism of Proposition 5.3 and the right column is the canonical distinguished triangle (4.1). The commutative square of Proposition 5.3 induces a quasi-isomorphism $\mathrm{RH}(b(F/A, E(K))) \xrightarrow{\sim}$

$C_{\exp}$ which makes the diagram

$$\begin{tikzcd}
\mathrm{RH}(a(F/A, E(K))) \arrow[r, "\sim"] \arrow[d] & E(K)[-1] \arrow[d] \\
\mathrm{RH}(b(F/A, E(K))) \arrow[r, dashed, "\simeq"] \arrow[d] & C_{\exp} \arrow[d] \\
\mathrm{RH}(g(F/A, E(K))) \arrow[r, "\sim"] \arrow[d] & \mathrm{Lie}_E(K)[0] \arrow[d, "\exp"] \\
{[1]} & E(K)[0]
\end{tikzcd}$$

into a morphism of distinguished triangles. Since $\mathrm{RH}(b(F/A, E(K))) \cong C_{\exp}$ Lemma 4.3 implies that the middle horizontal arrow in this diagram is the unique morphism which makes the top square commutative. So the result follows. ☐

Using an appropriate version of Lemma 4.3 one can show that the quasi-isomorphism

$$\mathrm{RH}(b(F/A, E(K))) \cong C_{\exp}$$

of Theorem 5.4 is natural in E. However we will not need this fact.

6. The main lemma

The following simple lemma is our main tool to compute the cohomology of Hom shtukas associated to Drinfeld modules.

Lemma 6.1. *Let A and B be associative unital $\mathbb{F}_q$-algebras. For all left $A\otimes B$-modules M and N there exists a natural quasi-isomorphism*

$$\mathrm{RHom}_{A\otimes B}(M, N) \cong \mathrm{RH}(A, \mathrm{RHom}_B(M, N)).$$

In degree 0 it coincides with the canonical isomorphism

$$\mathrm{Hom}_{A\otimes B}(M, N) \cong \mathrm{Hom}_{A\otimes A^\circ}(A, \mathrm{Hom}_B(M, N)).$$

In applications A will be commutative in which case the quasi-isomorphism will be A-linear.

Proof of Lemma 6.1. Let M be a projective left $A \otimes B$-module and N an injective left $A\otimes B$-module. We claim that $\mathrm{Hom}_B(M, N)$ is an injective (A, A)-bimodule. Indeed M is a flat A-module since it is a direct summand of a free $A \otimes B$-module and $A \otimes B$ is A-flat since B is $\mathbb{F}_q$-flat. So the functor

$$\mathrm{Hom}_{A\otimes A^\circ}(-, \mathrm{Hom}_B(M, N)) = \mathrm{Hom}_{A\otimes B}(- \otimes_A M, N).$$

is exact as the composition of exact functors $- \otimes_A M$ and $\mathrm{Hom}_{A\otimes B}(-, N)$.

Let M and N be arbitrary left $A \otimes B$-modules. Let $M^\bullet$ be a projective resolution of M and $N^\bullet$ an injective resolution of N. The argument above shows that $\mathrm{Hom}^\bullet_B(M^\bullet, N^\bullet)$ is a bounded below complex of injective (A, A)-bimodules. Hence

$$\begin{aligned}\mathrm{RHom}_{A\otimes A^\circ}(A, \mathrm{Hom}^\bullet_B(M^\bullet, N^\bullet)) &= \mathrm{Hom}_{A\otimes A^\circ}(A, \mathrm{Hom}^\bullet_B(M^\bullet, N^\bullet)) \\ &= \mathrm{Hom}^\bullet_{A\otimes B}(M^\bullet, N^\bullet) \\ &= \mathrm{RHom}_{A\otimes B}(M, N).\end{aligned}$$

Next we claim that $M^\bullet$ is a projective resolution of M as a B-module. Indeed if P is a projective left $A \otimes B$-module then it is a direct summand of a free $A \otimes B$-module. However $A \otimes B$ is a projective B-module since $\mathrm{Hom}_B(A \otimes B, -) = \mathrm{Hom}_{\mathbb{F}_q}(A, -)$. So P is a projective B-module and $M^\bullet$ is a projective B-module resolution of M. It follows that

$$\mathrm{RHom}_B(M, N) = \mathrm{Hom}^\bullet_B(M^\bullet, N[0]) = \mathrm{Hom}^\bullet_B(M^\bullet, N^\bullet).$$

Thus $\mathrm{RHom}_{A\otimes A^\circ}(A, \mathrm{RHom}_B(M, N)) = \mathrm{RHom}_{A\otimes A^\circ}(A, \mathrm{Hom}^\bullet_B(M^\bullet, N^\bullet))$. ☐

More generally this lemma applies to associative unital algebras A, B over an arbitrary commutative ring k provided that A is a projective k-module and B is a flat k-module.

7. Structure of the Hom shtuka

Let A be a coefficient ring, E a Drinfeld A-module over a reduced $\mathbb{F}_q$-algebra B and M the motive of E. We will mainly work with E by the means of the Hom shtuka $\mathcal{H}\mathrm{om}_{A\otimes B}(M, N)$ where N is a left $A \otimes B\{\tau\}$-module. To simplify the expressions we will generally write $\mathcal{H}\mathrm{om}(M, N)$ and $\mathrm{Hom}(M, N)$ in place of $\mathcal{H}\mathrm{om}_{A\otimes B}(M, N)$ and $\mathrm{Hom}_{A\otimes B}(M, N)$.

Lemma 7.1. *Let N be a left $A \otimes B\{\tau\}$-module. The shtuka $\mathcal{H}\mathrm{om}(M, N)$ is represented by the diagram*

$$\Big[\mathrm{Hom}(M, N) \underset{j}{\overset{i}{\rightrightarrows}} \mathrm{Hom}(M^{\geqslant 1}, N)\Big]$$

where i is the restriction to $M^{\geqslant 1}$ and j sends an element f to the map $\tau m \mapsto \tau f(m)$.

Proof. According to Proposition 7.1.10 the adjoint $\tau^* M \to M$ of the multiplication map $\tau\colon M \to M$ is injective with image $M^{\geqslant 1}$. So the result is a consequence of Lemma 1.12.2. ☐

8. Drinfeld modules and Hochschild cohomology

Let A be a coefficient ring in the sense of Definition 7.2.1. Let B be a reduced $\mathbb{F}_q$-algebra, the base ring. As usual we equip $A \otimes B$ with the τ-ring structure given by the endomorphism which acts as the identity on A and as the q-Frobenius on B.

Let E be a Drinfeld A-module over B and let $M = \mathrm{Hom}(E, \mathbb{G}_a)$ be its motive (see Definition 7.4.1). The motive M carries a natural structure of a left $A \otimes B\{\tau\}$-module with A acting on the right via E and $B\{\tau\}$ acting on the left via $\mathbb{G}_a$. In this section we study the cohomology of the shtuka $\mathcal{H}\mathrm{om}_{A\otimes B}(M, N)$ where N is an arbitrary left $A \otimes B\{\tau\}$-module (see Definition 1.12.1). To improve legibility we will write $\mathcal{H}\mathrm{om}(M, N)$ instead of $\mathcal{H}\mathrm{om}_{A\otimes B}(M, N)$.

Lemma 8.1. *For every left $A \otimes B\{\tau\}$-module N there exists a natural $A \otimes B$-linear quasi-isomorphism*

$$\mathrm{R}\Gamma(\nabla \mathcal{H}\mathrm{om}(M, N)) \cong \mathrm{RH}(A, \mathrm{Hom}_B(M/M^{\geqslant 1}, N)). \tag{8.1}$$

Moreover $\mathrm{H}^0(\nabla \mathcal{H}\mathrm{om}(M, N))$ *and* $\mathrm{Hom}_{A\otimes B}(M/M^{\geqslant 1}, N)$ *coincide as submodules of* $\mathrm{Hom}_{A\otimes B}(M, N)$ *so that in degree* 0 *the quasi-isomorphism* (8.1) *is given by the natural isomorphism*

$$\mathrm{Hom}_{A\otimes B}(M/M^{\geqslant 1}, N) \cong \mathrm{Hom}_{A\otimes A}(A, \mathrm{Hom}_B(M/M^{\geqslant 1}, N)).$$

Proof. By Lemma 7.1 the shtuka $\nabla \mathcal{H}\mathrm{om}(M, N)$ is given by the diagram

$$\mathrm{Hom}_{A\otimes B}(M, N) \underset{0}{\overset{i}{\rightrightarrows}} \mathrm{Hom}_{A\otimes B}(M^{\geqslant 1}, N)$$

where i is the restriction map. So by Theorem 1.8.1 we have

$$\mathrm{R}\Gamma(\nabla \mathcal{H}\mathrm{om}(M, N)) = \Big[\mathrm{Hom}_{A\otimes B}(M, N) \xrightarrow{i} \mathrm{Hom}_{A\otimes B}(M^{\geqslant 1}, N)\Big].$$

The complex on the right hand side computes $\mathrm{RHom}_{A\otimes B}(M/M^{\geqslant 1}, N)$ since M and $M^{\geqslant 1}$ are locally free $A \otimes B$-modules. As $M/M^{\geqslant 1}$ is an invertible B-module Lemma 6.1 shows that

$$\mathrm{RHom}_{A\otimes B}(M/M^{\geqslant 1}, N) = \mathrm{RH}(A, \mathrm{Hom}_B(M/M^{\geqslant 1}, N)). \quad \square$$

If N is a left $A \otimes B\{\tau\}$-module then the $\mathbb{F}_q$-vector space

$$\mathrm{Hom}_B(M^0, N) = \mathrm{Hom}_{B\{\tau\}}(M, N)$$

is in a natural way an (A, A)-bimodule: A acts on the right via M and on the left via N.

Lemma 8.2. *For every left $A \otimes B\{\tau\}$-module N there exists a natural A-linear quasi-isomorphism*

$$\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, N)) \cong \mathrm{RH}(A, \mathrm{Hom}_B(M^0, N)). \tag{8.2}$$

Moreover $\mathrm{H}^0(\mathcal{H}\mathrm{om}(M, N))$ *and* $\mathrm{Hom}_{A\otimes B\{\tau\}}(M, N)$ *coincide as submodules of* $\mathrm{Hom}_{A\otimes B}(M, N)$ *so that in degree* 0 *the quasi-isomorphism* (8.2) *is given by the natural isomorphism*

$$\mathrm{Hom}_{A\otimes B\{\tau\}}(M, N) \cong \mathrm{Hom}_{A\otimes A}(A, \mathrm{Hom}_{B\{\tau\}}(M, N)) \cong$$
$$\cong \mathrm{Hom}_{A\otimes A}(A, \mathrm{Hom}_B(M^0, N)).$$

Proof. By Theorem 1.12.5 we have

$$\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, N)) = \mathrm{RHom}_{A\otimes B\{\tau\}}(M, N).$$

Lemma 6.1 implies that

$$\mathrm{RHom}_{A\otimes B\{\tau\}}(M, N) = \mathrm{RH}(A, \mathrm{RHom}_{B\{\tau\}}(M, N)).$$

According to Proposition 7.1.6 the B-module $M^0 \subset M$ is invertible and $M = B\{\tau\} \otimes_B M^0$. So M is a projective $B\{\tau\}$-module. We conclude that

$$\mathrm{RHom}_{B\{\tau\}}(M, N) = \mathrm{Hom}_{B\{\tau\}}(M, N) = \mathrm{Hom}_B(M^0, N). \quad \square$$

9. Formulas of Barsotti-Weil type

As in the previous section A is a coefficient ring, B is a reduced $\mathbb{F}_q$-algebra, E is a Drinfeld A-module over B and M is the motive of E.

As in Section 7.1 given a map $m\colon E \to \mathbb{G}_a$ we denote dm the induced map from Lie_E to $\mathrm{Lie}_{\mathbb{G}_a}$. We denote ω the module of Kähler differentials of A over $\mathbb{F}_q$. Observe that $M/M^{\geqslant 1}$ and $\omega \otimes S$ are $A \otimes B$-modules.

Lemma 9.1. *For every B-algebra S the natural map*

$$\omega \otimes \mathrm{Lie}_E(S) \to \mathrm{Hom}_B(M/M^{\geqslant 1}, \omega \otimes S), \quad \eta \otimes \alpha \mapsto (m \mapsto \eta \otimes dm(\alpha))$$

is an isomorphism of (A, A)-bimodules.

Proof. Indeed $M/M^{\geqslant 1}$ is an invertible B-module so that $\mathrm{Hom}_B(M/M^{\geqslant 1}, \omega \otimes S) = \omega \otimes \mathrm{Hom}_B(M/M^{\geqslant 1}, S)$. Proposition 7.1.11 identifies $\mathrm{Hom}_B(M/M^{\geqslant 1}, S)$ and $\mathrm{Lie}_E(S)$. $\square$

Lemma 9.2. *For every B-algebra S the natural map*

$$\omega \otimes E(S) \to \mathrm{Hom}_B(M^0, \omega \otimes S), \quad \eta \otimes e \mapsto (m \mapsto \eta \otimes m(e))$$

is an isomorphism of (A, A)-bimodules.

Proof. M^0 is an invertible B-module by definition so that $\mathrm{Hom}_B(M^0, \omega \otimes S) = \omega \otimes \mathrm{Hom}_B(M^0, S)$. Proposition 7.1.9 identifies $\mathrm{Hom}_B(M^0, S)$ with $E(S)$. $\square$

Combining the lemmas above with the results of Section 8 and the isomorphism $\mathrm{RH}(A, \omega \otimes N) \cong N[-1]$ of Proposition 2.4 we obtain the formulas of Barsotti-Weil type as annouced in the introduction.

Theorem 9.3. *For every B-algebra S there is a natural $A \otimes S$-linear quasi-isomorphism*

$$\mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}(M, \omega \otimes S)) \cong \mathrm{Lie}_E(S)[-1].$$

It is given by the composition

$$\mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}(M, \omega \otimes S)) \cong \mathrm{RH}(A, \mathrm{Hom}_B(M/M^{\geqslant 1}, \omega \otimes S)) \cong \\ \cong \mathrm{RH}(A, \omega \otimes \mathrm{Lie}_E(S)) \cong \mathrm{Lie}_E(S)[-1]$$

of the quasi-isomorphism of Lemma 8.1, the quasi-isomorphism induced by Lemma 9.1 and the quasi-isomorphism of Proposition 2.4. $\square$

Theorem 9.4. *For every B-algebra S there is a natural A-linear quasi-isomorphism*

$$\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, \omega \otimes S)) \cong E(S)[-1].$$

It is given by the composition

$$\begin{aligned}\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M,\omega\otimes S)) &\cong \mathrm{RH}(A,\mathrm{Hom}_B(M^0,\omega\otimes S))\cong\\ &\cong \mathrm{RH}(A,\omega\otimes E(S))\cong E(S)[-1]\end{aligned}$$

of the quasi-isomorphism of Lemma 8.2, the quasi-isomorphism induced by Lemma 9.2 and the quasi-isomorphism of Proposition 2.4. □

In the case $A=\mathbb{F}_q[t]$ related statements were obtained in [18, 22].

10. Function spaces and the Hom shtuka

As before A is a coefficient ring and F the local field of A at infinity. Let K be a finite product of local fields containing $\mathbb{F}_q$. Fix a Drinfeld A-module E over K. We assume that the action of A on $\mathrm{Lie}_E(K)$ extends to a continuous action of F.

Our goal is to compute the cohomology of the shtuka $\mathcal{H}\mathrm{om}(M,N)$ where M is the motive of E and N is one of the function spaces $a(F/A,K)$, $b(F/A,K)$ or the germ space $g(F/A,K)$. Basically we repeat the results of Section 5 identifying Hochschild cohomology with the cohomology of the corresponding Hom shtukas using Lemmas 8.1 and 8.2.

Lemma 10.1. *The following maps are isomorphisms of $A\otimes K$-modules:*

$$\begin{aligned}a(F/A,\mathrm{Lie}_E(K))&\to\mathrm{Hom}_B(M/M^{\geqslant 1},a(F/A,K)),\quad f\mapsto(m\mapsto dm\circ f),\\ b(F/A,\mathrm{Lie}_E(K))&\to\mathrm{Hom}_B(M/M^{\geqslant 1},b(F/A,K)),\quad f\mapsto(m\mapsto dm\circ f),\\ g(F/A,\mathrm{Lie}_E(K))&\to\mathrm{Hom}_B(M/M^{\geqslant 1},g(F/A,K)),\quad f\mapsto(m\mapsto dm\circ f).\end{aligned}$$

Proof. As every invertible K-module is free we can assume without loss of generality that $E=\mathbb{G}_a$ as an $\mathbb{F}_q$-vector space scheme. In this case the result is clear. □

Lemma 10.2. *The following maps are isomorphisms of (A,A)-bimodules:*

$$\begin{aligned}a(F/A,E(K))&\to\mathrm{Hom}_B(M^0,a(F/A,K)),\quad f\mapsto(m\mapsto m\circ f),\\ b(F/A,E(K))&\to\mathrm{Hom}_B(M^0,b(F/A,K)),\quad f\mapsto(m\mapsto m\circ f),\\ g(F/A,E(K))&\to\mathrm{Hom}_B(M^0,g(F/A,K)),\quad f\mapsto(m\mapsto m\circ f).\quad\square\end{aligned}$$

Proposition 10.3. $\mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M,b(F/A,K)))=0$.

Proof. In view of Lemmas 10.1 and 8.1 we have $\mathrm{R}\Gamma(\nabla\,\mathcal{H}\mathrm{om}(M,b(F/A,K)))=\mathrm{RH}(b(F/A,\mathrm{Lie}_E(K)))$. So the result follows from Lemma 3.4. □

Lemma 10.4. *The natural sequence*

$$\begin{aligned}0\to\mathcal{H}\mathrm{om}(M,a(F/A,K))&\to\mathcal{H}\mathrm{om}(M,b(F/A,K))\to\\ &\to\mathcal{H}\mathrm{om}(M,g(F/A,K))\to 0\end{aligned}$$

is exact.

Proof. Indeed by Definition 7.4.1 the motive M is a locally free $A \otimes K$-module so the functor $\mathcal{H}\mathrm{om}(M, -)$ transforms exact sequences to exact sequences. $\square$

Applying $\mathrm{R}\Gamma$ to the triangle induced by the short exact sequence above we get canonical maps

$$\delta\colon \mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}(M, g(F/A, K))) \to \mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}(M, a(F/A, K)))[1],$$
$$\delta\colon \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, g(F/A, K))) \to \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, a(F/A, K)))[1].$$

in the derived category of A-modules.

Let ω be the module of Kähler differentials of A over $\mathbb{F}_q$. By Lemma 3.1 we have $a(F/A, K) = \omega \otimes K$ so that we can apply Theorems 9.3 and 9.4 to the shtuka $\mathcal{H}\mathrm{om}(M, a(F/A, K)) = \mathcal{H}\mathrm{om}(M, \omega \otimes K)$.

Proposition 10.5. *The map* $\mathrm{Lie}_E(K) \to \mathrm{Hom}(M, g(F/A, K))$, $\alpha \mapsto (m \mapsto (x \mapsto dm(\alpha x)))$ *induces a quasi-isomorphism*

$$\mathrm{Lie}_E(K)[0] \xrightarrow{\sim} \mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}(M, g(F/A, K))).$$

This quasi-isomorphism fits into a commutative square

$$\begin{tikzcd}
\mathrm{Lie}_E(K)[0] \arrow[r, equal] \arrow[d, "\wr"'] & \mathrm{Lie}_E(K)[0] \\
\mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}(M, g(F/A, K))) \arrow[r, "\delta"] & \mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}(M, a(F/A, K)))[1] \arrow[u, "\wr"']
\end{tikzcd}$$

where the right arrow is the quasi-isomorphism given by Theorem 9.3.

Proof. In view of Lemmas 10.1 and 8.1 the result is a consequence of Proposition 5.2. $\square$

Let $\exp\colon \mathrm{Lie}_E(K) \to E(K)$ be the exponential map.

Proposition 10.6. *The map* $\mathrm{Lie}_E(K) \to \mathrm{Hom}(M, g(F/A, K))$, $\alpha \mapsto (m \mapsto (x \mapsto m\exp(\alpha x)))$ *induces a quasi-isomorphism*

$$\mathrm{Lie}_E(K)[0] \xrightarrow{\sim} \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, g(F/A, K))).$$

This quasi-isomorphism fits into a commutative square

$$\begin{tikzcd}
\mathrm{Lie}_E(K)[0] \arrow[r, "\exp"] \arrow[d, "\wr"'] & E(K)[0] \\
\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, g(F/A, K))) \arrow[r, "\delta"] & \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, a(F/A, K)))[1] \arrow[u, "\wr"']
\end{tikzcd}$$

where the right arrow is the quasi-isomorphism given by Theorem 9.4.

Proof. In view of Lemmas 10.2 and 8.2 the result is a consequence of Proposition 5.3. $\square$

As before we denote $C_{\exp}$ the A-module complex

$$\Big[\operatorname{Lie}_E(K) \xrightarrow{\exp} E(K)\Big].$$

Theorem 10.7. *There exists a quasi-isomorphism*

$$\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, b(F/A, K))) \xrightarrow{\sim} C_{\exp}$$

with the following properties.

(1) *It is the unique map in the derived category of A-modules such that the square*

$$\begin{tikzcd}
\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, a(F/A, K))) \arrow[r] \arrow[d, "\wr"] & \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, b(F/A, K))) \arrow[d] \\
E(K)[-1] \arrow[r] & C_{\exp}
\end{tikzcd}$$

is commutative. Here the left arrow is the quasi-isomorphism of Theorem 9.4 and the bottom arrow is given by the identity in degree 1*.*

(2) *The square*

$$\begin{tikzcd}
\mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, b(F/A, K))) \arrow[r] \arrow[d, "\wr"'] & \mathrm{R}\Gamma(\mathcal{H}\mathrm{om}(M, g(F/A, K))) \\
C_{\exp} \arrow[r] & \operatorname{Lie}_E(K)[0] \arrow[u, "\wr"']
\end{tikzcd}$$

is commutative. Here the right arrow is the quasi-isomorphism of Proposition 10.6 and the bottom arrow is given by the identity in degree 0*.*

Proof. In view of Lemmas 10.2 and 8.2 the result is a consequence of Theorem 5.4. $\square$

11. Tate modules and Galois action

As usual A is a coefficient ring and ω is the module of Kähler differentials of A over $\mathbb{F}_q$. We work with a Drinfeld A-module E over a field k containing $\mathbb{F}_q$. We denote M the motive of E. In this section we study the Hom shtuka $\mathcal{H}\mathrm{om}(M, \omega \otimes k)$ and its relation to the Tate modules of E.

Lemma 11.1. *Let N be a left $A \otimes k\{\tau\}$-module. Consider the shtuka*

$$\mathcal{H}\mathrm{om}(M, N) = \Big[\operatorname{Hom}_{A\otimes k}(M, N) \underset{j}{\overset{i}{\rightrightarrows}} \operatorname{Hom}_{A\otimes k}(\tau^* M, N)\Big].$$

If the endomorphism $\tau\colon N \to N$ is a bijection then the following holds:

(1) *The arrow j is a bijection.*
(2) *The endomorphism $j^{-1}i$ of $\operatorname{Hom}_{A\otimes k}(M, N)$ sends a map f to the map $m \mapsto \tau^{-1}(f(\tau m))$.*

Proof. The proof is entirely formal and in fact the lemma holds for an arbitrary τ-ring k.

(1) Let $\tau^a \colon \tau^* N \to N$ be the adjoint of the multiplication map $\tau \colon N \to \tau_* N$. Observe that the square

$$\begin{tikzcd} \operatorname{Hom}_{A\otimes k}(M,N) \arrow[r, "f\mapsto \tau\circ f"] \arrow[d, "f\mapsto \tau^*(f)"'] & \operatorname{Hom}_{A\otimes k}(M,\tau_* N) \arrow[d, "\wr", "\text{adjunction}"'] \\ \operatorname{Hom}_{A\otimes k}(\tau^* M,\tau^* N) \arrow[r, "f\mapsto \tau^a\circ f"] & \operatorname{Hom}_{A\otimes k}(\tau^* M, N) \end{tikzcd}$$

is commutative. The top arrow is bijective since $\tau \colon N \to N$ is an automorphism by assumption. So the composition of the left and bottom arrows is bijective. However Lemma 1.12.2 shows that this composition is precisely the arrow j of $\mathcal{H}\mathrm{om}(M,N)$.

(2) Let $f \in \operatorname{Hom}_{A\otimes k}(M,N)$ and let f^τ be the map $m \mapsto \tau^{-1} f(\tau m)$. Lemma 7.1 shows that the shtuka $\mathcal{H}\mathrm{om}(M,N)$ is represented by the diagram

$$\operatorname{Hom}_{A\otimes k}(M,N) \underset{j}{\overset{i}{\rightrightarrows}} \operatorname{Hom}_{A\otimes k}(M^{\geqslant 1},N)$$

where i is the restriction to $M^{\geqslant 1}$ and j sends an element g to the map $\tau m \mapsto \tau g(m)$. So $j(f^\tau) = i(f)$ and we get the result. ☐

Let $\overline{k}$ be an algebraic closure of k and let $G_k = \operatorname{Aut}(\overline{k}/k)$ be the Galois group of k. We equip the field $\overline{k}$ with the discrete topology. Fix a maximal ideal $\mathfrak{p}$ of A. We denote $A_\mathfrak{p}$ the completion of A at $\mathfrak{p}$ and $F_\mathfrak{p}$ the fraction field of $A_\mathfrak{p}$. Set $\omega_\mathfrak{p} = \omega \otimes_A A_\mathfrak{p}$. The ring $A_\mathfrak{p}$, the field $F_\mathfrak{p}$ and the module $\omega_\mathfrak{p}$ are assumed to carry the $\mathfrak{p}$-adic topologies. We next study the $A_\mathfrak{p} \mathbin{\widehat{\otimes}} \overline{k}$-module $\omega_\mathfrak{p} \mathbin{\widehat{\otimes}} \overline{k}$.

Lemma 11.2. *The natural map $\omega \otimes_A (A_\mathfrak{p} \mathbin{\widehat{\otimes}} \overline{k}) \to \omega_\mathfrak{p} \mathbin{\widehat{\otimes}} \overline{k}$ is an isomorphism of $A_\mathfrak{p} \mathbin{\widehat{\otimes}} \overline{k}$-modules.* ☐

Lemma 11.3. *Let $\operatorname{res}_\mathfrak{p} \colon \omega \otimes_A F_\mathfrak{p} \to \mathbb{F}_q$ be the residue map at $\mathfrak{p}$. The map*

$$\omega_\mathfrak{p} \otimes_{\mathrm{c}} \overline{k} \to c(F_\mathfrak{p}/A_\mathfrak{p}, \overline{k}), \quad \eta \otimes \alpha \mapsto (x \mapsto \operatorname{res}_\mathfrak{p}(x\eta)\cdot\alpha)$$

extends to a unique topological isomorphism $\omega_\mathfrak{p} \mathbin{\widehat{\otimes}} \overline{k} \xrightarrow{\sim} c(F_\mathfrak{p}/A_\mathfrak{p}, \overline{k})$. This isomorphism is $A_\mathfrak{p}$-linear and natural in $\overline{k}$.

Proof. According to Theorem 3.10.1 the residue pairing identifies $\omega_\mathfrak{p}$ with $(F_\mathfrak{p}/A_\mathfrak{p})^*$, the continuous $\mathbb{F}_q$-linear dual of $F_\mathfrak{p}/A_\mathfrak{p}$. Next, Proposition 2.8.5 shows that the natural map $(F_\mathfrak{p}/A_\mathfrak{p})^* \otimes_{\mathrm{c}} \overline{k} \to c(F_\mathfrak{p}/A_\mathfrak{p}, \overline{k})$ extends to a unique topological isomorphism $(F_\mathfrak{p}/A_\mathfrak{p})^* \mathbin{\widehat{\otimes}} \overline{k} \xrightarrow{\sim} c(F_\mathfrak{p}/A_\mathfrak{p}, \overline{k})$. This isomorphism is natural with respect to continuous $\mathbb{F}_q$-linear endomorphisms of $F_\mathfrak{p}/A_\mathfrak{p}$ and $\overline{k}$. In particular it is $A_\mathfrak{p}$-linear. ☐

In the following we equip $\omega_{\mathfrak{p}} \,\widehat{\otimes}\, \overline{k}$ with an endomorphism τ given by the q-Frobenius of $\overline{k}$. So $\omega_{\mathfrak{p}} \,\widehat{\otimes}\, \overline{k}$ becomes a left $A_{\mathfrak{p}} \,\widehat{\otimes}\, \overline{k}\{\tau\}$-module.

As $\mathrm{Lie}_E(k)$ is a one-dimensional k-vector space the action of A on $\mathrm{Lie}_E(k)$ determines a homomorphism $A \to k$. The kernel of this homomorphism is called the *characteristic* of E.

Lemma 11.4. *If $\mathfrak{p}$ is different from the characteristic of E then*

$$\mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}(M,\, \omega_{\mathfrak{p}} \,\widehat{\otimes}\, \overline{k})) = 0.$$

Proof. Due to Lemma 11.2 it is enough to prove that the i-arrow of the shtuka $\mathcal{H}\mathrm{om}(M, \omega_{\mathfrak{p}} \otimes k)$ is an isomorphism. Observe that

$$\mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}(M,\, \omega_{\mathfrak{p}} \otimes k)) = \mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}(M,\, \omega \otimes k)) \otimes_A A_{\mathfrak{p}}.$$

Now $\mathrm{R}\Gamma(\nabla\, \mathcal{H}\mathrm{om}(M, \omega \otimes k)) = \mathrm{Lie}_E(k)[-1]$ by Theorem 9.3. The assumption on the characteristic of E means that $\mathrm{Lie}_E(k) \otimes_A A_{\mathfrak{p}} = 0$. Whence the result. □

We will also need an elementary lemma from Dieudonné-Manin theory.

Lemma 11.5. *Let N be a left $A_{\mathfrak{p}} \,\widehat{\otimes}\, \overline{k}\{\tau\}$-module which is finitely generated free as an $A_{\mathfrak{p}} \,\widehat{\otimes}\, \overline{k}$-module. If $\tau\colon N \to N$ is bijective then the natural map $N^{\tau=1} \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \,\widehat{\otimes}\, \overline{k}) \to N$ is an isomorphism.*

Proof. See [21, Proposition 4.4]. □

The remaining results of this section are essentially due to David Goss [10, Section 5.6]. He formulated them in terms of the motive M rather than the Hom shtuka. We give a different argument which uses the Hom shtuka approach.

Proposition 11.6. *There exists a G_k-equivariant $A_{\mathfrak{p}}$-module isomorphism*

$$\mathrm{Hom}_{A\otimes k\{\tau\}}(M,\, \omega_{\mathfrak{p}} \,\widehat{\otimes}\, \overline{k}) \cong T_{\mathfrak{p}}E.$$

The Galois group G_k acts on the left hand side via $\overline{k}$.

Proof. First we construct a G_k-equivariant (A, A)-bimodule isomorphism

$$\mathrm{Hom}_{k\{\tau\}}(M,\, \omega_{\mathfrak{p}} \,\widehat{\otimes}\, \overline{k}) \cong c(F_{\mathfrak{p}}/A_{\mathfrak{p}},\, E(\overline{k})). \tag{11.1}$$

By Proposition 7.1.6 the natural map $k\{\tau\} \otimes_k M^0 \to M$ is an isomorphism. As a consequence the map

$$c(F_{\mathfrak{p}}/A_{\mathfrak{p}},\, E(\overline{k})) \to \mathrm{Hom}_{k\{\tau\}}(M,\, c(F_{\mathfrak{p}}/A_{\mathfrak{p}}, \overline{k})), \quad f \mapsto (m \mapsto m \circ f)$$

is an isomorphism. Lemma 11.3 identifies $\omega_{\mathfrak{p}} \,\widehat{\otimes}\, \overline{k}$ with $c(F_{\mathfrak{p}}/A_{\mathfrak{p}}, \overline{k})$. Combining it with the isomorphism above we get (11.1).

Now (11.1) identifies $\mathrm{Hom}_{A\otimes k\{\tau\}}(M,\, \omega_{\mathfrak{p}} \,\widehat{\otimes}\, \overline{k})$ with the $A_{\mathfrak{p}}$-module of continuous A-linear maps from $F_{\mathfrak{p}}/A_{\mathfrak{p}}$ to $E(\overline{k})$. As $F_{\mathfrak{p}}/A_{\mathfrak{p}}$ is discrete it follows that the latter module is

$$\mathrm{Hom}_A(F_{\mathfrak{p}}/A_{\mathfrak{p}},\, E(\overline{k})) = T_{\mathfrak{p}}E.$$

We get the result since all the isomorphisms used above are G_k-equivariant by construction. $\square$

If $\mathfrak{p}$ is different from the characteristic of E then one can prove that

$$\mathrm{RHom}_{A\otimes k\{\tau\}}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k}) \cong T_{\mathfrak{p}}E[0].$$

We will only need the less precise result of Proposition 11.6.

Proposition 11.7. *If $\mathfrak{p}$ is different from the characteristic of E then the natural map*

$$\mathrm{Hom}_{A\otimes k\{\tau\}}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k}) \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k}) \to \mathrm{Hom}_{A\otimes k}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k})$$

is an isomorphism.

Proof. Consider the shtuka

$$\mathcal{H}\mathrm{om}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k}) = \Big[\mathrm{Hom}_{A\otimes k}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k}) \underset{j}{\overset{i}{\rightrightarrows}} \mathrm{Hom}_{A\otimes k}(\tau^* M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k})\Big].$$

Lemma 11.4 implies that the arrow i is bijective. Let us equip the $A_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k}$-module

$$H = \mathrm{Hom}_{A\otimes k}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k})$$

with the τ-linear endomorphism $i^{-1}j$.

The arrow j of the shtuka $\mathcal{H}\mathrm{om}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k})$ is a bijection by Lemma 11.1. Hence the endomorphism $i^{-1}j$ of H is in fact an automorphism. Furthermore H is a finitely generated free $A_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k}$-module since $\omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k}$ is a free $A_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k}$-module of rank 1. Now Lemma 11.5 shows that the natural map

$$H^{i^{-1}j=1} \otimes_{A_{\mathfrak{p}}} (A_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k}) \to H$$

is an isomorphism. It remains to describe the $A_{\mathfrak{p}}$-submodule $H^{i^{-1}j=1}$ of H.

Let $f \in \mathrm{Hom}_{A\otimes k}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k})$ be a map. Lemma 11.1 shows that $j^{-1}i(f)$ is the map $m \mapsto \tau^{-1}(f(\tau m))$. Hence $j^{-1}i(f) = f$ if and only if f commutes with multiplication by τ. So

$$H^{i^{-1}j=1} = H^{j^{-1}i=1} = \mathrm{Hom}_{A\otimes k\{\tau\}}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k})$$

and we get the result. $\square$

Proposition 11.8. *Assume that k is a finite extension of $\mathbb{F}_q$ of degree d. The shtuka*

$$\mathcal{H}\mathrm{om}(M, \omega \otimes k) = \Big[\mathrm{Hom}_{A\otimes k}(M,\, \omega \otimes k) \underset{j}{\overset{i}{\rightrightarrows}} \mathrm{Hom}_{A\otimes k}(\tau^* M,\, \omega \otimes k)\Big]$$

has the following properties.

(1) *The arrow j is a bijection.*
(2) *Let $\sigma \in G_k$ be the arithmetic Frobenius element. For every prime $\mathfrak{p}$ of A different from the characteristic of E we have an identity*

$$\det_{A\otimes k}\big(T - (j^{-1}i)^d \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \omega \otimes k)\big) = \det_{A_{\mathfrak{p}}}\big(T - \sigma \,\big|\, T_{\mathfrak{p}}E\big)$$

in $A_{\mathfrak{p}} \otimes k[T]$.

Proof. (1) follows immediately from Lemma 11.1. (2) Given an $A \otimes k$-module N we will denote ψ the endomorphism of $\mathrm{Hom}_{A\otimes k}(M, N)$ which sends a map f to the map $m \mapsto f(\tau^d m)$.

Proposition 11.6 identifies $T_{\mathfrak{p}}E$ and $\mathrm{Hom}_{A\otimes k\{\tau\}}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k})$ on which G_k acts via $\overline{k}$. So if f is an element of the latter module then $\sigma(f) = \tau^d \circ f$. As f is a homomorphism of left $A \otimes k\{\tau\}$-modules it follows that $\tau^d \circ f = \psi(f)$. We conclude that

$$\det\nolimits_{A_{\mathfrak{p}}}\big(T - \sigma \,\big|\, T_{\mathfrak{p}}E\big) = \det\nolimits_{A_{\mathfrak{p}}}\big(T - \psi \,\big|\, \mathrm{Hom}_{A\otimes k\{\tau\}}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k})\big).$$

Next, Proposition 11.7 implies that

$$\det\nolimits_{A_{\mathfrak{p}}}\big(T - \psi \,\big|\, \mathrm{Hom}_{A\otimes k\{\tau\}}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k})\big) = \\ \det\nolimits_{A_{\mathfrak{p}}\widehat{\otimes}\overline{k}}\big(T - \psi \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \omega_{\mathfrak{p}} \mathbin{\widehat{\otimes}} \overline{k})\big).$$

It follows from Lemma 11.2 that the natural map $(\omega\otimes k)\otimes_{A\otimes k}(A_{\mathfrak{p}}\widehat{\otimes}\overline{k}) \to \omega_{\mathfrak{p}}\widehat{\otimes}\overline{k}$ is an isomorphism. Hence

$$\det\nolimits_{A_{\mathfrak{p}}\widehat{\otimes}\overline{k}}\big(T-\psi \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \omega_{\mathfrak{p}}\widehat{\otimes}\overline{k})\big) = \det\nolimits_{A\otimes k}\big(T-\psi \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \omega\otimes k)\big).$$

Now Lemma 11.1 shows that the endomorphism $j^{-1}i$ of $\mathrm{Hom}_{A\otimes k}(M,\, \omega \otimes k)$ sends a map f to the map $m \mapsto \tau^{-1}(f(\tau m))$. As τ^{-d} is the identity automorphism of k we conclude that $(j^{-1}i)^d(f) = \psi(f)$ and the result follows. $\square$

Let F_0 be the fraction field of A and let $\Omega = \omega \otimes_A F_0$ be the module of Kähler differentials of F_0 over $\mathbb{F}_q$. The main result of this section is the following theorem.

Theorem 11.9. *Assume that k is a finite extension of $\mathbb{F}_q$ of degree d. The $F_0 \otimes k$-module shtuka*

$$\mathcal{H}\mathrm{om}(M,\, \Omega \otimes k) = \Big[\mathrm{Hom}_{A\otimes k}(M,\, \Omega \otimes k) \underset{j}{\overset{i}{\rightrightarrows}} \mathrm{Hom}_{A\otimes k}(\tau^* M,\, \Omega \otimes k)\Big]$$

has the following properties.

(1) *The arrow i is an isomorphism.*
(2) *Let $\sigma^{-1} \in G_k$ be the geometric Frobenius element. For every prime $\mathfrak{p}$ of A different from the characteristic of E we have an identity*

$$\det\nolimits_{F_0}\big(1 - T(i^{-1}j) \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \Omega \otimes k)\big) = \det\nolimits_{A_{\mathfrak{p}}}\big(1 - T^d\sigma^{-1} \,\big|\, T_{\mathfrak{p}}E\big)$$

in $F_{\mathfrak{p}}[T]$.

In particular both characteristic polynomials have coefficients in the ring $A_{\mathfrak{p}} \cap F_0 = A_{(\mathfrak{p})}$.

Proof of Theorem 11.9. (1) follows by the same argument as Lemma 11.4. (2) We have

$$\det\nolimits_{A_{\mathfrak{p}}}\big(1 - T^d\sigma^{-1} \,\big|\, T_{\mathfrak{p}}E\big) = \frac{\det_{A_{\mathfrak{p}}}\big(T^d - \sigma \,\big|\, T_{\mathfrak{p}}E\big)}{\det_{A_{\mathfrak{p}}}\big(-\sigma \,\big|\, T_{\mathfrak{p}}E\big)}.$$

Proposition 11.8 implies that

$$\frac{\det_{A_{\mathfrak{p}}}\big(T^d - \sigma \,\big|\, T_{\mathfrak{p}}E\big)}{\det_{A_{\mathfrak{p}}}\big(-\sigma \,\big|\, T_{\mathfrak{p}}E\big)} = \frac{\det_{A\otimes k}\big(T^d - (j^{-1}i)^d \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \omega\otimes k)\big)}{\det_{A\otimes k}\big(-(j^{-1}i)^d \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \omega\otimes k)\big)}.$$

The right hand side of this identity can be rewritten as

$$\det_{F_0\otimes k}\big(1 - T^d(i^{-1}j)^d \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \Omega\otimes k)\big).$$

Now [3, Lemma 8.1.4] shows that this polynomial coincides with

$$\det_{F_0}\big(1 - T(i^{-1}j) \,\big|\, \mathrm{Hom}_{A\otimes k}(M,\, \Omega\otimes k)\big).$$

and the result follows. □